**Nanopteron-Stegoton Traveling Waves in Mass and Spring Dimer Fermi-Pasta-Ulam-Tsingou Lattices**

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by

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## Dedications

To Mom and Dad, with all my love and thanks.

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## Abstract

Nanopteron-Stegoton Traveling Waves in Mass and Spring Dimer
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We study the existence of traveling waves in mass and spring dimer Fermi-Pasta-Ulam-Tsingou (FPUT) lattices. These are infinite, one-dimensional lattices of particles connected by nonlinear springs, in which either the masses alternate (the mass dimer or diatomic lattice) or the spring forces alternate (the spring dimer). Under the classical "long wave" scaling, the lattice equations of motion turn out to be singularly perturbed. In response to this complication, we apply a method of Beale to produce nanopteron traveling wave solutions with wave speed slightly greater than the lattice's speed of sound. The nanopteron wave profiles are the superposition of an exponentially decaying term (which itself is a small perturbation of a KdV $\text{sech}^2$-type soliton) and a periodic term of very small amplitude.

This dissertation builds on the previous work of Faver and Wright on mass dimer lattices to treat spring dimer lattices. Further generalizing the spring forces from the mass dimer case, we allow the springs' nonlinearity to contain higher order terms beyond the quadratic. This necessitates the use of composition operators to phrase the long wave problem, and these operators require delicate estimates due to the characteristic superposition of different function types from Beale's ansatz. Additionally, the value of the leading order term in the spring dimer traveling wave profiles alternates between particle sites, so that, unlike in the mass dimer, the spring dimer traveling waves are also "stegotons."

# Chapter 1. Introduction

**1.1. The Fermi-Pasta-Ulam-Tsingou lattice.** We assemble a one-dimensional lattice by placing infinitely many particles on a horizontal line and connecting each particle to the particles on its immediate left and right by springs. (That is, we assume that the only forces in the lattice will be due to "nearest-neighbor" interactions.) Such a construct is a Fermi-Pasta-Ulam-Tsingou (FPUT) lattice [FPU55].

Suppose that we index the particles by integers $j \in \mathbb{Z}$ and denote the mass of the $j$th particle by $m_j > 0$. We also label the springs by the same integers $j$ so that the $j$th spring connects the $j$th particle to the $(j+1)$st particle. We assume that the $j$th spring has length $\ell_j > 0$ when the lattice is at rest and that this spring exerts the force $F_j(r)$ when stretched a distance $r$ from this equilibrium length. Finally, let $u_j(t)$ be the position of the $j$th particle at time $t$. Newton's second law then implies that $u_j$ satisfies

$$m_j \frac{d^2 u_j}{dt^2} = F_j(r_j - \ell_j) - F_{j-1}(r_{j-1} - \ell_{j-1}), \qquad r_j := u_{j+1} - u_j. \tag{1.1.1}$$

Depending on the material properties that we ascribe to the lattice — how we choose the values of the masses $m_j$ and how we define the force functions $F_j$ — we can vary the lattice's behavior considerably. Our way of proceeding is to assume that these material properties vary periodically: there is an integer $N \geq 1$ such that $m_{j+N} = m_j$ and $F_{j+N} = F_j$ for all $j$. This is a *polyatomic* or *polymer* lattice. Special cases are the *monatomic* lattice, in which $N = 1$ and all the masses and springs are identical, and the *dimer*, in which $N = 2$ and the masses and springs alternate; Figure 1.1.1 contains a sketch of this lattice.

This dissertation discusses the existence and properties of certain kinds of traveling waves in the distinct cases of the mass and spring dimer lattices, which are

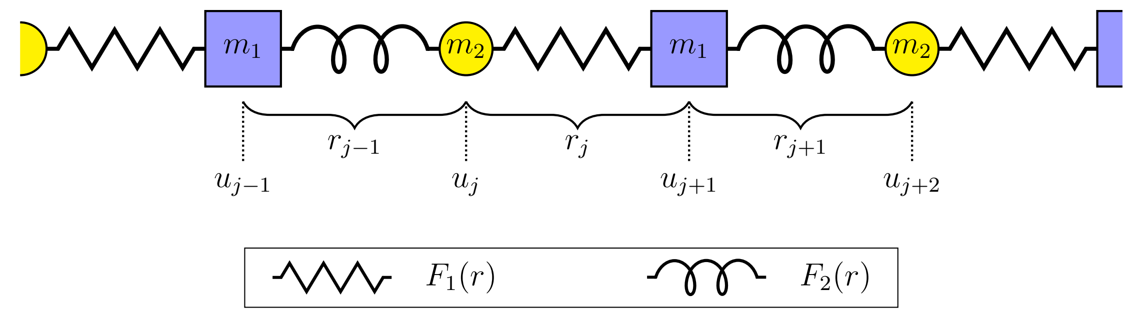


Figure 1.1.1: The general dimer

the simplest nontrivial generalizations of the well-understood monatomic lattice. In the mass dimer, which the mass values alternate but the spring forces are identical (Figure 1.4.2), and the spring dimer, in which the springs alternate but the masses are constant (Figure 1.4.1). The main result of this dissertation, stated somewhat informally in Theorem 1.1.1 below and more precisely in Theorems 1.4.1 and 1.4.2 is that the mass and spring dimer lattices possess *nanopteron* traveling waves, which are traveling waves whose profile is the superposition of an exponentially decaying function (the "core") and a small-amplitude periodic function (the "ripple") [Boy98].

Boyd introduced this terminology in [Boy90] and provides a rich, extensive discussion of nanopterons in mathematics and nature in [Boy98]. The word "nanopteron" emerges from the Greek for "dwarf-wing," [Boy89] the "wings" of the wave being the periodic ripples, which are extremely small compared to the core (the "body") of the wave. The nanopteron need not decay to zero at infinity but can instead asymptote to a nonvanishing oscillation (Figure 1.1.2). This differs from the well-established results for the monatomic lattice, which has *solitary* traveling waves that necessarily decay exponentially fast to zero at infinity [FW94], [FP99].

**1.1.1 Theorem.** *Under suitable conditions on the spring forces $F_j$, for each mass dimer and spring dimer lattice there exists a lower bound $c_s$ (the "speed of sound"), which depends on the lattice, such that for wave speeds $c$ slightly greater than $c_s$, there are traveling wave solutions for* (1.1.1) *of the form*

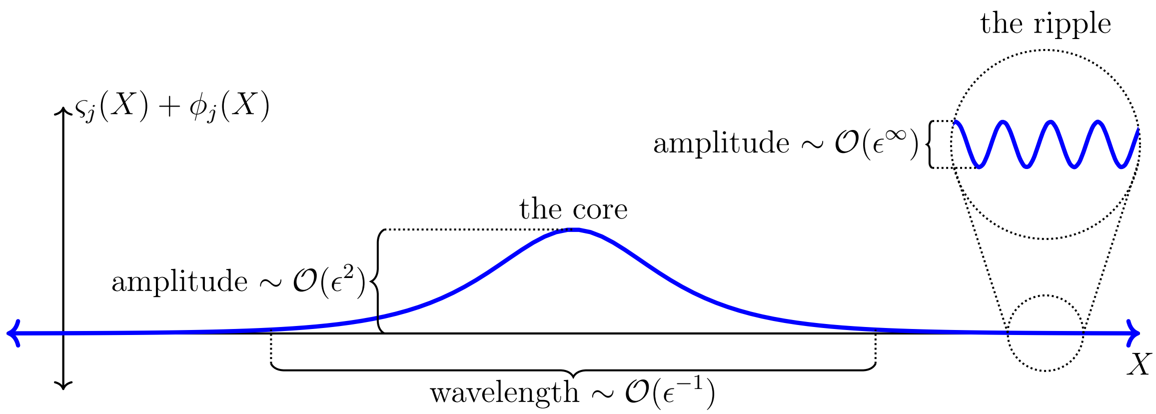


Figure 1.1.2: The nanopteron ($\mathcal{O}(\epsilon^\infty)$ = "small beyond all orders of $\epsilon$")

$$r_j(t) = \varsigma_j(j - ct) + \phi_j(j - ct).$$

*Both* $\varsigma_j$ *and* $\phi_j$ *are smooth and satisfy* $\varsigma_{j+2}(X) = \varsigma_j(X)$ *and* $\phi_{j+2}(X) = \phi_j(X)$ *for all* $X \in \mathbb{R}$. *Moreover,* $\varsigma_j$ *is exponentially decaying and has amplitude proportional to* $\epsilon^2 := c^2 - c_s^2$ *and wavelength proportional to* $\epsilon^{-1}$; $\phi_j$ *is periodic with amplitude small beyond all orders of* $\epsilon$ *and period* $\mathcal{O}(1)$ *in* $\epsilon$.

**1.2. History and applications of lattices.** Although the modern form of the FPUT lattice emerged with the advent of numerical computing in the 1950s, it appears, under various guises, in many historical and contemporary applications. Newton discretized air as a lattice of particles connected by springs to study the propagation of sound waves, and Cauchy did the same to study light waves [Bri53]; today, lattices model such diverse phenomena as DNA strand dynamics, electrical circuits, Bose-Einstein condensates, and the nonlinear dynamics of granular crystals and metamaterials [Kev11], [CPKD16].

The 1955 numerical study of Fermi, Pasta, Ulam [FPU55], and Tsingou [Dau08] reported unexpected behavior in the energy of a finite lattice of identical particles and nonlinear springs: energy did not "thermalize" or equidistribute among the modes of the system but instead exhibited "recurrence" and returned periodically to its initial conditions. Ten years later, a numerical report by Zabusky and Kruskal [ZK65] established the first link between the FPUT lattice and and the Korteweg-de Vries equation, given in (1.2.1). As outlined in their report and explored in more detail by Zabusky and Kruskal separately in [Zab62, Kru65, Kru74], let $u_j(t)$ be the position of the $j$th particle at time $t$, and consider the lattice as the discretization of a continuous string, so that $u_j(t) = u(jh, t)$. Here $h$ is the space between successive particles in the lattice and $u = u(x, t)$ is a function of space and time. Taylor-expanding the lattice equations (1.1.1) in $h$, making a few deft changes of variables, and ignoring most higher-order terms suggests that, at a highly formal level, $u$ will satisfy[1] the partial differential equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2.1}$$

This is the Korteweg-de Vries (KdV) equation, which is well-known as a model of small-amplitude water waves whose length is long compared to the depth of the water. It possesses traveling wave solutions of the form

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right), \tag{1.2.2}$$

with $c \in \mathbb{R}$ fixed. These are solitary waves in the classical sense of the ones first observed by Scott Russell [Rus44] in a shallow canal: they possess only one extremum and decay rapidly to zero at spatial infinity. Since the Taylor expansion above arises from thinking of the lattice as a continuous string, the resulting KdV equation is often called the "continuum limit" of the lattice.

[1] "If this story seems confusing, that is because it is." So claims Truesdall [Tru84] in an intriguing philosophical and historical reflection on the KdV limit. The historical sketch offered here is indeed a considerable simplification of a complicated derivation that only came to fully rigorous fruition in the results of Friesecke and Pego; in particular, see their remarks in [FP99] after Theorem 1.1. See also the remarks of Friesecke and Mikikits-Leitner [FML15] and James and Pelinovsky [JP14] for additional perspectives on the continuum limit with the long-wave scaling. Ablowitz [Abl11] offers a modern, systematic treatment of the transition from FPUT to KdV, and Porter et al. [PZHC09] describe the FPUT-KdV connection from a physics-grounded viewpoint that is accessible to the nonspecialist.

Much subsequent analysis has provided rigorous confirmation that solutions to the KdV equation are indeed valid approximations to the equations of motion for infinite lattices. Usually this work is done at the level of relative displacements, rather than position: set

$$r_j(t) := u_{j+1}(t) - u_j(t), \tag{1.2.3}$$

so that $r_j$ measures the distance between sucessive particles, and the equations of motion (1.1.1) become[2]

$$\ddot{r}_j = \frac{1}{m_{j+1}} F_{j+1}(r_{j+1}) - \left(\frac{1}{m_j} + \frac{1}{m_{j+1}}\right) F_j(r_j) + \frac{1}{m_j} F_{j-1}(r_{j-1}). \tag{1.2.4}$$

There are many possible expressions for the forces $F_j$; we will take them to be smooth functions with the Taylor expansions

$$F_j(r) = \varkappa_j r + \beta_j r^2 + \mathcal{O}(r^3), \ \ \varkappa_j > 0. \tag{1.2.5}$$

For the monatomic lattice ($m_j = m_{j+1}$ and $F_j = F_{j+1}$ for all $j$), Schneider and Wayne [SW00a] prove the estimate

$$\sup_{0 \le t \le \epsilon^3} \sup_{j \in \mathbb{Z}} \left| r_j(t) - \left(\epsilon^2 f(\epsilon(j - c_s t), \epsilon^3 t) + \epsilon^2 g(\epsilon(j + c_s t), \epsilon^3 t)\right) \right| = \mathcal{O}(\epsilon^{7/2}). \tag{1.2.6}$$

Here $f$ and $g$ solve a pair of KdV equations whose coefficients depend on the spring force of the lattice and $c_s = \sqrt{F'(0)}$ is the "speed of sound"[3] for the lattice. The scaling of $f$ and $g$ by $\epsilon$ in (1.2.6), which originates in a perturbation ansatz for the water wave problem [SW00b], is the "long wave scaling." This has consistently proved to be a highly useful change of variables.

This approximation result begs the following question: since the KdV equation has solitary wave solutions, and since KdV solutions are good approximations to traveling

[2]We may assume that the equilibrium spring lengths $\ell_j$ are all zero; see the nondimensionalization in (1.3.2).

[3]That is, $c_s$ is the maximum speed $\omega$ of any plane wave solution $r_j(t) = e^{i(kj - \omega t)}$ to the linearization of (1.2.4).

waves for monatomic FPUT lattices, do there exist solitary traveling wave solutions for the monatomic lattice? That is, can one find a function $p = p(x)$ that decays to 0 as $x \to \pm\infty$ and that satisfies $r_j(t) = p(j - ct)$ for all times $t$? This was first answered affirmatively by Friesecke and Wattis [FW94] using variational techniques.

The series of papers [FP99, FP02, FP04a, FP04b] by Friesecke and Pego offers a different approach to the problem of solitary traveling waves, which has directly motivated this research on dimer lattices. In [FP99], Friesecke and Pego prove the existence of traveling wave solutions for (1.2.4) of the form

$$r_j(t) = \epsilon^2 v_\epsilon(\epsilon(j - c_\epsilon t)). \tag{1.2.7}$$

The wave speed $c_\epsilon$ is "near-sonic" (more precisely, "supersonic") in that $c = c_s + \mathcal{O}(\epsilon^2)$. Evaluating the lattice equations (1.2.4) at the ansatz (1.2.7) and Taylor expanding in $\epsilon$ suggests that $v_\epsilon$ satisfies a differential equation that gives the traveling wave profiles for a KdV equation whose coefficients depend on constants in the lattice equation. Friesecke and Pego then rigorously reinterpret the problem in a functional analytic setting and use a quantitative inverse function theorem argument to prove that the profile $v_\epsilon$ satisfies the estimate

$$\|v_\epsilon - \sigma\|_{H^1(\mathbb{R})} = \mathcal{O}(\epsilon^2),$$

where $\sigma$ is the exact $\text{sech}^2$-type solution to this KdV equation.

All of this historical narration, so far, has concerned monatomic lattices. However, polymer lattices arise naturally and frequently in applications; for example, lattice models of DNA [Kev11], electrical lines [Bri53], and granular metamaterials [CPKD16] all require material heterogeneities. There are also substantial mathematical results on the dynamics of polymer lattices. Pankov [Pan05] conducts a thorough overview of the formulation of the general lattice equations (1.1.1) as a Hamiltonian system and a discussion of the related Cauchy problem in $\ell^2(\mathbb{Z})$ under even more general conditions on the masses and springs than the polymer set-up above. There are myriad contemporary studies of traveling waves in mass dimers, including [BP13] on the existence of periodic waves in granular crystals with a nonsmooth potential as the mass ratio $m_1/m_2$ tends to 0; [CBCPS12], which uses Bloch wave transforms to obtain KdV approximations; and [Qin15], which uses variational methods to construct periodic solutions for lattices whose forces contain linear and cubic, but not quadratic, terms. And using methods from homogenization theory, Gaison, Moskow, Wright, and Zhang [GMWZ14] have generalized the KdV approximation of monatomic lattices by Schneider and Wayne to polymer lattices. A natural question, then, is if there exist solitary traveling waves in polymer lattices, as Friesecke and Pego found for monatomic lattices. As discussed in Theorem 1.1.1, we do find traveling waves, but their nanopteron structure prevents them from necessarily being solitary waves.

**1.3. Equations of motion for the dimers.** We specialize the equations of motion (1.1.1) to the dimer case. For now, we allow both masses and spring forces to vary; in Section 1.4, we summarize results separately for mass and spring dimers; in Section 2.7, we explain why we specialize separately to mass and spring dimers; and beginning in Chapter 3, we will treat solely the spring dimer, the results for the mass dimer being contained in [FW18].

Our lattice consists of (potentially) alternating masses and spring forces indexed by integers $j \in \mathbb{Z}$. The $j$th mass is

$$m_j = \begin{cases} m_1 > 0, & j \text{ is odd} \\ m_2 > 0, & j \text{ is even;} \end{cases}$$

the $j$th spring length at equilibrium is

$$\ell_j = \begin{cases} \ell_1 > 0, & j \text{ is odd} \\ \ell_2 > 0, & j \text{ is even;} \end{cases}$$

and the $j$th spring exerts a force $F_j(r)$ when stretched a length $r$ from its equilibrium length, where

$$F_j(r) = \begin{cases} \varkappa_1 r + \beta_1 r^2 + r^3\overline{N}_1(r), & j \text{ is odd} \\ \varkappa_2 r + \beta_2 r^2 + r^3\overline{N}_2(r), & j \text{ is even.} \end{cases}$$

Here $\varkappa_1$, $\varkappa_2 > 0$ and we require at least one of $\beta_1$ and $\beta_2$ to be nonzero. We assume

$\overline{N}_j \in \mathcal{C}^\infty(\mathbb{R})$. Then with $\overline{u}_j(\overline{t})$ as the position of the $j$th particle at time $\overline{t}$, (1.1.1) becomes

$$\begin{cases} m_1 \dfrac{d^2\overline{u}_{2j+1}}{d\overline{t}^2} = F_1(\overline{u}_{2j+2} - \overline{u}_{2j+1} - \ell_1) - F_2(\overline{u}_{2j+1} - \overline{u}_{2j} - \ell_2) \\ \\ m_2 \dfrac{d^2\overline{u}_{2j}}{d\overline{t}^2} = F_2(\overline{u}_{2j+1} - \overline{u}_{2j} - \ell_2) - F_1(\overline{u}_{2j} - \overline{u}_{2j-1} - \ell_1). \end{cases} \tag{1.3.1}$$

We nondimensionalize this system in several ways. First, we can eliminate the spring lengths $\ell_1$ and $\ell_2$ by writing

$$\overline{u}_{2j+1} = \breve{u}_{2j+1} + j\ell_1 + j\ell_2 \quad \text{and} \quad \overline{u}_{2j} = \breve{u}_{2j} + j\ell_1 + (j-1)\ell_2. \tag{1.3.2}$$

Then

$$\overline{u}_{2j+1} - \overline{u}_{2j} - \ell_2 = \breve{u}_{2j+1} - \breve{u}_{2j} \quad \text{and} \quad \overline{u}_{2j} - \overline{u}_{2j-1} - \ell_1 = \breve{u}_{2j} - \breve{u}_{2j-1},$$

and clearly

$$\frac{d^2\overline{u}_j}{d\overline{t}^2} = \frac{d^2\breve{u}_j}{d\overline{t}^2}$$

for all $j$.

We next express the system in terms of relative displacements by setting

$$\breve{r}_j := \breve{u}_{j+1} - \breve{u}_j.$$

Then (1.3.1) becomes

$$\begin{cases} m_1 \dfrac{d^2\breve{u}_{2j+1}}{d\overline{t}^2} = F_1(\breve{r}_{2j+1}) - F_2(\breve{r}_{2j}) \\ \\ m_2 \dfrac{d^2\breve{u}_{2j}}{d\overline{t}^2} = F_2(\breve{r}_{2j}) - F_1(\breve{r}_{2j-1}). \end{cases} \tag{1.3.3}$$

Now we rescale the position functions $\breve{u}_j$. Let

$$\beta := \begin{cases} \beta_1/\beta_2, & \beta_2 \neq 0 \\ 0, & \beta_2 = 0 \end{cases} \quad \text{and} \quad a_1 := \sqrt{\frac{\varkappa_2}{m_1}}. \tag{1.3.4}$$

Suppose for the moment that $\beta_2 \neq 0$; we will address the (very similar) case $\beta_2 = 0$ in Appendix F.5. Then define

$$a_2 := \frac{\varkappa_2}{\beta_2} \tag{1.3.5}$$

and write

$$\breve{u}_j(\bar{t}) = a_2 u_j(a_1\bar{t}) \quad \text{and} \quad r_j := u_{j+1} - u_j, \tag{1.3.6}$$

where $u_j = u_j(t)$. After canceling some common factors and setting

$$w := \frac{m_1}{m_2}, \qquad \varkappa := \frac{\varkappa_1}{\varkappa_2}, \quad \text{and} \quad N_j(r) := \frac{a_2^2}{\varkappa_2}\overline{N}_j(a_1 r),\ j = 1,2,$$

we convert the system (1.3.3) to the nondimensionalized equations

$$\begin{cases} \ddot{u}_{2j+1} = \varkappa r_{2j+1} + \beta r_{2j+1}^2 + r_{2j+1}^3 N_1(r_{2j+1}) - r_{2j} - r_{2j}^2 - r_{2j}^3 N_2(r_{2j}) \\ \\ \dfrac{1}{w}\ddot{u}_{2j} = r_{2j} + r_{2j}^2 + r_{2j}^3 N_2(r_{2j}) - \varkappa r_{2j-1} - \beta r_{2j-1}^2 - r_{2j-1}^3 N_1(r_{2j-1}). \end{cases} \tag{1.3.7}$$

From these we can compute the equations of motion solely in terms of relative displacement:

$$\begin{cases} \ddot{r}_{2j+1} & = -(1+w)\left(\varkappa r_{2j+1} + \beta r_{2j+1}^2 + r_{2j+1}^3 N_1(r_{2j+1})\right) \\ & \quad +w\left((r_{2j+2} + r_{2j+2}^2 + r_{2j+2}^3 N_2(r_{2j+2})\right) \\ & \quad + \left(r_{2j} + r_{2j}^2 + r_{2j}^3 N_2(r_{2j})\right) \\ \\ \ddot{r}_{2j} & = -(1+w)\left(r_{2j} + r_{2j}^2 + r_{2j}^3 N_2(r_{2j})\right) + \left(\varkappa r_{2j+1} + \beta r_{2j+1}^2 + r_{2j+1}^3 N_1(r_{2j+1})\right) \\ & \quad +w\left(\varkappa r_{2j-1} + \beta r_{2j-1}^2 + r_{2j-1}^3 N_1(r_{2j-1})\right). \end{cases} \tag{1.3.8}$$

In Chapter 2 we will make a traveling wave ansatz on this system and proceed to construct its nanopteron solutions. In the next section, we state and discuss our main results.

**1.4. Main results.** We now translate the fixed point solutions of Theorem 1.1.1 and Beale's ansatz into the language of relative displacements for our lattice problem 1.3.8.

**1.4.1 Theorem (Existence of nanopterons in spring dimers).** *Let $w = 1$ and take $\varkappa > 1$ and $\beta \in \mathbb{R}$ to satisfy $\beta \neq -\varkappa^3$. There exist $\epsilon_\star$, $q_\star > 0$ such that for all $\epsilon \in$*

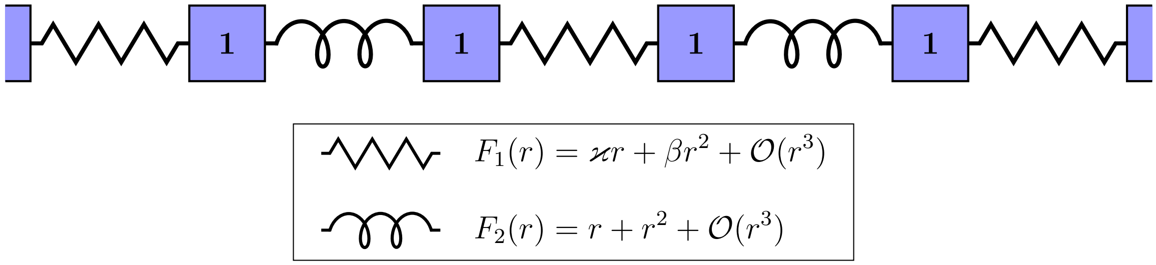

Figure 1.4.1: The spring dimer

$(0, \epsilon_\star)$, *there is a solution for the relative displacements* $r_j(t)$ *of the nondimensionalized lattice equations* (1.3.8) *in the form*

$$r_j(t) = \underbrace{\varkappa^{((-1)^j+1)/2} \frac{3\varkappa(\varkappa+1)}{c_\varkappa^2(\beta+\varkappa^3)} \epsilon^2 \operatorname{sech}^2\left(\frac{\epsilon(j-c_\epsilon t)}{2\sqrt{\alpha_\varkappa}}\right)}_{\textit{Note the extra factor of } \varkappa \textit{ for } j \textit{ even.}} + v_j^\epsilon(\epsilon(j-c_\epsilon t)) + p_j^\epsilon(j-c_\epsilon t), \tag{1.4.1}$$

*where* $c_\varkappa = c_\star(1,\varkappa)$ *from* (2.2.6), $\alpha_\varkappa = \alpha_\star(1,\varkappa)$ *from* (2.5.2), *and*

**(i)** $v_1^\epsilon,\ v_2^\epsilon \in \cap_{r=1}^\infty H_{q_\star}^r$.

**(ii)** $p_1^\epsilon,\ p_2^\epsilon \in \cap_{r=1}^\infty W^{r,\infty}$.

**(iii)** *For each* $r \geq 0$ *there is a constant* $C_r > 0$ *such that*

$$\left\|v_j^\epsilon\right\|_{H_{q_\star}^r} \leq C_r\epsilon^3 \quad \textit{and} \quad \left\|p_j^\epsilon\right\|_{W^{r,\infty}} \leq C_r\epsilon^r$$

*for all* $\epsilon \in (0,\epsilon_\star)$ *and* $j = 1,\ 2$.

**(iv)** $p_1^\epsilon$ *and* $p_2^\epsilon$ *are periodic with period* $P_\epsilon$, *and there is a constant* $C > 0$ *such that* $|P_\epsilon| \leq C$ *for all* $\epsilon \in (0,\epsilon_\star)$.

We prove this theorem in Appendix F.1 using the existence of periodic solutions from Theorem 3.1.1 and the existence of homoclinic connections to these solutions from Theorem 4.3.2. We discuss the case $\beta = \varkappa^3$ in Remark 3.1.2.

**1.4.2 Theorem (Existence of nanopterons in mass dimers).** *Let* $\varkappa = \beta = 1$

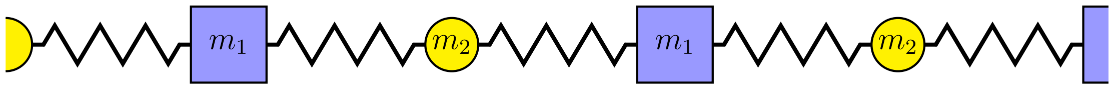


Figure 1.4.2: The mass dimer

*and take $w > 1$. There exist $\epsilon_\star$, $q_\star > 0$ such that for all $\epsilon \in (0, \epsilon_\star)$, there is a solution for the relative displacements $r_j(t)$ of the nondimensionalized lattice equations* (1.3.8) *in the form*

$$r_j(t) = \frac{3\epsilon^2}{2c_w^2} \operatorname{sech}^2\left(\frac{\epsilon}{2\sqrt{\alpha_w}}(j - c_\epsilon t)\right) + v_j^\epsilon(\epsilon(j - c_\epsilon t)) + p_j^\epsilon(j - c_\epsilon t), \qquad (1.4.2)$$

*where $c_w = c_\star(w, 1)$ from* (2.2.6)*, $\alpha_w = \alpha_\star(w, 1)$ from* (2.5.2)*, and $v_j^\epsilon$ and $p_j^\epsilon$ have the same properties as their spring dimer analogues in Theorem 1.4.1.*

This theorem was proved in [FW18] under the additional restriction $F_j(r) = r + r^2$ on the spring forces. The techniques in this dissertation developed for higher-order terms in the spring dimer's forces carry over easily to more complicated forces $F_j(r) = r + r^2 + \mathcal{O}(r^3)$ in the mass dimer.

**1.5. Stegotons.** We note that in the expression (1.4.1) for relative displacement, the leading order term in $\epsilon$ differs by a factor of $\varkappa$ depending on whether $j$ is even or odd. This is a feature not present in the relative displacements for the mass dimers, per (1.4.2), and if we fix time $t$ and graph the leading order terms in $r_j(t)$ over a range of lattice indices $j \in \mathbb{Z}$, it leads to the ridged graphs evocatively called "stegotons"[4] in [LY03a, LY03b]. We sketch these graphs in Figure 1.5.1 (with time fixed at $t = 0$, $w = 2$ for the mass dimer, and $\varkappa = 2$ for the spring dimer). These sketches suggest that successive pairs of particles in the mass dimer are roughly the same distance apart ($r_j(t) \approx r_{j+1}(t)$), while in the spring dimer successive pairs are at varying

[4]From [LY03b], this "com[es] from the Greek root 'stego-,' meaning roof or ridge, and suggested by the rough resemblance of these...waves... to the back of a stegosaurus."

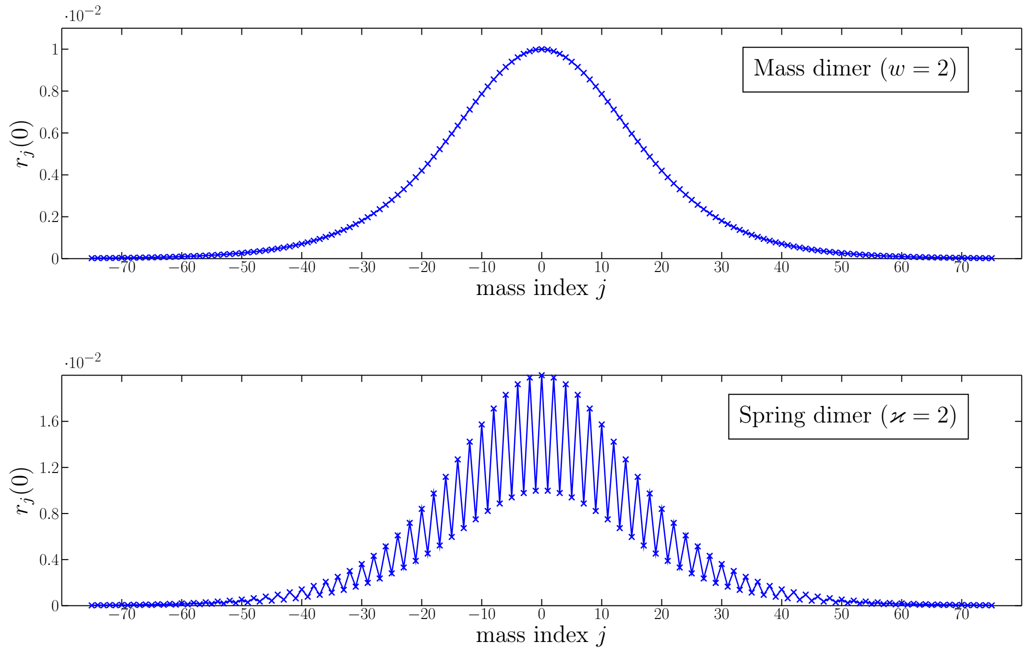


Figure 1.5.1: Relative displacements for mass and spring dimers

distances from each other. Such behavior for lattices with alternating spring forces was observed in [GMWZ14]; the leading order term in (1.4.1) confirms this behavior in the long wave limit.

Moreover, the estimates in Theorems 1.4.1 and 1.4.2 are consistent with the results of [GMWZ14], which establishes

$$r_j(t) = \frac{\epsilon^2}{\mathrm{K}_j}\big(U_-(\epsilon(j - c_\varkappa t), \epsilon^3 t) + U_+(\epsilon(j + c_\varkappa t), \epsilon^3 t)\big) + \mathcal{O}(\epsilon^{5/2}), \tag{1.5.1}$$

where

$$\mathrm{K}_j = \begin{cases} 1, & j \text{ is even} \\ \varkappa, & j \text{ is odd} \end{cases}$$

and $U_\pm = U_\pm(X, T)$ solve the KdV equations

$$\mp \frac{1}{c_\star}\partial_T[U_\pm] + \frac{\alpha_\star}{2c_\star}\partial_X^3[U_\pm] + \frac{\beta + \varkappa^3}{\varkappa^2(1+\varkappa)} U_\pm \partial_X[U_\pm] = 0. \tag{1.5.2}$$

We define $c_\star$ in (2.2.6) and $\alpha_\star$ in (2.5.2). We discuss this consistency further in Appendix F.2.

## Chapter 2. The Traveling Wave Problem in Dimers

**2.1. The traveling wave ansatz.** Set

$$r_j(t) = \begin{cases} p_1(j-ct), & j \text{ is odd} \\ p_2(j-ct), & j \text{ is even,} \end{cases} \tag{2.1.1}$$

where $p_1 = p_1(x)$ and $p_2 = p_2(x)$ are the traveling wave profiles and $c \in \mathbb{R}$ is the wave speed. With $S^d$ as the "shift by $d \in \mathbb{R}$" operator, i.e.,

$$(S^d f)(x) = f(x+d),$$

the system (1.3.8) becomes

$$\begin{cases} c^2 p_1'' & = -(1+w)(\varkappa p_1 + \beta p_1^2 + p_1^3 N_1(p_1)) + wS^1(p_2 + p_2^2 + p_2^3 N_2(p_2)) \\ & \quad + S^{-1}(p_2 + p_2^2 + p_2^3 N_2(p_2)) \\ \\ c^2 p_2'' & = -(1+w)(p_2 + p_2^2 + p_2^3 N_2(p_2)) + S^1(\varkappa p_1 + \beta p_1^2 + p_1^3 N_1(p_1)) \\ & \quad + wS^{-1}(\varkappa p_1 + \beta p_1^2 + p_1^3 N_1(p_1)). \end{cases}$$

We rewrite these equations for $p_1$ and $p_2$ in matrix-vector form. Let

$$\mathbf{p} := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad \text{and} \quad N(\mathbf{p}) := \begin{pmatrix} N_1(p_1) \\ N_2(p_2) \end{pmatrix}.$$

Then $\mathbf{p}$ satisfies

$$c^2 \mathbf{p}'' + L[\varkappa, w]\mathbf{p} + L[\beta, w]\mathbf{p}^{.2} + L[1,w](\mathbf{p}^{.3}.N(\mathbf{p})) = 0, \tag{2.1.2}$$

where, given $\alpha \in \mathbb{R}$, we set

$$L[\alpha, w] := \begin{bmatrix} \alpha(1+w) & -(wS^1 + S^{-1}) \\ -\alpha(S^1 + wS^{-1}) & (1+w) \end{bmatrix}.$$

Note that we have the useful factorization

$$L[\alpha, w] = \underbrace{\begin{bmatrix} (1+w) & -(wS^1 + S^{-1}) \\ -(S^1 + wS^{-1}) & (1+w) \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}}_{M_\alpha}. \tag{2.1.3}$$

The "dot" notation in (2.1.2) is componentwise squaring, cubing, or multiplication in the spirit of Matlab, e.g.,

$$\mathbf{p}^{.2} := \begin{pmatrix} p_1^2 \\ p_2^2 \end{pmatrix}.$$

We can rewrite (2.1.2) so that the operator $L[\varkappa, w]$ appears as the leading factor in the two nonlinear terms, which will be convenient in some later manipulations. The factorization (2.1.3) gives

$$L[\beta, w] = L[1, w]M_\beta = L[1, w]M_\varkappa M_\varkappa^{-1} M_\beta = L[\varkappa, w]M_{\beta/\varkappa}$$

and likewise $L[1, w] = L[\varkappa, w]M_{1/\varkappa}$. Then (2.1.2) is equivalent to

$$c^2\mathbf{p}'' + L[\varkappa, w]\mathbf{p} + L[\varkappa, w]M_{\beta/\varkappa}\mathbf{p}^{.2} + L[\varkappa, w]M_{1/\varkappa}(\mathbf{p}^{.3}.N(\mathbf{p})) = 0. \qquad (2.1.4)$$

**2.2. Diagonalization.** We treat the operator $L[\varkappa, w]$ as a Fourier multiplier (cf. Appendix D), so that for a function $\mathbf{f} = (f_1, f_2)$ we have

$$\mathfrak{F}[L[\varkappa, w]\mathbf{f}](k) = \widetilde{L}[\varkappa, w](k)\mathfrak{F}[\mathbf{f}](k),$$

with

$$\widetilde{L}[\varkappa, w](k) := \begin{bmatrix} \varkappa(1+w) & -(we^{ik} + e^{-ik}) \\ -\varkappa(e^{ik} + we^{-ik}) & (1+w) \end{bmatrix} \in \mathbb{C}^{2\times 2}.$$

Here $\mathfrak{F}[f](k) = \widehat{f}(k)$ is the Fourier transform of $f$ at $k$, defined in (B.1.1) for periodic functions and in (B.2.1) for $L^2$-functions.

We wish to diagonalize $L[\varkappa, w]$, and we begin by computing the eigenvalues of $\widetilde{L}[\varkappa, w](k)$; they are

$$\widetilde{\lambda}_\pm(k) := \frac{(1+\varkappa)(1+w)}{2} \pm \frac{\widetilde{\varrho}(k)}{2}, \qquad (2.2.1)$$

with

$$\widetilde{\varrho}(k) := \sqrt{(1+w)^2(1-\varkappa)^2 + 4\varkappa((1-w)^2 + 4w\cos^2(k))}.$$

In the language of [Bri53], we sometimes refer to the curve $\widetilde{\lambda}_-(k)$ as the "acoustic band" of the system and to $\widetilde{\lambda}_+(k)$ as the "optical band."

The function $\widetilde{\varrho}$ is "symmetric" in the parameters $w$ and $\varkappa$ in the sense that

$$(1+w)^2(1-\varkappa)^2+4\varkappa((1-w)^2+4w\cos^2(k)) = (1+\varkappa)^2(1-w)^2+4w((1-\varkappa)^2+4\varkappa\cos^2(k)).$$

This is a source of much convenience in many proofs to come. For legibility, though, we suppress further dependence on $\varkappa$ and $w$.

To find the corresponding eigenvectors, we set, in the case $w \neq 1$,

$$\widetilde{v}_-(k) := \frac{1+w-\widetilde{\lambda}_-(k)}{\varkappa(e^{ik}+we^{-ik})} \quad \text{and} \quad \widetilde{v}_+(k) := \frac{\varkappa(1+w)-\widetilde{\lambda}_+(k)}{we^{ik}+e^{-ik}} \tag{2.2.2}$$

and

$$\widetilde{\mathbf{v}}_-(k) := \begin{pmatrix} \widetilde{v}_-(k) \\ 1 \end{pmatrix} \quad \text{and} \quad \widetilde{\mathbf{v}}_+(k) := \begin{pmatrix} 1 \\ \widetilde{v}_+(k) \end{pmatrix}. \tag{2.2.3}$$

Then $\widetilde{\mathbf{v}}_\pm(k)$ are eigenvectors of $\widetilde{L}[\varkappa, w]$ corresponding to the eigenvalues $\widetilde{\lambda}_\pm(k)$.

It is also convenient to be able to scale the eigenvectors, at least in the case $\varkappa = 1$. For now, we let $\widetilde{\gamma}_\pm(k)$ be complex-valued functions that are analytic and vanish nowhere on some horizontal strip $|\operatorname{Im}(z)| \le \tau_0$. With

$$\widetilde{J}(k) := \begin{bmatrix} \widetilde{\gamma}_-(k)\widetilde{\mathbf{v}}_-(k) & \widetilde{\gamma}_+(k)\widetilde{\mathbf{v}}_+(k) \end{bmatrix}, \qquad \widetilde{J}_1(k) := \widetilde{J}(k)^{-1},$$
$$\text{and} \quad \widetilde{\Lambda}(k) := \begin{bmatrix} \widetilde{\lambda}_-(k) & 0 \\ 0 & \widetilde{\lambda}_+(k) \end{bmatrix}, \tag{2.2.4}$$

we have

$$\widetilde{L}[\varkappa, w](k) = \widetilde{J}(k)\widetilde{\Lambda}(k)\widetilde{J}_1(k).$$

Taking $J$, $J_1$, and $\Lambda$ to be the Fourier multipliers with the symbols $\widetilde{J}$, $\widetilde{J}_1$, and $\widetilde{\Lambda}$, we diagonalize the operator $L[\varkappa, w]$:

$$L[\varkappa, w] = J\Lambda J_1.$$

Before we exploit this diagonalization, we summarize the essential properties of these various Fourier multipliers.

**2.2.1 Proposition.** *Let $w$, $\varkappa > 0$ with either $w > 1$ or $\varkappa > 1$.*

**(i)** *For $q > 0$, let $\Sigma_q$ be the strip*

$$\Sigma_q := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < q\}$$

*and let $\overline{\Sigma}_q$ be its closure. There exists $q_0 > 0$ such that $\widetilde{\lambda}_\pm$ and $\widetilde{v}_\pm$ extend to $2\pi$-periodic, bounded, complex-valued analytic functions on $\overline{\Sigma}_{q_0}$. The functions $\widetilde{\lambda}_\pm$ are* *also even and real-valued at real inputs.*

**(ii)** $\widetilde{\lambda}_-(0) = 0$ *and* $\widetilde{\lambda}_+(0) = (1+w)(1+\varkappa)$.

**(iii)** *For all* $k \in \mathbb{R}$ *we have*

$$0 \le \widetilde{\lambda}_-(k) \le \widetilde{\lambda}_-\left(\frac{\pi}{2}\right) < \widetilde{\lambda}_+\left(\frac{\pi}{2}\right) \le \widetilde{\lambda}_+(k) \le (1+w)(1+\varkappa). \tag{2.2.5}$$

**(iv)** $\widetilde{\lambda}_\pm''(0) = \mp\dfrac{8\varkappa w}{(1+w)(1+\varkappa)}$.

**(v)** *For all* $k \in \mathbb{R}$ *we have*

$$|\widetilde{\lambda}_\pm'(k)| \le \min\left\{\frac{4w}{1+w}, \frac{4\varkappa}{1+\varkappa}, 2c_\star^2|k|\right\},$$

*where*

$$c_\star = c_\star(w,\varkappa) := \sqrt{\frac{\widetilde{\lambda}_-''(0)}{2}} = \sqrt{\frac{4\varkappa w}{(1+w)(1+\varkappa)}}. \tag{2.2.6}$$

*is the "speed of sound" of the lattice.*

**(vi)** *If* $c^2 > c_\star^2$, *then*

$$\widetilde{\xi}_c(k) := -c^2k^2 + \widetilde{\lambda}_-(k) < 0 \tag{2.2.7}$$

*for all* $k \neq 0$.

**(vii)** *There exist* $c_- \in (0,1)$ *and* $b_0 > 0$ *such that if* $c > c_-$, *then there is a unique* $\Omega_c \in (0,\infty)$ *such that* $c^2\Omega_c^2 - \widetilde{\lambda}_+(\Omega_c) = 0$. *Moreover,*

$$\frac{\sqrt{\widetilde{\lambda}_+(\pi/2)}}{c} \le \Omega_c \le \frac{\sqrt{(1+w)(1+\varkappa)}}{c} \tag{2.2.8}$$

*and*

$$|2c^2\Omega_c - \widetilde{\lambda}_+'(\Omega_c)| \ge b_0 > 0. \tag{2.2.9}$$

Now, using the factorization $L[\varkappa, w] = J\Lambda J_1$, we diagonalize the system (2.1.4). Set

$$\mathbf{h} = J_1\mathbf{p}, \qquad \mathbf{h}(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}. \tag{2.2.10}$$

Then (2.1.4) is equivalent to

$$(c^2\partial_x^2 + \Lambda)\mathbf{h} + \Lambda J_1 M_{\beta/\varkappa}[(J\mathbf{h})^{.2}] + \Lambda J_1 M_{1/\varkappa}[(J\mathbf{h})^{.3}.N(J\mathbf{h})] = 0 \tag{2.2.11}$$

With

$$B(\mathbf{h},\grave{\mathbf{h}}) = \begin{pmatrix} B_1(\mathbf{h},\grave{\mathbf{h}}) \\ B_2(\mathbf{h},\grave{\mathbf{h}}) \end{pmatrix} := J_1 M_{\beta/\varkappa}[(J\mathbf{h}).(J\grave{\mathbf{h}})], \tag{2.2.12}$$

$$\mathcal{N}(\mathbf{h}) := \mathbf{h}.N(\mathbf{h}), \tag{2.2.13}$$

and

$$\mathcal{Q}(\mathbf{h},\grave{\mathbf{h}},\breve{\mathbf{h}}) = \begin{pmatrix} \mathcal{Q}_1(\mathbf{h},\grave{\mathbf{h}},\breve{\mathbf{h}}) \\ \mathcal{Q}_2(\mathbf{h},\grave{\mathbf{h}},\breve{\mathbf{h}}) \end{pmatrix} := J_1 M_{1/\varkappa}[(J\mathbf{h}).(J\grave{\mathbf{h}}).\mathcal{N}(J\breve{\mathbf{h}})] \tag{2.2.14}$$

we further compress (2.2.11) to

$$(c^2\partial_x^2 + \Lambda)\mathbf{h} + \Lambda B(\mathbf{h},\mathbf{h}) + \Lambda\mathcal{Q}(\mathbf{h},\mathbf{h},\mathbf{h}) = 0. \tag{2.2.15}$$

**2.3. The Friesecke-Pego cancelation.** The first component of (2.2.15) is

$$(c^2\partial_x^2 + \lambda_-)h_1 + \lambda_- B_1(\mathbf{h},\mathbf{h}) + \lambda_-\mathcal{Q}_1(\mathbf{h},\mathbf{h},\mathbf{h}) = 0. \tag{2.3.1}$$

Applying the Fourier transform, this becomes

$$-(c^2k^2 - \widetilde{\lambda}_-(k))\widehat{h}_1(k) + \widetilde{\lambda}_-(k)\mathfrak{F}[B_1(\mathbf{h},\mathbf{h})](k) + \widetilde{\lambda}_-(k)\mathfrak{F}[\mathcal{Q}_1(\mathbf{h},\mathbf{h},\mathbf{h})](k) = 0. \tag{2.3.2}$$

By Proposition 2.2.1 (vi), we have $c^2k^2 - \widetilde{\lambda}_-(k) > 0$ for all $k \neq 0$, as long as $c^2 > c_\star^2$. We assume this lower bound on $c^2$ from now on, so that (2.3.2) becomes

$$\widehat{h}_1(k) + \widetilde{\varpi}_c(k)\mathfrak{F}[B_1(\mathbf{h},\mathbf{h})](k) + \widetilde{\varpi}_c(k)\mathfrak{F}[\mathcal{Q}_1(\mathbf{h},\mathbf{h},\mathbf{h})](k) = 0, \tag{2.3.3}$$

where

$$\widetilde{\varpi}_c(k) := -\frac{\widetilde{\lambda}_-(k)}{c^2k^2 - \widetilde{\lambda}_-(k)}.$$

Since $\widetilde{\lambda}_-''(0) \neq 0$ by Proposition 2.2.1 (iv) , we see that $\widetilde{\varpi}_c$ has a removable singularity at $k = 0$, so $\widetilde{\varpi}_c$ extends to an even, $\pi$-periodic, bounded complex-valued analytic function on the strip $\overline{\Sigma}_{q_0}$ from above.

We refer to the division by $c^2k^2 - \widetilde{\lambda}_-(k)$ that converts (2.3.2) into (2.3.3) as the

"Friesecke-Pego cancelation" because Friesecke and Pego [FP99] performed a similar division (cancelation) in setting up their fixed point problem for the monatomic lattice. Their analogue of our symbol $\widetilde{\varpi}_c(k)$ was $\sin^2(k/2)/(c^2k^2 - 4\sin^2(k/2))$, which is also analytic and has a removable singularity at $k = 0$.

Let $\varpi_c$ be the Fourier multiplier with symbol $\widetilde{\varpi}_c$. Then any function $\mathbf{h} = (h_1, h_2)$ that solves

$$h_1 + \varpi_c B_1(\mathbf{h}, \mathbf{h}) + \varpi_c \mathcal{Q}_1(\mathbf{h}, \mathbf{h}, \mathbf{h}) = 0,$$

will solve (2.3.1), and so we can find solutions to the entire system (2.2.15) by studying

$$\mathcal{H}_c(\mathbf{h}) := \begin{bmatrix} 1 & 0 \\ 0 & c^2\partial_x^2 + \lambda_+ \end{bmatrix} \mathbf{h} + \begin{bmatrix} \varpi_c & 0 \\ 0 & \lambda_+ \end{bmatrix} B(\mathbf{h}, \mathbf{h}) + \begin{bmatrix} \varpi_c & 0 \\ 0 & \lambda_+ \end{bmatrix} \mathcal{Q}(\mathbf{h}, \mathbf{h}, \mathbf{h}) = 0. \quad (2.3.4)$$

**2.4. The long wave scaling.** This is our final change of variables. We set

$$\mathbf{h}(x) = \epsilon^2 \boldsymbol{\theta}(\epsilon x), \quad (2.4.1)$$

where $\boldsymbol{\theta}(X) = (\theta_1(X), \theta_2(X))$, and we take the wave speed $c$ to satisfy

$$c^2 = c_\epsilon^2 := c_\star^2 + \epsilon^2.$$

That is, we intend to solve

$$\mathcal{H}_{c_\epsilon}(\epsilon^2 \boldsymbol{\theta}(\epsilon \cdot)) = 0 \quad (2.4.2)$$

for $\boldsymbol{\theta}$ with $\epsilon$ small.

Let $\varpi^\epsilon$ be the Fourier multiplier with symbol

$$\widetilde{\varpi^\epsilon}(K) := \epsilon^2 \widetilde{\varpi}_{c_\epsilon}(\epsilon k) = -\frac{\epsilon^2 \widetilde{\lambda}_-(\epsilon k)}{c_\epsilon^2(\epsilon k)^2 - \widetilde{\lambda}_-(\epsilon k)}. \quad (2.4.3)$$

Per the convention outlined in Appendix D, for any other Fourier multiplier $\mu$ in the definition of $\mathcal{H}_c$, let $\mu^\epsilon$ have the symbol $\widetilde{\mu^\epsilon}(k) = \widetilde{\mu}(\epsilon k)$, where of course $\widetilde{\mu}$ is the symbol of $\mu$. Let

$$B^\epsilon(\boldsymbol{\theta}, \grave{\boldsymbol{\theta}}) = \begin{pmatrix} B_1^\epsilon(\boldsymbol{\theta}, \grave{\boldsymbol{\theta}}) \\ B_2^\epsilon(\boldsymbol{\theta}, \grave{\boldsymbol{\theta}}) \end{pmatrix} := J_1^\epsilon M_{\beta/\varkappa}[(J^\epsilon \boldsymbol{\theta}).(J^\epsilon \grave{\boldsymbol{\theta}})] \quad (2.4.4)$$

and

$$\mathcal{Q}^{\epsilon}(\boldsymbol{\theta},\grave{\boldsymbol{\theta}},\breve{\boldsymbol{\theta}}) = \begin{pmatrix} \mathcal{Q}_1^{\epsilon}(\boldsymbol{\theta},\grave{\boldsymbol{\theta}},\breve{\boldsymbol{\theta}}) \\ \mathcal{Q}_2^{\epsilon}(\boldsymbol{\theta},\grave{\boldsymbol{\theta}},\breve{\boldsymbol{\theta}}) \end{pmatrix} := J_1^{\epsilon} M_{\beta/\varkappa}\left((J^{\epsilon}\boldsymbol{\theta}).(J^{\epsilon}\grave{\boldsymbol{\theta}}).\mathcal{N}(\epsilon^2 J^{\epsilon}\breve{\boldsymbol{\theta}})\right). \tag{2.4.5}$$

Then by the scaling properties of Fourier multipliers, our problem (2.4.2) is equivalent to

$$\boldsymbol{\Theta}_{\epsilon}(\boldsymbol{\theta}) := \mathcal{D}_1^{\epsilon}\boldsymbol{\theta} + \mathcal{D}_2^{\epsilon}B^{\epsilon}(\boldsymbol{\theta},\boldsymbol{\theta}) + \mathcal{D}_2^{\epsilon}\mathcal{Q}^{\epsilon}(\boldsymbol{\theta},\boldsymbol{\theta},\boldsymbol{\theta}) = 0, \tag{2.4.6}$$

where

$$\mathcal{D}_1^{\epsilon} := \begin{bmatrix} 1 & 0 \\ 0 & c_{\epsilon}^2\epsilon^2\partial_X^2 + \lambda_+^{\epsilon} \end{bmatrix} \quad \text{and} \quad \mathcal{D}_2^{\epsilon} := \begin{bmatrix} \varpi^{\epsilon} & 0 \\ 0 & \epsilon^2\lambda_+^{\epsilon} \end{bmatrix}. \tag{2.4.7}$$

Note that because the small parameter $\epsilon^2$ multiplies the second derivative operator $\partial_X^2$ in $\mathcal{D}_1^{\epsilon}$, our problem (2.4.2) is singularly perturbed. This was not a feature of the monatomic problem in [FP99], as the long wave problem there involved only an equation analogous to (2.3.1); a subsequent factoring and cancelation, which proceeded like ours in Section 2.3, removed that singularity.

**2.5. The formal long wave limit.** In this section we formally define what $\boldsymbol{\Theta}_0$ should be by assigning meaning to $\mathcal{D}_1^0$, $\mathcal{D}_2^0$, $B^0$, and $\mathcal{Q}^0$ in a natural way. As a first step toward solving (2.4.6), we then solve the equation $\boldsymbol{\Theta}_0(\boldsymbol{\theta}) = 0$ explicitly.

In general, if $\mu$ is a Fourier multiplier with symbol $\widetilde{\mu}(k)$, and $\mu^{\epsilon}$ is the multiplier with symbol $\widetilde{\mu}(\epsilon k)$, then $\mu^0$ should be the multiplier with "constant" symbol $\widetilde{\mu}(0)$, i.e., $\mu f = \widetilde{\mu}(0)f$ for any function $f$. Proposition 2.2.1 (ii) then implies

$$\mathcal{D}_1^0 = \begin{bmatrix} 1 & 0 \\ 0 & (1+w)(1+\varkappa) \end{bmatrix}.$$

To define $\mathcal{D}_2^0$, we need to specify $\varpi^0$, and here we need to be careful because the factor of $\epsilon$ appears in several places in (2.4.3). Since $\widetilde{\lambda}_-$ is analytic, we can write

$$\widetilde{\lambda}_-(k) = c_{\star}^2 k^2 - \frac{\widetilde{\lambda}_-^{(4)}(0)}{4!}k^4 + k^6\mathcal{R}(k), \qquad \mathcal{R} \in L^{\infty}, \tag{2.5.1}$$

and we have

$$\frac{\widetilde{\lambda}_-^{(4)}(0)}{4!} = \frac{1}{3}c_\star \frac{(\varkappa+1)^2 + 2(\varkappa^2 - 4\varkappa + 1)w + (\varkappa+1)^2 w^2}{(1+w)^2(1+\varkappa)^2} =: \alpha_\star(w,\varkappa) = \alpha_\star. \quad (2.5.2)$$

The minus sign on $k^4$ in (2.5.1) is for convenience, so that $\alpha_\star > 0$. Then (for $\epsilon, k \neq 0$) we can factor and simplify $\widetilde{\varpi}^\epsilon(k)$ as

$$\widetilde{\varpi}^\epsilon(k) = -\frac{\epsilon^2 c_\star^2(\epsilon k)^2 - \epsilon^2\alpha_\star(\epsilon k)^4 + \epsilon^2(\epsilon k)^6\mathcal{R}(\epsilon k)}{(c_\star^2+\epsilon^2)(\epsilon k)^2 - c_\star^2(\epsilon k)^2 + \alpha_\star(\epsilon k)^4 - (\epsilon k)^6\mathcal{R}(\epsilon k)}$$

$$= -\frac{c_\star^2 - \epsilon^2\alpha_\star k^2 + \epsilon^4 k^6\mathcal{R}(\epsilon k)}{1 + \alpha_\star k^2 + \epsilon^2 k^4\mathcal{R}(\epsilon k)}.$$

Setting $\epsilon = 0$, we find

$$\widetilde{\varpi}^0(k) = -\frac{c_\star^2}{1+\alpha_\star k^2},$$

and so we set

$$\varpi^0 := -c_\star^2(1 - \alpha_\star\partial_X^2)^{-1}.$$

Then

$$\mathcal{D}_2^0 := \begin{bmatrix} \varpi^0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We continue to use Proposition 2.2.1, along with (2.2.2) and (2.2.4), to compute

$$\widetilde{J}(0) = \begin{bmatrix} 1/\varkappa & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \widetilde{\gamma}_-(0) & 0 \\ 0 & \widetilde{\gamma}_+(0) \end{bmatrix}}_{\widetilde{\Gamma}(0)} \quad (2.5.3)$$

$$\widetilde{J}_1(0) = \frac{1}{\varkappa+1}\begin{bmatrix} 1/\widetilde{\gamma}_-(0) & 0 \\ 0 & 1/\widetilde{\gamma}_+(0) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \varkappa & -1 \end{bmatrix}.$$

This motivates the definition of the Fourier multipliers $J^0$ and $J_1^0$ as the "constant" operators $J^0 := \widetilde{J}(0)$ and $J_1^0 := \widetilde{J}_1(0)$. From this and (2.4.4) we have

$$B^0(\boldsymbol{\theta}, \grave{\boldsymbol{\theta}}) := J_1^0 M_{\beta/\varkappa}[(J^0\boldsymbol{\theta}).(J^0\grave{\boldsymbol{\theta}})]. \quad (2.5.4)$$

We could define $\mathcal{Q}^0$ in the same way using $J^0$ and $J_1^0$, but it is straightforward to see that $\mathcal{Q}^0$ will be identically zero thanks to the extra factor of $\epsilon^2$ that $\mathcal{Q}^\epsilon$ carries within $\mathcal{N}$, per (2.4.5) and (2.2.13). This factor of $\epsilon^2$ will resurface frequently in the depths of the estimates to come.

All together, this implies (formally) that

$$\boldsymbol{\Theta}_0(\boldsymbol{\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & (1+w)(1+\varkappa) \end{bmatrix} \boldsymbol{\theta} + \begin{bmatrix} \varpi^0 & 0 \\ 0 & 0 \end{bmatrix} B^0(\boldsymbol{\theta}, \boldsymbol{\theta}). \tag{2.5.5}$$

Now we consider the problem $\boldsymbol{\Theta}_0(\boldsymbol{\theta}) = 0$ with $\boldsymbol{\theta} = (\theta_1, \theta_2)$. We find from the second component of (2.5.5) that

$$(1+w)(1+\varkappa)\theta_2 = 0,$$

so $\theta_2 = 0$. Then using (2.5.4), the first component reduces to

$$0 = \theta_1 + \varpi^0 \left[ \widetilde{\gamma}_-(0) \frac{\beta + \varkappa^3}{\varkappa^2(1+\varkappa)} \theta_1^2 \right] = \theta_1 - c_\star^2 \widetilde{\gamma}_-(0) \frac{\beta + \varkappa^3}{\varkappa^2(1+\varkappa)} (1 - \alpha_\star \partial_X^2)^{-1} [\theta_1^2].$$

Applying $1 - \alpha_\star \partial_X^2$ to both sides, we get

$$\alpha_\star \theta_2'' - \theta_2 + c_\star^2 \frac{\widetilde{\gamma}_-(0)}{\varkappa^2(1+\varkappa)} (\beta + \varkappa^3) \theta_2^2 = 0. \tag{2.5.6}$$

This is a rescaling of the ordinary differential equation that gives the $\operatorname{sech}^2$-type traveling wave profiles for the KdV equation, provided that

$$\beta \neq -\varkappa^3 \tag{2.5.7}$$

to keep the nonlinear term $\theta_2^2$ present. If we require $\beta$ and $\varkappa$ to satisfy (2.5.7), then the solution to (2.5.6) is

$$\theta_2(X) = \sigma(X) := \frac{3\varkappa^2(1+\varkappa)}{2c_\star^2 \widetilde{\gamma}_-(0)(\beta + \varkappa^3)} \operatorname{sech}^2\left( \frac{X}{2\sqrt{\alpha_\star}} \right). \tag{2.5.8}$$

**2.6. Motivation for Beale's nanopteron ansatz.** So far, we have followed the set-up of Friesecke and Pego [FP99] for the monatomic lattice: we converted the equations of motion for position into equations for relative displacement; we made a traveling wave ansatz; and we introduced the long wave scaling. Friesecke and Pego used the same long wave scaling as well as a cancelation analogous to the one in Section 2.3 for the acoustic band and converted their traveling wave problem into a fixed point equation of the form $\phi = \Phi_\epsilon(\phi)$; they subsequently applied a quantitative inverse function theorem argument to the map $\phi \mapsto \phi - \Phi_\epsilon(\phi)$ to solve this fixed point problem. Our attempt to turn the dimer's long wave problem (2.4.6) into a fixed point problem will fail, but this failure is quite instructive. Write (2.4.6) as

$$\boldsymbol{\Theta}_\epsilon(\boldsymbol{\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}_\epsilon \end{bmatrix} \boldsymbol{\theta} - \begin{pmatrix} \mathcal{R}_1^\epsilon(\boldsymbol{\theta}) \\ \mathcal{R}_2^\epsilon(\boldsymbol{\theta}) \end{pmatrix}, \qquad \mathcal{T}_\epsilon := c_\epsilon^2 \epsilon^2 \partial_X^2 + \lambda_+^\epsilon. \tag{2.6.1}$$

Here we have collected all of the nonlinear terms into $-\mathcal{R}_1^\epsilon$ and $-\mathcal{R}_2^\epsilon$ for convenience.

We are interested in the long wave problem for $\epsilon$ small, so we expect our solution $\boldsymbol{\theta}$ to be close to the sech$^2$-type solution $\boldsymbol{\sigma}$, which satisfies $\boldsymbol{\Theta}_0(\boldsymbol{\sigma}) = 0$ from the preceding section. We can quantify this by looking for $\boldsymbol{\theta}$ in the form $\boldsymbol{\theta} = \boldsymbol{\sigma} + \boldsymbol{\eta}$, where $\boldsymbol{\eta} = (\eta_1, \eta_2) \in H_q^2 \times H_q^2$ for a suitable $q > 0$; the Sobolev space $H_q^2$ of exponentially decaying $H^2$-functions is discussed in Appendix C.3. Then (2.6.1) is equivalent to

$$\begin{cases} \eta_1 = \mathcal{R}_1^\epsilon(\boldsymbol{\sigma} + \boldsymbol{\eta}) - \sigma \\ \\ \mathcal{T}_\epsilon \eta_2 = \mathcal{R}_2^\epsilon(\boldsymbol{\sigma} + \boldsymbol{\eta}). \end{cases} \tag{2.6.2}$$

If we could invert $\mathcal{T}_\epsilon$, then (2.6.1) would be equivalent to a fixed point problem, to which we could conceivably apply some kind of contraction mapping method. However, Theorem 2.2.1 furnishes a (unique) $\Omega_{c_\epsilon} > 0$ such that

$$-c_\epsilon^2(\pm\Omega_{c_\epsilon})^2 + \widetilde{\lambda}_+(\pm\Omega_{c_\epsilon}) = 0.$$

Set

$$\omega_\epsilon := \frac{\Omega_{c_\epsilon}}{\epsilon}. \tag{2.6.3}$$

Then with

$$\widetilde{\mathcal{T}}_\epsilon(k) = -c_\epsilon^2(\epsilon k)^2 + \widetilde{\lambda}_+(\epsilon k)$$

as the symbol of $\mathcal{T}_\epsilon$, we have

$$\widetilde{\mathcal{T}}_\epsilon(\pm\omega_\epsilon) = 0. \tag{2.6.4}$$

We also have

$$\widetilde{\mathcal{T}}_\epsilon'(\pm\omega_\epsilon) = \epsilon \frac{d}{dk}\left(-2c_\epsilon^2(\epsilon k) + \widetilde{\lambda}_+'(\epsilon k)\right)\bigg|_{k=\omega_\epsilon} \neq 0,$$

and so we conclude that $1/\widetilde{\mathcal{T}}_\epsilon$ has poles of order one at $k = \pm\omega_\epsilon$. We therefore cannot define a Fourier multiplier with symbol $1/\widetilde{\mathcal{T}}_\epsilon$ on $H_q^2$, and so it is not immediately clear how to proceed in the vein of Friesecke and Pego.

Another, somewhat more informal, way to view (2.6.2) is that it consists ostensibly of two equations in the two unknowns $\eta_1$ and $\eta_2$, but the second equation $\mathcal{T}_\epsilon\eta_2 = \mathcal{R}_2^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})$ forces two more equations:

$$\mathcal{T}_\epsilon\eta_2 = \mathcal{R}_2^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta}) \Longrightarrow \widehat{\mathcal{R}_2^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})}(\pm\omega_\epsilon) = \widehat{\mathcal{T}_\epsilon\eta_2}(\pm\omega_\epsilon) = \widetilde{\mathcal{T}}_\epsilon(\pm\omega_\epsilon)\widehat{\eta}_2(\pm\omega_\epsilon) = 0.$$

(Note that since $\eta_2 \in H_q^1 \subseteq L^1$ for $q > 0$ by Proposition C.3.12, the pointwise Fourier transform of $\eta_2$ at $\pm\omega_\epsilon$ is unambiguously defined.) That is, we have the "solvability conditions" $\widehat{\mathcal{R}_2^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})}(\pm\omega_\epsilon) = 0$. It turns out that, at least in the cases of the mass and spring dimers, we can remove one of these equations by exploiting a "symmetry" in the underlying lattice's structure and restricting $\eta_1$ and $\eta_2$ to be even or odd. We would still be left, though, with solving the three equations

$$\begin{cases} \eta_1 = \mathcal{R}_1^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta}) \\ \\ \mathcal{T}_\epsilon\eta_2 = \mathcal{R}_2^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta}) \\ \\ \widehat{\mathcal{R}_2^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})}(\omega_\epsilon) = 0 \end{cases} \tag{2.6.5}$$

in only the two unknowns $\eta_1$ and $\eta_2$. A solution, then, is to introduce a third unknown into the problem, making for the classically optimistic situation of "three equations in three unknowns."

We make this precise by following the nanopteron ansatz introduced by Beale [Bea91a] for the water waves problem and subsequently adapted by Amick and Toland [AT92] for a singularly perturbed model equation derived from the Euler equations for water waves. That is, we revise the ansatz $\boldsymbol{\theta} = \boldsymbol{\sigma}+\boldsymbol{\eta}$ to

$$\boldsymbol{\theta} = \boldsymbol{\sigma} + a\boldsymbol{\varphi}_a + \boldsymbol{\eta},$$

where now $\boldsymbol{\eta} \in H_q^2 \times H_q^2$, $a \in \mathbb{R}$, and $\boldsymbol{\varphi}_a(X) = \boldsymbol{\phi}_a(\omega_a X)$ for some $\boldsymbol{\phi}_a \in H_{\mathrm{per}}^2 \times H_{\mathrm{per}}^2$ and

$\omega_a \in \mathbb{R}$ with $\mathbf{\Theta}_\epsilon(a\boldsymbol{\varphi}_a) = 0$. The periodic Sobolev space $H^2_{\text{per}}$ is defined in Appendix C.2.

The success of this ansatz first hinges, of course, on the existence of such $a$ and $\boldsymbol{\varphi}_a$ satisfying $\mathbf{\Theta}_\epsilon(a\boldsymbol{\varphi}_a) = 0$. The existence of such periodic solutions is the content of Chapter 3. Then we rearrange the problem $\mathbf{\Theta}_\epsilon(\boldsymbol{\sigma} + a\boldsymbol{\varphi}_a + \boldsymbol{\eta}) = 0$ into a fixed point problem for $a$ and $\boldsymbol{\eta}$, which incorporates the solvability condition from (2.6.5) so that we have three equations in three unknowns. This is the content of Chapter 4.

**2.7. Specialization to mass and spring dimers.** We need to make one further specialization before implementing Beale's ansatz. This chapter has derived the long wave problem for a general dimer lattice, which allows both masses and springs to vary. The original intention of this dissertation was to address the nanopteron problem in the general dimer, requiring only $w > 1$ and $\varkappa \neq 0$ or $\varkappa > 1$ and $w \neq 0$. To date, this remains an open problem, chiefly due to the following technical difficulty.

The existence of periodic solutions is based on a fixed point argument originating from the proof of the Crandall-Rabinowitz-Zeidler theorem on bifurcation from a simple eigenvalue (stated as Theorem 3.2.1). To apply the theorem or to construct the fixed point equations as we do in Section 3.3 requires, fundamentally, that the linearization of the nonlinear operator $\mathbf{\Theta}_\epsilon$ from the long wave equations in (2.4.6) have a one-dimensional kernel. The Friesecke-Pego cancelation is a key step in this direction, as it removes "constant" elements from the kernel. We discuss this in more precise detail in Section 3.2 and especially in Remark 3.2.2. Merely removing these constant solutions from the kernel of $\mathcal{D}_1^\epsilon$, however, does not reduce the kernel's dimension sufficiently, as (2.6.4) turns out to guarantee that without further restrictions on our function space, $\mathcal{D}_1^\epsilon$ will always be at least two dimensional. Instead, we will be able to decrease the kernel's dimension down to one by restricting to spaces of certain even and/or odd functions.

**2.7.1 Lemma.** *Recall the definition of $\mathcal{H}_c$ from* (2.3.4) *and the eigenvector scalings $\widetilde{\gamma}_{\pm}(k)$ from* (2.2.3).

**(i)** *(Mass dimer) Suppose $w > 1$ and $\varkappa = 1$. Let*

$$\widetilde{\gamma}_+(k) = e^{-ik} + \frac{e^{ik}\widetilde{\varrho}(k)}{we^{ik} + e^{-ik}} \qquad \text{and} \qquad \widetilde{\gamma}_-(k) = \frac{we^{ik} + e^{-ik}}{\widetilde{\varrho}(k)}\widetilde{\gamma}_+(k). \tag{2.7.1}$$

*If $h_1$ is even and $h_2$ is odd and $\mathbf{h} = (h_1, h_2)$, then $\mathcal{H}_c(\mathbf{h}) \cdot \mathbf{i}$ is even and $\mathcal{H}_c(\mathbf{h}) \cdot \mathbf{j}$ is odd. Furthermore, in the long wave coordinates, if $\theta_1$ is even and $\theta_2$ is odd and $\boldsymbol{\theta} = (\theta_1, \theta_2)$, then $\boldsymbol{\Theta}_{\epsilon,1}(\boldsymbol{\theta})$ is even and $\boldsymbol{\Theta}_{\epsilon,2}(\boldsymbol{\theta})$ is odd.*

**(ii)** *(Spring dimer) Suppose $w = 1$ and $\varkappa > 1$. Let $\widetilde{\gamma}_{\pm}(k) = 1$. If $h_1$ and $h_2$ are both even and $\mathbf{h} = (h_1, h_2)$, then the components of $\mathcal{H}_c(\mathbf{h})$ are also both even. If $\theta_1$ and $\theta_2$ are even and $\boldsymbol{\theta} = (\theta_1, \theta_2)$, then $\boldsymbol{\Theta}_{\epsilon,1}(\boldsymbol{\theta})$ and $\boldsymbol{\Theta}_{\epsilon,2}(\boldsymbol{\theta})$ are odd.*

**Proof. (i)** This was proved as Lemma 2.8 in [FW18].

**(ii)** We use the fact that if the symbol of a Fourier multiplier $\mu$ is even and $f$ is an even function, then $\mu f$ is even. Observe that the Fourier multipliers in the definition of $\mathcal{H}_c$ — which are $c^2\partial_x^2 + \lambda_+$, $\lambda_+$, $\varpi_c$, $J_1$, and $J$ — all have even symbols. Moreover, multiplication and composition of even functions of course preserves evenness. Together with the structure of $\mathcal{H}_c$, this proves the parity result for $\mathcal{H}_c(\mathbf{h})$. Likewise, every Fourier multiplier in the definition of $\boldsymbol{\Theta}_\epsilon$ has an even symbol, since the rescaling of $\widetilde{\mu}(k)$ to $\widetilde{\mu}(\epsilon k)$ does not change the parity of an arbitrary symbol $\widetilde{\mu}$, and so the components of $\boldsymbol{\Theta}_\epsilon(\boldsymbol{\theta})$ are also both even when $\theta_1$ and $\theta_2$ are even. ■

We have found no such "symmetry" condition for the general dimer. There, the components (2.2.2) of the eigenvectors of $L[w, \varkappa]$ are neither even (nor odd) as in the spring dimer. Likewise, a rescaling of the eigenvectors in the vein of (2.7.1) for the mass dimer fails to yield any symmetry in the general dimer due to the presence of the matrix $M_{\beta/\varkappa}$, which in the general dimer's traveling wave equations is no longer the identity matrix. It remains entirely possible that there do not exist periodic solutions to the long wave problem (2.4.6), in which case Beale's ansatz is no longer valid, and we would not expect nanopteron solutions to exist.

At this point it is appropriate to mention a different formulation of the lattice nanopteron problem. Making a clever change of variables, Iooss and Kirchgässner convert the traveling wave problems for the monatomic lattice [Ioo00] and a monatomic lattice with coupling [IK00] to an evolution equation of the form

$$\frac{dU}{dt} = \mathcal{L}(U,\mu) + \mathcal{M}(U,\mu), \tag{2.7.2}$$

where $U$ belongs to a Hilbert space $\mathcal{X}$, $\mu \in \mathbb{R}$, $\mathcal{L}$ is linear in $U$, and $\mathcal{M}$ is quadratic in $U$. They then appeal to the work of Lombardi [Lom00], who has proved the existence of nanopteron solutions for a broad class of equations like (2.7.2). One of Lombardi's key hypotheses is the existence of a "symmetry" $S \in \mathbf{B}(\mathcal{X})$ that anticommutes with $\mathcal{L}$ and $\mathcal{M}$ and satisfies $S^2 = \mathbb{1}$. This is the so-called "reversibility" of the system (2.7.2).

Venney and Zimmer [VZ14] have used the Iooss-Kirchgässner variables and Lombardi's results to prove the existence of nanopterons in lattices with both nearest and next-to-nearest neighbor spring connections, and we expect that the methods of Venney and Zimmer will carry over to the mass and spring dimers. It is relatively straightforward to identify the appropriate symmetry operator $S$ for the mass and spring dimers when their traveling wave problems are written in the Iooss-Kirchgässner variable, although the symmetry is a different operator for each dimer. It is not apparent at all, however, that the general dimer's equation has any such anticommuting symmetry. While this is certainly not a proof of the nonexistence of nanopteron solutions, it certainly adds to the complications in finding them.

For these reasons, we restrict ourselves to the mass and spring dimers only. The main existence theorem for mass dimers, Theorem 1.4.1, was proved in [FW18], and so we will focus on the case of the spring dimer in this dissertation. These results, leading to the proof of Theorem 1.4.2, are also contained in the forthcoming [Fav].

# Chapter 3. Existence of Periodic Solutions

**3.1. The periodic existence theorem.** The goal of this chapter is to prove the following existence theorem for periodic solutions to (2.4.6) in the spring dimer case.

**3.1.1 Theorem.** *There exist $\epsilon_{\text{per}}$, $a_{\text{per}} > 0$ such that for all $\epsilon \in (0, \epsilon_{\text{per}})$, there are maps*

$$\begin{aligned}
&[-a_{\text{per}}, a_{\text{per}}] \to \mathbb{R} \colon a \mapsto \omega_\epsilon^a \\
&[-a_{\text{per}}, a_{\text{per}}] \to \mathcal{C}_{\text{per}}^\infty \cap \{\text{even functions}\} \colon a \mapsto \psi_{\epsilon,1}^a \\
&[-a_{\text{per}}, a_{\text{per}}] \to \mathcal{C}_{\text{per}}^\infty \cap \{\text{even functions}\} \colon a \mapsto \psi_{\epsilon,2}^a
\end{aligned}$$

*such that the following hold.*

**(i)** *If*

$$\boldsymbol{\nu} := \cos(\cdot)\mathbf{j}, \qquad \boldsymbol{\psi}_\epsilon^a := \begin{pmatrix} \psi_{\epsilon,1}^a \\ \psi_{\epsilon,2}^a \end{pmatrix}, \qquad \text{and} \qquad \boldsymbol{\varphi}_\epsilon^a(X) := \boldsymbol{\nu}(\omega_\epsilon^a X) + \boldsymbol{\psi}_\epsilon^a(\omega_\epsilon^a X),$$

*then $\boldsymbol{\theta} := a\boldsymbol{\varphi}_\epsilon^a$ solves (2.4.6) for all $|a| \le a_{\text{per}}$ and $\epsilon \in (0, \epsilon_{\text{per}})$.*

**(ii)** *The frequency $\omega_\epsilon^0$ satisfies $\omega_\epsilon^0 = \omega_\epsilon$ as defined in (2.6.3) above. We say that $\omega_\epsilon = \mathcal{O}(1/\epsilon)$ in the sense that there are constants $C_1$, $C_2 > 0$ such that*

$$\frac{C_1}{\epsilon} < \omega_\epsilon < \frac{C_2}{\epsilon}$$

*for all $\epsilon \in (0, \epsilon_{\text{per}})$.*

**(iii)** $\psi_{\epsilon,1}^0 = \psi_{\epsilon,2}^0 = 0$.

**(iv)** *For all $r \ge 0$, there is $C_r > 0$ such that*

$$|\epsilon\omega_\epsilon^a| + \|\boldsymbol{\psi}_\epsilon^a\|_{\mathcal{C}_{\text{per}}^r \times \mathcal{C}_{\text{per}}^r} \le C_r$$

*and*

$$|\omega_\epsilon^a - \omega_\epsilon^{\grave{a}}| + \left\|\boldsymbol{\psi}_\epsilon^a - \boldsymbol{\psi}_\epsilon^{\grave{a}}\right\|_{\mathcal{C}_{\text{per}}^r \times \mathcal{C}_{\text{per}}^r} \le C_r|a - \grave{a}|$$

*for all* $|a|$, $|\grave{a}| \le a_{\text{per}}$ *and* $0 < \epsilon < \epsilon_{\text{per}}$.

**3.1.2 Remark.** *We wish to point out that the restriction* $\beta \neq -\varkappa^3$ *in Theorem 1.4.1 is absent from the hypotheses of Theorem 3.1.1, which gives the existence of exact periodic traveling wave solutions. That is, periodic solutions exist under the more general conditions* $\varkappa > 1$ *and* $\beta \neq 0$*; the condition* $\beta \neq -\varkappa^3$ *only enters from the formal long wave limit in Section 2.5, from which the useful* $\operatorname{sech}^2$*-type KdV soliton* $\boldsymbol{\sigma}$ *arises and without which Beale's ansatz simply does not make sense.s When* $\beta = -\varkappa^3$, *the KdV equation* (2.5.6) *reduces to*

$$\alpha_{\varkappa}\theta_2'' - \theta_2 = 0,$$

*which has no nontrivial exponentially decaying solutions, and so we have no natural analogue for* $\boldsymbol{\sigma}$.

**3.2. Bifurcation from a simple eigenvalue.** We introduce a periodic profile and a frequency scaling: let $\boldsymbol{\theta}(X) = \boldsymbol{\phi}(\omega X)$, where $\boldsymbol{\phi} = \boldsymbol{\phi}(Y)$ is $2\pi$-periodic and $\omega \in \mathbb{R}$. Then the problem of (2.4.6), $\boldsymbol{\Theta}_\epsilon(\boldsymbol{\phi}(\omega\cdot)) = 0$, converts to

$$\begin{aligned}\boldsymbol{\Phi}_\epsilon(\boldsymbol{\phi},\omega) := \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2\omega^2(c_\varkappa^2+\epsilon^2)\partial_Y^2 + \lambda_+^{\epsilon\omega}\end{bmatrix}\boldsymbol{\phi} + \begin{bmatrix}\varpi^{\epsilon,\omega} & 0 \\ 0 & \epsilon^2\lambda_+^{\epsilon\omega}\end{bmatrix} B^{\epsilon\omega}(\boldsymbol{\phi},\boldsymbol{\phi}) \\ + \begin{bmatrix}\varpi^{\epsilon,\omega} & 0 \\ 0 & \epsilon^2\lambda_+^{\epsilon\omega}\end{bmatrix} \mathcal{Q}^{\epsilon\omega}(\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}) = 0, \quad (3.2.1)\end{aligned}$$

where $\varpi^{\epsilon,\omega}$ has the symbol

$$\widetilde{\varpi^{\epsilon,\omega}}(k) = \widetilde{\varpi}^{\epsilon,\omega}(k) := \widetilde{\varpi}^{\epsilon}(\omega k)$$

with $\widetilde{\varpi}^\epsilon$ defined in (2.4.3) and the other Fourier multipliers are defined per Section 2.4 and the scaling properties in Appendix D.

At this point we could appeal to the following version of the Crandall-Rabinowitz-Zeidler theorem (stated, with some modifications, in [AAW13]) to prove, for each fixed $\epsilon$, the existence of a family of nontrivial solutions to (3.2.1).

**3.2.1 Theorem (Crandall-Rabinowitz-Zeidler).** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and $\Phi\colon \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ be a function with the following properties. Suppose that*

**(i)** *The partial derivatives $D_{\boldsymbol{\phi}}\Phi$, $D_\omega\Phi$, and $D_{\boldsymbol{\phi}\omega}\Phi$ exist and are continuous.*

**(ii)** $\Phi(0,\omega) = 0$ *for all* $\omega \in \mathbb{R}$.

**(iii)** *There is $\omega_\star \in \mathbb{R}$ such that $\mathcal{D} := D_\phi\Phi(0,\omega_\star)$ has a one-dimensional kernel and has Fredholm index equal to zero.*

**(iv)** *If $\phi_\star \in \mathcal{X}$ spans $\ker(\mathcal{D})$ and $\psi_\star \in \mathcal{Y}$ spans $\ker(\mathcal{D}^*)$, where $\mathcal{D}^*\colon \mathcal{Y} \to \mathcal{X}$ is the adjoint of $\mathcal{D}$, then $\langle D_{\phi\omega}\Phi(0,\omega_\star)\phi_\star, \psi_\star\rangle_{\mathcal{Y}} \neq 0$.*

*Then there is a sequence $\big((\phi_n,\omega_n)\big)$ in $\mathcal{X} \times \mathbb{R}$ such that $\Phi(\phi_n,\omega_n) = 0$ with $\phi_n \neq 0$ for all $n$ and $\lim_{n\to\infty}(\phi_n,\omega_n) = (\phi_\star,\omega_\star)$.*

To apply this theorem, we would work in the Hilbert spaces $E^2_{\text{per}} \times E^2_{\text{per}}$ and $E^0_{\text{per}} \times E^0_{\text{per}}$, where

$$E^r_{\text{per}} := \left\{ f \in H^r_{\text{per}} \;\middle|\; f \text{ is even} \right\}.$$

The necessary regularity on $\mathbf{\Phi}_\epsilon$ follows from Proposition D.3.1. We have $\mathbf{\Phi}_\epsilon(0,\omega) = 0$ for all $\omega$ and we noted in Section 3 that $D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\cos(\cdot)\mathbf{j} = 0$.

Now, take $\omega_\star = \omega_\epsilon$, where $\omega_\epsilon > 0$ was defined in (2.6.3) and satisfies

$$c_\epsilon^2(\pm\epsilon\omega_\epsilon)^2 - \widetilde{\lambda}_+(\pm\epsilon\omega_\epsilon) = 0. \tag{3.2.2}$$

Moreover, $\omega_\epsilon$ is the unique (positive) root of

$$c_\epsilon^2(\pm\epsilon k)^2 - \widetilde{\lambda}_+(\pm\epsilon k) = 0.$$

Suppose

$$D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\phi} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2\omega_\epsilon^2(c_\varkappa^2+\epsilon^2)\partial_Y^2 + \lambda_+^{\epsilon\omega_\epsilon} \end{bmatrix}\boldsymbol{\phi} = 0.$$

Then $\phi_1 = 0$ and

$$\big(c_\epsilon^2(\epsilon\omega_\epsilon k)^2 - \widetilde{\lambda}_+(\epsilon\omega_\epsilon k)\big)\widehat{\phi}_2(k) = 0$$

for all $k \in \mathbb{Z}$. We know

$$c_\epsilon^2(\epsilon\omega_\epsilon k)^2 - \widetilde{\lambda}_+(\epsilon\omega_\epsilon k) \neq 0$$

for $|k| \neq 1$, so $\widehat{\phi}_2(k) = 0$ for $|k| \neq 1$. By (3.2.2) we need not have $\widehat{\phi}_2(\pm 1) = 0$, and so

$$\ker D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_\epsilon(0,\omega_\epsilon) = \mathrm{span}(\{\cos(\cdot),\sin(\cdot)\}) \cap \left(E^2_{\mathrm{per}} \times E^2_{\mathrm{per}}\right) = \mathrm{span}(\{\cos(\cdot)\}).$$

A similar calculation shows that $\ker D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_\epsilon(0,\omega_\epsilon) = \mathrm{span}(\{\cos(\cdot)\})$ as well, and checking condition (iii) in the Crandall-Rabinowitz-Zeidler theorem ultimately rests on the inequality (2.2.9). So, for each $\epsilon > 0$, there is a family $\{(\phi_n^\epsilon, \omega_n^\epsilon)\}$ of solutions to $\boldsymbol{\Phi}_\epsilon(\boldsymbol{\phi},\epsilon) = 0$ satisfying the properties in Theorem 3.2.1.

**3.2.2 Remark.** *We can now explain the full utility of the Friesecke-Pego cancelation and the restriction to supersonic wave speeds and "even $\times$ even" functions. We could have looked for periodic solutions to* (2.2.15) *before the Friesecke-Pego cancelation and without the hypothesis $c^2 > c_\varkappa^2$. The linearization of* (2.2.15) *is*

$$\mathcal{D}_c := \begin{bmatrix} c^2\partial_x^2 + \lambda_- & 0 \\ 0 & c^2\partial_x^2 + \lambda_+ \end{bmatrix}.$$

*Studying the dimension of $\ker \mathcal{D}_c$ amounts to determining the number of intersections of the curves $c^2k^2$ and $\widetilde{\lambda}_+(k)$ and $c^2k^2$ and $\widetilde{\lambda}_-(k)$. By Proposition 2.2.1, the parabola $c^2k^2$ will always intersect the acoustic band $\widetilde{\lambda}_-(k)$ at $k = 0$, and so $\mathcal{D}_c\mathbf{i} = 0$. This parabola will also intersect the optical band $\widetilde{\lambda}_+(k)$ at least twice; this is unavoidable since $c^2k^2$ is strictly increasing and unbounded. For $c^2 < c_\varkappa^2$, the parabola $c^2k^2$ can intersect the acoustic band $\widetilde{\lambda}_-(k)$ too many times; effectively, when $c^2$ is "too small," the parabola $c^2k^2$ is "too wide." All together, without further modifications of our system and our function spaces, there will be at least five intersections, as sketched in Figure 3.2.1.*

*However, after the Friesecke-Pego cancelation, it is clear that $D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_\epsilon(0,\omega)\mathbf{i} \neq 0$ for all $\omega$. More broadly, $\mathcal{D}_1^\epsilon\mathbf{i} \neq 0$, where $\mathcal{D}_1^\epsilon$ is defined in* (2.4.7). *At the same time, the hypothesis $c^2 > c_\varkappa^2$ — which we also needed for the Friesecke-Pego cancelation, due*

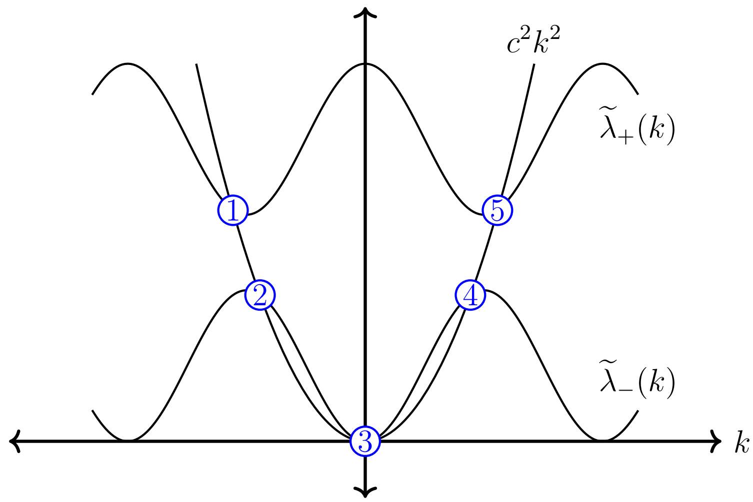


Figure 3.2.1: Intersections of $c^2k^2$ with $\widetilde{\lambda}_\pm(k)$ when $c^2 < c_\varkappa^2$

*to the division in* (2.4.3) *— ensures that the parabola $c^2k^2$ intersects $\widetilde{\lambda}_-(k)$ only at the origin. We label this as #2 in Figure 3.2.2. The Friesecke-Pego cancelation then removes this intersection from "counting" toward the dimension of the kernel.*

*Last, the restriction to "even $\times$ even" functions removes one of the intersections #1 and #3 in Figure 3.2.2, which correspond to the roots $\pm\epsilon\omega_\epsilon$ in* (3.2.2)*. Then we are down to one intersection and a one-dimensional kernel.*

**3.3. Conversion to a fixed-point problem.** We seek estimates on the solutions to (3.2.1) that are uniform in $\epsilon$ for use in the subsequent nanopteron problem of Chapter 4. The Crandall-Rabinowitz-Zeidler approach given above will only work for a single fixed $\epsilon$ at a time, and this will not ostensibly provide the desired uniformity. Instead, we follow the proof of the original version of Theorem 3.2.1 given in [CR71], and we rewrite the problem (3.2.1) in fixed point form. With $\boldsymbol{\nu} = \cos(\cdot)\mathbf{j}$ as in Theorem 3.1.1, let

$$\mathcal{Z} := \left\{\boldsymbol{\psi} \in E^2_{\text{per}} \times E^2_{\text{per}} \;\middle|\; \widehat{\psi}_1(\pm 1) = 0\right\} = \{\boldsymbol{\nu}\}^\perp.$$

In other words, $\mathcal{Z} = \{\boldsymbol{\nu}\}^\perp$, the orthogonal complement of $\{\boldsymbol{\nu}\}$ in $E^2_{\text{per}} \times E^2_{\text{per}}$. We recall the definitions and properties of periodic Sobolev spaces from Appendix C.2.

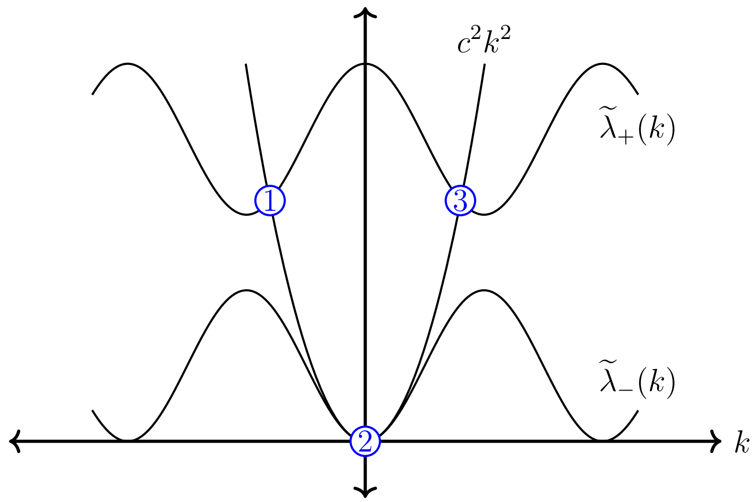


Figure 3.2.2: Intersections of $c^2k^2$ with $\widetilde{\lambda}_\pm(k)$ when $c^2 > c_\varkappa^2$

Then with the ansatz

$$\boldsymbol{\phi} = a\boldsymbol{\nu} + a\boldsymbol{\psi},\ \ \boldsymbol{\psi} \in \mathcal{Z} \quad \text{ and } \quad \omega = \omega_\epsilon + t,\ \ t \in \mathbb{R},$$

which is inspired by the proof in [CR71], the system (3.2.1) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2(\omega_\epsilon + t)^2 c_\epsilon^2 \partial_Y^2 + \lambda_+^{\epsilon(\omega_\epsilon + t)} \end{bmatrix} (\boldsymbol{\nu} + \boldsymbol{\psi}) + a \begin{bmatrix} \varpi_{c_\epsilon}^{\epsilon,\omega_\epsilon+t} & 0 \\ 0 & \epsilon^2 \lambda_+^{\epsilon(\omega_\epsilon+t)} \end{bmatrix} \mathcal{B}^\epsilon(\boldsymbol{\psi}, t) \\ + a \begin{bmatrix} \varpi_{c_\epsilon}^{\epsilon,\omega_\epsilon+t} & 0 \\ 0 & \epsilon^2 \lambda_+^{\epsilon(\omega_\epsilon+t)} \end{bmatrix} \mathcal{E}^\epsilon(\boldsymbol{\psi}, t, a) = 0. \quad (3.3.1)$$

Here we have abbreviated

$$\mathcal{B}^\epsilon(\boldsymbol{\psi}, t) = \begin{pmatrix} \mathcal{B}_1^\epsilon(\boldsymbol{\psi}, t) \\ \mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) \end{pmatrix} := B^{\epsilon(\omega_\epsilon+t)}(\boldsymbol{\nu} + \boldsymbol{\psi}, \boldsymbol{\nu} + \boldsymbol{\psi})$$

and

$$\mathcal{E}^\epsilon(\boldsymbol{\psi}, t, a) = \begin{pmatrix} \mathcal{E}_1^\epsilon(\boldsymbol{\psi}, t, a) \\ \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a) \end{pmatrix}$$

$$:= a\epsilon^2 J_1^{\epsilon(\omega_\epsilon+t)} M_{1/\varkappa} \left[ \left( J^{\epsilon(\omega_\epsilon+t)}(\boldsymbol{\nu} + \boldsymbol{\psi}) \right)^{.3} . N(a\epsilon^2 J^{\epsilon(\omega_\epsilon+t)}(\boldsymbol{\nu} + \boldsymbol{\psi})) \right].$$

Let $\Pi_1$ be the multiplier with symbol

$$\widetilde{\Pi}_1(k) := \delta_{|k|,1} = \begin{cases} 1, & |k| = 1 \\ 0, & |k| \neq 1 \end{cases}$$

and let $\Pi_2 := \mathbb{1} - \Pi_1$. That is, $\Pi$ projects onto $\operatorname{span}(\{\cos(\cdot)\})$. Then

$$\Pi_1 \cos(\cdot) = \cos(\cdot), \qquad \Pi_2 \cos(\cdot) = 0, \qquad \Pi_1 \psi = 0, \quad \text{and} \quad \Pi_2 \psi \tag{3.3.2}$$

for any $\psi \in E^2_{\text{per}}$ with $\widehat{\psi}(1) = 0$.

Let $\xi_c$ be the multiplier with symbol

$$\widetilde{\xi}_c(k) := -c^2k^2 + \widetilde{\lambda}_+(k),$$

which we first encountered in (2.2.7), and let

$$\xi^{\epsilon,t} := \xi_{c_\epsilon}^{\epsilon(\omega_\epsilon+t)} = \epsilon^2(\omega_\epsilon + t)^2 c_\epsilon^2 \partial_Y^2 + \lambda_+^{\epsilon(\omega_\epsilon+t)},$$

so $\xi^{\epsilon,t}$ has the symbol

$$\widetilde{\xi^{\epsilon,t}}(k) = \widetilde{\xi}_{c_\epsilon}(\epsilon(\omega_\epsilon + t)k).$$

After we apply $\Pi_1$ and $\Pi_2$ to the second component of (3.3.1) and use (3.3.2), we see that (3.3.1) is equivalent to the three equations

$$\psi_1 + a\varpi^{\epsilon,\omega_\epsilon+t}\left(\mathcal{B}_1^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_1^\epsilon(\boldsymbol{\psi}, t, a)\right) = 0. \tag{3.3.3}$$

$$\xi^{\epsilon,t}\psi_2 + a\epsilon^2\Pi_2\lambda_+^{\epsilon(\omega_\epsilon+t)}\left(\mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a)\right) = 0, \tag{3.3.4}$$

and

$$\xi^{\epsilon,t}\cos(\cdot) + a\epsilon^2\Pi_1\lambda_+^{\epsilon(\omega_\epsilon+t)}\left(\mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a)\right) = 0. \tag{3.3.5}$$

The first equation, (3.3.3), immediately converts to the fixed-point form

$$\psi_1 = -a\varpi^{\epsilon,\omega_\epsilon+t}\left(\mathcal{B}_1^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_1^\epsilon(\boldsymbol{\psi}, t, a)\right) =: \Psi_1^\epsilon(\boldsymbol{\psi}, t, a). \tag{3.3.6}$$

The invertibility of $\xi^{\epsilon,t}$ on the range of $\Pi_2$, as detailed in Proposition G.1.1, means that (3.3.4) is equivalent to

$$\psi_2 = -a\epsilon^2\left(\xi^{\epsilon,t}\right)^{-1}\Pi_2\lambda_+^{\epsilon(\omega_\epsilon+t)}\left(\mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a)\right) =: \Psi_2^\epsilon(\boldsymbol{\psi}, t, a). \tag{3.3.7}$$

Last, if $\boldsymbol{\psi}$ is even, then (3.3.5) holds if and only if the Fourier transform of its left side evaluated at $k=1$ is equal to zero. We use (G.1.1) below to write

$$\mathfrak{F}[\xi^{\epsilon,t}\cos(\cdot)](1) = \frac{\widetilde{\xi}_{c_\epsilon}(\epsilon\omega_\epsilon + \epsilon t)}{2} = \frac{(\epsilon t)\Upsilon_\epsilon}{2} + \frac{(\epsilon t)^2\mathcal{R}_\epsilon(\epsilon t)}{2},$$

where $\Upsilon_\epsilon$ is bounded away from zero. We conclude that (3.3.3) is equivalent to

$$t = -\frac{\epsilon}{\Upsilon_\epsilon}\mathcal{R}_\epsilon(\epsilon t)t^2 - \frac{2\epsilon a}{\Upsilon_\epsilon}\mathfrak{F}\left[\lambda_+^{\epsilon(\omega_\epsilon+t)}\left(\mathcal{B}_2^\epsilon(\boldsymbol{\psi},t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi},t,a)\right)\right](1) =: \Psi_3^\epsilon(\boldsymbol{\psi},t,a). \quad (3.3.8)$$

Now we are ready to pose our fixed point problem. Let

$$\mathcal{W}^r = \left(E^r_{\text{per}} \times E^r_{\text{per}}\right) \cap \mathcal{Z} \quad \text{and} \quad \|\boldsymbol{\psi}\|_r := \|\boldsymbol{\psi}\|_{\mathcal{W}^r} = \|\psi_1\|_{H^r_{\text{per}}} + \|\psi_2\|_{H^r_{\text{per}}}$$

We will find an interval $[-a_{\text{per}}, a_{\text{per}}] \subseteq \mathbb{R}$ and maps

$$[-a_{\text{per}}, a_{\text{per}}] \to \mathcal{W}^2 \colon a \mapsto \boldsymbol{\psi}_\epsilon^a = (\psi_{\epsilon,1}^a, \psi_{\epsilon,2}^a) \quad \text{and} \quad [-a_{\text{per}}, a_{\text{per}}] \to \mathbb{R} \colon t \mapsto t_\epsilon^a$$

such that

$$\begin{pmatrix} \psi_{1,\epsilon}^a \\ \psi_{2,\epsilon}^a \\ t_\epsilon^a \end{pmatrix} = \begin{pmatrix} \Psi_1^\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a) \\ \Psi_2^\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a) \\ \Psi_3^\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a) \end{pmatrix} =: \boldsymbol{\Psi}^\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a)$$

for all $\epsilon$ in the interval $(0, \epsilon_{\text{per}})$, where $\epsilon_{\text{per}}$ comes from Proposition G.1.1. Once we show the existence of these maps, we will prove the additional properties and estimates in Theorem 3.1.1.

**3.4. The fixed point lemma.** We will use the following lemma, proved in Appendix F.4, to solve the fixed point problem and obtain the various estimates and smoothness properties.

**3.4.1 Lemma.** *Let $\mathcal{X}$ be a Banach space and and let $F_\epsilon \colon \mathcal{X} \times \mathbb{R} \to \mathcal{X}$, $0 < \epsilon < \epsilon_0$, be a family of maps with the following properties: there exist continuous maps $\mathcal{M}_{\text{map}} \colon \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{M}_{\text{lip}} \colon \mathbb{R}_+^2 \to \mathbb{R}_+$, and $\mathcal{M}_{\max} \colon \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $a_1 > 0$ such that if $|a|$, $|\grave{a}| \le a_1$, then*

$$\sup_{0<\epsilon<\epsilon_0} \|F_\epsilon(x,a)\| \le \mathcal{M}_{\text{map}}[\|x\|]\left(|a| + \|x\|^2\right), \quad (3.4.1)$$

$$\sup_{0<\epsilon<\epsilon_0} \|F_\epsilon(x,a) - F_\epsilon(\grave{x},a)\| \leq \mathcal{M}_{\mathrm{lip}}[\|x\|\,,\|\grave{x}\|]\,(|a| + \|x\| + \|\grave{x}\|)\,\|x-\grave{x}\|\,, \tag{3.4.2}$$

*and*

$$\sup_{0<\epsilon<\epsilon_0} \|F_\epsilon(x,a) - F_\epsilon(x,\grave{a})\| \leq \mathcal{M}_{\max}[\|x\|]|a-\grave{a}| \tag{3.4.3}$$

*for any* $x, \grave{x} \in \mathcal{X}$. *Then there are constants* $a_0$, $r_0$ *such that if* $|a| \leq a_0$ *and* $0 < \epsilon < \epsilon_0$, *there exists a unique* $x_\epsilon^a \in \mathfrak{B}(r_0) := \{x \in \mathcal{X} \mid \|x\| \leq r_0\}$ *such that*

$$\|x_\epsilon^a\| \leq r_0 \quad \textit{and} \quad F_\epsilon(x_\epsilon^a, a) = x_\epsilon^a. \tag{3.4.4}$$

*Moreover, there is a constant* $C > 0$ *such that if* $|a|$, $|\grave{a}| \leq a_0$, *then*

$$\sup_{0<\epsilon<\epsilon_0} \left\|x_\epsilon^a - x_\epsilon^{\grave{a}}\right\| \leq C|a-\grave{a}|. \tag{3.4.5}$$

**3.5. Solution of the fixed-point problem.** It is convenient to introduce some new notation. Let

$$\mathcal{L}_1^\epsilon(t) := \begin{bmatrix} \varpi^{\epsilon,\omega_\epsilon+t} & 0 \\ 0 & \epsilon^2\left(\xi^{\epsilon,t}\right)^{-1}\Pi_2 \end{bmatrix},$$

$$\mathcal{L}_2^\epsilon(t) := -\begin{bmatrix} 1 & 0 \\ 0 & \lambda_+^{\epsilon(\omega_\epsilon+t)} \end{bmatrix},$$

$$\mathcal{L}_3^\epsilon(t) := J^{\epsilon(\omega_\epsilon+t)},$$

$$\mathcal{L}_4^\epsilon(t) := J_1^{\epsilon(\omega_\epsilon+t)},$$

$$\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a) := \mathcal{L}_2^\epsilon(t)\left(\mathcal{B}^\epsilon(\boldsymbol{\psi},t) + \mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)\right),$$

and

$$\mathcal{F}_\epsilon(\boldsymbol{\psi},t,a) := a\mathcal{L}_1^\epsilon(t)\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a) = \begin{pmatrix} \Psi_1^\epsilon(\boldsymbol{\psi},t;a) \\ \Psi_2^\epsilon(\boldsymbol{\psi},t;a) \end{pmatrix}.$$

We obtain the estimates (3.4.1), (3.4.2), and (3.4.3) for the function $\boldsymbol{\Psi}^\epsilon$ by showing they hold for the function $\mathcal{F}_\epsilon$ just defined and then, separately, for $\Psi_3^\epsilon$, for which the map $\mathcal{G}_\epsilon$ will be useful. In our application of Lemma 3.4.1 we will take the Banach space to be $\mathcal{X} = \mathcal{W}^2$. However, we will prove various estimates in the spaces $\mathcal{W}^r$ for the sake of the subsequent bootstrap arguments.

*3.5.1. Estimates for $\mathcal{G}_\epsilon$.*

**3.5.1 Proposition.** *Let $\epsilon_{\mathrm{per}} > 0$ be as in Proposition G.1.1. We have the following estimates for all $r \geq 2$.*

**(i)** *There exists an increasing function $\mathcal{M}^{\mathcal{G}}_{\mathrm{map},r} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\sup_{\substack{0<\epsilon<\epsilon_{\mathrm{per}} \\ |t|\leq 1}} \|\mathcal{G}_\epsilon(\boldsymbol{\psi}, t, a)\|_r \leq \mathcal{M}^{\mathcal{G}}_{\mathrm{map},r}[\|\boldsymbol{\psi}\|_r] \tag{3.5.1}$$

*for any $\boldsymbol{\psi} \in \mathcal{W}^r$ and $|a| \leq 1$.*

**(ii)** *There exists a radially increasing[5] function $\mathcal{M}^{\mathcal{G}}_{\mathrm{lip},r} \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ such that*

$$\sup_{0<\epsilon<\epsilon_{\mathrm{per}}} \left\|\mathcal{G}_\epsilon(\boldsymbol{\psi}, t, a) - \mathcal{G}_\epsilon(\grave{\boldsymbol{\psi}}, \grave{t}, a)\right\|_{r-1} \leq \mathcal{M}^{\mathcal{G}}_{\mathrm{lip},r}[\|\boldsymbol{\psi}\|_r, \|\grave{\boldsymbol{\psi}}\|_r] \left(\|\boldsymbol{\psi} - \grave{\boldsymbol{\psi}}\|_r + |t - \grave{t}|\right) \tag{3.5.2}$$

*for any $\boldsymbol{\psi}$, $\grave{\boldsymbol{\psi}} \in \mathcal{W}^r$, $|t|$, $|\grave{t}| \leq 1$, and $|a| \leq 1$.*

In the proof of this proposition and others in this section, we will often rely on the Lipschitz estimates for operator norms of Fourier multipliers on periodic spaces given in Proposition D.3.1.

**Proof.** Throughout this proof, we assume $\epsilon$, $|t|$, $|\grave{t}|$, $|a| \leq 1$ and $\boldsymbol{\psi} \in \mathcal{W}^r$.

**(i)** A first pass using (G.1.4) shows

$$\begin{aligned}
\|\mathcal{G}_\epsilon(\boldsymbol{\psi}, t, a)\|_r &\leq |a| \, \|\mathcal{L}^\epsilon_2(t)\|_{\mathbf{B}(\mathcal{W}^r, \mathcal{W}^r)} \left(\|\mathcal{B}^\epsilon(\boldsymbol{\psi}, t)\|_r + \|\mathcal{E}^\epsilon(\boldsymbol{\psi}, t, a)\|_r\right) \\
&\leq |a| C^2_{\mathrm{map}} \left(\|\mathcal{B}^\epsilon(\boldsymbol{\psi}, t)\|_r + \|\mathcal{E}^\epsilon(\boldsymbol{\psi}, t, a)\|_r\right).
\end{aligned}$$

Since $r \geq 2$, we can use the Sobolev embedding to estimate the products in $\mathcal{B}^\epsilon$ and $\mathcal{E}^\epsilon$:

$$\|\mathcal{B}^\epsilon(\boldsymbol{\psi}, t)\|_r = \left\|B^{\epsilon(\omega_\epsilon + t)}(\boldsymbol{\nu} + \boldsymbol{\psi}, \boldsymbol{\nu} + \boldsymbol{\psi})\right\|_r$$

[5] See Remark E.1.2.

$$
\begin{aligned}
&= \left\| \mathcal{L}_4^{\epsilon(\omega_\epsilon+t)} M_{\beta/\varkappa}[\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi})]^{.2} \right\|_r \\
&\leq C_{\mathrm{map}} \frac{\beta}{\varkappa} \left\| \mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi}) \right\|_r^2 \\
&\leq C_{\mathrm{map}}^3 \frac{\beta}{\varkappa} \left( \|\boldsymbol{\nu}\|_r^2 + 2\|\boldsymbol{\nu}\|_r \|\boldsymbol{\psi}\|_r + \|\boldsymbol{\psi}\|_r^2 \right) \\
&\leq C_{\mathrm{map}}^3 C_r \frac{\beta}{\varkappa} \left( 1 + \|\boldsymbol{\psi}\|_r + \|\boldsymbol{\psi}\|_r^2 \right)
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\|\mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)\|_r &= \left\| \mathcal{L}_4^\epsilon(t) M_{1/\varkappa}[(\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi}))^{.3}.N(a\epsilon^2\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi}))] \right\|_r \\
&\leq C_{\mathrm{map}}^2 \frac{1}{\varkappa} \|\boldsymbol{\nu}+\boldsymbol{\psi}\|_r^3 \left\| N(a\epsilon^2\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi})) \right\|_r .
\end{aligned}
$$

We apply Proposition E.5.1 to estimate

$$
\left\| N(a\epsilon^2\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi})) \right\|_r \leq \mathcal{M}_r[|a|\epsilon^2 \|\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi})\|_r]
$$

for some increasing function $\mathcal{M}_r \colon \mathbb{R}_+ \to \mathbb{R}_+$. Since

$$
\sup_{\substack{|a|,|t|\leq 1 \\ 0<\epsilon<\epsilon_{\mathrm{per}}}} |a|\epsilon^2 \|\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi})\|_{r-1} \leq C_r \left(1 + \|\boldsymbol{\psi}\|_r\right)
$$

and $\mathcal{M}_r$ is increasing, we have

$$
\left\| N(a\epsilon^2\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi})) \right\|_r \leq \mathcal{M}_r[\|\boldsymbol{\psi}\|_r].
$$

This bound, together with the estimate on $\|\mathcal{B}^\epsilon(\boldsymbol{\psi},t)\|_r$ above, produces (3.5.1).

**(ii)** We have

$$
\begin{aligned}
\left\| \mathcal{G}_\epsilon(\boldsymbol{\psi},t,a) - \mathcal{G}_\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a) \right\|_{r-1} &= \underbrace{\left\| (\mathcal{L}_2^\epsilon(t) - \mathcal{L}_2^\epsilon(\grave{t}))(\mathcal{B}^\epsilon(\boldsymbol{\psi},t) + \mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)) \right\|_{r-1}}_{\Delta_1} \\
&\quad + \underbrace{\left\| \mathcal{L}_2^\epsilon(\grave{t})(\mathcal{B}^\epsilon(\boldsymbol{\psi},t) - \mathcal{B}^\epsilon(\grave{\boldsymbol{\psi}},\grave{t})) \right\|_{r-1}}_{\Delta_2}
\end{aligned}
$$

$$+\underbrace{\left\|\mathcal{L}_2^\epsilon(\grave{t})(\mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)-\mathcal{E}^\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a))\right\|_{r-1}}_{\Delta_3}.$$

It is straightforward to estimate $\Delta_1$ using (G.1.7) and Proposition E.5.1:

$$\begin{aligned}\Delta_1 &\leq \left\|\mathcal{L}_2^\epsilon(t)-\mathcal{L}_2^\epsilon(\grave{t})\right\|_{\mathbf{B}(\mathcal{W}^r,\mathcal{W}^{r-1})}\left\|\mathcal{B}^\epsilon(\boldsymbol{\psi},t)+\mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)\right\|_r \\ &\leq C_{\mathrm{lip}}|t-\grave{t}|\left\|\mathcal{B}^\epsilon(\boldsymbol{\psi},t)+\mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)\right\|_r \\ &\leq C_{\mathrm{lip}}|t-\grave{t}|\mathcal{M}_r[\|\boldsymbol{\psi}\|_r].\end{aligned}$$

We handle $\Delta_2$ and $\Delta_3$ in essentially the same way; we estimated

$$\Delta_2 \leq C_{\mathrm{lip}}\big(|t-\grave{t}|+\left\|\boldsymbol{\psi}-\grave{\boldsymbol{\psi}}\right\|_{r-1}\big)$$

in [FW18], so we provide some more detail only for $\Delta_3$ here. First,

$$\begin{aligned}\Delta_3 &\leq \left\|\mathcal{L}_2^\epsilon(\grave{t})\right\|_{\mathbf{B}(\mathcal{W}^{r-1},\mathcal{W}^{r-1})}\left\|\mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)-\mathcal{E}^\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a)\right\|_{r-1} \\ &\leq C_{\mathrm{map}}\underbrace{\left\|\mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)-\mathcal{E}^\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a)\right\|_{r-1}}_{\Delta_4},\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}^\epsilon(\boldsymbol{\psi},t,a)-\mathcal{E}^\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a) &= \mathcal{L}_4^\epsilon(t)M_{1/\varkappa}[(\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi}))^{.3}.N(a\epsilon^2\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi}))] \\ &\quad -\mathcal{L}_4^\epsilon(\grave{t})M_{1/\varkappa}[(\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\nu}+\grave{\boldsymbol{\psi}}))^{.3}.N(a\epsilon^2\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\nu}+\grave{\boldsymbol{\psi}}))].\end{aligned}$$

Then, adding a number of zeroes, we can bound $\Delta_4$ in the natural way, and the only term the likes of which we have not seen before, either here or in [FW18], will be

$$\left\|\mathcal{L}_4^\epsilon(\grave{t})M_{1/\varkappa}[(\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\nu}+\grave{\boldsymbol{\psi}}))^{.3}.(N(a\epsilon^2\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi}))-N(a\epsilon^2\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\nu}+\grave{\boldsymbol{\psi}})))]\right\|_{r-1}.$$

After factoring out the operators $\mathcal{L}_4^\epsilon(\grave{t})M_{1/\varkappa}$ and using the Sobolev inequality (C.2.3) for products, we invoke Proposition E.5.2 to bound

$$\left\|N(a\epsilon^2\mathcal{L}_3(t)(\boldsymbol{\nu}+\boldsymbol{\psi}))-N(a\epsilon^2\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\nu}+\grave{\boldsymbol{\psi}}))\right\|_{r-1}$$

$$\leq \mathcal{M}_{r-1}[\left\|a\epsilon^2\mathcal{L}_3^\epsilon(t)(\boldsymbol{\nu}+\boldsymbol{\psi})\right\|_{r-1},\left\|a\epsilon^2\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\nu}+\grave{\boldsymbol{\psi}})\right\|_{r-1}]\left\|\mathcal{L}_3(t)(\boldsymbol{\nu}+\boldsymbol{\psi})-\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\nu}+\grave{\boldsymbol{\psi}})\right\|_{r-1}.$$

Since $\mathcal{M}_{r-1}$ is radially increasing, the uniform bound on $\mathcal{L}_3^\epsilon$ and the triangle inequality allow us to bound this by

$$\mathcal{M}_{r-1}[\|\boldsymbol{\psi}\|_r,\|\grave{\boldsymbol{\psi}}\|_r]\Big(\left\|(\mathcal{L}_3^\epsilon(t)-\mathcal{L}_3^\epsilon(\grave{t}))\boldsymbol{\nu}\right\|_{r-1}+\left\|(\mathcal{L}_3^\epsilon(t)-\mathcal{L}_3^\epsilon(\grave{t}))\boldsymbol{\psi}\right\|_{r-1}$$
$$+\left\|\mathcal{L}_3^\epsilon(\grave{t})(\boldsymbol{\psi}-\grave{\boldsymbol{\psi}})\right\|_{r-1}\Big),$$

and we estimate these terms easily enough using (G.1.7) to achieve (3.5.2). ■

*3.5.2. Estimates for $\mathcal{F}_\epsilon$.* We will obtain the estimates (3.4.1), (3.4.2), and (3.4.3) for $\mathcal{F}_\epsilon$ directly from the following proposition.

**3.5.2 Proposition.** *Let $\epsilon_{\text{per}}>0$ be as in Proposition G.1.1. The following estimates hold for all $r\geq 2$.*

**(i)** *There exists an increasing function $\mathcal{M}_{\text{map},r}^{\mathcal{F}}\colon\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$\sup_{\substack{0<\epsilon<\epsilon_{\text{per}}\\|t|\leq 1}}\|\mathcal{F}_\epsilon(\boldsymbol{\psi},t,a)\|_r\leq|a|\mathcal{M}_{\text{map},r}^{\mathcal{F}}[\|\boldsymbol{\psi}\|_{r-1}] \tag{3.5.3}$$

*for any $\boldsymbol{\psi}\in\mathcal{W}^r$ and $|a|\leq 1$.*

**(ii)** *There exists a radially increasing function $\mathcal{M}_{\text{lip},r}^{\mathcal{F}}\colon\mathbb{R}_+^2\to\mathbb{R}_+$ such that*

$$\sup_{0<\epsilon<\epsilon_{\text{per}}}\left\|\mathcal{F}_\epsilon(\boldsymbol{\psi},t,a)-\mathcal{F}_\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a)\right\|_r\leq|a|\mathcal{M}_{\text{lip},r}^{\mathcal{F}}[\|\boldsymbol{\psi}\|_r,\|\grave{\boldsymbol{\psi}}\|_r]\Big(\left\|\boldsymbol{\psi}-\grave{\boldsymbol{\psi}}\right\|_{r-1}+|t-\grave{t}|\Big) \tag{3.5.4}$$

*for any $\boldsymbol{\psi}$, $\grave{\boldsymbol{\psi}}\in\mathcal{W}^r$, $|t|$, $|\grave{t}|\leq 1$, and $|a|\leq 1$.*

**(iii)** *There exists a continuous function $\mathcal{M}_{\text{max},r}^{\mathcal{F}}\colon\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$\sup_{\substack{0<\epsilon<\epsilon_{\text{per}}\\|t|\leq 1}}\|\mathcal{F}_\epsilon(\boldsymbol{\psi},t,a)-\mathcal{F}_\epsilon(\boldsymbol{\psi},t,\grave{a})\|_r\leq\mathcal{M}_{\text{max},r}^{\mathcal{G}}[\|\boldsymbol{\psi}\|_r]|a-\grave{a}|. \tag{3.5.5}$$

**Proof.** **(i)** We use the smoothing property of $\mathcal{L}_1^\epsilon(t)$, (G.1.3), and (3.5.1) to find

$$\begin{aligned}\|\mathcal{F}_\epsilon(\boldsymbol{\psi},t,a)\|_r &= |a|\,\|\mathcal{L}_1^\epsilon(t)\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a)\|_r \\ &\le |a|\,\|\mathcal{L}_1^\epsilon(t)\|_{\mathbf{B}(\mathcal{W}^{r-1},\mathcal{W}^r)}\,\|\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a)\|_{r-1} \\ &\le C_{\mathrm{map}}|a|\mathcal{M}^{\mathcal{G}}_{\mathrm{map},r-1}[\|\boldsymbol{\psi}\|_{r-1}].\end{aligned}$$

**(ii)** First, we have

$$\begin{aligned}\left\|\mathcal{F}_\epsilon(\boldsymbol{\psi},t,a)-\mathcal{F}_\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a)\right\|_r &= |a|\left\|\mathcal{L}_1^\epsilon(t)\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a)-\mathcal{L}_1^\epsilon(\grave{t})\mathcal{G}_\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a)\right\|_r \\ &\le |a|\underbrace{\left\|(\mathcal{L}_1^\epsilon(t)-\mathcal{L}_1^\epsilon(\grave{t}))\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a)\right\|_r}_{\Delta_1} \\ &\quad+|a|\underbrace{\left\|\mathcal{L}_1^\epsilon(\grave{t})(\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a)-\mathcal{G}_\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a))\right\|_r}_{\Delta_2}.\end{aligned}$$

We apply (G.1.6) to $\Delta_1$ and then (3.5.1) to find

$$\begin{aligned}\Delta_1 &\le |a|\left\|\mathcal{L}_1^\epsilon(t)-\mathcal{L}_1^\epsilon(\grave{t})\right\|_{\mathbf{B}(\mathcal{W}^r,\mathcal{W}^r)}\|\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a)\|_r \\ &\le |a|C_{\mathrm{lip}}|t-\grave{t}|\mathcal{M}^{\mathcal{G}}_{\mathrm{map},r}[\|\boldsymbol{\psi}\|_r].\end{aligned}$$

For $\Delta_2$, we need the smoothing property of $\mathcal{L}_1^\epsilon(\grave{t})$:

$$\begin{aligned}\Delta_2 &\le |a|\left\|\mathcal{L}_1^\epsilon(\grave{t})\right\|_{\mathbf{B}(\mathcal{W}^{r-2},\mathcal{W}^r)}\left\|\mathcal{G}_\epsilon(\boldsymbol{\psi},t,a)-\mathcal{G}_\epsilon(\grave{\boldsymbol{\psi}},\grave{t},a)\right\|_{r-2} \\ &\le C_{\mathrm{map}}|a|\mathcal{M}^{\mathcal{G}}_{\mathrm{lip},r}[\|\boldsymbol{\psi}\|_r,\|\grave{\boldsymbol{\psi}}\|_r]\left(\left\|\boldsymbol{\psi}-\grave{\boldsymbol{\psi}}\right\|_{r-1}+|t-\grave{t}|\right).\end{aligned}$$

**(iii)** The necessary estimates are straightfoward and similar enough to the proof of (3.5.4) that we omit them; they rely fundamentally on the mapping and Lipschitz estimates (3.5.1) and (3.5.2) for $\mathcal{G}_\epsilon$ and on the Lipschitz composition estimate in Proposition E.5.2. ■

*3.5.3. Estimates for $\Psi_3^\epsilon$.* We take $r = 2$ and $\epsilon \in (0, \epsilon_{\mathrm{per}})$.

*Proof of* (3.4.1) *for* $\Psi_3^\epsilon$. The triangle inequality gives

$$|\Psi_3^\epsilon(\boldsymbol{\psi}, t, a)| \le \frac{1}{\Upsilon_\epsilon}|\mathcal{R}_\epsilon(\epsilon t)|t^2 + \frac{2|a|}{\Upsilon_\epsilon}\left|\mathfrak{F}\left[\lambda_+^{\epsilon(\omega_\epsilon+t)}(\mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a))\right](1)\right|$$

The bound (G.1.2) on $\Upsilon_\epsilon$ and (G.1.5) bound the first term above by $t^2/b_0$, and elementary properties of the Fourier transform, the Sobolev embedding, and (3.5.1) imply

$$\begin{aligned}\left|\mathfrak{F}\left[\lambda_+^{\epsilon(\omega_\epsilon+t)}(\mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a))\right](1)\right| &\le \left\|\lambda_+^{\epsilon(\omega_\epsilon+t)}(\mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a))\right\|_{L^\infty} \\ &\le C\left\|\lambda_+^{\epsilon(\omega_\epsilon+t)}(\mathcal{B}_2^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}_2^\epsilon(\boldsymbol{\psi}, t, a))\right\|_2 \\ &\le C\left\|\mathcal{L}_2^\epsilon(t)(\mathcal{B}^\epsilon(\boldsymbol{\psi}, t) + \mathcal{E}^\epsilon(\boldsymbol{\psi}, t, a))\right\|_2 \\ &= C\left\|\mathcal{G}_\epsilon(\boldsymbol{\psi}, t, a)\right\|_2 \\ &\le C\mathcal{M}_{\mathrm{map},2}^{\mathcal{G}}[\|\boldsymbol{\psi}\|_2].\end{aligned}$$

All together, we have

$$|\Psi_3^\epsilon(\boldsymbol{\psi}, t, a)| \le C_{\mathrm{map}}t^2 + 2C|a|\mathcal{M}_{\mathrm{map},2}^{\mathcal{G}}[\|\boldsymbol{\psi}\|_2].$$

*Proof of* (3.4.2) *for* $\Psi_3^\epsilon$. We use the triangle inequality to estimate

$$\begin{aligned}|\Psi_3^\epsilon(\boldsymbol{\psi}, t, a) - \Psi_3^\epsilon(\grave{\boldsymbol{\psi}}, \grave{t}, a)| &\le Ct^2|\mathcal{R}_\epsilon(\epsilon t) - \mathcal{R}_\epsilon(\epsilon\grave{t})| + |\mathcal{R}_\epsilon(\epsilon\grave{t})||t^2 - \grave{t}^2| \\ &\quad + C|a|\left\|\mathcal{G}_\epsilon(\boldsymbol{\psi}, t, a) - \mathcal{G}_\epsilon(\grave{\boldsymbol{\psi}}, \grave{t}, a)\right\|_1.\end{aligned}$$

Here we have used the Sobolev embedding inequality (C.2.5) to bound the Fourier transform terms by their $\mathcal{W}^1$-norm instead of the $\mathcal{W}^2$-norm as we did above with the mapping estimate. This allows us to use (3.5.2) to bound

$$\left\|\mathcal{G}_\epsilon(\boldsymbol{\psi}, t, a) - \mathcal{G}_\epsilon(\grave{\boldsymbol{\psi}}, \grave{t}, a)\right\|_1 \le \mathcal{M}_{\mathrm{lip},2}^{\mathcal{G}}[\|\boldsymbol{\psi}\|_2, \|\grave{\boldsymbol{\psi}}\|_2]\left(\|\boldsymbol{\psi} - \grave{\boldsymbol{\psi}}\|_2 + |t - \grave{t}|\right).$$

We estimate the first term with (G.1.8):

$$|\mathcal{R}_\epsilon(\epsilon t) - \mathcal{R}_\epsilon(\epsilon \grave{t})| \le C_{\text{lip}}|t - \grave{t}|$$

and the second by the difference of squares and (G.1.5).

*Proof of* (3.4.3) *for* $\Psi_3^\epsilon$. The proof is the same as that of (3.5.5) above.

*3.5.4. Proof of the remainder of Theorem 3.1.1.* Taking $r = 2$ in Proposition 3.5.2 and using the estimates for $\Psi_3^\epsilon$ in Section 3.5.3, we find that the map $\mathbf{\Psi}_\epsilon$ satisfies the estimates of Lemma 3.4.1 on the space $\mathcal{W}^2 = \left(H^2_{\text{per}} \times H^2_{\text{per}}\right) \cap \mathcal{Z}$. So, there exist $r_0$, $\epsilon_{\text{per}}$, $a_{\text{per}} \in (0,1)$ such that for all $0 < \epsilon < \epsilon_{\text{per}}$ and $|a| \le a_{\text{per}}$, there is a unique $(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a) = (\psi_{1,\epsilon}^a, \psi_{2,\epsilon}^a, t_\epsilon^a) \in \mathcal{W}^2 \times \mathbb{R}$ satisfying

$$\mathbf{\Psi}_\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a) = (\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a)$$

and

$$\|(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a)\|_{\mathcal{W}^2\times\mathbb{R}} = \left\|\psi_{\epsilon,1}^a\right\|_{H^2_{\text{per}}} + \left\|\psi_{\epsilon,2}^a\right\|_{H^2_{\text{per}}} + |t_\epsilon^a| \le r_0.$$

There is also a constant $C_2 > 0$ such that for $|a|$, $|\grave{a}| \le a_{\text{per}}$ and $0 < \epsilon < \epsilon_{\text{per}}$, we have

$$\left\|(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a) - (\boldsymbol{\psi}_\epsilon^{\grave{a}}, t_\epsilon^{\grave{a}})\right\|_{\mathcal{W}^2\times\mathbb{R}} \le C_2|a - \grave{a}| \tag{3.5.6}$$

Set

$$\omega_\epsilon^a := \omega_\epsilon + t_\epsilon^a.$$

We are ready to prove the rest of Theorem 3.1.1.

*Proof of (i).* Undoing the fixed point set-up of Section 3.3, we find that

$$\boldsymbol{\theta}(X) := a\boldsymbol{\varphi}_\epsilon^a(X) := a\boldsymbol{\nu}(\omega_\epsilon^a X) + a\boldsymbol{\psi}_\epsilon^a(\omega_\epsilon^a X)$$

solves (2.4.6).

*Proof of (ii) and (iii).* Recalling the definitions of the components of $\mathbf{\Psi}_\epsilon$ in (3.3.6), (3.3.7), and (3.3.8), we compute

$$(\psi_{1,\epsilon}^0, \psi_{2,\epsilon}^0, t_\epsilon^0) = \mathbf{\Psi}_\epsilon(\boldsymbol{\psi}_\epsilon^0, t_\epsilon^0, 0) = (0,0,0),$$

and so $\boldsymbol{\psi}_\epsilon^0 = 0$ and $t_\epsilon^0 = 0$, hence $\omega_\epsilon^0 = \omega_\epsilon + t_\epsilon^0 = \omega_\epsilon$.

For the $\mathcal{O}(\epsilon)$ bound on $\omega_\epsilon$, we know from (2.2.8) that with $c_\epsilon = \sqrt{c_\varkappa^2 + \epsilon^2}$ and $\omega_\epsilon = \Omega_{c_\epsilon}/\epsilon$, we have

$$\frac{1}{\epsilon}\left(\frac{\sqrt{2\varkappa}}{\sqrt{c_\varkappa^2+\epsilon_{\mathrm{per}}^2}}\right) \le \frac{1}{\epsilon}\frac{\sqrt{2\varkappa}}{\sqrt{c_\varkappa^2+\epsilon^2}} \le \omega_\epsilon \le \frac{1}{\epsilon}\frac{\sqrt{2+2\varkappa}}{\sqrt{c_\varkappa^2+\epsilon^2}} \le \frac{1}{\epsilon}\left(\frac{\sqrt{2+2\varkappa}}{c_\varkappa}\right).$$

*Proof of (iv).* Since $\boldsymbol{\psi}_\epsilon^a = \mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a)$, we use (3.5.3) to estimate

$$\|\boldsymbol{\psi}_\epsilon^a\|_r \le \mathcal{M}_{\mathrm{map},r}^{\mathcal{F}}[\|\boldsymbol{\psi}_\epsilon^a\|_{r-1}].$$

When $r = 3$, we know that $\|\boldsymbol{\psi}_\epsilon^a\|_2 \le r_0$, so the continuity of $\mathcal{M}_{\mathrm{map},2}^{\mathcal{F}}$ implies

$$\|\boldsymbol{\psi}_\epsilon^a\|_3 \le a_{\mathrm{per}} \max_{0\le s\le r_0} \mathcal{M}_{\mathrm{map},2}^{\mathcal{F}}[s] < \infty.$$

Induction on $r$ then furnishes a constant $C_r > 0$ such that

$$\|\boldsymbol{\psi}_\epsilon^a\|_r \le C_r, \ \ r \ge 3.$$

Next,

$$\begin{aligned}
\left\|\boldsymbol{\psi}_\epsilon^a - \boldsymbol{\psi}_\epsilon^{\grave{a}}\right\|_r &= \left\|\mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a) - \mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^{\grave{a}}, t_\epsilon^{\grave{a}}, \grave{a})\right\|_r \\
&\le \underbrace{\left\|\mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^a, t_\epsilon^a, a) - \mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^{\grave{a}}, t_\epsilon^{\grave{a}}, a)\right\|_r}_{\Delta_{1,r}} + \underbrace{\left\|\mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^{\grave{a}}, t_\epsilon^{\grave{a}}, a) - \mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^{\grave{a}}, t_\epsilon^{\grave{a}}, \grave{a})\right\|_r}_{\Delta_{2,r}}.
\end{aligned}$$

We bound $\Delta_{1,r}$ using (3.5.4):

$$\Delta_{1,r} \le \mathcal{M}_{\mathrm{lip},r}^{\mathcal{F}}[\|\boldsymbol{\psi}_\epsilon^a\|_r, \left\|\boldsymbol{\psi}_\epsilon^{\grave{a}}\right\|_r]\left(\left\|\boldsymbol{\psi}_\epsilon^a - \boldsymbol{\psi}_\epsilon^{\grave{a}}\right\|_{r-1} + |t_\epsilon^a - t_\epsilon^{\grave{a}}|\right)$$

and $\Delta_{2,r}$ using (3.5.5):

$$\Delta_{2,r} \le \mathcal{M}_{\max,r}^{\mathcal{F}}[\left\|\boldsymbol{\psi}_\epsilon^{\grave{a}}\right\|_r]|a - \grave{a}|.$$

The uniform bounds on $\|\boldsymbol{\psi}_\epsilon^a\|_r$ and $\|\boldsymbol{\psi}_\epsilon^a\|_r$ and the continuity of $\mathcal{M}_{\mathrm{lip},r}^{\mathcal{F}}$ and $\mathcal{M}_{\max,r}^{\mathcal{F}}$ then allow us to bound

$$\Delta_{1,r} + \Delta_{2,r} \le C_r\left(\left\|\boldsymbol{\psi}_\epsilon^a - \boldsymbol{\psi}_\epsilon^{\grave{a}}\right\|_{r-1} + |t_\epsilon^a - t_\epsilon^{\grave{a}}| + |a - \grave{a}|\right).$$

All that remains is to induct on $r$ using the base case (3.5.6).

# Chapter 4. The Nanopteron Equations

**4.1. Beale's ansatz.** As detailed in Appendix C.3, we define $E_q^1$ to be the space of even, exponentially decaying functions in $H^1$, i.e.,

$$E_q^1 := \left\{ f \in H^1 \;\middle|\; f \text{ is even and} \cosh(q\cdot)f \in H^1 \right\},$$

with norm

$$\|f\|_{1,q} := \|\cosh(q\cdot)f\|_{H^1}.$$

We return to our main problem $\mathbf{\Theta}_\epsilon(\boldsymbol{\theta}) = 0$ from (2.4.6) and make Beale's ansatz:

$$\boldsymbol{\theta} = \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a) := \boldsymbol{\sigma} + a\boldsymbol{\varphi}_\epsilon^a + \boldsymbol{\eta}, \tag{4.1.1}$$

where

- $\boldsymbol{\eta} = (\eta_1, \eta_2) \in E_q^1 \times E_q^1$;
- $a \in \mathbb{R}$;
- $\boldsymbol{\varphi}_\epsilon^a$ is periodic and $a\boldsymbol{\varphi}_\epsilon^a$ satisfies $\mathbf{\Theta}_\epsilon(a\boldsymbol{\varphi}_\epsilon^a) = 0$, per Theorem 3.1.1;
- $\boldsymbol{\sigma} := (\sigma, 0)$, where $\sigma$ solves the KdV profile equation (2.5.6).

Beale's ansatz introduces three unknowns into our problem: the amplitude $a$ of the periodic ripple and the decaying terms $\eta_1$ and $\eta_2$.

We find that $\mathbf{\Theta}_\epsilon(\mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) = 0$ is componentwise equivalent to

$$\begin{cases} \eta_1 = -\displaystyle\sum_{k=1}^{5} \big(\mathcal{j}_{k1}^\epsilon(\boldsymbol{\eta}, a) + \mathcal{j}_{k2}^\epsilon(\boldsymbol{\eta}, a)\big) - \mathcal{j}_6^\epsilon(\boldsymbol{\eta}, a) =: \mathfrak{R}_1^\epsilon(\boldsymbol{\eta}, a) \\ \mathcal{T}_\epsilon\eta_2 = -\displaystyle\sum_{k=1}^{5} (\ell_{k1}^\epsilon(\boldsymbol{\eta}, a) + \ell_{k2}^\epsilon(\boldsymbol{\eta}, a)) - \ell_6^\epsilon(\boldsymbol{\eta}, a) =: \mathfrak{R}_2^\epsilon(\boldsymbol{\eta}, a). \end{cases} \tag{4.1.2}$$

where we have used the bilinearity of $B^\epsilon$ and of $\mathcal{Q}^\epsilon(\cdot, \cdot, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a))$ in its first two arguments to break up the terms as

$$\mathcal{j}_{11}^\epsilon(\boldsymbol{\eta}, a) := \sigma + \varpi^\epsilon B_1^\epsilon(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \qquad \mathcal{j}_{12}^\epsilon(\boldsymbol{\eta}, a) := \varpi^\epsilon \mathcal{Q}_2^\epsilon(\boldsymbol{\sigma}, \boldsymbol{\sigma}, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a))$$

$$\begin{aligned}
\jmath^\epsilon_{21}(\boldsymbol{\eta},a) &:= 2\varpi^\epsilon B^\epsilon_1(\boldsymbol{\sigma},\boldsymbol{\eta}) & \jmath^\epsilon_{22}(\boldsymbol{\eta},a) &:= \varpi^\epsilon \mathcal{Q}^\epsilon_1(\boldsymbol{\sigma},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) \\
\jmath^\epsilon_{31}(\boldsymbol{\eta},a) &:= 2a\varpi^\epsilon B^\epsilon_1(\boldsymbol{\sigma},\boldsymbol{\varphi}^a_\epsilon) & \jmath^\epsilon_{32}(\boldsymbol{\eta},a) &:= 2a\varpi^\epsilon \mathcal{Q}^\epsilon_1(\boldsymbol{\sigma},\boldsymbol{\varphi}^a_\epsilon,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) \\
\jmath^\epsilon_{41}(\boldsymbol{\eta},a) &:= 2a\varpi^\epsilon B^\epsilon_1(\boldsymbol{\eta},\boldsymbol{\varphi}^a_\epsilon) & \jmath^\epsilon_{42}(\boldsymbol{\eta},a) &= 2a\varpi^\epsilon \mathcal{Q}^\epsilon_1(\boldsymbol{\eta},\boldsymbol{\varphi}^a_\epsilon,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) \\
\jmath^\epsilon_{51}(\boldsymbol{\eta},a) &:= 2\varpi^\epsilon B^\epsilon_1(\boldsymbol{\eta},\boldsymbol{\eta}) & \jmath^\epsilon_{52}(\boldsymbol{\eta},a) &:= 2\varpi^\epsilon \mathcal{Q}^\epsilon_1(\boldsymbol{\eta},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)),
\end{aligned}$$

$$\begin{aligned}
\ell^\epsilon_{11}(\boldsymbol{\eta},a) &:= \lambda^\epsilon_+ B^\epsilon_2(\boldsymbol{\sigma},\boldsymbol{\sigma}) & \ell^\epsilon_{12}(\boldsymbol{\eta},a) &:= \lambda^\epsilon_+ \mathcal{Q}^\epsilon_2(\boldsymbol{\sigma},\boldsymbol{\sigma},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) \\
\ell^\epsilon_{21}(\boldsymbol{\eta},a) &:= 2\lambda^\epsilon_+ B^\epsilon_2(\boldsymbol{\sigma},\boldsymbol{\eta}) & \ell^\epsilon_{22}(\boldsymbol{\eta},a) &:= 2\lambda^\epsilon_+ \mathcal{Q}^\epsilon_2(\boldsymbol{\sigma},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) \\
\ell^\epsilon_{31}(\boldsymbol{\eta},a) &:= 2a\lambda^\epsilon_+ B^\epsilon_2(\boldsymbol{\sigma},\boldsymbol{\varphi}^a_\epsilon) & \ell^\epsilon_{32}(\boldsymbol{\eta},a) &:= 2a\lambda^\epsilon_+ \mathcal{Q}^\epsilon_2(\boldsymbol{\sigma},\boldsymbol{\varphi}^a_\epsilon,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) \\
\ell^\epsilon_{41}(\boldsymbol{\eta},a) &:= 2a\lambda^\epsilon_+ B^\epsilon_2(\boldsymbol{\eta},\boldsymbol{\varphi}^a_\epsilon) & \ell^\epsilon_{42}(\boldsymbol{\eta},a) &:= 2a\lambda^\epsilon_+ \mathcal{Q}^\epsilon_2(\boldsymbol{\eta},\boldsymbol{\varphi}^a_\epsilon,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) \\
\ell^\epsilon_{51}(\boldsymbol{\eta},a) &:= \lambda^\epsilon_+ B^\epsilon_2(\boldsymbol{\eta},\boldsymbol{\eta}) & \ell^\epsilon_{52}(\boldsymbol{\eta},a) &:= \lambda^\epsilon_+ \mathcal{Q}^\epsilon_2(\boldsymbol{\eta},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)),
\end{aligned}$$

$$\mathcal{T}_\epsilon := c_\epsilon^2\partial_X^2 + \lambda^\epsilon_+$$

and

$$\begin{aligned}
\jmath^\epsilon_6(\boldsymbol{\eta},a) &:= a^2\varpi^\epsilon\left[\mathcal{Q}^\epsilon_1(\boldsymbol{\varphi}^a_\epsilon,\boldsymbol{\varphi}^a_\epsilon,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) - \mathcal{Q}^\epsilon_1(\boldsymbol{\varphi}^a_\epsilon,\boldsymbol{\varphi}^a_\epsilon,a\boldsymbol{\varphi}^a_\epsilon)\right] \\
\ell^\epsilon_6(\boldsymbol{\eta},a) &:= a^2\lambda^\epsilon_+\left[\mathcal{Q}^\epsilon_2(\boldsymbol{\varphi}^a_\epsilon,\boldsymbol{\varphi}^a_\epsilon,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) - \mathcal{Q}^\epsilon_2(\boldsymbol{\varphi}^a_\epsilon,\boldsymbol{\varphi}^a_\epsilon,a\boldsymbol{\varphi}^a_\epsilon)\right].
\end{aligned}$$

**4.2. Adjustments to the nanopteron equations.** We need to modify the system (4.1.2) in several ways before it becomes amenable to our intended quantitative contraction mapping argument. First, as it stands, the term $\jmath^\epsilon_{21}$ is $\mathcal{O}(1)$ in $\epsilon$, which will be inadequate for our later estimates. So, we add the term $2\varpi^0 B^0_1(\boldsymbol{\sigma},\boldsymbol{\eta})$ to both sides of the equation for $\eta_1$. Expanding $B^0_1(\boldsymbol{\sigma},\boldsymbol{\eta})$ from its definition in (2.5.4), we have

$$2\varpi^0 B^0_1(\boldsymbol{\sigma},\boldsymbol{\eta}) = \underbrace{\frac{2(\beta+\varkappa^3)}{\varkappa^2(\varkappa+1)}\varpi^0(\sigma\eta_1)}_{\mathcal{K}_1\eta_1} + \underbrace{\frac{2(\beta-\varkappa^2)}{\varkappa(\varkappa+1)}\varpi^0(\sigma\eta_2)}_{\mathcal{K}_2\eta_2}. \tag{4.2.1}$$

Then subtracting $\mathcal{K}_1\eta_1$ from both sides, we find

$$\underbrace{\eta_1 - \mathcal{K}_1\eta_1}_{\mathcal{A}\eta_1} = \mathfrak{R}^{\epsilon,\mathrm{mod}}_1(\boldsymbol{\eta},a) - \mathcal{K}_2\eta_2, \tag{4.2.2}$$

where

$$\mathfrak{R}^{\epsilon,\mathrm{mod}}_1(\boldsymbol{\eta},a) := -\sum_{k=1}^5\left(\jmath^{\epsilon,\mathrm{mod}}_{k1}(\boldsymbol{\eta},a) + \jmath^\epsilon_{k2}(\boldsymbol{\eta},a)\right) - \jmath^\epsilon_6(\boldsymbol{\eta},a)$$

and

$$j_{1k}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a) := \begin{cases} 2\varpi^{\epsilon}B_1^{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\eta}) - 2\varpi^0 B_1^0(\boldsymbol{\sigma},\boldsymbol{\eta}), k=2 \\ \\ j_{1k}^{\epsilon}(\boldsymbol{\eta},a), k \neq 2. \end{cases}$$

The operator $\mathcal{A}$ given in (4.2.2) is invertible on $E_q^1$ by Proposition D.5.2 for $q$ sufficiently small, and so we may solve for $\eta_1$:

$$\eta_1 = \mathcal{A}^{-1}\mathfrak{R}_1^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a) - \mathcal{A}^{-1}\mathcal{K}_2\eta_2. \tag{4.2.3}$$

The term $\mathcal{A}^{-1}\mathcal{K}_2\eta_2$ is still $\mathcal{O}(1)$ in $\epsilon$, which will ruin our contraction estimates. However, once we establish our fixed point equation for $\eta_2$, we will rewrite the system yet again in a manner that eliminates this difficulty.

Next, we know from Section 2.6 that the operator $\mathcal{T}_\epsilon$ is not invertible since (2.6.4) implies

$$\widehat{\mathcal{T}_\epsilon f}(\pm\omega_\epsilon) = 0$$

for any $f \in E_q^1$. That is, $\mathcal{T}_\epsilon$ is not surjective. Equivalently, if $g$ is an even function in the range of $\mathcal{T}_\epsilon$, then

$$\underbrace{\int_{-\infty}^{\infty} g(X)\cos(\omega_\epsilon X)\ dX}_{\iota_\epsilon[g]} = \widehat{g}(\pm\omega_\epsilon) = 0. \tag{4.2.4}$$

So, if $\boldsymbol{\Theta}_\epsilon(\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) = 0$, then $(\boldsymbol{\eta},a)$ must meet the "solvability condition"

$$\iota_\epsilon[\mathfrak{R}_2^\epsilon(\boldsymbol{\eta},a)] = 0.$$

Then since we cannot merely invert $\mathcal{T}_\epsilon$ to solve for $\eta_2$ in (4.1.2), we instead follow the route established and motivated in [FW18] to convert the pair of equations

$$\begin{cases} \mathcal{T}_\epsilon\eta_2 = \mathfrak{R}_2^\epsilon(\boldsymbol{\eta},a) \\ \\ \iota_\epsilon[\mathfrak{R}_2^\epsilon(\boldsymbol{\eta},a)] = 0 \end{cases} \tag{4.2.5}$$

into a pair of fixed point equations for $\eta_2$ and $a$.

Let

$$\begin{aligned}
&\boldsymbol{\nu}_\epsilon := \boldsymbol{\varphi}_\epsilon^0 := \cos(\omega_\epsilon \cdot)\mathbf{j} \\
&\chi_\epsilon := \lambda_+^\epsilon J_1^\epsilon[(J^0\boldsymbol{\sigma}).(J^\epsilon\boldsymbol{\nu}_\epsilon)] \cdot \mathbf{j} \qquad (4.2.6) \\
&\upsilon_\epsilon := \iota_\epsilon[\chi_\epsilon].
\end{aligned}$$

Subsequent estimates, which we detail in Proposition G.2.1, reveal that $\upsilon_\epsilon$ is bounded away from zero and that $\mathcal{T}_\epsilon$ is invertible on $E_q^1 \cap \ker(\iota_\epsilon)$ for $q$ sufficiently small. It is an easy calculation that if $f \in E_q^1$, then

$$\iota_\epsilon\left[f - \frac{1}{\upsilon_\epsilon}\iota_\epsilon[f]\chi_\epsilon\right] = 0,$$

and so we may define

$$\mathcal{P}_\epsilon f := \mathcal{T}_\epsilon^{-1}\left(f - \frac{1}{\upsilon_\epsilon}\iota_\epsilon[f]\chi_\epsilon\right) \qquad (4.2.7)$$

on $E_q^1$. Then with

$$\ell_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) := \begin{cases} -2a\chi_\epsilon + \ell_{31}^\epsilon(\boldsymbol{\eta}, a), k = 3 \\ \\ \ell_{k1}^\epsilon(\boldsymbol{\eta}, a), k \neq 3, \end{cases}$$

and

$$\mathfrak{R}_2^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) := \sum_{k=1}^5 \ell_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) + \sum_{k=1}^5 \ell_{k2}^\epsilon(\boldsymbol{\eta}, a) + \ell_6^\epsilon(\boldsymbol{\eta}, a),$$

the pair of equations (4.2.5) is equivalent to

$$\begin{cases} \eta_2 = \epsilon^2 \mathcal{P}_\epsilon \mathfrak{R}_2^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) =: \mathfrak{N}_2^\epsilon(\boldsymbol{\eta}, a) \\ \\ a = \dfrac{1}{2\upsilon_\epsilon}\iota_\epsilon\left[\mathfrak{R}_2^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a)\right] =: \mathfrak{N}_3^\epsilon(\boldsymbol{\eta}, a). \end{cases} \qquad (4.2.8)$$

Combining (4.2.3) and (4.2.8), our problem $\boldsymbol{\Theta}_\epsilon(\mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) = 0$ is equivalent to the fixed point problem

$$\begin{cases} \eta_1 = \mathcal{A}^{-1}\mathfrak{R}_1^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) - \mathcal{A}^{-1}\mathcal{K}_2\eta_2 \\ \\ \eta_2 = \mathfrak{N}_2^\epsilon(\boldsymbol{\eta}, a) \\ \\ a = \mathfrak{N}_3^\epsilon(\boldsymbol{\eta}, a). \end{cases} \qquad (4.2.9)$$

Now we can eliminate the difficulty with the term $\mathcal{A}^{-1}\mathcal{K}_2\eta_2$. A pair $(\boldsymbol{\eta}, a) \in$

$E_q^1 \times E_q^1 \times \mathbb{R}$ is a fixed point solution to (4.2.9) if and only if $(\boldsymbol{\eta}, a)$ solves

$$\begin{cases} \eta_1 = \mathcal{A}^{-1}\mathfrak{R}_1^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) - \mathcal{A}^{-1}\mathcal{K}_2\mathfrak{N}_2^\epsilon(\boldsymbol{\eta}, a) =: \mathfrak{N}_1^\epsilon(\boldsymbol{\eta}, a) \\ \eta_2 = \mathfrak{N}_2^\epsilon(\boldsymbol{\eta}, a) \\ a = \mathfrak{N}_3^\epsilon(\boldsymbol{\eta}, a). \end{cases} \tag{4.2.10}$$

The term $\mathcal{A}^{-1}\mathcal{K}_2\mathfrak{N}_2^\epsilon(\boldsymbol{\eta}, a)$ in the revised equation for $\eta_1$ turns out to have the "right" estimates in $\epsilon$ for our contraction mapping argument below. We conclude that $\boldsymbol{\Theta}_\epsilon(\mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) = 0$ if and only if

$$(\boldsymbol{\eta}, a) = (\mathfrak{N}_1^\epsilon(\boldsymbol{\eta}, a), \mathfrak{N}_2^\epsilon(\boldsymbol{\eta}, a), \mathfrak{N}_3^\epsilon(\boldsymbol{\eta}, a)) =: \boldsymbol{\mathfrak{N}}^\epsilon(\boldsymbol{\eta}, a), \ (\boldsymbol{\eta}, a) \in E_q^1 \times E_q^1 \times \mathbb{R}, \tag{4.2.11}$$

and we will solve this fixed point problem in the following section.

**4.2.1 Remark.** *That $\mathfrak{N}_j^\epsilon(\boldsymbol{\eta}, a) \in E_q^1$ for $j = 1, 2$ is ultimately a consequence of Lemma D.4.1 applied to all of the Fourier multipliers that constitute $\mathfrak{N}_j^\epsilon$; to invoke Lemma D.4.1, we use the boundedness of the symbols of these operators on strips as detailed in Lemma 2.2.1.*

**4.3. Existence and properties of solutions.** We model our existence proof on the approach of [HW17] for the small mass ratio, which in turn is a refinement of the contraction mapping proof in [FW18]. Let $q_\star > 0$ be as in Appendix G.2. For $r \geq 0$, set

$$\mathcal{X}^r := \begin{cases} E_{q_\star/2}^1 \times E_{q_\star/2}^1 \times \mathbb{R}, & r = 0 \\ E_{q_\star}^r \times E_{q_\star}^r \times \mathbb{R}, & r > 1. \end{cases}$$

Note that the $\mathcal{X}^r$ spaces are Hilbert spaces and that $\mathcal{X}^s \subseteq \mathcal{X}^r$ for any $0 \leq r \leq s$. Next, for $r$, $\epsilon$, $\tau > 0$, let

$$\mathcal{U}_{\epsilon,\tau}^r := \left\{ (\boldsymbol{\eta}, a) \in \mathcal{X}^r \ \middle| \ \|\boldsymbol{\eta}\|_{r,q_\star} \leq \tau\epsilon, \ |a| \leq \tau\epsilon^r \right\}.$$

We base our contraction mapping argument on the following collection of estimates, which are proved in Section 4.4.

**4.3.1 Proposition.** *There exists $\epsilon_\star > 0$ with the following properties.*

**(i)** *There exists $\tau_\star > 0$ such that if $\epsilon \in (0, \epsilon_\star)$, then*

$$(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\epsilon,\tau_\star} \Longrightarrow \mathfrak{N}^\epsilon(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\epsilon,\tau_\star} \tag{4.3.1}$$

*and*

$$(\boldsymbol{\eta}, a), (\grave{\boldsymbol{\eta}}, \grave{a}) \in \mathcal{U}^1_{\epsilon,\tau_\star} \Longrightarrow \|\mathfrak{N}^\epsilon(\boldsymbol{\eta}, a) - \mathfrak{N}^\epsilon(\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0} \le \frac{1}{2}\|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0}. \tag{4.3.2}$$

**(ii)** *For all $r \ge 1$ and $\tau > 0$, there exists $\overline{\tau} = \overline{\tau}(\tau, r) > 0$ such that if $\epsilon \in (0, \epsilon_\star)$, then*

$$(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\epsilon,\tau_\star} \cap \mathcal{U}^r_{\epsilon,\tau} \Longrightarrow \mathfrak{N}^\epsilon(\boldsymbol{\eta}, a) \in \mathcal{U}^{r+1}_{\epsilon,\overline{\tau}}. \tag{4.3.3}$$

These estimates are essentially the same as the ones achieved in Lemma 8.1 of [HW17], and a proof similar to that of their principal result, Theorem 8.2, produces the following solution to our ultimate fixed point problem (4.2.11). For completeness, we include the proof below.

**4.3.2 Theorem.** *Let $\epsilon \in (0, \epsilon_\star)$. There exists a unique pair $(\boldsymbol{\eta}_\epsilon, a_\epsilon) \in \mathcal{U}^1_{\epsilon,\tau_\star}$ such that $\mathfrak{N}^\epsilon(\boldsymbol{\eta}_\epsilon, a_\epsilon) = (\boldsymbol{\eta}_\epsilon, a_\epsilon)$. This solution $(\boldsymbol{\eta}_\epsilon, a_\epsilon)$ has the following additional properties:*

**(i)** $\boldsymbol{\eta}_\epsilon \in \cap_{r=1}^\infty E^r_{q_\star} \times E^r_{q_\star}$;

**(ii)** *For all $r \ge 0$, there is $C_r > 0$ such that*

$$\|\boldsymbol{\eta}_\epsilon\|_{r,q_\star} \le C_r\epsilon \quad \text{and} \quad |a_\epsilon| \le C_r\epsilon^r \tag{4.3.4}$$

*for all $\epsilon \in (0, \epsilon_\star)$.*

**Proof.** Let $\epsilon \in (0, \epsilon_\star)$ and define $\boldsymbol{\eta}_0 := 0$ (i.e., $\boldsymbol{\eta}_0$ is the zero vector in $E^r_q \times E^r_q$ for any $q$ and $r$) and $a_0 = 0 \in \mathbb{R}$. Set

$$(\boldsymbol{\eta}_{n+1}, a_{n+1}) := \mathfrak{N}^\epsilon(\boldsymbol{\eta}_n, a_n), \ n \ge 0.$$

Property (4.3.1) above tells us that $(\boldsymbol{\eta}_n, a_n) \in \mathcal{U}^1_{\epsilon,\tau_\star}$ for all $n$, and so

$$\|(\boldsymbol{\eta}_n, a_n)\|_{\mathcal{X}^1} \le 2\tau_\star\epsilon \tag{4.3.5}$$

for all $n$. That is, the sequence $\big((\boldsymbol{\eta}_n, a_n)\big)$ is bounded in $\mathcal{X}^1$. Since $\mathcal{X}^1$ is a Hilbert space, $\big((\boldsymbol{\eta}_n, a_n)\big)$ has a subsequence that converges weakly to $(\boldsymbol{\eta}_\epsilon, a_\epsilon) \in \mathcal{X}^1$. We denote this subsequence by $\big((\boldsymbol{\eta}_n, a_n)\big)$ as well.

That is, $(\boldsymbol{\eta}_n)$ converges weakly to $\boldsymbol{\eta}_\epsilon$ in $E^1_{q_\star} \times E^1_{q_\star}$ and $(a_n)$ simply converges to $a_\epsilon$ in $\mathbb{R}$. It follows from the definition of weak convergence that

$$\|\boldsymbol{\eta}_\epsilon\|_{E^1_{q_\star} \times E^1_{q_\star}} \le \limsup_{n\to\infty} \|\boldsymbol{\eta}_n\|_{E^1_{q_\star} \times E^1_{q_\star}} \le \tau_\star \epsilon$$

and from properties of regular convergence in $\mathbb{R}$ that $|a_\epsilon| \le \tau_\star \epsilon$. Consequently, $(\boldsymbol{\eta}_\epsilon, a_\epsilon) \in \mathcal{U}^1_{\epsilon,\tau_\star}$.

Next, we induct with property (4.3.2) to show

$$\big\|(\boldsymbol{\eta}_{n+1}, a_{n+1}) - (\boldsymbol{\eta}_n, a_n)\big\|_{\mathcal{X}^0} \le \frac{1}{2^n} \|(\boldsymbol{\eta}_1, a_1)\|_{\mathcal{X}^0}, \ n \ge 1.$$

Then standard arguments imply that $\big((\boldsymbol{\eta}_n, a_n)\big)$ is Cauchy in $\mathcal{X}^0$. Since $\mathcal{X}^0$ is a Hilbert space, $\big((\boldsymbol{\eta}_n, a_n)\big)$ converges to some $(\grave{\boldsymbol{\eta}}_\epsilon, \grave{a}_\epsilon) \in \mathcal{X}^0$. It then follows from Lemma C.3.3 that $(\boldsymbol{\eta}_\epsilon, a_\epsilon) = (\grave{\boldsymbol{\eta}}_\epsilon, \grave{a}_\epsilon)$.

Using this equality, the convergence of $\big((\boldsymbol{\eta}_n, a_n)\big)$ to $(\grave{\boldsymbol{\eta}}_\epsilon, \grave{a}_\epsilon)$ in $\mathcal{X}^0$, and the continuity of the norm, we have

$$\begin{aligned}
\|(\boldsymbol{\eta}_\epsilon, a_\epsilon) - \mathfrak{N}^\epsilon(\boldsymbol{\eta}_\epsilon, a_\epsilon)\|_{\mathcal{X}^0} &= \lim_{n\to\infty} \|(\boldsymbol{\eta}_n, a_n) - \mathfrak{N}^\epsilon(\boldsymbol{\eta}_\epsilon, a_\epsilon)\|_{\mathcal{X}^0} \\
&= \lim_{n\to\infty} \big\|\mathfrak{N}^\epsilon(\boldsymbol{\eta}_{n-1}, a_{n-1}) - \mathfrak{N}^\epsilon(\boldsymbol{\eta}_\epsilon, a_\epsilon)\big\|_{\mathcal{X}^0} \text{ by (4.3.5)} \\
&\le \limsup_{n\to\infty} \frac{1}{2} \big\|(\boldsymbol{\eta}_{n-1}, a_{n-1}) - (\boldsymbol{\eta}_\epsilon, a_\epsilon)\big\|_{\mathcal{X}^0} \text{ by (4.3.2)} \\
&= \frac{1}{2} \lim_{n\to\infty} \|(\boldsymbol{\eta}_n, a_n) - (\boldsymbol{\eta}_\epsilon, a_\epsilon)\|_{\mathcal{X}^0} \\
&= 0.
\end{aligned}$$

Uniqueness follows from (4.3.2), as usual in a contraction mapping argument. For the bootstrapping and small beyond all orders estimate of (4.3.4), we can use (4.3.3) and induct on $r$, since we have $(\boldsymbol{\eta}_\epsilon, a_\epsilon) \in \mathcal{U}^1_{\epsilon,\tau_\star}$. ■

**4.3.3 Remark.** *The estimate* (4.3.4) *shows that the amplitude* $a_\epsilon$ *of the nanopteron's ripple is small beyond all orders of* $\epsilon$*. We may ask for more refined estimates on* $a_\epsilon$ *in two ways.*

**1.** *First, is* $a_\epsilon$ *exponentially small in* $\epsilon$*? That is, do we have*

$$|a_\epsilon| \le C \exp\left(\frac{q}{\epsilon^r}\right)$$

*for some* $C$*,* $q$*,* $r > 0$*? This is true for the ripple in Beale's water wave problem [Bea91a] as proved later by Sun and Shen [SS93].*

**2.** *Second, are there any values of* $\epsilon$ *for which* $a_\epsilon = 0$*? In such cases, the ripple is not present, and so the nanopteron reduces to the classical solitary wave. Intriguingly, asymptotics in [VSWP16] for the mass dimer and [SV17] for the spring dimer suggest that, when the wave speed* $c$ *is fixed, the traveling wave for each kind of dimer is almost always a nanopteron when the ratio* $1/w$ *for the mass dimer and* $\varkappa$ *for the spring dimer is sufficiently small. But these asymptotics also indicate a countable number of ratios* $1/w$ *and* $\varkappa$ *for which the ripple vanishes and the nanopteron becomes a solitary wave.*

*These questions are common to many nanopteron problems; see Sections 6.4 and 6.5 of [Boy98] for a comprehensive account of their prevalence. While the methods of this dissertation do not yet extend to answer either of them, we expect that the alternative framework provided by the Iooss-Kirchgässner variables in conjunction with Lombardi's theory, as detailed at the end of Section 2.7, will address both problems for the separate cases of mass and spring dimers. If successful, we will then need to translate our results from the Iooss-Kirchgässner-Lombardi framework of reversible systems into the language of our fixed-point, functional analytic approach.*

**4.4. Proof of Proposition 4.3.1.** We first need a new collection of estimates.

**4.4.1 Proposition.** *Let $\overline{\epsilon} \in (0, \epsilon_{\mathrm{per}})$ be as in Proposition G.2.1.*

**(i)** *There exists an increasing function $\mathcal{M}_{\mathrm{map}} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathfrak{N}^\epsilon(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1} \le \mathcal{M}_{\mathrm{map}}[\|\boldsymbol{\eta}\|_{1,q_\star}] \left(\epsilon + \epsilon \|(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1} + \|(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1}^2\right) \tag{4.4.1}$$

*for all $\boldsymbol{\eta} \in E^1_{q_\star} \times E^1_{q_\star}$, $|a| \le a_{\mathrm{per}}$, and $0 < \epsilon < \overline{\epsilon}$.*

**(ii)** *There exists a radially increasing function $\mathcal{M}_{\mathrm{lip}} \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ such that*

$$\begin{aligned}
&\|\mathfrak{N}^\epsilon(\boldsymbol{\eta}, a) - \mathfrak{N}^\epsilon(\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0} \\
&\quad \le \mathcal{M}_{\mathrm{lip}}[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}] \left(\epsilon + \|(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1} + \|(\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^1}\right) \|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0}
\end{aligned} \tag{4.4.2}$$

*for all $\boldsymbol{\eta}, \grave{\boldsymbol{\eta}} \in E^1_{q_\star} \times E^1_{q_\star}$, $|a| \le a_{\mathrm{per}}$, and $0 < \epsilon < \overline{\epsilon}$.*

**(iii)** *For all integers $r \ge 1$ there exists an increasing function $\mathcal{M}_{\mathrm{boot},r} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\begin{aligned}
&\|\mathfrak{N}^\epsilon(\boldsymbol{\eta}, a)\|_{r+1,q_\star} \\
&\quad \le \mathcal{M}_{\mathrm{boot},r}[\|\boldsymbol{\eta}\|_{r,q_\star}] \left(\epsilon + \|\boldsymbol{\eta}\|_{r,q_\star} + |a|\epsilon^{-r} \|\boldsymbol{\eta}\|_{r,q_\star} + |a|\epsilon^{1-r} + a^2\epsilon^{1-2r} + |a|^3\epsilon^{1-3r}\right)
\end{aligned} \tag{4.4.3}$$

*and*

$$|\mathfrak{N}_3^\epsilon(\boldsymbol{\eta}, a)| \le \mathcal{M}_{\mathrm{boot},r}[\|\boldsymbol{\eta}\|_{r,q_\star}] \left(\epsilon^{r+1} + \epsilon^r \|\boldsymbol{\eta}\|_{r,q_\star} + |a| \|\boldsymbol{\eta}\|_{r,q_\star} + \epsilon|a| + a^2\epsilon^{1-r} + |a|^3\epsilon^{1-2r}\right) \tag{4.4.4}$$

*for all $\boldsymbol{\eta} \in E^r_{q_\star} \times E^r_{q_\star}$, $|a| \le a_{\mathrm{per}}$, and $0 < \epsilon < \overline{\epsilon}$.*

The proof of (4.4.1) is developed in Section 4.5, of (4.4.2) in Section 4.6, and of (4.4.3) and (4.4.4) in Section 4.7. Now we are ready to to prove Proposition 4.3.1.

Let $M_\star > 0$ be such that

$$|\mathcal{M}_{\mathrm{map}}[\|\boldsymbol{\eta}\|_{1,q_\star}]| \le M_\star \quad \text{and} \quad |\mathcal{M}_{\mathrm{lip}}[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]| \le M_\star$$

whenever $\|\boldsymbol{\eta}\|_{1,q_\star} \le 1$ and $\|\grave{\boldsymbol{\eta}}\|_{1,q_\star} \le 1$. Set $\tau_\star = M_\star + 2$ and

$$\epsilon_\star = \min\left\{\frac{1}{M_\star(M_\star+2)}, \frac{1}{M_\star(M_\star+2)^2}, \frac{1}{2M_\star(M_\star+5)}, \frac{1}{M_\star+2}, \overline{\epsilon}, \frac{a_{\text{per}}}{M_\star+2}, \frac{1}{2}\right\} \tag{4.4.5}$$

Observe that if $(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\epsilon,\tau_\star}$, then

$$\|\boldsymbol{\eta}\|_{1,q_\star} \le \|(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1} \le \epsilon(M_\star+2) \le \epsilon_\star(M_\star+2) \le 1. \tag{4.4.6}$$

Moreover if $(\boldsymbol{\eta}, a) \in \mathcal{U}^r_{\epsilon,\tau_\star}$ and $r \ge 1$, then

$$|a| \le \tau_\star\epsilon^r \le (M_\star+2)\epsilon_\star^r \le (M_\star+2)\epsilon_\star \le (M_\star+2)\frac{a_{\text{per}}}{M_\star+2} = a_{\text{per}}. \tag{4.4.7}$$

This estimate is important because the four estimates in Proposition 4.4.1 only hold when $|a| \le a_{\text{per}}$.

*Proof of* (4.3.1). Let $0 < \epsilon < \epsilon_\star$ and $(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\epsilon,\tau_\star}$. Then (4.4.6), (4.4.7), and (4.4.1) allow us to estimate

$$\begin{aligned}
\|\mathfrak{N}^\epsilon(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1} &\le M_\star\left(\epsilon + \epsilon^2(M_\star+2) + \epsilon^2(M_\star+2)^2\right) \\
&= \epsilon\left(M_\star + M_\star(M_\star+2)\epsilon + M_\star(M_\star+2)^2\epsilon\right) \\
&\le \epsilon\left(M_\star + M_\star(M_\star+2)\epsilon_\star + M_\star(M_\star+2)^2\epsilon_\star\right) \\
&\le \epsilon\left(M_\star + 1 + 1\right) \\
&= (M_\star+2)\epsilon.
\end{aligned}$$

Hence $\mathfrak{N}^\epsilon(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\epsilon,\tau_\star}$.

*Proof of* (4.3.2). Take $0 < \epsilon < \epsilon_\star$ and $(\boldsymbol{\eta}, a)$, $(\grave{\boldsymbol{\eta}}, \grave{a}) \in \mathcal{U}^1_{\epsilon,\tau_\star}$. Then (4.4.6), (4.4.7), and (4.4.2) imply

$$\begin{aligned}
\|\mathfrak{N}^\epsilon(\boldsymbol{\eta}, a) - \mathfrak{N}^\epsilon(\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0} &\le M_\star\left(\epsilon + \|(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1} + \|(\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^1}\right)\|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0} \\
&\le M_\star\left(\epsilon + 2\epsilon(M_\star+2)\right)\|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0}
\end{aligned}$$

$$= \epsilon M_\star(5+M_\star)\left\|(\boldsymbol{\eta},a)-(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{\mathcal{X}^0}$$

$$\le \epsilon_\star M_\star(5+M_\star)\left\|(\boldsymbol{\eta},a)-(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{\mathcal{X}^0}$$

$$\le \frac{1}{2}\left\|(\boldsymbol{\eta},a)-(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{\mathcal{X}^0}.$$

*Proof of* (4.3.3). That $\mathfrak{N}^\epsilon(\boldsymbol{\eta},a)\in\mathcal{X}^{r+1}$ if $(\boldsymbol{\eta},a)\in\mathcal{X}^r$ follows from the smoothing properties of $\mathcal{P}_\epsilon$ and $\varpi^\epsilon$. Given $\tau>0$, let

$$M_{r,\tau}=\max_{\|\boldsymbol{\eta}\|_{r,q_\star}\le\tau\epsilon_\star}\mathcal{M}_{\mathrm{boot},r}[\boldsymbol{\eta}].$$

Let $0<\epsilon<\epsilon_\star$ and take $(\boldsymbol{\eta},a)\in\mathcal{U}^1_{\epsilon,\tau_\star}\cap\mathcal{U}^r_{\epsilon,\tau}$, so that $\|\boldsymbol{\eta}\|_{r,q_\star}\le\tau\epsilon\le\tau\epsilon_\star$ and $|a|\le\min\{\tau\epsilon^r,a_{\mathrm{per}}\}$. Then (4.4.3) and (4.4.7) imply

$$\begin{aligned}
\|\mathfrak{N}^\epsilon(\boldsymbol{\eta},a)\|_{\mathcal{X}^{r+1}} &\le M_{r,\tau}\left(\epsilon+\tau\epsilon+\tau\epsilon^r(\epsilon^{1-r})+(\tau\epsilon)^2\epsilon^{1-2r}+(\tau\epsilon^r)\epsilon^{-r}(\tau\epsilon)+(\tau\epsilon^r)^3\epsilon^{1-3r}\right)\\
&= M_{r,\tau}\left(\epsilon+2\tau\epsilon+2\tau^2\epsilon+\tau^3\epsilon\right)\\
&= M_{r,\tau}\left(1+2\tau+2\tau^3+\tau^3\right)\epsilon.
\end{aligned}$$

In particular, we find

$$\|(\mathfrak{N}^\epsilon_1(\boldsymbol{\eta},a),\mathfrak{N}^\epsilon_2(\boldsymbol{\eta},a))\|_{r+1,q_\star}\le\|\mathfrak{N}^\epsilon(\boldsymbol{\eta},a)\|_{\mathcal{X}^{r+1}}\le M_{r,\tau}\left(1+2\tau+2\tau^3+\tau^3\right)\epsilon.$$

We need to refine this estimate, however, for $\mathfrak{N}^\epsilon_3(\boldsymbol{\eta},a)$. Using (4.4.4), we find

$$\begin{aligned}
|\mathfrak{N}^\epsilon_3(\boldsymbol{\eta},a)| &\le M_{\tau,r}\left(\epsilon^{r+1}+\epsilon^r(\tau\epsilon)+(\tau\epsilon^r)(\tau\epsilon)+\epsilon(\tau\epsilon^r)+(\tau\epsilon^r)^2\epsilon^{1-r}+(\tau\epsilon)^3\epsilon^{1-2r}\right)\\
&= M_{\tau,r}\left(\epsilon^{r+1}+2\tau\epsilon^{r+1}+2\tau^2\epsilon^{r+1}+\tau^3\epsilon^{r+1}\right)\\
&= M_{\tau,r}\left(1+2\tau+2\tau^2+\tau^3\right)\epsilon^{r+1}.
\end{aligned}$$

So, we take

$$\overline{\tau}:=M_{\tau,r}\left(1+2\tau+2\tau^2+\tau^3\right),$$

to conclude that if $(\boldsymbol{\eta},a)\in\mathcal{U}^1_{\epsilon,\tau_\star}\cap\mathcal{U}^r_{\epsilon,\tau}$, then $\mathfrak{N}^\epsilon(\boldsymbol{\eta},a)\in\mathcal{U}^{r+1}_{\epsilon,\overline{\tau}}$.

This concludes the proof of Proposition 4.3.1. In the remaining sections in this chapter we prove the estimates in Proposition 4.4.1.

**4.5. The mapping estimates.** Let

$$\mathcal{R}^{\epsilon}_{\text{map}}(\boldsymbol{\eta}, a) = \epsilon + \epsilon \|\boldsymbol{\eta}\|_{1,q_\star} + \epsilon|a| + \|\boldsymbol{\eta}\|^2_{1,q_\star} + |a|^2. \tag{4.5.1}$$

Observe that

$$\mathcal{R}^{\epsilon}_{\text{map}}(\boldsymbol{\eta}, a) \le \epsilon + \epsilon \|(\boldsymbol{\eta}, a)\|_{1,q_\star} + \|(\boldsymbol{\eta}, a)\|^2_{1,q_\star},$$

so to obtain (4.4.1) it suffices to prove the existence of an increasing function $\mathcal{M}_{\text{map}} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\boldsymbol{\mathfrak{N}}^{\epsilon}(\boldsymbol{\eta}, a)\|_{\mathcal{X}^1} \le \mathcal{M}_{\text{map}}[\|\boldsymbol{\eta}\|_{1,q_\star}]\mathcal{R}^{\epsilon}_{\text{map}}(\boldsymbol{\eta}, a).$$

*4.5.1. General strategy.* The bounds on $\mathcal{A}^{-1}$, $\mathcal{K}_2$, $\mathcal{P}_\epsilon$, $\iota_\epsilon$, and $\upsilon_\epsilon$ from Proposition G.2.1 let us estimate

$$\left\|\mathfrak{R}^{\epsilon,\text{mod}}_2(\boldsymbol{\eta}, a)\right\| \le \sum_{k=1}^{5} \left\|\mathfrak{l}^{\epsilon,\text{mod}}_{k1}(\boldsymbol{\eta}, a)\right\|_{1,q_\star} + \sum_{k=1}^{5} \left\|\mathfrak{l}^{\epsilon}_{k,2}(\boldsymbol{\eta}, a)\right\|_{1,q_\star} + \|\mathfrak{l}^{\epsilon}_6(\boldsymbol{\eta}, a)\|_{1,q_\star},$$

$$\begin{aligned}
\|\mathfrak{N}^{\epsilon}_2(\boldsymbol{\eta}, a)\|_{1,q_\star} &= \epsilon^2 \left\|\mathcal{P}_\epsilon \mathfrak{R}^{\epsilon,\text{mod}}_2(\boldsymbol{\eta}, a)\right\|_{1,q_\star} \\
&\le C \sum_{k=1}^{5} \epsilon \left\|\mathfrak{l}^{\epsilon,\text{mod}}_{k1}(\boldsymbol{\eta}, a)\right\|_{1,q_\star} + C \sum_{k=1}^{5} \epsilon \left\|\mathfrak{l}^{\epsilon}_{k,2}(\boldsymbol{\eta}, a)\right\|_{1,q_\star} + C\epsilon \|\mathfrak{l}^{\epsilon}_6(\boldsymbol{\eta}, a)\|_{1,q_\star} \\
|\mathfrak{N}^{\epsilon}_3(\boldsymbol{\eta}, a)| &= \frac{1}{2\upsilon_\epsilon} \left\|\iota_\epsilon\left[\mathfrak{R}^{\epsilon,\text{mod}}_2(\boldsymbol{\eta}, a)\right]\right\|_{1,q_\star} \\
&\le C \sum_{k=1}^{5} \epsilon \left\|\mathfrak{l}^{\epsilon,\text{mod}}_{k1}(\boldsymbol{\eta}, a)\right\|_{1,q_\star} + C \sum_{k=1}^{5} \epsilon \left\|\mathfrak{l}^{\epsilon}_{k,2}(\boldsymbol{\eta}, a)\right\|_{1,q_\star} + C\epsilon \|\mathfrak{l}^{\epsilon}_6(\boldsymbol{\eta}, a)\|_{1,q_\star},
\end{aligned}$$

and

$$\|\mathfrak{N}^{\epsilon}_1(\boldsymbol{\eta}, a)\|_{1,q_\star} \le \left\|\mathcal{A}^{-1}\mathfrak{R}^{\epsilon,\text{mod}}_1(\boldsymbol{\eta}, a)\right\|_{1,q_\star} + \left\|\mathcal{A}^{-1}\mathcal{K}_2[\mathfrak{N}^{\epsilon}_2(\boldsymbol{\eta}, a)\right\|_{1,q_\star}$$

$$\le C\sum_{k=1}^{5}\left(\left\|j_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{1,q_\star}+\epsilon\left\|\ell_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{1,q_\star}\right)$$

$$+C\sum_{k=1}^{5}\left(\left\|j_{k2}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star}+\epsilon\left\|\ell_{k,2}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star}\right)$$

$$+C\left(\left\|j_{6}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star}+\epsilon\left\|\ell_{6}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star}\right).$$

We saw in [FW18] that the terms

$$\left\|j_{1k}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{1,q_\star} \quad \text{and} \quad \epsilon\left\|\ell_{1k}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{1,q_\star},\ k=1,\ldots,5$$

are bounded above, up to a constant, by $\mathcal{R}_{\mathrm{map}}^{\epsilon}(\boldsymbol{\eta},a)$. (To be fair, of course the mass dimer versions of $j_{1k}^{\epsilon,\mathrm{mod}}$ and $\ell_{1k}^{\epsilon,\mathrm{mod}}$ were entirely different functions, but the structure of the necessary estimates is exactly the same.) Now we show that the terms

$$\left\|j_{2k}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star}, \quad \epsilon\left\|\ell_{2k}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star},\ k=1,\ldots,5, \quad \left\|j_{6}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star}, \quad \text{and} \quad \epsilon\left\|\ell_{6}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star}$$

are bounded above by $\mathcal{M}_{\mathrm{map}}[\|\boldsymbol{\eta}\|_{1,q_\star}]\mathcal{R}_{\mathrm{map}}^{\epsilon}(\boldsymbol{\eta},a)$.

**4.5.1 Remark.** *Here and elsewhere in these appendices, we will write*

$$\|\boldsymbol{\varphi}\|_{W^{r,\infty}} := \|\varphi_1\|_{W^{r,\infty}}+\|\varphi_2\|_{W^{r,\infty}}$$

*for a function* $\boldsymbol{\varphi}=(\varphi_1,\varphi_2)\in W^{r,\infty}\times W^{r,\infty}$.

*4.5.2. Mapping estimates for $j_{12}$ and $\ell_{12}$.* We present this first series of estimates in detail to show the general techniques and reliance on Proposition E.1.1 that will permeate the subsequent mapping estimates. We have

$$\begin{aligned}\left\|j_{12}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{1,q_\star} &= \left\|\varpi^{\epsilon}\mathcal{Q}_2^{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\sigma},\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{1,q_\star}\\ &\le C\left\|\mathcal{Q}_2^{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\sigma},\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{1,q_\star}\\ &\le C\left\|\mathcal{Q}^{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\sigma},\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{1,q_\star}\end{aligned}$$

$$= C \left\| J_1^\epsilon M_{1/\varkappa} \left( (J^\epsilon \boldsymbol{\sigma})^{.2} . \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right) \right\|_{1,q_\star}$$

$$\leq C \left\| (J^\epsilon \boldsymbol{\sigma})^{.2} . \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}.$$

We remark that

$$(J^\epsilon \boldsymbol{\sigma})^{.2} . \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) = \underbrace{(J^\epsilon \boldsymbol{\sigma})^{.2}}_{H^1_{q_\star}} . \mathcal{N}\Bigg( \epsilon \big( \underbrace{\epsilon J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta})}_{H^1_{q_\star}} + a \underbrace{(\epsilon J^\epsilon \boldsymbol{\varphi}^a_\epsilon)}_{W^{1,\infty}} \big) \Bigg),$$

and so we could control this quantity with Proposition E.1.1. However, that would be premature, as the resulting estimate would not have the conducive form of $\mathcal{R}^\epsilon_{\text{map}}(\boldsymbol{\eta}, a)$. Instead, in order to factor out an all-important power of $\epsilon$ (a recurring theme in these estimates), we expand $\mathcal{N}$ and find

$$C \left\| (J^\epsilon \boldsymbol{\sigma})^{.2} . \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star} = C \left\| (J^\epsilon \boldsymbol{\sigma})^{.2} . (\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) . N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}$$

$$= C\epsilon \left\| (J^\epsilon \boldsymbol{\sigma})^{.2} . (\epsilon J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) . N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}.$$

Next, we use the definition of $\mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)$ and the triangle inequality to break this norm into two terms:

$$C\epsilon \left\| (J^\epsilon \boldsymbol{\sigma})^{.2} . (\epsilon J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) . N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}$$

$$\leq \underbrace{C\epsilon \left\| (J^\epsilon \boldsymbol{\sigma})^{.2} . (\epsilon J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta})) . N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}}_{\Pi_1}$$

$$+ \underbrace{C\epsilon \left\| (J^\epsilon \boldsymbol{\sigma})^{.2} . (\epsilon J^\epsilon (a \boldsymbol{\varphi}^a_\epsilon)) . N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}}_{\Pi_2}.$$

Observe that the product in $\Pi_1$ really has the form of the factors in the estimate in Proposition E.1.1:

$$(J^\epsilon \boldsymbol{\sigma})^{.2} . (\epsilon J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta})) . N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a))$$

$$= \underbrace{(J^\epsilon \boldsymbol{\sigma})^{.2} . (\epsilon J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta}))}_{H^1_q} . N\Bigg( \epsilon \big( \underbrace{J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta})}_{H^1_q} + a \underbrace{(\epsilon J^\epsilon \boldsymbol{\varphi}^a_\epsilon)}_{W^{1,\infty}} \big) \Bigg).$$

That proposition implies

$$\Pi_1 \leq C\epsilon\mathcal{M}[\|J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})\|_{1,q_\star}]\left(1+|a|\epsilon^{1-1}\right)\left\|(J^\epsilon\boldsymbol{\sigma})^{.2}.(\epsilon J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta}))\right\|_{1,q_\star},$$

for some increasing function $\mathcal{M}\colon \mathbb{R}_+ \to \mathbb{R}_+$. The triangle inequality and the uniform bound on $J^\epsilon$ from Proposition G.2.1 give constants $C_1$, $C_2 > 0$ such that

$$\mathcal{M}[\|J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})\|_{1,q_\star}]\left\|(J^\epsilon\boldsymbol{\sigma})^{.2}.(\epsilon J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta}))\right\|_{1,q_\star} \leq \underbrace{\mathcal{M}[C_1\|\boldsymbol{\eta}\|_{1,q_\star}]C_2(1+\|\boldsymbol{\eta}\|_{1,q_\star}}_{\mathcal{M}_{\ell_{12}}[\|\boldsymbol{\eta}\|_{1,q_\star}]}.$$

That is, $\mathcal{M}_{\ell_{12}}$ is increasing with

$$\Pi_1 \leq \mathcal{M}_{\ell_{12}}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon,$$

and this bound has the desired form of $\mathcal{R}^\epsilon_{\mathrm{map}}(\boldsymbol{\eta},a)$ from (4.5.1).

The estimate on $\Pi_2$ proceeds just as the one for $\Pi_1$, except first we factor

$$\begin{aligned}\Pi_2 &\leq C\epsilon\,\|\epsilon J^\epsilon(a\boldsymbol{\varphi}^a_\epsilon)\|_{W^{1,\infty}}\left\|(J^\epsilon\boldsymbol{\sigma})^{.2}.N(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{1,q_\star}\\ &\leq C\epsilon\left\|(J^\epsilon\boldsymbol{\sigma})^{.2}.N(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{1,q_\star}\end{aligned}$$

and then apply Proposition E.1.1 to the term on the right. The whole estimate for $\ell_{12}$ follows in an identical way, so we omit it.

*4.5.3. Mapping estimates for $j_{22}$ and $\ell_{22}$.* These estimates are essentially the same as the ones for $j_{12}$ and $\ell_{12}$, except wherever we had a factor of $(J^\epsilon\boldsymbol{\sigma})^{.2}$ in the previous section, now we have a factor of $(J^\epsilon\boldsymbol{\sigma}).(J^\epsilon\boldsymbol{\eta})$. We omit the details.

*4.5.4. Mapping estimates for $j_{32}$ and $\ell_{32}$.* To handle the new presence of $\boldsymbol{\varphi}^a_\epsilon$, which costs a factor of $\epsilon$ each time we estimate its $W^{1,\infty}$-norm, we need to expose an additional factor of $\epsilon$ in the estimates. We achieve this via the smoothing property of $\varpi^\epsilon$ for $j_{32}$ and the extra $\epsilon$ that naturally comes along with $\ell_{32}$. We have

$$\left\|j^\epsilon_{32}(\boldsymbol{\eta},a)\right\|_{1,q_\star}2|a|\,\|\varpi^\epsilon\mathcal{Q}^\epsilon_2(\boldsymbol{\sigma},\boldsymbol{\varphi}^a_\epsilon,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\|_{1,q_\star}$$

$$\leq C|a| \left\| \mathcal{Q}_2^\epsilon(\boldsymbol{\sigma}, \boldsymbol{\varphi}_\epsilon^a, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{0,q}$$

$$\leq C|a| \left\| \mathcal{Q}^\epsilon(\boldsymbol{\sigma}, \boldsymbol{\varphi}_\epsilon^a, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{0,q}$$

$$\leq C|a| \left\| (J^\epsilon \boldsymbol{\sigma}).(J^\epsilon \boldsymbol{\varphi}_\epsilon^a).\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{0,q}$$

$$\leq C\epsilon|a| \left\| J^\epsilon \boldsymbol{\varphi}_\epsilon^a \right\|_{W^{0,\infty}} \left\| (J^\epsilon \boldsymbol{\sigma}).(\epsilon J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)).N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{0,q}$$

$$\leq C\epsilon|a| \left\| (J^\epsilon \boldsymbol{\sigma}).(\epsilon J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)).N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{0,q}.$$

We estimate the factor in the $H_q^0$-norm above with Proposition E.1.1 and bound that as in the case of $\jmath_{12}$ to find, ultimately,

$$\left\| \jmath_{32}^\epsilon(\boldsymbol{\eta}, a) \right\|_{1,q_\star} \leq \mathcal{M}_{\jmath_{32}}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon|a|.$$

For the $\ell_{32}$ estimate, we stay in the $H_q^1$ norm but use our extra factor of $\epsilon$ to counterbalance the $J^\epsilon \boldsymbol{\varphi}_\epsilon^a$ factor. Specifically,

$$\epsilon \left\| \ell_{32}^\epsilon(\boldsymbol{\eta}, a) \right\|_{1,q_\star} \leq C\epsilon|a| \left\| (J^\epsilon \boldsymbol{\sigma}).(J^\epsilon \boldsymbol{\varphi}_\epsilon^a).\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}$$

$$\leq C\epsilon|a| \left\| J^\epsilon \boldsymbol{\varphi}_\epsilon^a \right\|_{W^{1,\infty}} \left\| (J^\epsilon \boldsymbol{\sigma}).\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}$$

$$\leq C\epsilon|a| \left\| (J^\epsilon \boldsymbol{\sigma}).(\epsilon J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)).N(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{1,q_\star}.$$

We finish by applying Proposition E.1.1 to the last term above in the $H_q^1$ norm and find

$$\left\| \ell_{32}^\epsilon(\boldsymbol{\eta}, a) \right\|_{1,q_\star} \leq \mathcal{M}_{\ell_{32}}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon|a|.$$

*4.5.5. Mapping estimates for $\jmath_{42}$ and $\ell_{42}$.* These estimates are the same as those for $\jmath_{32}$ and $\ell_{32}$, except all factors of $J^\epsilon \boldsymbol{\sigma}$ are replaced by $J^\epsilon \boldsymbol{\eta}$.

*4.5.6. Mapping estimates for $\jmath_{52}$ and $\ell_{52}$.* These estimates are the same as those for $\jmath_{12}$ and $\ell_{12}$ with the factor of $(J^\epsilon \boldsymbol{\sigma})^{.2}$ replaced by $(J^\epsilon \boldsymbol{\eta})^{.2}$.

*4.5.7. Mapping estimates for $\jmath_6$ and $\ell_6$.* It is for these terms that we designed the estimate in Proposition E.2.1. We begin with a straightforward estimate on $\jmath_6$:

$$\begin{aligned}\left\|\jmath_6^\epsilon(\boldsymbol{\eta},a)\right\|_{1,q_\star} &\leq Ca^2\left\|\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a,\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a,\boldsymbol{\varphi}_\epsilon^a,a\boldsymbol{\varphi}_\epsilon^a)\right\|_{1,q_\star} \\ &= Ca^2\left\|(J^\epsilon\boldsymbol{\varphi}_\epsilon^a)^{.2}.(\mathcal{N}(\epsilon^2J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{N}(\epsilon^2J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a)))\right\|_{1,q_\star} \\ &\leq Ca^2\left\|J^\epsilon\boldsymbol{\varphi}_\epsilon^a\right\|_{W^{1,\infty}}^2\left\|\mathcal{N}(\epsilon^2J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{N}(\epsilon^2J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a))\right\|_{1,q_\star}.\end{aligned}$$

Since

$$\mathcal{N}(\epsilon^2J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{N}(\epsilon^2J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a))=\mathcal{N}(\epsilon^2J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})+\epsilon^2J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a))-\mathcal{N}(0+\epsilon^2J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a)),$$

Proposition E.2.1 applies to give

$$\begin{aligned}&\left\|\mathcal{N}(\epsilon^2J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{N}(\epsilon^2J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a))\right\|_{1,q_\star} \\ &\qquad\leq\mathcal{M}[\|\epsilon J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})\|_{1,q_\star}]\left(1+|a|\epsilon^{1-1}\right)\left\|\epsilon^2J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})-0\right\|_{1,q_\star}.\end{aligned}$$

Then estimating $\|J^\epsilon\boldsymbol{\varphi}_\epsilon^a\|_{W^{1,\infty}}\leq C\epsilon^{-2}$, we find

$$\begin{aligned}\left\|\jmath_6^\epsilon(\boldsymbol{\eta},a)\right\|_{1,q_\star} &\leq Ca^2\epsilon^{-2}\mathcal{M}[\|\epsilon J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})\|_{1,q_\star}]\left(1+|a|\epsilon^{1-1}\right)\left\|\epsilon^2J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})-0\right\|_{1,q_\star} \\ &\leq Ca^2\mathcal{M}[\|\epsilon J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})\|_{1,q_\star}]\left\|J^\epsilon(\boldsymbol{\sigma}+\boldsymbol{\eta})\right\|_{1,q_\star}\end{aligned}$$

Taking the supremum over $0<\epsilon<\overline{\epsilon}$ gives a bound of the form

$$\left\|\jmath_6^\epsilon(\boldsymbol{\eta},a)\right\|_{1,q_\star}\leq\mathcal{M}_{\jmath_6}[\|\boldsymbol{\eta}\|_{1,q_\star}]a^2.$$

The estimate for $\ell_6$ follows in the same way.

**4.6. The Lipschitz estimates.** Let

$$\mathcal{R}_{\mathrm{lip}}^\epsilon(\boldsymbol{\eta},\grave{\boldsymbol{\eta}},a,\grave{a})=(\epsilon+\|(\boldsymbol{\eta},a)\|_{\mathcal{X}^1}+\|(\grave{\boldsymbol{\eta}},\grave{a})\|_{\mathcal{X}^1})\left\|(\boldsymbol{\eta},a)-(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{\mathcal{X}^0}. \tag{4.6.1}$$

We prove the existence of an increasing function $\mathcal{M}_{\mathrm{lip}}\colon\mathbb{R}_+^2\to\mathbb{R}_+$ such that

$$\|\mathfrak{N}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{N}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0} \le \mathcal{M}_{\mathrm{lip}}[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\mathcal{R}^{\epsilon}_{\mathrm{lip}}(\boldsymbol{\eta}, \grave{\boldsymbol{\eta}}, a, \grave{a}) \tag{4.6.2}$$

for all $\boldsymbol{\eta}, \grave{\boldsymbol{\eta}} \in E^1_{q_\star} \times E^1_{q_\star}$, $|a| \le a_{\mathrm{per}}$, and $0 < \epsilon < \overline{\epsilon}$.

*4.6.1. General strategy.* A first pass using the estimates in Proposition G.2.1 gives

$$\begin{aligned}
\|\mathfrak{N}^{\epsilon}_1(\boldsymbol{\eta}, a) - \mathfrak{N}^{\epsilon}_1(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,q_\star/2} \le{}& C\sum_{k=1}^{5}\left\|\mathfrak{j}^{\epsilon,\mathrm{mod}}_{k1}(\boldsymbol{\eta}, a) - \mathfrak{j}^{\epsilon,\mathrm{mod}}_{k1}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2} \\
&+ C\sum_{k=1}^{5}\epsilon\left\|\mathfrak{l}^{\epsilon,\mathrm{mod}}_{k1}(\boldsymbol{\eta}, a) - \mathfrak{l}^{\epsilon,\mathrm{mod}}_{k1}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2} \\
&+ C\sum_{k=1}^{5}\left\|\mathfrak{j}^{\epsilon}_{k2}(\boldsymbol{\eta}, a) - \mathfrak{j}^{\epsilon}_{k2}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2} \\
&+ C\sum_{k=1}^{5}\epsilon\left\|\mathfrak{l}^{\epsilon}_{k2}(\boldsymbol{\eta}, a) - \mathfrak{l}^{\epsilon}_{k2}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2} \\
&+ C\left\|\mathfrak{j}^{\epsilon}_{6}(\boldsymbol{\eta}, a) - \mathfrak{j}^{\epsilon}_{6}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2} \\
&+ C\epsilon\left\|\mathfrak{l}^{\epsilon}_{6}(\boldsymbol{\eta}, a) - \mathfrak{l}^{\epsilon}_{6}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2},
\end{aligned}$$

$$\begin{aligned}
\|\mathfrak{N}^{\epsilon}_2(\boldsymbol{\eta}, a) - \mathfrak{N}^{\epsilon}_1(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,q_\star/2} \le{}& C\sum_{k=1}^{5}\epsilon\left\|\mathfrak{l}^{\epsilon,\mathrm{mod}}_{k1}(\boldsymbol{\eta}, a) - \mathfrak{l}^{\epsilon,\mathrm{mod}}_{k1}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2} \\
&+ C\sum_{k=1}^{5}\epsilon\left\|\mathfrak{l}^{\epsilon}_{k2}(\boldsymbol{\eta}, a) - \mathfrak{l}^{\epsilon}_{k2}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2} \\
&+ C\epsilon\left\|\mathfrak{l}^{\epsilon}_{6}(\boldsymbol{\eta}, a) - \mathfrak{l}^{\epsilon}_{6}(\grave{\boldsymbol{\eta}}, \grave{a})\right\|_{1,q_\star/2},
\end{aligned}$$

and

$$
\begin{aligned}
|\mathfrak{N}_3^\epsilon(\boldsymbol{\eta}, a) - \mathfrak{N}_3^\epsilon(\grave{\boldsymbol{\eta}}, \grave{a})| &\le C \sum_{k=1}^{5} \epsilon \left\| \mathfrak{l}_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) - \mathfrak{l}_{k1}^{\epsilon,\mathrm{mod}}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2} \\
&\quad + C \sum_{k=1}^{5} \epsilon \left\| \mathfrak{l}_{k2}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{l}_{k2}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2} \\
&\quad + C\epsilon \left\| \mathfrak{l}_{6}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{l}_{6}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2}.
\end{aligned}
$$

In [FW18] we bounded the differences

$$
\left\| \mathfrak{j}_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) - \mathfrak{j}_{k1}^{\epsilon,\mathrm{mod}}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2}, \quad \text{and} \quad \left\| \mathfrak{l}_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta}, a) - \mathfrak{l}_{k1}^{\epsilon,\mathrm{mod}}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2}, \ k = 1, \ldots, 5
$$

by $\mathcal{R}_{\mathrm{lip}}^\epsilon(\boldsymbol{\eta}, \grave{\boldsymbol{\eta}}, a, \grave{a})$. We just need to show that

$$
\left\| \mathfrak{j}_{k2}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{j}_{k2}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2}, \qquad \epsilon \left\| \mathfrak{l}_{k2}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{j}_{k2}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2}, \ k = 1, \ldots, 5,
$$

$$
\left\| \mathfrak{j}_{6}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{j}_{6}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2}, \quad \text{and} \quad \epsilon \left\| \mathfrak{l}_{6}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{l}_{6}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2}
$$

are all bounded by $\mathcal{M}_{\mathrm{lip}}[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\mathcal{R}_{\mathrm{lip}}^\epsilon(\boldsymbol{\eta}, \grave{\boldsymbol{\eta}}, a, \grave{a})$.

We will use a particular consequence of the essential estimate (G.2.2) often enough that it is worthwhile to single it out here.

**4.6.1 Lemma.** *For each $r \ge 0$ there is $C_r > 0$ such that for all $\epsilon \in (0, \overline{\epsilon})$ and $|a|$, $|\grave{a}| \le a_{\mathrm{per}}$, we have*

$$
\left\| \operatorname{sech}\left(\frac{q_\star}{2}\cdot\right) J^\epsilon (a\boldsymbol{\varphi}_\epsilon^a - \grave{a}\boldsymbol{\varphi}_\epsilon^{\grave{a}}) \right\|_{W^{r,\infty}} \le C_r \epsilon^{-r} |a - \grave{a}|.
$$

*4.6.2. Lipschitz estimates for $\mathfrak{j}_{12}$ and $\mathfrak{l}_{12}$.* As with the mapping estimates, we spell out this first estimate in detail to show our reliance on the general Lipschitz estimates of Appendices E.2, E.3, and E.4. We begin with

$$
\left\| \mathfrak{j}_{12}^{\epsilon}(\boldsymbol{\eta}, a) - \mathfrak{j}_{12}^{\epsilon}(\grave{\boldsymbol{\eta}}, \grave{a}) \right\|_{1,q_\star/2} \le C \left\| \mathcal{Q}^\epsilon(\boldsymbol{\sigma}, \boldsymbol{\sigma}, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\sigma}, \boldsymbol{\sigma}, \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}}, \grave{a})) \right\|_{1,q_\star/2}
$$

$$\leq C \left\| (J^\epsilon \boldsymbol{\sigma})^{.2}.(\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}}, \grave{a}))) \right\|_{1,q_\star/2}$$

$$\leq \underbrace{C \left\| (J^\epsilon \boldsymbol{\sigma})^{.2}.(\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon (\grave{\boldsymbol{\eta}}, a))) \right\|_{1,q_\star/2}}_{\Delta_1}$$

$$+ \underbrace{C \left\| (J^\epsilon \boldsymbol{\sigma})^{.2}.(\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}}, \grave{a}))) \right\|_{1,q_\star/2}}_{\Delta_2}$$

Since

$$\begin{aligned} \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon (\grave{\boldsymbol{\eta}}, a)) &= \mathcal{N}\Bigg(\epsilon\big(\underbrace{\epsilon J^\epsilon (\boldsymbol{\sigma} + \grave{\boldsymbol{\eta}})}_{H^1_{q_\star}} + a(\underbrace{\epsilon J^\epsilon \boldsymbol{\varphi}^a_\epsilon}_{W^{1,\infty}})\big)\Bigg) \\ &\quad - \mathcal{N}\Bigg(\epsilon\big(\underbrace{\epsilon J^\epsilon (\boldsymbol{\sigma} + \grave{\boldsymbol{\eta}})}_{H^1_{q_\star}} + a(\underbrace{\epsilon J^\epsilon \boldsymbol{\varphi}^a_\epsilon}_{W^{1,\infty}})\big)\Bigg), \end{aligned}$$

we can use Proposition E.2.1 to bound

$$\begin{aligned} \Delta_1 \leq \mathcal{M}_1[\left\| \epsilon^2 J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta}) \right\|_{1,q_\star}, \left\| \epsilon^2 J^\epsilon (\boldsymbol{\sigma} + \grave{\boldsymbol{\eta}}) \right\|_{1,q_\star}] \left(1 + |a| \epsilon^{1-1}\right) \\ \times \left\| \epsilon^2 J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta}) - \epsilon^2 J^\epsilon (\boldsymbol{\sigma} + \grave{\boldsymbol{\eta}}) \right\|_{1,q_\star/2}, \end{aligned} \tag{4.6.3}$$

where $\mathcal{M}_1 \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ is radially increasing. We have

$$\left\| \epsilon^2 J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta}) \right\|_{1,q_\star} \leq C(1 + \|\boldsymbol{\eta}\|)_{1,q_\star},$$

and so

$$\begin{aligned} &\mathcal{M}_1[\left\| \epsilon^2 J^\epsilon (\boldsymbol{\sigma} + \boldsymbol{\eta}) \right\|_{1,q_\star}, \left\| \epsilon^2 J^\epsilon (\boldsymbol{\sigma} + \grave{\boldsymbol{\eta}}) \right\|_{1,q_\star}](1 + |a|) \\ &\qquad \leq \underbrace{2\mathcal{M}_1[C(1 + \|\boldsymbol{\eta}\|)_{1,q_\star}, C(1 + \|\grave{\boldsymbol{\eta}}\|)_{1,q_\star}]}_{\mathcal{M}_2[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]}. \end{aligned}$$

We see that $\mathcal{M}_2$ is also radially increasing. Then

$$\Delta_1 \leq \mathcal{M}_2[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}] \epsilon^2 \|\boldsymbol{\eta} - \grave{\boldsymbol{\eta}}\|_{1,q_\star/2}.$$

For $\Delta_2$, we first note

$$
\begin{aligned}
&(J^\epsilon\boldsymbol{\sigma})^{.2}.(\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))-\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a})))\\
&\qquad=\underbrace{(J^\epsilon\boldsymbol{\sigma})^{.2}}_{H^1_{q_\star}}.\Big(\mathcal{N}\big(\underbrace{\epsilon^2 J^\epsilon(\boldsymbol{\sigma}+\grave{\boldsymbol{\eta}})}_{H^1_{q_\star}}+\underbrace{\epsilon^2 J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a)}_{W^{1,\infty}}\big)-\mathcal{N}\big(\underbrace{\epsilon^2 J^\epsilon(\boldsymbol{\sigma}+\grave{\boldsymbol{\eta}})}_{H^1_{q_\star}}+\underbrace{\epsilon^2 J^\epsilon(\grave{a}\boldsymbol{\varphi}_\epsilon^{\grave{a}})}_{W^{1,\infty}}\big)\Big).
\end{aligned}
$$

Then we can use the estimate of Proposition E.4.1:

$$
\begin{aligned}
\Delta_2 \le \mathcal{M}_3[\left\|\epsilon^2 J^\epsilon(\boldsymbol{\sigma}+\grave{\boldsymbol{\eta}})\right\|_{1,q_\star}, \left\|\epsilon^2 J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a)\right\|_{W^{1,\infty}}, \left\|\epsilon^2 J^\epsilon(\grave{a}\boldsymbol{\varphi}_\epsilon^{\grave{a}})\right\|_{W^{1,\infty}}]&\\
\times\, \epsilon^2 \left\|\operatorname{sech}\left(\frac{q_\star}{2}\cdot\right)(aJ^\epsilon\boldsymbol{\varphi}_\epsilon^a-\grave{a}J^\epsilon\boldsymbol{\varphi}_\epsilon^{\grave{a}})\right\|_{W^{1,\infty}}\left\|(J^\epsilon\boldsymbol{\sigma})^{.2}\right\|_{1,q_\star},&
\end{aligned}
$$

where $\mathcal{M}_3\colon \mathbb{R}^3_+ \to \mathbb{R}_+$ is radially increasing. Taking the supremum over $\epsilon$, $a$, and $\grave{a}$ and using the estimate in Lemma 4.6.1 on the $W^{1,\infty}$-factor, we find

$$
\Delta_2 \le \mathcal{M}_4[\|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\epsilon|a-\grave{a}|.
$$

for an increasing function $\mathcal{M}_4$. We conclude that the sum $\Delta_1+\Delta_2$ has an upper bound of the form $\mathcal{R}^\epsilon_{\text{lip}}(\boldsymbol{\eta},\grave{\boldsymbol{\eta}},a,\grave{a})$ from (4.6.1), and the estimate for $\ell_{12}$ is the same.

*4.6.3. Lipschitz estimates for $j_{22}$ and $\ell_{22}$.* These estimates are mostly the same as the ones for $j_{12}$ and $\ell_{12}$ with a few small changes that are worth pointing out. We estimate $j_{22}$ to illustrate them:

$$
\begin{aligned}
\left\|j^\epsilon_{22}(\boldsymbol{\eta},a)-j^\epsilon_{22}(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{1,q_\star/2} &\le C\left\|\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{1,q_\star/2}\\
&\le \underbrace{\left\|\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{1,q_\star/2}}_{\Delta_1}\\
&\quad+\underbrace{\left\|\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))\right\|_{1,q_\star/2}}_{\Delta_2}\\
&\quad+\underbrace{\left\|\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))-\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{1,q_\star/2}}_{\Delta_3}
\end{aligned}
$$

We have

$$\Delta_1 \le C\left\|(J^\epsilon\boldsymbol{\sigma}).(J^\epsilon(\boldsymbol{\eta}-\grave{\boldsymbol{\eta}})).(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)).N(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{1,q_\star/2},$$

and using the algebra property of $H_q^1 \times H_q^1$, this factors as

$$\Delta_1 \le C\epsilon\left\|J^\epsilon(\boldsymbol{\eta}-\grave{\boldsymbol{\eta}})\right\|_{1,q_\star/2}\underbrace{\left\|(J^\epsilon\boldsymbol{\sigma}).(\epsilon J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)).N(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{1,q_\star/2}}_{\Pi}.$$

We can bound the factor $\Pi$ using Proposition E.1.1 effectively as we did in Appendix 4.5.2 for the mapping estimates on $j_{12}$ and $\ell_{12}$. Then

$$\Delta_1 \le \mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon\left\|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\right\|_{1,q_\star/2}.$$

The estimate for $\Delta_2$ again uses the algebra property of $H_q^1 \times H_q^1$ and then Proposition E.2.1 as we did in (4.6.3) for the Lipschitz estimates on $j_{21}$ and $\ell_{21}$. Finally, the estimate for $\Delta_3$ uses Proposition E.4.1.

The estimate for $\ell_{22}$ is identical.

*4.6.4. Lipschitz estimates for $j_{32}$ and $\ell_{33}$.* As with the mapping estimates, we need to exploit the smoothing operator $\varpi^\epsilon$ on $j_{32}$ and the extra $\epsilon$ on $\ell_{33}$ to manage the presence of $\boldsymbol{\varphi}_\epsilon^a$ and $\boldsymbol{\varphi}_\epsilon^{\grave{a}}$. We have

$$\begin{aligned}
\left\|j_{32}^\epsilon(\boldsymbol{\eta},a)-j_{32}^\epsilon(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{1,q_\star/2} &= 2\left\|\varpi^\epsilon(a\mathcal{Q}_2^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\grave{a}\mathcal{Q}_2^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a})))\right\|_{1,q_\star/2} \\
&\le C\left\|a\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\grave{a}\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{0,q_\star/2} \\
&\le \underbrace{|a-\grave{a}|\left\|\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)\right\|_{0,q_\star/2}}_{\Delta_1} \\
&\quad+\underbrace{|\grave{a}|\left\|\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)-\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{0,q_\star/2}}_{\Delta_2} \\
&\quad+\underbrace{|\grave{a}|\left\|\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\boldsymbol{\sigma},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))\right\|_{0,q_\star/2}}_{\Delta_3}
\end{aligned}$$

$$+\underbrace{|\grave{a}|\left\|\mathcal{Q}^{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\varphi}_{\epsilon}^{\grave{a}},\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},a))-\mathcal{Q}^{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\varphi}_{\epsilon}^{\grave{a}},\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{0,q_\star/2}}_{\Delta_4}.$$

We handle $\Delta_1$ easily using the mapping estimate of Proposition E.1.1:

$$\begin{aligned}
\Delta_1 &\leq C\epsilon|a-\grave{a}|\left\|(J^{\epsilon}\boldsymbol{\sigma}).(J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}).(\epsilon J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a)).N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\leq C\epsilon|a-\grave{a}|\left\|J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}\right\|_{W^{0,\infty}}\left\|(J^{\epsilon}\boldsymbol{\sigma}).(\epsilon J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a)).N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\leq \mathcal{M}_{\Delta_1}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon|a-\grave{a}|.
\end{aligned}$$

For $\Delta_2$ we need both Proposition E.1.1 and Lemma 4.6.1:

$$\begin{aligned}
\Delta_2 &\leq C|\grave{a}|\left\|(J^{\epsilon}\boldsymbol{\sigma}).(J^{\epsilon}(\boldsymbol{\varphi}_{\epsilon}^{a}-\boldsymbol{\varphi}_{\epsilon}^{\grave{a}})).\mathcal{N}(\epsilon^2\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\leq C\epsilon|\grave{a}|\left\|J^{\epsilon}(\boldsymbol{\varphi}_{\epsilon}^{a}-\boldsymbol{\varphi}_{\epsilon}^{\grave{a}})\right\|_{W^{0,\infty}}\left\|(J^{\epsilon}\boldsymbol{\sigma}).(\epsilon J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a)).N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\leq \mathcal{M}_{\Delta_2}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon|\grave{a}||a-\grave{a}|.
\end{aligned}$$

We factor $\Delta_3$ and then employ Proposition E.2.1:

$$\begin{aligned}
\Delta_3 &\leq C|\grave{a}|\left\|(J^{\epsilon}\boldsymbol{\sigma}).(J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{\grave{a}}).(\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))-\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},a)))\right\|_{0,q_\star/2} \\
&\leq C|\grave{a}|\left\|J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}\right\|_{W^{0,\infty}}\left\|J^{\epsilon}\boldsymbol{\sigma}\right\|_{W^{0,\infty}}\left\|\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))-\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},a))\right\|_{0,q_\star/2} \\
&\leq \mathcal{M}_{\Delta_3}[\|\boldsymbol{\eta}\|_{1,q_\star},\|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\epsilon^2|\grave{a}|\left\|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\right\|_{0,q_\star/2}.
\end{aligned}$$

Finally, $\Delta_4$ uses Proposition E.4.1:

$$\begin{aligned}
\Delta_4 &\leq C|\grave{a}|\left\|(J^{\epsilon}\boldsymbol{\sigma}).(J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}).(\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},a))-\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},\grave{a})))\right\|_{0,q_\star/2} \\
&\leq C|\grave{a}|\left\|J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}\right\|_{W^{0,\infty}}\left\|(J^{\epsilon}\boldsymbol{\sigma}).(\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},a))-\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\grave{\boldsymbol{\eta}},\grave{a})))\right\|_{0,q_\star/2} \\
&\leq \mathcal{M}[\|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\epsilon^2|\grave{a}|\left\|\operatorname{sech}\left(\frac{q_\star}{2}\cdot\right)J^{\epsilon}(a\boldsymbol{\varphi}_{\epsilon}^{a}-\grave{a}\boldsymbol{\varphi}_{\epsilon}^{\grave{a}})\right\|_{W^{0,\infty}}\left\|J^{\epsilon}\boldsymbol{\sigma}\right\|_{1,q_\star}
\end{aligned}$$

$$\leq \mathcal{M}_{\Delta_4}[\|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\epsilon^2|\grave{a}||a-\grave{a}|.$$

*4.6.5. Lipschitz estimates for $\jmath_{42}$ and $\ell_{42}$.* Again, we smooth with $\varpi^\epsilon$ on $\jmath_{42}$ and use the extra $\epsilon$ on $\ell_{42}$ to our advantage; the mechanics are the same as the estimates for $\jmath_{32}$ and $\ell_{32}$ with one exception, which we highlight below. We bound

$$\begin{aligned}
\left\|\jmath_{42}^\epsilon(\boldsymbol{\eta},a)-\jmath_{42}^\epsilon(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{1,q_\star/2} &\leq C\left\|a\mathcal{Q}^\epsilon(\boldsymbol{\eta},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\grave{a}\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{0,q_\star/2} \\
&\leq |a-\grave{a}|\left\|\mathcal{Q}^\epsilon(\boldsymbol{\eta},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\quad \underbrace{+|\grave{a}|\left\|\mathcal{Q}^\epsilon(\boldsymbol{\eta},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{0,q_\star/2}}_{\Delta} \\
&\quad +|\grave{a}|\left\|\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^a,\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\quad +|\grave{a}|\left\|\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))\right\|_{0,q_\star/2} \\
&\quad +|\grave{a}|\left\|\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\boldsymbol{\varphi}_\epsilon^{\grave{a}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{0,q_\star/2}
\end{aligned}$$

The first term above can be bounded with our standard mapping estimate from Proposition E.1.1. The last three terms above are analogous to $\Delta_2$, $\Delta_3$, and $\Delta_4$ in Appendix 4.6.4. The exception is the term that we have labeled $\Delta$ above:

$$\begin{aligned}
\Delta &\leq C|\grave{a}|\left\|(J^\epsilon(\boldsymbol{\eta}-\grave{\boldsymbol{\eta}})).(J^\epsilon\boldsymbol{\varphi}_\epsilon^a).\mathcal{N}(\epsilon^2J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\leq C\epsilon|\grave{a}|\left\|J^\epsilon\boldsymbol{\varphi}_\epsilon^a\right\|_{W^{0,\infty}}\left\|\epsilon J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)\right\|_{W^{0,\infty}}\left\|(J^\epsilon(\boldsymbol{\eta}-\grave{\boldsymbol{\eta}})).N(\epsilon^2J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\leq C\epsilon|\grave{a}|\left\|(J^\epsilon(\boldsymbol{\eta}-\grave{\boldsymbol{\eta}})).N(\epsilon^2J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{0,q_\star/2} \\
&\leq C\epsilon|\grave{a}|\mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}]\left\|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\right\|_{0,q_\star/2}.
\end{aligned}$$

For the last inequality we have used Proposition E.3.1. The estimate for $\mathfrak{l}_{42}$ goes through in the same way.

*4.6.6. Lipschitz estimates for $\mathfrak{j}_{52}$ and $\mathfrak{l}_{52}$.* As we have no factor of $\boldsymbol{\varphi}_\epsilon^a$ here, we do not need to use smoothing or an extra factor of $\epsilon$ to avoid problems. We bound

$$\begin{aligned}
\left\|\mathfrak{j}_{52}^\epsilon(\boldsymbol{\eta},a)-\mathfrak{j}_{52}^\epsilon(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{1,q_\star/2} &\le C\left\|\mathcal{Q}^\epsilon(\boldsymbol{\eta},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{1,q_\star/2} \\
&\le \underbrace{C\left\|\mathcal{Q}^\epsilon(\boldsymbol{\eta},\boldsymbol{\eta},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{1,q_\star/2}}_{\Delta} \\
&\quad + C\left\|\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))\right\|_{1,q_\star/2} \\
&\quad + C\left\|\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a))-\mathcal{Q}^\epsilon(\grave{\boldsymbol{\eta}},\grave{\boldsymbol{\eta}},\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a}))\right\|_{1,q_\star/2}.
\end{aligned}$$

Of the three terms above, we know how to estimate the second and third using Propositions E.2.1 and E.4.1; we bound $\Delta$ by

$$\begin{aligned}
\Delta &\le C\left\|((J^\epsilon\boldsymbol{\eta})^{.2}-(J^\epsilon\grave{\boldsymbol{\eta}})^{.2}).\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)\right\|_{1,q_\star/2} \\
&\le C\epsilon\left\|J^\epsilon(\boldsymbol{\eta}+\grave{\boldsymbol{\eta}})\right\|_{1,q_\star/2}\left\|(J^\epsilon(\boldsymbol{\eta}-\grave{\boldsymbol{\eta}})).(\epsilon J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a)).N(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta},a))\right\|_{1,q_\star/2} \\
&\le \mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star},\|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\epsilon\left\|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\right\|_{1,q_\star/2}.
\end{aligned}$$

For the second inequality, we factored the difference of squares and used the algebra property of $H^1_{q_\star/2}$ and for the third we used Proposition E.3.1. The estimate for $\mathfrak{l}_{52}$ is the same.

*4.6.7. Lipschitz estimates for $\mathfrak{j}_6$ and $\mathfrak{l}_6$.* We work on $\mathfrak{j}_6$; the strategy for $\mathfrak{l}_6$ is the same. We have

$$\left\|\mathfrak{j}_6^\epsilon(\boldsymbol{\eta},a)-\mathfrak{j}_6^\epsilon(\grave{\boldsymbol{\eta}},\grave{a})\right\|_{1,q_\star/2} \le C\sum_{k=1}^{5}\left\|\Delta_k\right\|_{1,q_\star/2},$$

where

$$\Delta_1 = (a^2 - \grave{a}^2)(\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a, \boldsymbol{\varphi}_\epsilon^a, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a, \boldsymbol{\varphi}_\epsilon^a, a\boldsymbol{\varphi}_\epsilon^a))$$

$$\begin{aligned}\Delta_2 &= \grave{a}^2(\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a, \boldsymbol{\varphi}_\epsilon^a, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a, \boldsymbol{\varphi}_\epsilon^a, a\boldsymbol{\varphi}_\epsilon^a) \\ &\quad - (\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^a, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^a, a\boldsymbol{\varphi}_\epsilon^a)))\end{aligned}$$

$$\begin{aligned}\Delta_3 &= \grave{a}^2((\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^a, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^a, a\boldsymbol{\varphi}_\epsilon^a))) \\ &\quad - (\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, a\boldsymbol{\varphi}_\epsilon^a))\end{aligned}$$

$$\begin{aligned}\Delta_4 &= \grave{a}^2((\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, a\boldsymbol{\varphi}_\epsilon^a)) \\ &\quad - (\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, a\boldsymbol{\varphi}_\epsilon^a)))\end{aligned}$$

$$\begin{aligned}\Delta_5 &= \grave{a}^2((\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, a\boldsymbol{\varphi}_\epsilon^a)) \\ &\quad - (\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}}, \grave{a})) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, \grave{a}\boldsymbol{\varphi}_\epsilon^{\grave{a}})))\end{aligned}$$

Proposition E.2.1 lets us bound $\Delta_1$ as

$$\|\Delta_1\|_{1,q_\star/2} \le \mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon|a - \grave{a}|.$$

We use this proposition again on $\Delta_2$:

$$\begin{aligned}\|\Delta_2\|_{1,q_\star/2} &\le C\grave{a}^2 \left\|(J^\epsilon(\boldsymbol{\varphi}_\epsilon^a - \boldsymbol{\varphi}_\epsilon^{\grave{a}})).(J^\epsilon\boldsymbol{\varphi}_\epsilon^a).(\mathcal{N}(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a)))\right\|_{1,q_\star/2} \\
&\le C\grave{a}^2 \left\|J^\epsilon(\boldsymbol{\varphi}_\epsilon^a - \boldsymbol{\varphi}_\epsilon^{\grave{a}})\right\|_{W^{1,\infty}} \|J^\epsilon\boldsymbol{\varphi}_\epsilon^a\|_{W^{1,\infty}} \\
&\quad \times \left\|\mathcal{N}(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a)))\right\|_{1,q_\star/2} \\
&\le \grave{a}^2\epsilon^{-2}|a - \grave{a}|\mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}]\epsilon^2 \\
&= \mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}]|\grave{a}|^2|a - \grave{a}| \\
&\le \mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}]|a - \grave{a}|\end{aligned}$$

since $|\grave{a}| \le a_{\mathrm{per}} < 1$.

The estimate for $\Delta_3$ is exactly the same as the one for $\Delta_2$, while for $\Delta_4$ the $\pm\mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^{\grave{a}}, \boldsymbol{\varphi}_\epsilon^{\grave{a}}, a\boldsymbol{\varphi}_\epsilon^a)$ terms nicely cancel to give us

$$\|\Delta_4\|_{1,q_\star/2} \le C\grave{a}^2 \left\|(J^\epsilon \boldsymbol{\varphi}_\epsilon^{\grave{a}})^{.2}.(\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta},a)) - \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a)))\right\|_{1,q_\star/2}$$

$$\le \grave{a}^2\epsilon^{-2}\mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]\epsilon^2 \|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\|_{1,q_\star/2}$$

$$\le \mathcal{M}[\|\boldsymbol{\eta}\|_{1,q_\star}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]|\grave{a}| \|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\|_{1,q_\star/2}.$$

After factoring out $(J^\epsilon \boldsymbol{\varphi}_\epsilon^{\grave{a}})^{.2}$, we bound $\|\Delta_5\|_{1,q_\star/2}$ above by a form amenable to Proposition E.4.2:

$$C\grave{a}^2\epsilon^{-2}\left\|(\mathcal{N}(\epsilon^2 J^\epsilon\mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},a)) - \mathcal{N}(\epsilon^2 J^\epsilon(a\boldsymbol{\varphi}_\epsilon^a))) - (\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\grave{\boldsymbol{\eta}},\grave{a})) - \mathcal{N}(\epsilon^2 J^\epsilon(\grave{a}\boldsymbol{\varphi}_\epsilon^{\grave{a}})))\right\|_{1,q_\star/2}$$

$$\le \mathcal{M}[\|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]|\grave{a}|\epsilon^{-2}\left\|\operatorname{sech}\left(\frac{q_\star}{2}\cdot\right)\epsilon^2(aJ^\epsilon\boldsymbol{\varphi}_\epsilon^a - \grave{a}J^\epsilon\boldsymbol{\varphi}_\epsilon^{\grave{a}})\right\|_{W^{1,\infty}}\left\|\epsilon^2 J^\epsilon(\boldsymbol{\sigma}+\grave{\boldsymbol{\eta}})\right\|_{1,q_\star}$$

$$\le \mathcal{M}[\|\grave{\boldsymbol{\eta}}\|_{1,q_\star}]|\grave{a}|\epsilon|a-\grave{a}|.$$

**4.7. The bootstrap estimates.** Let

$$\mathcal{R}_{\mathrm{boot},r}^\epsilon(\boldsymbol{\eta},a) := \epsilon + |a|\epsilon^{1-r} + a^2\epsilon^{1-2r} + |a|^3\epsilon^{1-3r} + \|\boldsymbol{\eta}\|_{r,\star} + |a|\,\|\boldsymbol{\eta}\|_{r,q_\star}\epsilon^{-r}. \quad (4.7.1)$$

We prove the existence of increasing functions $\mathcal{M}_{\mathrm{boot},r}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|(\mathfrak{N}_1^\epsilon(\boldsymbol{\eta},a), \mathfrak{N}_2^\epsilon(\boldsymbol{\eta},a))\|_{r+1,q_\star} \le \mathcal{M}_{\mathrm{boot},r}[\|\boldsymbol{\eta}\|_{r,q_\star}]\mathcal{R}_{\mathrm{boot},r}^\epsilon(\boldsymbol{\eta},a) \quad (4.7.2)$$

and

$$|\mathfrak{N}_3^\epsilon(\boldsymbol{\eta},a)| \le \mathcal{M}_{\mathrm{boot},r}[\|\boldsymbol{\eta}\|_{r,q_\star}]\epsilon^r\mathcal{R}_{\mathrm{boot},r}^\epsilon(\boldsymbol{\eta},a) \quad (4.7.3)$$

for all $\boldsymbol{\eta} \in \mathcal{X}^r$ $|a| \le a_{\mathrm{per}}$, and $0 < \epsilon < \overline{\epsilon}$.

*4.7.1. General strategy.* It is for the sake of these bootstrap estimates that we proved Propositions E.1.1 and E.2.1 for arbitrary $r$ (thereby complicating the proofs considerably as opposed to doing them for just $r = 1$). In these sections $r$ will always be a positive integer.

The smoothing property (G.2.6) of $\mathcal{P}_\epsilon$ gives

$$\|\mathfrak{N}_2^\epsilon(\boldsymbol{\eta},a)\|_{r+1,q_\star} \le C_r \sum_{k=1}^{5} \left\|\ell_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{r+1,q_\star} + C_r \sum_{k=1}^{5} \|\ell_{k2}^\epsilon(\boldsymbol{\eta},a)\|_{r+1,q_\star} + \|\ell_6^\epsilon(\boldsymbol{\eta},a)\|_{r+1,q_\star},$$

and this along with the boundedness of $\mathcal{A}^{-1}$ implies

$$\begin{aligned}\|\mathfrak{N}_1^\epsilon(\boldsymbol{\eta},a)\|_{r+1,q_\star} &\le C_r \sum_{k=1}^{5} \left(\left\|j_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{r+1,q_\star} + \left\|\ell_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{r,q_\star}\right) \\ &\quad + C_r \sum_{k=1}^{5} \left(\left\|j_{k2}^\epsilon(\boldsymbol{\eta},a)\right\|_{r+1,q_\star} + \|\ell_{k2}^\epsilon(\boldsymbol{\eta},a)\|_{r,q_\star}\right) \\ &\quad + C_r \left(\left\|j_6^\epsilon(\boldsymbol{\eta},a)\right\|_{r+1,q_\star} + \|\ell_6^\epsilon(\boldsymbol{\eta},a)\|_{r,q_\star}\right).\end{aligned}$$

Last, the estimate (G.2.3) on $\iota_\epsilon$ and the boundedness of $\upsilon_\epsilon$ from (G.2.4) give

$$|\mathfrak{N}_3^\epsilon(\boldsymbol{\eta},a)| \le C\epsilon^r \sum_{k=1}^{5} \left(\left\|\ell_{k1}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{r,q_\star} + \|\ell_{k2}^\epsilon(\boldsymbol{\eta},a)\|_{r,q_\star}\right) + C\epsilon^r \|\ell_6^\epsilon(\boldsymbol{\eta},a)\|_{r,q_\star}.$$

In [FW18] we saw that the terms

$$\left\|j_{1k}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{r+1,q_\star} \quad \text{and} \quad \left\|\ell_{1k}^{\epsilon,\mathrm{mod}}(\boldsymbol{\eta},a)\right\|_{r,q_\star}, \ k=1,\ldots,5$$

are all bounded by $\mathcal{R}_{\mathrm{boot},r}^\epsilon(\boldsymbol{\eta},a)$. Now we bound the remaining terms

$$\left\|j_{2k}^\epsilon(\boldsymbol{\eta},a)\right\|_{r+1,q_\star}, \quad \|\ell_{2k}^\epsilon(\boldsymbol{\eta},a)\|_{r,q_\star}, \ k=1,\ldots,5, \quad \left\|j_6^\epsilon(\boldsymbol{\eta},a)\right\|_{r+1,q_\star}, \quad \text{and} \quad \|\ell_6^\epsilon(\boldsymbol{\eta},a)\|_{r,q_\star}$$

by $\mathcal{M}_{\mathrm{boot},r}[\|\boldsymbol{\eta}\|_{r,q_\star}]\mathcal{R}_{\mathrm{boot},r}^\epsilon(\boldsymbol{\eta},a)$.

We will only show the estimates for the $j$ terms, as once we have smoothed by $\varpi^\epsilon$ on a $j$ term, the resulting upper bound is a constant multiple of the upper bound for the corresponding $\ell$ term.

*4.7.2. Bootstrap estimates for $j_{12}$.* As with the mapping and Lipschitz estimates, we write this section in particular detail to illustrate our techniques. Smoothing by $\varpi^\epsilon$

per (G.2.8), we have

$$\begin{aligned}\left\|j_{12}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{r+1,q_\star} &\le C\left\|\mathcal{Q}^{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\sigma},\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{r,q_\star} \\ &\le C\left\|(J^{\epsilon}\boldsymbol{\sigma})^{.2}.\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{r,q_\star} \\ &\le \underbrace{C\epsilon^2\left\|(J^{\epsilon}\boldsymbol{\sigma})^{.2}.(J^{\epsilon}(\boldsymbol{\sigma}+\boldsymbol{\eta})).N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{r,q_\star}}_{\Pi_1} \\ &+ \underbrace{C\epsilon^2\left\|(J^{\epsilon}\boldsymbol{\sigma})^{.2}.(aJ^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}).N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{r,q_\star}}_{\Pi_2}.\end{aligned}$$

Then Proposition E.1.1 with $C_\star = \max\{C_1,\ldots,C_r\}$ and $C_1,\ldots,C_r$ satisfying the estimate (G.2.1) implies

$$\Pi_1 \le \epsilon^2\mathcal{M}_{1,r}[\epsilon^2\left\|J^{\epsilon}(\boldsymbol{\sigma}+\boldsymbol{\eta})\right\|_{r,q_\star}]\left(1+|a|\epsilon^{1-r}\right)\left\|(J^{\epsilon}\boldsymbol{\sigma})^{.2}.(J^{\epsilon}(\boldsymbol{\sigma}+\boldsymbol{\eta}))\right\|_{r,q_\star}.$$

After factoring, the same proposition gives

$$\begin{aligned}\Pi_2 &\le \epsilon^2|a|\left\|J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}\right\|_{W^{r,\infty}}\left\|(J^{\epsilon}\boldsymbol{\sigma})^{.2}.N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta},a))\right\|_{r,q_\star} \\ &\le C_r\epsilon^{2-r}|a|\mathcal{M}_{2,r}[\left\|\epsilon^2 J^{\epsilon}(\boldsymbol{\sigma}+\boldsymbol{\eta})\right\|_{r,q_\star}]\left(1+|a|\epsilon^{1-r}\right)\left\|(J^{\epsilon}\boldsymbol{\sigma})^{.2}\right\|_{r,q_\star}.\end{aligned}$$

Since we can assume $\mathcal{M}_{1,r}$ and $\mathcal{M}_{2,r}$ are increasing, we take the supremum over $0<\epsilon<\overline{\epsilon}<1$ and find

$$\Pi_1 \le \widetilde{\mathcal{M}}_{1,r}[\|\boldsymbol{\eta}\|_{r,q_\star}]\epsilon\left(1+|a|\epsilon^{1-r}\right) \tag{4.7.4}$$

and

$$\Pi_2 \le \widetilde{\mathcal{M}}_{2,r}[\|\boldsymbol{\eta}\|_{r,q_\star}]\epsilon^{2-r}|a|\left(1+|a|\epsilon^{1-r}\right) \le \widetilde{\mathcal{M}}_{2,r}[\|\boldsymbol{\eta}\|_{r,q_\star}]|a|\epsilon^{1-r}+a^2\epsilon^{1-2r} \tag{4.7.5}$$

for increasing functions $\widetilde{\mathcal{M}}_{1,r}$ and $\widetilde{\mathcal{M}}_{2,r}$. All together, these estimates give an upper bound on $\left\|j_{12}^{\epsilon}(\boldsymbol{\eta},a)\right\|_{r+1,q_\star}$ of the form $\mathcal{M}[\|\boldsymbol{\eta}\|_{r,q_\star}]\mathcal{R}_{\mathrm{boot},r}^{\epsilon}(\boldsymbol{\eta},a)$ given in (4.7.1).

*4.7.3. Bootstrap estimates for $j_{22}$.* We have

$$
\begin{aligned}
\left\| j_{22}^{\epsilon}(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} &\leq C \left\| \mathcal{Q}^{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\eta}, \mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&\leq C \left\| (J^{\epsilon}\boldsymbol{\sigma}).(J^{\epsilon}\boldsymbol{\eta}).\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&\leq C \left\| J^{\epsilon}\boldsymbol{\eta} \right\|_{r,q_\star} \left\| (J^{\epsilon}\boldsymbol{\sigma}).\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&\leq \mathcal{M}_r[\|\boldsymbol{\eta}\|_{r,q_\star}] \left(1 + |a|\epsilon^{1-r}\right) \|\boldsymbol{\eta}\|_{r,q_\star},
\end{aligned}
$$

using Proposition E.1.1 for the last inequality.

*4.7.4. Bootstrap estimates for $j_{32}$.* We begin with

$$
\begin{aligned}
\left\| j_{32}^{\epsilon}(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} &\leq C|a| \left\| \mathcal{Q}^{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\varphi}_{\epsilon}^{a}, \mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&\leq C|a| \left\| (J^{\epsilon}\boldsymbol{\sigma}).(J^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}).\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&\leq C|a|\epsilon^{-r} \left\| (J^{\epsilon}\boldsymbol{\sigma}).\mathcal{N}(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star}.
\end{aligned}
$$

Expanding $\mathcal{N}$ into its product form, we find

$$
\begin{aligned}
\left\| j_{32}^{\epsilon}(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} \leq\ & C|a|\epsilon^{2-r} \left\| (J^{\epsilon}\boldsymbol{\sigma}).(J^{\epsilon}(\boldsymbol{\sigma} + \boldsymbol{\eta})).N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&+ C|a|\epsilon^{2-r} \left\| (J^{\epsilon}\boldsymbol{\sigma}).(aJ^{\epsilon}\boldsymbol{\varphi}_{\epsilon}^{a}).N(\epsilon^2 J^{\epsilon}\mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star}.
\end{aligned}
$$

We estimate these terms as we did $\Pi_1$ and $\Pi_2$ in (4.7.4) and (4.7.5) and find

$$
\left\| j_{32}^{\epsilon}(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} \leq \mathcal{M}_r[\|\boldsymbol{\eta}\|_{r,q_\star}] \left( |a|\epsilon^{1-r} + a^2\epsilon^{1-2r} \right).
$$

*4.7.5. Bootstrap estimates for $j_{42}$.* Routine estimates give

$$
\left\| j_{42}^{\epsilon}(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} \leq C|a| \left\| \mathcal{Q}^{\epsilon}(\boldsymbol{\eta}, \boldsymbol{\varphi}_{\epsilon}^{a}, \mathfrak{A}_{\epsilon}(\boldsymbol{\eta}, a)) \right\|_{r,q_\star}
$$

$$\leq C|a| \left\| (J^\epsilon \boldsymbol{\eta}).(J^\epsilon \boldsymbol{\varphi}_\epsilon^a).\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{r,q_\star}$$

$$\leq C|a|\epsilon^{-r} \left\| (J^\epsilon \boldsymbol{\eta}).\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{r,q_\star}.$$

From here we follow the $j_{32}$ estimates with $\boldsymbol{\sigma}$ replaced by $\boldsymbol{\eta}$.

*4.7.6. Bootstrap estimates for $j_{52}$.* Straightforward estimates and one invocation of Proposition E.1.1 give

$$\begin{aligned}
\left\| j_{52}^\epsilon(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} &\leq C \left\| \mathcal{Q}^\epsilon(\boldsymbol{\eta}, \boldsymbol{\eta}, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&\leq C \left\| (J^\epsilon \boldsymbol{\eta})^{.2}.\mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) \right\|_{r,q_\star} \\
&\leq \mathcal{M}_r[\epsilon^2 \left\| J^\epsilon(\boldsymbol{\sigma} + \boldsymbol{\eta}) \right\|_{r,q_\star}] \left(1 + |a|\epsilon^{1-r}\right) \left\| (J^\epsilon \boldsymbol{\eta})^{.2} \right\|_{r,q_\star} \\
&\leq \mathcal{M}_r[\left\| \boldsymbol{\eta} \right\|_{r,q_\star}] \left(1 + |a|\epsilon^{1-r}\right) \left\| \boldsymbol{\eta} \right\|_{r,q_\star}^2.
\end{aligned}$$

*4.7.7. Bootstrap estimates for $j_6$.* We rely on Proposition E.2.1:

$$\begin{aligned}
\left\| j_6^\epsilon(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} &\leq Ca^2 \left\| \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a, \boldsymbol{\varphi}_\epsilon^a, \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{Q}^\epsilon(\boldsymbol{\varphi}_\epsilon^a, \boldsymbol{\varphi}_\epsilon^a, a\boldsymbol{\varphi}_\epsilon^a) \right\|_{r,q_\star} \\
&\leq Ca^2 \left\| (J^\epsilon \boldsymbol{\varphi}_\epsilon^a)^{.2}. \left( \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon (a\boldsymbol{\varphi}_\epsilon^a)) \right) \right\|_{r,q_\star} \\
&\leq Ca^2\epsilon^{-2r} \left\| \mathcal{N}(\epsilon^2 J^\epsilon \mathfrak{A}_\epsilon(\boldsymbol{\eta}, a)) - \mathcal{N}(\epsilon^2 J^\epsilon (a\boldsymbol{\varphi}_\epsilon^a)) \right\|_{r,q_\star} \\
&\leq Ca^2\epsilon^{2-2r}\mathcal{M}_r[\left\| J^\epsilon(\boldsymbol{\sigma} + \boldsymbol{\eta}) \right\|_{r,q_\star}] \left(1 + |a|\epsilon^{1-r}\right) \left\| J^\epsilon(\boldsymbol{\sigma} + \boldsymbol{\eta}) \right\|_{r,q_\star}.
\end{aligned}$$

Taking the supremum over $\epsilon \in (0, \overline{\epsilon})$ in the $\mathcal{M}_r$ factor, we conclude

$$\left\| j_6^\epsilon(\boldsymbol{\eta}, a) \right\|_{r+1,q_\star} \leq \mathcal{M}_r[\left\| \boldsymbol{\eta} \right\|_{r,q_\star}] a^2\epsilon^{2-2r} \left(1 + |a|\epsilon^{1-r}\right) \leq \mathcal{M}_r[\left\| \boldsymbol{\eta} \right\|_{r,q_\star}] \left(a^2\epsilon^{1-2r} + |a|^3\epsilon^{1-3r}\right).$$

# Appendix A. Calculus

**A.1. Leibniz's rule.** We will often use Leibniz's rule for an arbitrary derivative of a product:

$$\partial_X^r[fg] = \sum_{k=0}^{r} \binom{r}{k} \partial_X^k[f]\partial_X^{r-k}[g]. \tag{A.1.1}$$

**A.2. Faá di Bruno's formula.** We employ the convenient expression of Faá di Bruno's formula for the chain rule found in [Mor13]. For $k$, $r \in \mathbb{N}$ with $k \leq r$, let

$$\Sigma_k^r = \left\{ \boldsymbol{\sigma} \in \mathbb{N}^k \;\middle|\; \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 1,\ |\boldsymbol{\sigma}| = r \right\},$$

where

$$|\boldsymbol{\sigma}| := \sum_{j=1}^{k} \sigma_j.$$

**A.2.1 Remark.** **(i)** *It is apparent from the definition that* $\Sigma_1^r = \{r\}$.

**(ii)** *If* $\boldsymbol{\sigma} \in \Sigma_k^r$ *with* $2 \leq k \leq r$, *then* $\sigma_j < r$ *for all* $j$.

**A.2.2 Theorem (Faá di Bruno).** *Let* $N, f \in \mathcal{C}^r(\mathbb{R})$. *Then*

$$\partial_X^r[N(f)] = \sum_{k=1}^{r} \partial_X^k[N](f) \sum_{\boldsymbol{\sigma} \in \Sigma_k^r} C_{\boldsymbol{\sigma}} \prod_{j=1}^{k} \partial_X^{\sigma_j}[f],$$

*where the* $C_{\boldsymbol{\sigma}}$ *are positive constants that depend on* $r$ *and* $k$ *but are independent of* $f$ *and* $X$.

## Appendix B. The Fourier Transform

**B.1. The periodic Fourier transform.** For an integrable, $2P$-periodic function $f$ defined on $\mathbb{R}$, we set

$$\mathfrak{F}[f](k) = \widehat{f}(k) =: \frac{1}{\sqrt{2P}} \int_{-P}^{P} f(X) e^{ik\pi X/P} \, dX. \tag{B.1.1}$$

Then

$$f(X) = \sum_{k=-\infty}^{\infty} e^{ik\pi X/P} \widetilde{\mu}\left(\frac{k\pi}{P}\right) \widehat{f}(k),$$

where the sum converges in the $L^2_{\text{per}}$-norm defined in (C.2.1).

**B.2. The Fourier transform on $\mathbb{R}$.** For a function $f \in L^1$, we set

$$\mathfrak{F}[f](k) = \widehat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X) e^{-ikX} \, dX \tag{B.2.1}$$

and

$$\mathfrak{F}^{-1}[f](k) = \check{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X) e^{ikX} \, dX. \tag{B.2.2}$$

We extend the Fourier transform to $L^2$ by the density of $L^1 \cap L^2$ in $L^2$, cf. [Eva10].

**B.3. The complex Fourier transform.** We will need the Fourier transform for a function of a complex variable. Our preferred development of these results is contained in Chapter 5 of [Fis99] and Chapter 8 of [Det84]. Let $q > 0$ and

$$L^p_q := \{ f \in L^p(\mathbb{R}) \mid \cosh(q \cdot) f \in L^p(\mathbb{R}) \}. \tag{B.3.1}$$

Let $\Sigma_q$ be the strip

$$\Sigma_q := \{ z \in \mathbb{C} \mid |\operatorname{Im}(z)| < q \}.$$

Then for any $f \in L^1_q$ and $z \in \Sigma_q$, we define the Fourier transform of $f$ at $z$ to be

$$\mathfrak{F}[f](z) = \widehat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X) e^{-izX} \, dX. \tag{B.3.2}$$

It follows that $\widehat{f}$ is analytic on $\Sigma_q$, and it is a direct calculation that if $|y| < q$, then

$$\mathfrak{F}[e^{y\cdot}f](k) = \widehat{f}(k+iy) \tag{B.3.3}$$

for any $k \in \mathbb{R}$.

The integral in (B.3.2) need not converge when $|\operatorname{Im}(z)| = q$. However, we can still ascribe meaning to $\widehat{f}(k \pm iq)$ for a function $f \in L^2_q \cap \left(\cap_{0 \le y < q} L^1_y\right)$, which will be our concern later. In this case, we have the estimate

$$e^{yX}|f(X)| \le e^{qX}|f(X)| = \cosh(qX)|f(X)|\operatorname{sech}(qX)|e^{\pm qX} \le \cosh(qX)|f(X)|, \tag{B.3.4}$$

so that $e^{y\cdot}f \in L^2$ for $|y| \le q$ since $f \in L^2_q$. By (B.3.3), we have $\widehat{f}(\cdot + iy) \in L^2$ for $|y| < q$.

Then motivated by (B.3.3), we *define*

$$\widehat{f}(k \pm iq) := \mathfrak{F}[e^{q\cdot}f](k). \tag{B.3.5}$$

To see that this is the "correct" definition, observe that by Plancherel's theorem, if $|y| < q$, then

$$\|\mathfrak{F}[e^{y\cdot}f] - \mathfrak{F}[e^{q\cdot}f]\|_{L^2} = \|(e^{y\cdot} - e^{q\cdot}f)\|_{L^2}$$

and clearly we have the pointwise convergence

$$\lim_{y\to q}\left(e^{yX} - e^{qX}\right)f(X) = 0.$$

We can also dominate pointwise for $|y| < q$ by (B.3.4):

$$\left|\left(e^{yX} - e^{qX}\right)f(X)\right| \le 2|\cosh(qX)f(X)|.$$

Since $\cosh(q\cdot)f \in L^2$, we conclude

$$\lim_{y\to q}\|\mathfrak{F}[e^{y\cdot}f] - \mathfrak{F}[e^{q\cdot}f]\|_{L^2} = \lim_{y\to q}\|e^{y\cdot}f - e^{q\cdot}f\|_{L^2} = 0. \tag{B.3.6}$$

That is,

$$\lim_{y\to q}\widehat{f}(\cdot + iy) = \widehat{f}(\cdot + iq) \tag{B.3.7}$$

in $L^2$, which is certainly a reasonable expectation of how (B.3.5) should behave.

## Appendix C. Sobolev Spaces

**C.1. Basic definitions.** We follow the treatment of [Ada75] for this material; our other standard, preferred references are [Eva10] and [EG16]. Let $U \subseteq \mathbb{R}$. We denote by $W^{m,p}$ the the set of all functions $f \in L^p(\mathbb{R})$ such that the weak partial derivatives $\partial_X^j[f]$ exist for all $0 \leq j \leq m$. We have

$$\|f\|_{W^{m,p}} := \left( \sum_{j=0}^{m} \left\| \partial_X^j[f] \right\|_{L^p}^p \right)^{1/p}.$$

The space $W_0^{m,p}$ is the closure of $\mathcal{C}_c^\infty(\mathbb{R})$ in the $W^{m,p}$-norm, where $\mathcal{C}_c^\infty$ is the subspace of functions in $\mathcal{C}^\infty(\mathbb{R})$ with compact support. By Corollary 3.19 in [Ada75], $W_0^{m,p} = W^{m,p}$.

We set

$$|f|_{j,p} := \left\| \partial_X^j[f] \right\|_{L^p}$$

for $f \in W^{m,p}, 1 \leq p < \infty$ and $0 \leq j \leq m$. Then

$$\|f\|_{W^{m,p}} = \left( \sum_{j=0}^{m} |f|_{j,p}^p \right)^{1/p} \qquad \text{and} \qquad |f|_{0,p} = \|f\|_{W^{0,p}} = \|f\|_{L^p}.$$

The mappings $|\cdot|_j, j = 1, \ldots, r$, are seminorms on $W^{m,p}$.

By Corollary 4.16 in [Ada75], the map

$$|||f|||_{m,p} := \left(|f|_{0,p}^p + |f|_{m,p}^p\right)^{1/p} = \left( \|f\|_{L^p}^p + \sum_{|\alpha|=m} \|\partial_X^\alpha[f]\|_{L^p} \right)^{1/p}$$

is a norm on $W_0^{m,p}$ that is equivalent to the $W_0^{m,p}$-norm, i.e., the $W^{m,p}$-norm on $W_0^{m,p}$. In particular, the norms $|||\cdot|||_{m,p}$ and $\|\cdot\|_{W^{m,p}}$ are equivalent since $W_0^{m,p} = W^{m,p}$.

We also set $W^{m,\infty}$ to be the space of all functions $f \in L^\infty(\mathbb{R})$ whose weak derivatives $\partial_X^j[f]$ exist for $j = 0, \ldots, m$ with $\partial_X^j[f] \in L^\infty(\mathbb{R})$ for each $j$. Its norm is

$$\|f\|_{W^{r,\infty}} := \sum_{j=0}^{m} \left\| \partial_X^j[f] \right\|_{L^\infty}.$$

Specializing to the case of $H^r := W^{r,2}$, we see that the norms

$$f \mapsto \left(\sum_{j=0}^{r} \left\|\partial_X^j[f]\right\|_{L^2}^2\right)^{1/2} \qquad \text{and} \qquad f \mapsto \|f\|_{L^2} + \|\partial_X^r[f]\|_{L^2}$$

are equivalent. Here we have used the equivalence of the norms

$$(x_1, x_2) \mapsto \left(x_1^2 + x_2^2\right)^{1/2} \qquad \text{and} \qquad (x_1, x_2) \mapsto |x_1| + |x_2|$$

on $\mathbb{R}^2$. We write

$$\|f\|_{H^r} = \|f\|_{L^2} + \|\partial_X^r[f]\|_{L^2} \tag{C.1.1}$$

from now on. Note that because we only include the lowest (zeroth) and highest ($r$th) derivatives in our definition of $\|\cdot\|_{H^r}$, we must keep track of a constant when bounding lower Sobolev norms by higher ones: if $r \le s$, then

$$\|f\|_{H^r} \le C_r \left(\sum_{j=0}^{r} \left\|\partial_X^j[f]\right\|_{L^2}^2\right)^{1/2} \le C_r \left(\sum_{j=0}^{s} \left\|\partial_X^j[f]\right\|_{L^2}^2\right)^{1/2} \le C_{r,s} \|f\|_{H^s}. \tag{C.1.2}$$

We also mention the Fourier transform characterization of $H^r$: given $f \in L^2(\mathbb{R})$, we have $f \in H^r$ if and only if

$$\int_{-\infty}^{\infty} (1 + k^2)^r |\widehat{f}(k)|^2 \, dk < \infty,$$

and $\|\cdot\|_{H^r}$ is equivalent to the norm $f \mapsto \int_{-\infty}^{\infty} (1 + k^2)^r |\widehat{f}(k)|^2 \, dk$.

**C.2. Periodic Sobolev spaces.** Our preferred references are [Kre89] and [HN01]. We set

$$\mathcal{C}_{\text{per}}^r := \{f \in \mathcal{C}^r([-\pi, \pi]) \mid f(-\pi) = f(\pi)\} \qquad \text{and} \qquad \|f\|_{\mathcal{C}_{\text{per}}^r} := \sum_{j=0}^{r} \left\|\partial_X^j[f]\right\|_{L^\infty}.$$

Then we define

$$\mathcal{C}_{\text{per}}^\infty := \cap_{r=0}^\infty \mathcal{C}_{\text{per}}^r$$

and, for $f \in \mathcal{C}_{\text{per}}^\infty$,

$$\|f\|_{L_{\text{per}}^2} := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f(x)|^2 \, dx. \tag{C.2.1}$$

We denote by $L^2_{\mathrm{per}}$ the completion of $\mathcal{C}^\infty_{\mathrm{per}}$ in the norm $\|\cdot\|_{L^2_{\mathrm{per}}}$. For $f \in L^2_{\mathrm{per}}$ and $k \in \mathbb{Z}$, the $k$th Fourier coefficient of $f$ is

$$\widehat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(X) e^{-ikX}\, dX.$$

Then for $f \in L^2_{\mathrm{per}}$, we set

$$\|f\|_{H^r_{\mathrm{per}}} := \left( \sum_{k\in\mathbb{Z}} (1+k^2)^r |\widehat{f}(k)|^2 \right)^{1/2},$$
$$\langle f, g\rangle_{H^r_{\mathrm{per}}} := \sum_{k\in\mathbb{Z}} (1+k^2)^r \widehat{f}(k)\overline{\widehat{g}(k)},$$

and

$$H^r_{\mathrm{per}} := \left\{ f \in L^2_{\mathrm{per}} \;\middle|\; \|f\|_{H^r_{\mathrm{per}}} < \infty \right\}.$$

For $r = 0$ we let $H^0_{\mathrm{per}} = L^2_{\mathrm{per}}$.

Theorem 8.4 in [Kre89] states that $\|\cdot\|_{H^r_{\mathrm{per}}}$ is equivalent to

$$f \mapsto \left( \|f\|^2_{L^2_{\mathrm{per}}} + \|\partial^r_X[f]\|^2_{L^2_{\mathrm{per}}} \right)^{1/2},$$

which is in turn equivalent to

$$f \mapsto \|f\|_{L^2_{\mathrm{per}}} + \|\partial^r_X[f]\|_{L^2_{\mathrm{per}}}. \tag{C.2.2}$$

This is our preferred norm for $H^r_{\mathrm{per}}$ due to its similarity to the convenient structure of (C.1.1).

We will also need three familiar estimates:

**(i)** The Sobolev product estimate

$$\|fg\|_{H^r_{\mathrm{per}}} \le C_r \|f\|_{H^r_{\mathrm{per}}} \|g\|_{H^r_{\mathrm{per}}}; \tag{C.2.3}$$

**(ii)** The Sobolev embedding estimate

$$\|f\|_{\mathcal{C}^{r-1}_{\mathrm{per}}} \le C_r \|f\|_{H^r_{\mathrm{per}}} \tag{C.2.4}$$

for $r \ge 1$;

**(iii)** And the Fourier transform estimate

$$|\widehat{f}(k)| \le \sqrt{2\pi}\, \|f\|_{L^\infty} \le C_r \|f\|_{H^r_{\mathrm{per}}}. \tag{C.2.5}$$

**C.3. Sobolev spaces of exponentially decaying functions.** For $r$, $q \geq 0$ we set

$$H_q^r := \{f \in H^r \mid \cosh(q\cdot)f \in H^r\}$$

and, for integer $r$ (which is the only kind of $r$ we consider), we use the norm

$$\|f\|_{r,q} := \|\cosh(q\cdot)f\|_{L^2} + \|\cosh(q\cdot)\partial_X^r[f]\|_{L^2} . \tag{C.3.1}$$

That $\cosh(q\cdot)f \in L^2$ if $f \in H_q^r$ is the result of Lemma C.3.9. See Appendix C.3.1 for a thorough discussion of several other equivalent norms on and definitions of $H_q^r$. Set $H_q^0 = L_q^2$ as in (B.3.1).

Each $H_q^r$ space is a Hilbert space with inner product

$$\langle f, g\rangle_{r,q} := \langle \cosh(q\cdot)f, \cosh(q\cdot)g\rangle_{H^r} ,$$

which we now demonstrate.

**C.3.1 Lemma.** *Let $f \in H_q^r$ for some $q \geq 0$ and $r \geq 1$. Let $(f_n)$ be a sequence in $H_q^r$ such that $f_n \to f$ in $L_q^2$ and $\partial_X^j[f_n] \to g_j$ in $L_q^2$ for some $0 < j \leq r$ and some $g_j \in L_q^2$. Then $g_j = \partial_X^j[f]$.*

**Proof.** Let $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty} \varphi(X)(\partial_X^j[f](X) - g_j(X))\, dX = \underbrace{\int_{-\infty}^{\infty} \varphi(X)(\partial_X^j[f](X) - \partial_X^j[f_n](X))\, dX}_{\mathcal{I}_1} + \underbrace{\int_{-\infty}^{\infty} \varphi(X)(\partial_X^j[f_n](X) - g_j(X))\, dX}_{\mathcal{I}_2} .$$

The definition of the weak derivative gives

$$\int_{-\infty}^{\infty} \varphi(X)\partial_X^j[f](X)\, dX = (-1)^j \int_{-\infty}^{\infty} \partial_X^j[\varphi](X) f_n(X)\, dX$$

and

$$\int_{-\infty}^{\infty} \varphi(X)\partial_X^j[f_n](X) = (-1)^j \int_{-\infty}^{\infty} \partial_X^j[\varphi](X) f_n(X)\, dX.$$

Hence

$$\mathcal{I}_1 = (-1)^j \int_{-\infty}^{\infty} \partial_X^j[\varphi](f - f_n),$$

and so, using the Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \varphi(X)(\partial_X^j[f](X) - g_j(X))\, dX\right| &\le |\mathcal{I}_1| + |\mathcal{I}_2| \\ &\le \left\|\partial_X^j[\varphi]\right\|_{L^2} \|f_n - f\|_{L^2} + \|\varphi\|_{L^2} \left\|\partial_X^j[f_n] - g_j\right\|_{L^2} \\ &\le \left\|\partial_X^j[\varphi]\right\|_{L^2} \|f_n - f\|_{L^2_q} + \|\varphi\|_{L^2} \left\|\partial_X^j[f_n] - g_j\right\|_{L^2_q}.\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} \varphi(X)(\partial_X^j[f](X) - g_j(X))\, dX = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R})$, and so $\partial_X^j[f] = g_j$. ■

**C.3.2 Proposition.** *Each space $H_q^r$ is a Banach space.*

**Proof.** Let $(f_n)$ be a Cauchy sequence in $H_q^r$. It follows that the sequences $(f_n)$, $(\cosh(q\cdot)f_n)$, $(\partial_X^r[f_n])$, and $(\cosh(q\cdot)\partial_X^r[f_n])$ are all $L^2$-Cauchy, and so they have $L^2$-limits $F_1$, $F_2$, $F_3$, and $F_4$, respectively. We may extract subsequences (which we still denote by $f_n$) so that the convergence is also pointwise a.e; this is Theorem VII.1.4 in [Lan93]. By Lemma C.3.1 with $q = 0$, we have $\partial_X^r[F_1] = F_3$. Then, pointwise a.e. in $X$, we have both

$$\cosh(qX)F_1(X) = \cosh(qX) \lim_{n\to\infty} f_n(X) = \lim_{n\to\infty} \cosh(qX)f_n(X) = F_2(X)$$

and likewise

$$\begin{aligned}\cosh(qX)\partial_X^r[F_1](X) = \cosh(qX)F_3(X) &= \cosh(qX) \lim_{n\to\infty} \partial_X^r[f_n](X) \\ &= \lim_{n\to\infty} \cosh(qX)\partial_X^r[f_n](X) = F_4(X).\end{aligned}$$

So, $\cosh(q\cdot)F_1 = F_2 \in L^2$ and $\cosh(q\cdot)\partial_X^r[F_1] = F_4 \in L^2$. Finally,

$$\|f_n - F_1\|_{r,q} = \|\cosh(q\cdot)f_n - \cosh(q\cdot)F_1\|_{L^2} + \|\cosh(q\cdot)\partial_X^r[f_n] - \cosh(q\cdot)\partial_X^r[F_1]\|_{L^2}$$

$$= \|\cosh(q\cdot)f_n - F_2\|_{L^2} + \|\cosh(q\cdot)\partial_X^r[f_n] - F_4\|_{L^2} \to 0. \quad \blacksquare$$

That is, $f_n \to F_1$ in $H_q^r$, so $H_q^r$ is complete.

We also record here a lemma that we need for the proof of Theorem 4.3.2.

**C.3.3 Lemma.** *Let $(f_n)$ be a sequence in $H_{q_1}^{r_1}$ that converges weakly in $H_{q_1}^{r_1}$ to $F_1$ and strongly in $H_{q_2}^{r_2}$ to $F_2$. Suppose $q_1 \geq q_2$ and $r_1 \geq r_2$. Then $F_1 = F_2$.*

**Proof.** Fix $h \in L^2$ and define

$$\varphi_h \colon H_{q_1}^{r_1} \to \mathbb{C} \colon f \mapsto \langle f, h\rangle_{L^2}.$$

Then $\varphi_h$ is a bounded linear functional on $H_{q_1}^{r_1}$, as

$$|\varphi_h(f)| = |\langle f, h\rangle_{L^2}| \leq \|f\|_{L^2}\|h\|_{L^2} \leq C_{r_1,q_1}\|f\|_{r_1,q_1}\|h\|_{L^2}.$$

Hence

$$0 = \lim_{n\to\infty}\varphi_h(f_n - F_1) = \lim_{n\to\infty}\langle f_n - F_1, h\rangle_{L^2}$$

in $\mathbb{C}$. Since $h \in L^2$ was arbitrary, we conclude that $f_n \to F_1$ weakly in $L^2$. But since $\|f\|_{L^2} \leq C_{r_2,q_2}\|f\|_{r_2,q_2}$, we know that $f_n \to F_2$ strongly in $L^2$ and so also weakly. By the uniqueness of weak limits, we have $F_1 = F_2$ in $L^2$, which means $F_1 = F_2$ a.e., and so $\|F_1 - F_2\|_{r_1,q_1} = \|F_1 - F_2\|_{r_2,q_2} = 0$. $\blacksquare$

*C.3.1. Equivalent definitions of some weighted Sobolev spaces.* We can replace the weight $\cosh(q\cdot)$ with $\cosh^q(\cdot)$ in the definition of $H_q^r$ and retain the same space and an equivalent norm. This turns out to be convenient for a variety of proofs and estimates.

For $q > 0$ and $r \in \mathbb{N}$, let

$$\mathcal{X}_q^r = \{f \in H^r \mid \cosh(q\cdot)f \in H^r\},$$

$$\mathcal{Y}_q^r = \{f \in H^r \mid \cosh^q(\cdot)f \in H^r\},$$

$$\|f\|_{\mathcal{X}_q^r} = \|\cosh(q\cdot)f\|_{H^r},$$

$$\|f\|_{\mathcal{Y}_q^r} = \|\cosh^q(\cdot) f\|_{H^r} \,,$$

$$|||f|||_{\mathcal{X}_q^r} = \|\cosh(q\cdot) f\|_{L^2} + \|\cosh(q\cdot)\partial_X^r[f]\|_{L^2} \,,$$

and

$$|||f|||_{\mathcal{Y}_q^r} = \|\cosh^q(\cdot) f\|_{L^2} + \|\cosh^q(\cdot)\partial_X^r[f]\|_{L^2} \,.$$

Note that $|||f|||_{\mathcal{X}_q^r} = \|f\|_{r,q}$ from (C.3.1).

**C.3.4 Proposition.** $\mathcal{X}_q^r = \mathcal{Y}_q^r$ *and the norms* $\|\cdot\|_{\mathcal{X}_q^r}$, $\|\cdot\|_{\mathcal{Y}_q^r}$, $|||\cdot|||_{\mathcal{X}_q^r}$, *and* $|||\cdot|||_{\mathcal{Y}_q^r}$ *are all equivalent.*

We prove this in Lemmas C.3.5, C.3.6, and C.3.7 below. We use whichever of the four norms $\|\cdot\|_{\mathcal{X}_q^r}$, $\|\cdot\|_{\mathcal{Y}_q^r}$, $|||\cdot|||_{\mathcal{X}_q^r}$, and $|||\cdot|||_{\mathcal{Y}_q^r}$ we find convenient and always denote this norm by $\|\cdot\|_{r,q}$ as before.

**C.3.5 Lemma.** *The norms* $\|\cdot\|_{\mathcal{X}_q^r}$ *and* $|||\cdot|||_{\mathcal{X}_q^r}$ *are equivalent on* $H_q^r$.

**Proof.** First we show there is $C_{r,q} > 0$ such that

$$\|f\|_{\mathcal{X}_q^r} \le C_{r,q} \, |||f|||_{\mathcal{X}_q^r} \,, \ f \in \mathcal{X}_q^r.$$

We have

$$\|\partial_X^r[\cosh(q\cdot) f]\|_{L^2} = \|\mathfrak{F}[\partial_X^r[\cosh(q\cdot) f]]\|_{L^2}$$

and[6]

$$\begin{aligned}
\mathfrak{F}[\partial_X^r[\cosh(q\cdot) f]](k) &= (ik)^r \mathfrak{F}[\cosh(q\cdot) f](k) \\
&= (ik)^r \frac{\widehat{f}(k+iq) + \widehat{f}(k-iq)}{2} \\
&= (ik - q + q)^r \frac{\widehat{f}(k+iq)}{2} + (ik + q - q)\frac{\widehat{f}(k-iq)}{2}
\end{aligned}$$

[6] We interpret the following calculations as equalities in $L^2(\mathbb{R})$, per Theorem 3.4.5 in [Mik98], which gives the Fourier transform relation $\mathfrak{F}[\partial_X[f]](k) = (ik)\widehat{f}(k)$ when $\partial_X[f]$ is the weak derivative of $f$.

$$= \sum_{j=0}^{r} \binom{r}{j} (ik - q)^j q^{r-j} \frac{\widehat{f}(k + iq)}{2}$$

$$+ \sum_{j=0}^{r} \binom{r}{j} (ik + q)^j (-q)^{r-j} \frac{\widehat{f}(k - iq)}{2}.$$

Using (C.3.9) and Lemma C.3.10, we find

$$\left| (k \pm iq)^j \frac{\widehat{f}(k \pm iq)}{2} \right| \leq \frac{|\widehat{f}(k \pm iq)|}{2} + \frac{|(k \pm iq)^r \widehat{f}(k \pm iq)|}{2}$$

$$= \frac{|\widehat{e^{\pm q \cdot} f}(k)|}{2} + \frac{|\widehat{\partial_X^r[f]}(k \pm iq)|}{2}$$

$$= \frac{|\widehat{e^{\pm q \cdot} f}(k)|}{2} + \frac{|\widehat{e^{\pm q} \partial_X^r[f]}(k)|}{2},$$

and so

$$\|\partial_X^r[\cosh(q \cdot) f]\|_{L^2} \leq C_{r,q} \left( \left\| \widehat{e^{q \cdot} f} \right\|_{L^2} \left\| \widehat{e^{-q \cdot} f} \right\|_{L^2} + \left\| \widehat{e^{q \cdot} \partial_X^r[f]} \right\|_{L^2} + \left\| \widehat{e^{-q \cdot} \partial_X^r[f]} \right\|_{L^2} \right)$$

$$= C_{r,q} \left( \|e^{q \cdot} f\|_{L^2} + \left\|e^{-q \cdot} f\right\|_{L^2} + \|e^{q \cdot} \partial_X^r[f]\|_{L^2} + \left\|e^{-q \cdot} \partial_X^r[f]\right\|_{L^2} \right)$$

$$\leq C_{r,q} \left( \|\cosh(q \cdot) f\|_{L^2} + \|\cosh(q \cdot) \partial_X^r[f]\|_{L^2} \right)$$

$$= C_{r,q} \left\| f \right\|_{r,q}.$$

Hence

$$\|f\|_{\mathcal{X}_q^r} = \|\cosh(q \cdot) f\|_{H^r} = \|\cosh(q \cdot) f\|_{L^2} + \|\partial_X^r[\cosh(q \cdot) f]\|_{L^2} \leq C_{r,q} \left\| f \right\|_{r,q}.$$

Now we show

$$\left\| f \right\|_{\mathcal{X}_q^r} \leq C_{r,q} \|f\|_{\mathcal{X}_q^r} \tag{C.3.2}$$

by induction on $r$. We will use the essential inequality

$$|\sinh(X)| = \left|\frac{e^X - e^{-X}}{2}\right| \leq \frac{e^X + e^{-X}}{2} = \cosh(X) \tag{C.3.3}$$

and the equality

$$\partial_X[\cosh(\cdot)] = \sinh(\cdot). \tag{C.3.4}$$

When $r = 1$,

$$\begin{aligned}
\|\cosh(q\cdot)\partial_X[f]\|_{L^2} &\leq \|\partial_X[\cosh(q\cdot)f]\|_{L^2} + q\,\|\sinh(q\cdot)f\|_{L^2} \\
&\leq \|\cosh(q\cdot)f\|_{H^1} + q\,\|\cosh(q\cdot)f)\|_{L^2} \\
&\leq (1+q)\,\|\cosh(q\cdot)f\|_{H^1} \\
&= (1+q)\,\|f\|_{\mathcal{X}_q^1}\,,
\end{aligned}$$

and so

$$|||f|||_{\mathcal{X}_q^1} \leq (1+q)\,\|f\|_{\mathcal{X}_q^1}\,.$$

Assume that (C.3.2) holds for some $r \geq 1$. Then Leibniz's rule (A.1.1) gives

$$\cosh(q\cdot)\partial_X^{r+1}[f] = \partial_X^{r+1}[\cosh(q\cdot)f] - \sum_{j=1}^{r+1}\binom{r+1}{j}\partial_X^j[\cosh(q\cdot)]\partial_X^{r+1-j}[f],$$

where $\partial_X^{r+1}[\cosh(q\cdot)f] \in L^2$ and, for $1 \leq j \leq r+1$, we have $0 \leq r+1-j \leq r$, hence

$$\begin{aligned}
\left\|\partial_X^j[\cosh(q\cdot)]\partial_X^{r+1-j}[f]\right\|_{L^2} &\leq q^j\left\|\cosh(q\cdot)\partial_X^{r+1-j}[f]\right\|_{L^2} \text{ by (C.3.3) and (C.3.4)} \\
&\leq q^j\,|||f|||_{\mathcal{X}_q^{r+1-j}} \\
&\leq C_{r,q,j}\,\|f\|_{\mathcal{X}_q^{r+1-j}} \text{ by the induction hypothesis} \\
&= C_{r,q,j}\,\|\cosh(q\cdot)f\|_{H^{r+1-j}} \\
&\leq C_{r,q,j}\,\|\cosh(q\cdot)f\|_{H^{r+1}} \text{ by (C.1.2)} \\
&\leq C_{r,q}\,\|f\|_{\mathcal{X}_q^{r+1}}\,.
\end{aligned}$$

Thus

$$\begin{aligned}\left\|\cosh(q\cdot)\partial_X^{r+1}[f]\right\|_{L^2} &\leq \left\|\partial_X^{r+1}[\cosh(q\cdot)f]\right\|_{L^2} + C_{r,q}\left\|f\right\|_{\mathcal{X}_q^{r+1}} \\ &\leq \left\|\cosh(q\cdot)f\right\|_{H^{r+1}} + C_{r,q}\left\|f\right\|_{\mathcal{X}_q^{r+1}} \\ &\leq C_{r,q}\left\|f\right\|_{\mathcal{X}_q^{r+1}},\end{aligned}$$

and so we conclude

$$\|f\|_{\mathcal{X}_q^{r+1}} = \left\|\cosh(q\cdot)f\right\|_{L^2} + \left\|\cosh(q\cdot)\partial_X^{r+1}[f]\right\|_{L^2} \leq C_{r,q}\left\|f\right\|_{\mathcal{X}_q^{r+1}}. \tag{C.3.5}$$

■

**C.3.6 Lemma.** *For all $q > 0$ there exists a constant $C_q > 0$ such that*

$$\frac{1}{C_q}\cosh(qX) \leq \cosh^q(X) \leq 2\cosh(qX), \quad X \in \mathbb{R}. \tag{C.3.6}$$

**Proof.** For the second inequality, recall that if $A, B > 0$, then

$$(A+B)^q \leq 2^q(A^q + B^q). \tag{C.3.7}$$

Then

$$\cosh^q(X) = \frac{1}{2^q}(e^X + e^{-X})^q \leq e^{qX} + e^{-qX} = 2\cosh(qX).$$

For the first inequality, set

$$f(X) = \cosh^q(X) - \frac{1}{c}\cosh(qX),$$

where

$$c = q2^{q-1} + 2 > 1.$$

Then

$$f(0) = 1 - \frac{1}{c} > 0$$

since $c > 1$. We will show that $f'(X) > 0$ for $X > 0$, so that $f(X) \geq f(0) > 0$ for $X > 0$. And since $f$ is even, this means $f(X) \geq 0$ for all $X$. Then we will have the first inequality in (C.3.6).

We compute

$$f'(X) = q\cosh^{q-1}(X)\sinh(X) - \frac{q}{c}\sinh(qX),$$

so $f'(X) > 0$ if and only if

$$0 < \left(\frac{e^X + e^{-X}}{2}\right)^{q-1}(e^X - e^{-X}) - \frac{1}{c}(e^{qX} - e^{-qX})$$

Since $e^X - e^{-X} > 0$ for $X > 0$, this rearranges to

$$f'(X) > 0 \iff \frac{2^{q-1}}{c}\frac{e^{qX} - e^{-qX}}{e^X - e^{-X}} < (e^X + e^{-X})^{q-1}. \tag{C.3.8}$$

Set

$$g(Y) = Y^q.$$

The mean value theorem implies

$$\begin{aligned}\frac{e^{qX} - e^{-qX}}{e^X - e^{-X}} &= \frac{g(e^X) - g(e^{-X})}{e^X - e^{-X}} \\ &\leq \max_{e^{-X} \leq Y \leq e^X} g'(Y) \\ &= q \max_{e^{-X} \leq Y \leq e^X} Y^{q-1} \\ &= \begin{cases} qe^{(1-q)X}, & 0 < q < 1 \\ qe^{(q-1)X}, & q \geq 1. \end{cases}\end{aligned}$$

When $0 < q < 1$, we have

$$e^{(1-q)X} = (e^X)^{1-q} < (e^X + e^{-X})^{1-q}$$

and likewise

$$e^{(q-1)X} < (e^X + e^{-X})^{q-1}$$

when $q \geq 1$. So, for any $q > 1$, we find

$$\frac{e^{qX} - e^{-qX}}{e^X - e^{-X}} \leq q(e^X + e^{-X})^{q-1}.$$

Then

$$
\begin{aligned}
\frac{2^{q-1}}{c}\frac{e^{qX}-e^{-qX}}{e^{X}-e^{-X}} &\leq \frac{2^{q-1}q}{c}(e^{X}+e^{-X})^{q-1} \\
&= \frac{q2^{q-1}}{q2^{q-1}+2}(e^{X}+e^{-X})^{q-1} \\
&< (e^{X}+e^{-X})^{q-1},
\end{aligned}
$$

and this is (C.3.8), which implies $f'(X) > 0$ for any $X > 0$. ■

An immediate consequence of this lemma is that the norms $\|\!|\cdot|\!\|_{\mathcal{X}_q^r}$ and $\|\!|\cdot|\!\|_{\mathcal{Y}_q^r}$ are equivalent.

**C.3.7 Lemma.** *The norms $\|\cdot\|_{\mathcal{Y}_q^r}$ and $\|\!|\cdot|\!\|_{\mathcal{Y}_q^r}$ are equivalent on $H_q^r$.*

**Proof.** We rely on

$$|\partial_X^r[\cosh^q(\cdot)]| \leq C_{r,q}\cosh^q(\cdot),$$

per Lemma C.3.11 below. First,

$$
\begin{aligned}
\|\partial_X^r[\cosh^q(\cdot)f]\|_{L^2} &\leq C_r\sum_{j=0}^{r}\left\|\partial_X^j[\cosh^q(\cdot)]\partial_X^{r-j}[f]\right\|_{L^2} \\
&\leq C_{r,q}\sum_{j=0}^{r}\left\|\cosh^q(\cdot)\partial_X^j[f]\right\|_{L^2} \\
&\leq C_{r,q}\sum_{j=0}^{r}\left\|\cosh(q\cdot)\partial_X^j[f]\right\|_{L^2} \text{ by Lemma (C.3.6)} \\
&\leq C_{r,q}\sum_{j=0}^{r}\|\!|f|\!\|_{\mathcal{X}_q^j} \\
&\leq C_{r,q}\sum_{j=0}^{r}\|f\|_{\mathcal{X}_q^j} \text{ by Lemma C.3.5}
\end{aligned}
$$

$$\leq C_{r,q} \|f\|_{\mathcal{X}_q^r} \text{ by (C.1.2)}$$

$$\leq C_{r,q} \vert\!\vert\!\vert f \vert\!\vert\!\vert_{\mathcal{X}_q^r}$$

$$\leq C_{r,q} \vert\!\vert\!\vert f \vert\!\vert\!\vert_{\mathcal{Y}_q^r} \text{ by the equivalence of } \|\cdot\|_{\mathcal{X}_q^r} \text{ and } \|\cdot\|_{\mathcal{Y}_q^r}.$$

Hence there is $C_{r,q} > 0$ such that

$$\|f\|_{\mathcal{Y}_q^r} = \|\cosh^q(\cdot)f\|_{H^r} = \|\cosh^q(\cdot)f\|_{L^2} + \|\partial_X^r[\cosh^q(\cdot)f]\|_{L^2} \leq C_{r,q} \vert\!\vert\!\vert f \vert\!\vert\!\vert_{\mathcal{Y}_q^r}, \ f \in H_q^r.$$

For the other inequality, we induct. When $r = 1$, we have

$$\begin{aligned}
\|\cosh^q(\cdot)f'\|_{L^2} &\leq \|\partial_X[\cosh^q(\cdot)f]\|_{L^2} + \|\partial_X[\cosh^q(\cdot)]f\|_{L^2} \\
&\leq \|\partial_X[\cosh^q(\cdot)f]\|_{L^2} + C_q \|\cosh^q(\cdot)f\|_{L^2} \text{ by (C.3.11)} \\
&\leq C_q \|f\|_{\mathcal{Y}_q^1}.
\end{aligned}$$

Assume there are $r \geq 1$ and $C_{j,q} > 0$ such that

$$\vert\!\vert\!\vert f \vert\!\vert\!\vert_{\mathcal{Y}_q^j} \leq C_{j,q} \|f\|_{\mathcal{Y}_q^j}, \ j = 0, \ldots, r.$$

Then, just as with the induction in the proof of Lemma C.3.5, we have

$$\begin{aligned}
\left\|\cosh^q(\cdot)\partial_X^{r+1}[f]\right\|_{L^2} &\leq \left\|\partial_X^{r+1}[\cosh(q\cdot)f]\right\|_{L^2} + C_r \sum_{j=1}^{r+1} \left\|\partial_X[\cosh^q(\cdot)]\partial_X^{r+1-j}[f]\right\|_{L^2} \\
&\leq \|\cosh(q\cdot)f\|_{H^{r+1}} + C_{r,q} \sum_{j=0}^{r} \left\|\cosh^q(\cdot)\partial_X^j[f]\right\|_{L^2} \\
&\leq \|f\|_{\mathcal{Y}_q^{r+1}} + C_{r,q} \sum_{j=0}^{r} \vert\!\vert\!\vert f \vert\!\vert\!\vert_{\mathcal{Y}_q^j} \\
&\leq C_{r,q} \sum_{j=0}^{r+1} \|f\|_{\mathcal{Y}_q^j}
\end{aligned}$$

$$\leq C_{r,q} \|f\|_{\mathcal{Y}_q^{r+1}} \text{ by (C.1.2).} \qquad \blacksquare$$

*C.3.2. Lemmas for the proof of the equivalent norms.* Here we collect several lemmas that played a support role in Appendix C.3.1.

**C.3.8 Lemma.** *Let $r > 0$. Then*

$$|k|^j < 1 + |k|^r \tag{C.3.9}$$

*for all $k \in \mathbb{C}$ and $j \in [0, r]$.*

**Proof.** If $|k| < 1$, then

$$|k|^j < 1 < 1 + |k|^r,$$

and if $|k| \geq 1$, then

$$|k|^j \leq |k|^r < 1 + |k|^r. \qquad \blacksquare$$

**C.3.9 Lemma.** *Let* $\cosh(q\cdot)f \in H^r$. *Then*

$$\lim_{y \to q} \|(e^{y\cdot} - e^{q\cdot})f\|_{H^r} = 0.$$

**Proof.** We need to take limits on two $L^2$-norms:

$$\|(e^{y\cdot} - e^{q\cdot})f\|_{H^r} = \|(e^{y\cdot} - e^{q\cdot})f\|_{L^2} + \|\partial_X^r[(e^{y\cdot} - e^{q\cdot})f]\|_{L^2}.$$

The pointwise convergence to 0 as $y \to q$ is obvious, so we check for domination: first,

$$\begin{aligned} \|(e^{y\cdot} - e^{q\cdot})f\|_{L^2} &\leq \|e^{y\cdot}f\|_{L^2} + \|e^{q\cdot}f\|_{L^2} \\ &\leq 2\|\cosh(y\cdot)f\|_{L^2} + 2\|\cosh(q\cdot)f\|_{L^2} \\ &\leq 4\|\cosh(q\cdot)f\|_{L^2}, \end{aligned}$$

where we are using the inequality

$$e^{yx} = 2\frac{e^{yx}}{2} \leq 2\frac{e^{yx} + e^{-yx}}{2} = 2\cosh(yx) \leq 2\cosh(qx).$$

Also, and similarly,

$$\begin{aligned}\|\partial_X^r[(e^{y\cdot} - e^{q\cdot})f]\|_{L^2} &\leq \sum_{j=0}^r \binom{r}{j} \left\|(y^j e^{y\cdot} - q^j e^{q\cdot})\partial_X^{r-j}[f]\right\|_{L^2} \\ &\leq C_r \sum_{j=0}^r \left(|y|^j \left\|e^{y\cdot}\partial_X^{r-j}[f]\right\|_{L^2} + q^j \left\|e^{q\cdot}\partial_X^{r-j}[f]\right\|_{L^2}\right) \\ &\leq C_r \sum_{j=0}^r q^j \left\|\cosh(q\cdot)\partial_X^{r-j}[f]\right\|_{L^2}.\end{aligned}$$

So, the limit follows from the dominated convergence theorem. ■

With these lemmas, we can show that we preserve the familiar identity for the derivative when we extend the Fourier transform to complex values: if $f \in H_q^r$ for some integer $r \geq 1$ and $q \geq 0$, then

$$\mathfrak{F}[\partial_X^r f](z) = (iz)^r \widehat{f}(z),\ z \in \overline{\Sigma}_q, .$$

where $\overline{\Sigma}_q$ is the closure of $\Sigma_q$. The proof for $|\operatorname{Im}(z)| < q$ is Theorem 8.4.4 in [Det84]. We prove the $|\operatorname{Im}(z)| = q$ case here as an equality in $L^2$.

**C.3.10 Lemma.** *Let $f \in H_q^r$. Then $\widehat{\partial_X^j[f]}(k \pm iq) = (i(k \pm iq))^j \widehat{f}(k \pm iq)$.*

**Proof.** Since $H_q^r \subseteq H_q^{r-1}$, we give the proof just for $r = r$. Let $|y| < q$. We have

$$\begin{aligned}\widehat{\partial_X^r f}(k+iq) - (i(k+iq))^r \widehat{f}(k+iq) = &\underbrace{\widehat{\partial_X^r f}(k+iq) - \widehat{\partial_X^r f}(k+iy)}_{\Delta_1(k,y)} \\ &+ \underbrace{\widehat{\partial_X^r f}(k+iy) - (i(k+iq))^r \widehat{f}(k+iq)}_{\Delta_2(k,y)}.\end{aligned}$$

Immediately (B.3.7) gives

$$\lim_{y \to q} \|\Delta_1(\cdot, y)\|_{L^2} = 0.$$

Next, rewrite $\Delta_2(y)$ as

$$\Delta_2(y) = \underbrace{(i(k+iy))^r \widehat{f}(k+iy) - (i(k+iy))^r \widehat{f}(k+iq)}_{\Delta_3(k,y)} + \underbrace{(i(k+iy))^r \widehat{f}(k+iq) - (i(k+iq))^r \widehat{f}(k+iq)}_{\Delta_4(k,y)}.$$

Then (C.3.9) implies

$$\begin{aligned}
|\Delta_3(k,y)| &= i^r \sum_{\ell=0}^{r} \binom{r}{\ell} k^\ell (-y)^{r-\ell} \left( \widehat{e^{y\cdot} f}(k) - \widehat{e^{q\cdot} f}(k) \right) \\
&\le C_{r,q} \left( \left| \widehat{e^{y\cdot} f}(k) - \widehat{e^{q\cdot} f}(k) \right| + \left| (ik)^r \left( \widehat{e^{y\cdot} f}(k) - \widehat{e^{q\cdot} f}(k) \right) \right| \right) \\
&= C_{r,q} \left( |\mathfrak{F}[(e^{y\cdot} - e^{q\cdot}) f](k)| + |\mathfrak{F}[\partial_X^r[(e^{y\cdot} - e^{q\cdot}) f]](k)| \right),
\end{aligned}$$

and so

$$\begin{aligned}
\|\Delta_3(\cdot,y)\|_{L^2} &\le C_{r,q} \left( \|\mathfrak{F}[(e^{y\cdot} - e^{q\cdot}) f]\|_{L^2} + \|\mathfrak{F}[\partial_X^r[(e^{y\cdot} - e^{q\cdot}) f]]\|_{L^2} \right) \\
&= C_{r,q} \left( \|(e^{y\cdot} - e^{q\cdot}) f\|_{L^2} + \|\partial_X^r[(e^{y\cdot} - e^{q\cdot})]\|_{L^2} \right) \\
&= C_{r,q} \|(e^{y\cdot} - e^{q\cdot}) f\|_{H^r}.
\end{aligned}$$

Hence $\|\Delta_3(\cdot,y)\|_{L^2} \to 0$ as $y \to q$ by Lemma C.3.9.

Last, we rewrite

$$\begin{aligned}
\Delta_4(k,y) &= i^r \left( (k+iy)^r - (k+iq)^r \right) \widehat{f}(k+iq) \\
&= i^r \sum_{\ell=0}^{r} \left( k^\ell (iy)^{r-\ell} - k^\ell (iq)^{r-\ell} \right) \widehat{e^{q\cdot} f}(k)
\end{aligned}$$

and so

$$\|\Delta_4(\cdot,y)\|_{L^2} \le C_r \sum_{\ell=0}^{r} |y^{r-\ell} - q^{r-\ell}| \int_{-\infty}^{\infty} (1+k^2)^r \left| \widehat{e^{q\cdot} f}(k) \right|^2 \, dk$$

$$= C_r \sum_{\ell=0}^{r} |y^{r-\ell} - q^{r-\ell}| \, \|e^{q\cdot} f\|_{H^r}$$

Then $\|\Delta_4(\cdot, y)\|_{L^2} \to 0$ as $y \to q$. ■

**C.3.11 Lemma.** *For all $r, q \geq 0$, there is a constant $C_{r,q} > 0$ such that*

$$|\partial_X^r[\cosh^q(\cdot)]| \leq C_{r,q} \cosh^q(\cdot). \tag{C.3.10}$$

**Proof.** Use Faá di Bruno's rule with $N(X) = X^q$ to write

$$\partial_X^r[\cosh^q(\cdot)] = \sum_{k=1}^{r} \left( \prod_{j=0}^{k-1} q - k \right) \cosh^{q-k}(\cdot) \sum_{\boldsymbol{\sigma} \in \Sigma_k^r} \prod_{j=1}^{k} \partial_X^{\sigma_j}[\cosh(\cdot)] \tag{C.3.11}$$

and observe that since $\cosh(X) \geq 0$,

$$\left| \prod_{\ell=1}^{k} \partial_X^{\sigma_j}[\cosh(\cdot)] \right| \leq \prod_{\ell=1}^{k} \cosh(\cdot) = \cosh^k(\cdot).$$

This and (C.3.11) imply the desired inequality (C.3.10). ■

*C.3.3. Estimates for $H_q^r$.*

**C.3.12 Proposition.** *Let $r \geq 1$ be an integer and $q > 0$.*

**(i)** $\|f\|_{L^\infty} \leq C_r \|f\|_{H^r}$.

**(ii)** *If $f \in H_q^r$, then $|f(X)| \leq C_r \|f\|_{r,q} \operatorname{sech}^q(X)$.*

**(iii)** $\|fg\|_{H^r} \leq C_r \|f\|_{W^{r,\infty}} \|g\|_{H^r}$.

**(iv)** $\left\|\partial_X^k[\cosh(q\cdot)f]\right\|_{L^\infty} \leq C_{r-k} \|f\|_{r,q}$.

**(v)** $\|f\|_{H^r} \leq C_{r,q} \|f\|_{r,q}$.

**(vi)** $\|fg\|_{r,q} \leq C_{r,q} \|f\|_{r,q} \|g\|_{r,q}$.

**(vii)** $\|fg\|_{r,q} \leq C_r \|g\|_{W^{r,\infty}} \|f\|_{r,q}$.

**(viii)** $\|f\|_{r,q} \leq C_{r,q'-q} \|f\|_{r,q'}$, $q \leq q'$.

**Proof.** **(i)** This is the Sobolev embedding.

**(ii)** We have

$$|f(X)| = \operatorname{sech}^q(X)\cosh^q(X)|f(X)| \leq C_r \operatorname{sech}^q(X)\left\|\cosh^q(\cdot)f\right\|_{H^r} = C_r \left\|f\right\|_{r,q} \operatorname{sech}^q(X).$$

**(iii)** We compute

$$\begin{aligned}
\left\|fg\right\|_{H^r} &= \sum_{k=0}^{r} \left\|\partial_X^r[fg]\right\|_{L^2} \\
&\leq \sum_{k=0}^{r}\sum_{j=0}^{k}\binom{k}{j}\left\|\partial_X^j[f]\partial_X^{k-j}[g]\right\|_{L^2} \\
&\leq \sum_{k=0}^{r}\sum_{j=0}^{k}\binom{k}{j}\left\|\partial_X^j[f]\right\|_{L^\infty}\left\|\partial_X^{k-j}[g]\right\|_{L^2} \\
&\leq \left\|f\right\|_{W^{r,\infty}}\sum_{k=0}^{r}\sum_{j=0}^{k}\binom{k}{j}\left\|\partial_X^{k-j}[g]\right\|_{L^2} \\
&\leq \left\|f\right\|_{W^{r,\infty}}\left\|g\right\|_{H^r}\underbrace{\sum_{k=0}^{r}\sum_{j=0}^{k}\binom{k}{j}}_{C_r}
\end{aligned}$$

**(iv)** This is essentially the Sobolev embedding:

$$\begin{aligned}
\left\|\partial_X^k[\cosh(q\cdot)f]\right\|_{L^\infty} &\leq C_{r-k}\left\|\partial_X^k[\cosh(q\cdot)f]\right\|_{H^{r-k}} \\
&\leq C_{r-k}\left\|\cosh(q\cdot)f\right\|_{H^r} \\
&= C_{r-k}\left\|f\right\|_{r,q}.
\end{aligned}$$

**(v)** Here we use (ii):

$$\begin{aligned}
\|f\|_{H^r} &= \|\operatorname{sech}^q(\cdot) f \cosh^q(\cdot) f\|_{H^r} \\
&\le C_r \|\operatorname{sech}^q(\cdot)\|_{W^{r,\infty}} \|\cosh^q(\cdot) f\|_{H^r} \\
&= \underbrace{C_r \|\operatorname{sech}(\cdot)\|^q_{W^{r,\infty}}}_{C_{r,q}} \|f\|_{r,q} \,.
\end{aligned}$$

**(vi)** This relies on the fact that $H^r$ is an algebra:

$$\begin{aligned}
\|fg\|_{r,q} &= \|\cosh(q\cdot) fg\|_{H^r} \\
&\le C_r \|\cosh(q\cdot) f\|_{H^r} \|g\|_{H^r} \\
&\le C_{r,q} \|f\|_{r,q} \|g\|_{r,q} \,.
\end{aligned}$$

**(vii)** Again we use (ii):

$$\begin{aligned}
\|fg\|_{r,q} &= \|\cosh(q\cdot) fg\|_{H^r} \\
&\le C_r \|f\|_{W^{r,\infty}} \|\cosh(q\cdot) g\|_{H^r} \\
&= C_r \|f\|_{W^{r,\infty}} \|g\|_{r,q} \,.
\end{aligned}$$

**(viii)** We have

$$\begin{aligned}
\|f\|_{r,q} &= \|\cosh^q(\cdot) f\|_{H^r} \\
&= \left\|\cosh^{q-q'}(\cdot)\cosh^{q'}(\cdot) f\right\|_{H^r} \\
&= \left\|\operatorname{sech}^{q'-q}(\cdot)\cosh^{q'}(\cdot) f\right\|_{H^r}
\end{aligned}$$

$$\leq \underbrace{\left\|\operatorname{sech}^{q'-q}(\cdot)\right\|_{W^{r,\infty}}}_{C_{r,q'-q}} \left\|\cosh^{q'}(\cdot)f\right\|_{H^r}$$

$$= C_{r,q'-q} \|f\|_{r,q'} .$$

Note that since $q' \geq q$, we do indeed have $\operatorname{sech}^{q'-q}(\cdot) \in W^{r,\infty}$. ■

*C.3.4. The compact embedding of $H^r_q$ into $H^{r-1}_q$.*

**C.3.13 Remark.** *We adopt the following notation for sequences: a sequence in a set $\mathcal{X}$ is a function $f\colon \mathbb{N} \to \mathcal{X}$, and a subsequence of $f$ is a sequence $g$ of the form $g(n) = f(\varphi(n))$, where $\varphi\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing. We will use the following familiar construction of "diagonal" sequences: let $f\colon \mathbb{N}^2 \to X$ be a function such that $f(\cdot, k+1)$ is a subsequence of $f(\cdot, k)$ for each $k \in \mathbb{N}$. Then the function $g(n) := f(n, n)$ is a subsequence of $f(\cdot, 1)$. Indeed, suppose that for each $k \in \mathbb{N}$ we have a strictly increasing function $\iota_k\colon \mathbb{N} \to \mathbb{N}$ with $f(n, k+1) = f(\iota_k(n), k)$. Note that $\iota_k(n) \geq n$ for all $k$ and $n$.*

*We need to write $g(n) = f(\iota(n), 1)$ for $\iota\colon \mathbb{N} \to \mathbb{N}$ strictly increasing. The illustrative calculations*

$$g(2) = f(2, 2) = f(2, 1 + 1) = f(\iota_1(2), 1)$$

*and*

$$g(3) = f(3, 3) = f(3, 2 + 1) = f(\iota_2(3), 2) = f(\iota_2(3), 1 + 1) = f(\iota_1(\iota_2(3)), 1)$$

*suggest that we define*

$$\iota(n) = \iota_1 \circ \iota_2 \circ \cdots \circ \iota_{n-1}(n);$$

*we prove by induction that $\iota$ is increasing.*

**C.3.14 Lemma.** *Let $b$, $q > 0$. The space $H^1_{q+b}$ is compactly embedded in $L^2_q$.*

**Proof.** Let $(f_n)$ be a bounded sequence in $H^1_{q+b}$, so $(f_n)$ is a bounded sequence in $H^1$ with

$$M := \sup_{n \in \mathbb{N}} \|f_n\|_{1,q+b} < \infty.$$

Then

$$|f_n(X)| \le C \|f_n\|_{1,q+b} \le C_\star \operatorname{sech}^{q+b}(X), \qquad C_\star := CM. \tag{C.3.12}$$

Fix $N \in \mathbb{N}$. Since $H^1([-N,N])$ embeds compactly into $L^2([-N,N])$, there exists a subsequence $(f_{\varphi_N(n)})$ of $(f_n)$ such that $(f_{\varphi_N(n)})$ converges to some function $F_{(N)}$ in $L^2([-N,N])$. Here, each map $\varphi_N \colon \mathbb{N} \to \mathbb{N}$ is strictly increasing. We may select these subsequences inductively so that $(f_{\varphi_{N+1}(n)})$ is a subsequence of $(f_{\varphi_N(n)})$ and $(f_{\varphi_N(n)})$ also converges pointwise a.e. on $[-N,N]$ to $F_{(N)}$. Observe the following.

**(i)** Since $(f_{\varphi_{N+1}(n)})$ is a subsequence of $(f_{\varphi_N(n)})$, we have $F_{(N+1)} = F_{(N)}$ on $[-N,N]$.

**(ii)** The pointwise convergence on $[-N,N]$ combines with (C.3.12) to produce

$$|F_{(N)}(X)| = \lim_{n \to \infty} |f_{\varphi_N(n)}(X)| \le C_\star \operatorname{sech}^{q+b}(X) \tag{C.3.13}$$

a.e. on $[-N,N]$.

**(iii)** Since $\cosh^q(\cdot)$ is bounded on $[-N,N]$, there is an integer $\psi(N)$ large enough that if $n \ge \psi(N)$, then

$$\left\| \cosh^q(\cdot)(f_{\varphi_N(n)} - F_{(N)}) \right\|_{L^2([-N,N])} < \frac{1}{\sqrt{N}}. \tag{C.3.14}$$

We may take $\psi(N) < \psi(N+1)$ for each $N$.

Let $\Phi(N) = \varphi_N(\psi(N))$, so that by the diagonal construction in Remark C.3.13 $(f_{\Phi(N)})$ is a subsequence of $(f_n)$. Now let

$$f(X) = F_{(|\lceil X \rceil|)}(X),$$

where $\lceil X \rceil$ is the least integer greater than or equal to $X$. That is,

$$f(X) = F_{(N)}(X) \quad \text{if} \quad |X| \le N.$$

We see from (C.3.13) that

$$|f(X)| \le C_\star \operatorname{sech}^{q+b}(X) \tag{C.3.15}$$

a.e. on $\mathbb{R}$. If we know that $f$ is measurable, then we will have $\cosh^q(\cdot)f \in L^2$ from this inequality. Given $E \subseteq \mathbb{R}$ measurable, we have

$$f^{-1}(E) = \bigcup_{N=1}^{\infty} f^{-1}(E) \cap [-N,N] = \bigcup_{N=1}^{\infty} F_{(N)}^{-1}(E) \cap [-N,N].$$

Each set $F_{(N)}^{-1}(E \cap [-N,N])$ is measurable since $F_{(N)} \in L^2([-N,N])$. And so $f$ is measurable. Hence $f \in L^2_b$.

Now we can show that

$$\lim_{N\to\infty} \left\| f_{\Phi(N)} - f \right\|_{L^2_b} = 0. \tag{C.3.16}$$

Let $\epsilon > 0$ and take $N_\epsilon \in \mathbb{N}$ so large that

$$\max\left\{ \frac{1}{N_\epsilon}, 8C_\star^2 \int_{N_\epsilon}^{\infty} \operatorname{sech}^{2b}(x)\, dx \right\} < \frac{\epsilon}{2}.$$

Then for $N \ge N_\epsilon$, we have

$$\begin{aligned}
\left\| \cosh^q(\cdot)(f_{\Phi(N)} - f) \right\|_{L^2}^2 &= \int_{-\infty}^{-N} \cosh^{2q}(X) |f_{\Phi(N)}(X) - f(X)|^2 \, dX \\
&\quad + \int_{-N}^{N} \cosh^{2q}(X) |f_{\Phi(N)}(X) - f(X)|^2 \, dX \\
&\quad + \int_{N}^{\infty} \cosh^{2q}(X) |f_{\Phi(N)}(X) - f(X)|^2 \, dX.
\end{aligned}$$

We estimate the second integral using (C.3.14):

$$\begin{aligned}
\int_{-N}^{N} \cosh^{2q}(X) |f_{\Phi(N)}(X) - f(X)|^2 \, dX &= \left\| \cosh^q(\cdot)(f_{\Phi(N)} - f) \right\|_{L^2([-N,N])}^2 \\
&< \frac{1}{N} \le \frac{1}{N_\epsilon} < \frac{\epsilon}{2}.
\end{aligned} \tag{C.3.17}$$

For the first and third integrals, we use (C.3.12) and (C.3.15):

$$\begin{aligned}
\cosh^{2q}(X)|f_{\Phi(N)}(X) - f(X)|^2 \le \cosh^{2q}(X)|f_{\Phi(N)}(X)|^2 &+ 2\cosh^{2q}(X)|f_{\Phi(N)}(X)||f(X)| \\
&+ \cosh^{2q}(X)|f(X)|^2
\end{aligned}$$

$$\leq C_\star^2 \cosh^{2q}(X) \operatorname{sech}^{2q+2b}(X) + 2C_\star^2 \cosh^{2q} \operatorname{sech}^{2q+2b}(X)$$
$$+ C_\star^2 \cosh^{2q}(X) \operatorname{sech}^{2q+2b}(X)$$

$$= 4C_\star^2 \operatorname{sech}^{2b}(X).$$

Thus

$$\int_{-\infty}^{-N} \cosh^{2b}(X)|f_{\Phi(N)}(X) - f(X)|^2 \, dx + \int_N^\infty \cosh^{2b}(X)|f_{\Phi(N)}(X) - f(X)|^2 \, dX$$

$$\leq 4C_\star^2 \int_{-\infty}^{-N} \operatorname{sech}^{2\epsilon}(X) \, dX + 4C_\star^2 \int_N^\infty \operatorname{sech}^{2\epsilon}(X) \, dX$$

$$= 8C_\star^2 \int_N^\infty \operatorname{sech}^{2b}(X) \, dX \leq 8C_\star^2 \int_{N_\epsilon}^\infty \operatorname{sech}^{2b}(X) \, dX < \frac{\epsilon}{2}. \quad \text{(C.3.18)}$$

We combine this with (C.3.17) to conclude that for $N \geq N_\epsilon$, we have

$$\left\|\cosh^q(\cdot)(f_{\Phi(N)} - f)\right\|_{L^2}^2 < \epsilon,$$

and so the limit (C.3.16) holds. ∎

**C.3.15 Proposition.** *Let $b > 0$, $q \geq 0$ and $r \in \mathbb{N}$. Then $H_{q+b}^{r+1}$ is compactly embedded in $H_q^r$.*

**Proof.** We have proved the $r = 0$ case in Lemma C.3.14 and so we induct on $r$. Suppose the proposition is true for $k = 0, \ldots, r$. Let $(f_n)$ be a bounded sequence in $H_{q+b}^{r+1}$, so $(\partial_X^k[f_n])$ is bounded in $H_{q+b}^1$ for $k = 0, \ldots, r$. Passing to subsequences and relabeling as needed, by Lemma C.3.14 there exist a subsequence $(f_{n_j})$ of $(f_n)$ and functions $g_0, \ldots, g_r \in L_q^2$ such that $\partial_X^k[f_{n_j}] \to g_k$ in $L_q^2$. By Lemma C.3.1, we have $g_k = \partial_X^k[g_0]$. Hence

$$\lim_{j \to \infty} \left\|f_{n_j} - g_0\right\|_{H_q^r} = \lim_{j \to \infty} \sum_{k=0}^r \left\|\partial_X^k[f_{n_j} - g_0]\right\|_{L_q^2} = 0,$$

and so $f_{n_j} \to g_0$ in $H_q^r$. ∎

It will be useful to know that a certain composition of smoothing and multiplication operators is compact when the multiplication induces decay.

**C.3.16 Lemma.** *Let $r \geq 1$ be an integer, $q \geq 0$, $b > 0$, and $\varsigma \in H_q^r$. There exists a constant $C > 0$ such that*

$$\|\varsigma f\|_{r,q+b} \leq C \, \|f\|_{r,q} \tag{C.3.19}$$

*for all $f \in H^r$. That is, multiplication by $\varsigma$ is a bounded operator from $H_q^r$ to $H_{q+b}^r$.*

**Proof.** We have

$$\|\varsigma f\|_{r,q+b} = \left\|\cosh^{q+b}(\cdot)\varsigma f\right\|_{L^2} + \left\|\cosh^{q+b}(\cdot)\partial_X^r[\varsigma f]\right\|_{L^2},$$

with

$$\left\|\cosh^{q+b}(\cdot)\varsigma f\right\|_{L^2} \leq \left\|\cosh^b(\cdot)\varsigma\right\|_{L^\infty} \|\cosh^q(\cdot)f\|_{L^2} \tag{C.3.20}$$

and

$$\left\|\cosh^{q+b}(\cdot)\partial_X^r[\varsigma f]\right\|_{L^2} \leq C_{r,q+b} \sum_{k=0}^{r} \left\|\cosh^{q+b}(\cdot)\partial_X^k[\varsigma]\partial_X^{r-k}[f]\right\|_{L^2}.$$

Here we have used Leibniz's rule and (C.3.10). When $k = 0$, this is

$$\left\|\cosh^{q+b}(\cdot)\varsigma\partial_X^r[f]\right\|_{L^2} \leq \left\|\cosh^b(\cdot)\varsigma\right\|_{L^\infty} \|\cosh^q(\cdot)\partial_X^r[f]\|_{L^2} \tag{C.3.21}$$

and for $1 \leq k \leq r$,

$$\left\|\cosh^{q+b}(\cdot)\partial_X^k[\varsigma]\partial_X^{r-k}[f]\right\|_{L^2} \leq \left\|\cosh^q(\cdot)\partial_X^{r-k}[f]\right\|_{L^\infty} \left\|\cosh^b(\cdot)\partial_X^k[\varsigma]\right\|_{L^2}. \tag{C.3.22}$$

Combining (C.3.20), (C.3.21), and (C.3.22), we arrive at the bound (C.3.19). ■

**C.3.17 Proposition.** *Let $b > 0$, $q \geq 0$, $\varpi \in \mathbf{B}(H_q^{r-1}, H_q^r)$, and $\varsigma \in H_b^r$. The operator $f \mapsto \varpi(\varsigma f)$ is compact from $H_q^r$ to $H_q^r$.*

**Proof.** The following diagram summarizes the proof:

$$H_q^r \xrightarrow{f \mapsto \varsigma f} H_{q+b}^r \xhookrightarrow{\varsigma f \mapsto \varsigma f} H_q^{r-1} \xrightarrow{\varsigma f \mapsto \varpi(\varsigma f)} H_q^r.$$

By Lemma C.3.16, we know that $f \mapsto \varsigma f$ is a bounded operator from $H_q^r$ to $H_{q+b}^r$. Then Proposition C.3.15 implies that the identity mapping $\varsigma f \mapsto \varsigma f$ is compact from

$H^r_{q+b}$ to $H^{r-1}_q$. Hence $f \mapsto \varpi(\varsigma f)$ is also compact from $H^r_q$ to $H^r_q$. ■

# Appendix D. Fourier Multipliers

**D.1. Definitions.** Let $\widetilde{\mu} \in L^\infty$. For $f \in L^2$, we define the Fourier multiplier operator $\mu$ with symbol $\widetilde{\mu}$ by

$$\mu f := \mathfrak{F}^{-1}[\widetilde{\mu}\widehat{f}]. \tag{D.1.1}$$

That is, $\widehat{\mu f}(k) = \widetilde{\mu}(k)\widehat{f}(k)$ for all $k \in \mathbb{R}$. Since $\widetilde{\mu} \in L^\infty$, we have $\widetilde{\mu}\widehat{f} \in L^2$, so (D.1.1) is defined.

Similarly, for an integrable, $2P$-periodic $f$ on $\mathbb{R}$, we define $\mu f$ to be the function whose Fourier coefficients are $\widetilde{\mu}(k\pi/P)\widehat{f}(k)$, which is to say,

$$\mu f(X) := \sum_{k \in \mathbb{Z}} e^{ik\pi X/P}\widetilde{\mu}\left(\frac{k\pi}{P}\right)\widehat{f}(k). \tag{D.1.2}$$

The series converges in $L^2_{\text{per}}$ since $\widetilde{\mu}$ is bounded.

Finally, if $f \in L^2$ and $g$ is integrable and $2P$-periodic, then we set

$$\mu(f+g) := \mu f + \mu g,$$

where $\mu f$ is defined per (D.1.1) and $\mu g$ by (D.1.2).

We will use the following properties of Fourier multipliers frequently; their proofs are straightforward computations with the definitions (D.1.1) and (D.1.2).

**D.1.1 Proposition.** *Let $\mu$ be the Fourier multiplier with symbol $\widetilde{\mu} \in L^\infty$. Let $f$ be a function so that $\mu f$ is defined (either in $L^2$ or $L^2_{\text{per}}$).*

**(i)** *If $\widetilde{\mu}$ is even and $f$ is even (odd), then $\mu f$ is even (odd).*

**(ii)** *If $\overline{\widetilde{\mu}(k)} = \widetilde{\mu}(-k)$ and $f$ is real-valued, then $\mu f$ is real-valued.*

**(iii)** *Let $\omega \in \mathbb{R}$ and let $\mu^\omega$ be the Fourier multiplier with symbol $\widetilde{\mu^\omega}(k) := \widetilde{\mu}(\omega k)$. Then*

$$\mu[f(\omega\cdot)] = (\mu^\omega f)(\omega\cdot),$$

**D.2. Operator norms of Fourier multipliers.** Next we calculate exactly[7] the operator norm of a Fourier multiplier between Sobolev spaces.

**D.2.1 Lemma.** *Let $\mu$ be a Fourier multiplier with symbol $\widetilde{\mu}$. Then*

**(i)** $\|\mu\|_{\mathbf{B}(H^r_{\mathrm{per}},H^s_{\mathrm{per}})} = \sup_{k\in\mathbb{Z}} \dfrac{|\widetilde{\mu}(k)|}{(1+k^2)^{(r-s)/2}}$

**(ii)** $\|\mu\|_{\mathbf{B}(H^r,H^s)} = \sup_{k\in\mathbb{R}} \dfrac{|\widetilde{\mu}(k)|}{(1+k^2)^{(r-s)/2}}$

**Proof. (i)** One direction of this inequality is a direct computation. For $f \in H^r_{\mathrm{per}}$, we have

$$\begin{aligned}
\|\mu f\|^2_{H^s_{\mathrm{per}}} &= \sum_{k\in\mathbb{Z}} (1+k^2)^s |\widetilde{\mu}(k)|^2 |\widehat{f}(k)|^2 \\
&= \sum_{k\in\mathbb{Z}} \big((1+k^2)^{s-r}|\widetilde{\mu}(k)|^2\big)\big((1+k^2)^r|\widehat{f}(k)|^2\big) \\
&\le \left(\sup_{k\in\mathbb{Z}} \frac{|\widetilde{\mu}(k)|^2}{(1+k^2)^{r-s}}\right) \sum_{k\in\mathbb{Z}} (1+k^2)^r |\widehat{f}(k)|^2 \\
&= \left(\sup_{k\in\mathbb{Z}} \frac{|\widetilde{\mu}(k)|}{(1+k^2)^{(r-s)/2}}\right)^2 \|f\|^2_{H^r_{\mathrm{per}}}.
\end{aligned}$$

To get the reverse inequality, let

$$L = \sup_{k\in\mathbb{Z}} \frac{|\widetilde{\mu}(k)|}{(1+k^2)^{(r-s)/2}}.$$

and let $(k_j)$ be a sequence in $\mathbb{Z}$ such that

$$\lim_{j\to\infty} \frac{|\widetilde{\mu}(k_j)|}{(1+k_j^2)^{(r-s)/2}} = L.$$

Set

$$\widehat{f_j}(X) = e^{ik_jX},$$

[7] I am grateful to the authors of [FML15] for stating the full equality, rather than just the easy "$\le$" inequality. This led me on the amusing path of proving the reverse inequalities.

so $\widehat{f}_j(k) = \delta_{k,k_j}$. Then

$$\|f_j\|^2_{H^r_{\text{per}}} = (1+k_j)^2 \quad \text{and} \quad \|\mu f_j\|^2_{H^s_{\text{per}}} = (1+k_j)^2|\widetilde{\mu}(k_j)|^2,$$

and so

$$\lim_{j\to\infty} \frac{\|\mu f_j\|^2_{H^s_{\text{per}}}}{\|f_j\|^2_{H^r_{\text{per}}}} = \lim_{j\to\infty} \frac{|\widetilde{\mu}(k_j)|^2}{(1+k_j^2)^{r-s}} = L^2.$$

Since we already know $\|\mu\|_{\mathbf{B}(H^r_{\text{per}},H^s_{\text{per}})} \le L$, this forces equality.

**(ii)** First we need a straightforward fact: if $E \subseteq \mathbb{R}$ is bounded and measurable, then $\mathfrak{F}^{-1}[\mathbb{1}_E] \in H^r \setminus \{0\}$ (although $\mathbb{1}_E \notin H^r$, in general). This is true because

$$\begin{aligned}\left\|\mathfrak{F}^{-1}[\mathbb{1}_E]\right\|^2_{H^r} = \int_{-\infty}^{\infty} (1+|k|^2)^r|\widehat{\mathfrak{F}^{-1}[\mathbb{1}_E]}(k)|^2\ dk &= \int_{-\infty}^{\infty} (1+|k|^2)^r|\mathbb{1}_E(k)|^2\ dk \\ &= \int_E (1+|k|^2)^r\ dk < \infty.\end{aligned}$$

We have the last inequality because $E$ is bounded.

Now let

$$M(k) = \frac{|\widetilde{\mu}(k)|}{(1+k^2)^{(r-s)/2}}.$$

From here we follow the proof of Theorem 8.14 in [Kna05] to show $\|M\|_{L^\infty} = \|\mu\|_{\mathbf{B}(H^r,H^s)}$. First observe that if $c > 0$ and $M(k) \ge c$ on a bounded, measurable set $E$, then Then $\mathfrak{F}^{-1}[\mathbb{1}_E] \in H^r$ and

$$\begin{aligned}\left\|\mu\mathfrak{F}^{-1}[\mathbb{1}_E]\right\|^2_{H^s} = \int_{-\infty}^{\infty} (1+k^2)^r|M(k)|^2\mathbb{1}_E(k)\ dk &\ge c^2 \int_{-\infty}^{\infty} (1+k^2)^r\mathbb{1}_E(k)\ dk \\ &= c^2 \left\|\mathfrak{F}^{-1}[\mathbb{1}_E]\right\|^2_{H^r}.\end{aligned}$$

Hence

$$c \le \frac{\|\mu\mathfrak{F}^{-1}[\mathbb{1}_E]\|_{H^s}}{\|\mathfrak{F}^{-1}[\mathbb{1}_E]\|_{H^r}} \le \sup_{f\in H^r\setminus\{0\}} \frac{\|\mu f\|_{H^s}}{\|f\|_{H^r}} = \|\mu\|_{\mathbf{B}(H^r,H^s)}.$$

Now consider the case $\|M\|_{L^\infty} = \infty$. Then for all $c > 0$, there is a set $E_c \subseteq \mathbb{R}$ of positive measure such that $M(k) \ge c$ for $k \in E_c$. Using the calculation above, we conclude that $c \le \|\mu\|_{\mathbf{B}(H^r,H^s)}$ for all $c > 0$, hence $\|\mu\|_{\mathbf{B}(H^r,H^s)} = \infty$.

Last, suppose $\|M\|_{L^\infty} < \infty$. Take $0 < c < \|M\|_{L^\infty}$ and let

$$E_c = \left\{k \in \mathbb{R}^d \;\middle|\; |M(k)| > c\right\}.$$

If $E_c$ has measure 0, then $|M(k)| \le c$ a.e., and so $\|M\|_{L^\infty} \le c$. So, $E_c$ has positive measure, and, repeating the estimates above, we find $c \le \|\mu\|_{\mathbf{B}(H^r,H^s)}$. Since this was true for all $0 < c < \|M\|_{L^\infty}$, we must have $\|M\|_{L^\infty} \le \|\mu\|_{\mathbf{B}(H^r,H^s)}$. Conversely, if $f \in H^r$, then

$$\|\mu f\|_{H^s}^2 = \int_{-\infty}^{\infty} M(k)^2(1+k^2)^r|\widehat{f}(k)|^2 \; dk \le \|M\|_{L^\infty}^2 \|f\|_{H^r}^2 ,$$

and so $\|\mu\|_{\mathbf{B}(H^r,H^s)} \le \|M\|_{L^\infty}$. ■

**D.3. Calculus on Fourier multipliers.** For a function $f \colon \mathcal{X} \to \mathcal{Y}$ between normed spaces $\mathcal{X}$ and $\mathcal{Y}$, we set

$$\mathrm{Lip}(f) := \sup_{\substack{x,\grave{x} \in \mathcal{X} \\ x \neq \grave{x}}} \frac{\|f(x) - f(\grave{x}\|}{\|x - \grave{x}\|}.$$

We recall

$$\mathcal{C}^{0,1}(\mathcal{X},\mathcal{Y}) := \{f \in \mathcal{C}(\mathcal{X},\mathcal{Y}) \mid \mathrm{Lip}(f) < \infty\}$$

and

$$\mathcal{C}^{1,1}(\mathcal{X},\mathcal{Y}) := \left\{f \in \mathcal{C}^1(\mathcal{X},\mathcal{Y}) \;\middle|\; \mathrm{Lip}(Df) < \infty\right\},$$

with $Df \in \mathbf{B}(\mathcal{X},\mathcal{Y})$ denoting the derivative of $f$.

**D.3.1 Proposition.** *Let $\mu$ be a Fourier multiplier with symbol $\widetilde{\mu} \in L^\infty$ and, for $\omega \in \mathbb{R}$, let $\mu^\omega$ be the Fourier multiplier with symbol $\widetilde{\mu}^\omega(k) := \widetilde{\mu}(\omega k)$. Define*

$$\mu^{(\cdot)} \colon \mathbb{R} \to \mathbf{B}(H^r_{\mathrm{per}}, H^s_{\mathrm{per}}) \colon \omega \mapsto \mu^\omega.$$

**(i)** $\sup_{r,\omega \in \mathbb{R}} \|\mu^\omega\|_{\mathbf{B}(H^r_{\mathrm{per}})} \le \|\widetilde{\mu}\|_{L^\infty}$.

**(ii)** *If $r \ge s+1$ and $\widetilde{\mu} \in \mathcal{C}(\mathbb{R})$, then $\mu^{(\cdot)} \in \mathcal{C}(\mathbb{R}, \mathbf{B}(H^r_{\mathrm{per}}, H^s_{\mathrm{per}}))$.*

**(iii)** *If $r \ge s+1$ and $\widetilde{\mu} \in \mathcal{C}^{0,1}(\mathbb{R})$, then $\mu^{(\cdot)} \in \mathcal{C}^{0,1}(\mathbb{R}, \mathbf{B}(H^r_{\mathrm{per}}, H^s_{\mathrm{per}}))$ with* $\mathrm{Lip}(\mu^{(\cdot)}) \le$

$\mathrm{Lip}(\widetilde{\mu})$.

**(iv)** *If $r \geq s+2$ and $\widetilde{\mu}$ is differentiable with $\|\widetilde{\mu}'\|_{L^\infty} < \infty$, then $\mu^{(\cdot)}$ is differentiable as a function $\mathbb{R} \to \mathbf{B}(H^r_{\text{per}}, H^s_{\text{per}})$ and its derivative $\left.\frac{d}{d\omega}\mu^\omega\right|_{\omega=\omega_*} \in \mathbf{B}(H^r_{\text{per}}, H^s_{\text{per}})$ satisfies*

$$\mathfrak{F}\left[\left(\frac{d}{d\omega}\mu^\omega\right)\Big|_{\omega=\omega_*} f\right](k) = k\widetilde{\mu}'(\omega_* k\widehat{f}(k).$$

**(v)** *If $r \geq s+2$ and $\widetilde{\mu} \in \mathcal{C}^1(\mathbb{R})$ with $\|\widetilde{\mu}'\|_{L^\infty} < \infty$, then $\mu^{(\cdot)} \in \mathcal{C}^1(\mathbb{R}, \mathbf{B}(H^r_{\text{per}}, H^s_{\text{per}}))$.*

**(vi)** *If $r \geq s+2$ and $\widetilde{\mu} \in \mathcal{C}^{1,1}(\mathbb{R})$, then $\mu^{(\cdot)} \in \mathcal{C}^{1,1}(\mathbb{R}, \mathbf{B}(H^r_{\text{per}}, H^s_{\text{per}}))$ with* $\mathrm{Lip}\left(\frac{d}{d\omega}\mu^\omega\right) \leq$ $\mathrm{Lip}(\widetilde{\mu}')$.

**Proof. (i)** From Lemma D.2.1,

$$\sup_{r,\omega\in\mathbb{R}} \|\mu^\omega\|_{\mathbf{B}(H^r_{\text{per}})} \leq \sup_{r,\omega\in\mathbb{R}} \sup_{k\in\mathbb{Z}} \frac{|\widetilde{\mu}(\omega k)|}{(1+k^2)^{(r-r)/2}} = \sup_{r,\omega\in\mathbb{R}} \sup_{k\in\mathbb{Z}} |\widetilde{\mu}(\omega k)| \leq \|\mu\|_{L^\infty}.$$

**(ii)** Fix $\omega_* \in \mathbb{R}$ and let $\epsilon > 0$. Since $r - s \geq 1$, there is $K \in \mathbb{N}$ such that

$$\frac{1}{(1+k^2)^{(r-s)/2}} < \frac{\epsilon}{2\|\widetilde{\mu}\|_{L^\infty}}. \tag{D.3.1}$$

when $|k| > K$. Since $\widetilde{\mu}$ is continuous, there exist $\delta_{-K}, \ldots, \delta_K > 0$ such that if $|\omega - \omega_*| < \delta_k$, then $|\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)| < \epsilon$, for $k = -K, \ldots, K$. Set $\delta = \min_{-K\leq k\leq K} \delta_k$ and let $|\omega - \omega_*| < \delta$. Lemma D.2.1 implies

$$\|\mu^\omega - \mu^{\omega_*}\|_{\mathbf{B}(H^r_{\text{per}}, H^s_{\text{per}})} \leq \sup_{k\in\mathbb{Z}} \frac{|\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)|}{(1+k^2)^{(r-s)/2}}.$$

If $|k| \leq K$, then our choice of $\delta$ and the hypothesis $r - s \geq 1$ imply

$$\frac{|\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)|}{(1+k^2)^{(r-s)/2}} \leq |\widetilde{\mu}(\omega k - \widetilde{\mu}(\omega_* k)| < \epsilon,$$

while for $|k| > K$, (D.3.1) gives

$$\frac{|\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)|}{(1+k^2)^{(r-s)/2}} \leq \frac{2\|\widetilde{\mu}\|_{L^\infty}}{(1+k^2)^{(r-s)/2}} < \epsilon.$$

**(iii)** We compute

$$\left\|\mu^\omega - \mu^{\grave{\omega}}\right\|_{\mathbf{B}(H^r_{\text{per}}, H^s_{\text{per}})} \leq \sup_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^{r-s}} |\widetilde{\mu}(\omega k) - \widetilde{\mu}(\grave{\omega} k)|$$

$$\leq \sup_{k \in \mathbb{Z}} \frac{\operatorname{Lip}(\widetilde{\mu})|\omega k - \grave{\omega} k|}{(1+k^2)^{(r-s)/2}}$$

$$= \operatorname{Lip}(\widetilde{\mu})|\omega - \grave{\omega}| \sup_{k \in \mathbb{Z}} \frac{|k|}{(1+k^2)^{(r-s)/2}}$$

$$\leq \operatorname{Lip}(\widetilde{\mu})|\omega - \grave{\omega}| \sup_{k \in \mathbb{Z}} \frac{|k|}{\sqrt{1+k^2}}$$

$$\leq \operatorname{Lip}(\widetilde{\mu})|\omega - \grave{\omega}|.$$

**(iv)** Fix $\omega_* \in \mathbb{R}$. We begin by observing that if $\nu^\omega$ is the multiplier with symbol $\widetilde{\nu}^\omega := k\widetilde{\mu}'(\omega k)$, then Lemma D.2.1 implies

$$\left\| \frac{\mu^\omega - \mu^{\omega_*}}{\omega - \omega_*} - \nu^\omega \right\|_{\mathbf{B}(H^r_{\text{per}}, H^s_{\text{per}})} \leq \sup_{k \in \mathbb{Z}} \frac{1}{(1+k^2)^{(r-s)/2}} \left| \frac{\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)}{\omega - \omega_*} - k\widetilde{\mu}'(\omega_* k) \right|. \tag{D.3.2}$$

Also, the mean value theorem implies

$$\sup_{\substack{k_1, k_2, k_3 \in \mathbb{R} \\ k_1 \neq k_2}} \left| \frac{\widetilde{\mu}(k_1) - \widetilde{\mu}(k_2)}{k_1 - k_2} - \widetilde{\mu}'(k_3) \right| \leq 2 \left\| \widetilde{\mu}' \right\|_{L^\infty}. \tag{D.3.3}$$

Choose $\epsilon > 0$ and $K \in \mathbb{N}$ such that if $|k| > K$ then

$$\frac{|k|}{(1+k^2)^{(r-s)/2}} \leq \frac{\epsilon}{2 \left\| \widetilde{\mu}' \right\|_{L^\infty}}. \tag{D.3.4}$$

The differentiability of $\widetilde{\mu}$ implies, for $-K \leq k \leq K$, the existence of $\delta_k > 0$ such that if $|\omega - \omega_*| < \delta_k$, then

$$\left| \frac{\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)}{\omega - \omega_*} - k\widetilde{\mu}'(\omega_* k) \right| < \epsilon.$$

Take $\delta = \min_{-K \leq k \leq K} \delta_k$ and $|\omega - \omega_*| < \delta$. If $|k| \leq K$, then the choice of $\delta$ implies

$$\frac{1}{(1+k^2)^{(r-s)/2}} \left| \frac{\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)}{\omega - \omega_*} - k\widetilde{\mu}'(\omega_* k) \right| \leq \left| \frac{\widetilde{\mu}(\omega k) - \widetilde{\mu}(\omega_* k)}{\omega - \omega_*} - k\widetilde{\mu}'(\omega_* k) \right| < \epsilon.$$

If $|k| > K$, then $k \neq 0$ and we can combine (D.3.3) and (D.3.4) along with the hypothesis $r - s - 2 \geq 0$ to reach

$$\frac{1}{(1+k^2)^{(r-s)/2}}\left|\frac{\widetilde{\mu}(\omega k)-\widetilde{\mu}(\omega_* k)}{\omega-\omega_*}-k\widetilde{\mu}'(\omega_* k)\right|$$

$$=\frac{|k|}{(1+k^2)^{(r-s)/2}}\left|\frac{\widetilde{\mu}(\omega k)-\widetilde{\mu}(\omega_* k)}{\omega k-\omega_* k}-\widetilde{\mu}'(\omega_* k)\right|$$

$$\leq\frac{|k|}{(1+k^2)^{(r-s)/2}}2\left\|\widetilde{\mu}'\right\|_{L^\infty}<\epsilon.$$

**(v)** Fix $\omega_* \in \mathbb{R}$. We again denote by $\nu^\omega$ the multiplier with symbol $\widetilde{\nu}^\omega := k\widetilde{\mu}'(\omega k)$, take $\epsilon > 0$, and choose $K \in \mathbb{N}$ such that (D.3.4) holds for $|k| > K$. Since $\widetilde{\mu}'$ is continuous, there are $\delta_k > 0, |k| \leq K$, such that if $|\omega - \omega_*| < \delta_k$, then $|\widetilde{\mu}'(\omega k) - \widetilde{\mu}'(\omega_* k)| < \epsilon$. Take $\delta = \min_{-K\leq k\leq K} \delta_k$. Lemma D.2.1 provides

$$\|\nu^\omega-\nu^{\omega_*}\|_{\mathbf{B}(H^r_{\mathrm{per}},H^s_{\mathrm{per}})}\leq\sup_{k\in\mathbb{Z}}\frac{|k|}{(1+k^2)^{(r-s)/2}}|\widetilde{\mu}'(\omega k)-\widetilde{\mu}'(\omega_* k)|.$$

Take $|\omega - \omega_*| < \delta$. For $|k| \leq K$, we have

$$\frac{|k|}{(1+k^2)^{(r-s)/2}}|\widetilde{\mu}'(\omega k)-\widetilde{\mu}'(\omega_* k)|<\frac{|k|}{(1+k^2)^{(r-s)/2}}\epsilon\leq\epsilon$$

by choice of $\delta$ and the hypothesis $r - s \geq 2$. For $|k| > K$, we have

$$\frac{|k|}{(1+k^2)^{(r-s)/2}}|\widetilde{\mu}'(\omega k)-\widetilde{\mu}'(\omega_* k)|\leq\frac{|k|}{(1+k^2)^{(r-s)/2}}2\left\|\widetilde{\mu}'\right\|_{L^\infty}<\epsilon$$

by (D.3.4).

**(vi)** Using the notation $\nu^\omega$ from part (v), we compute

$$\left\|\nu^\omega-\nu^{\grave{\omega}}\right\|_{\mathbf{B}(H^r_{\mathrm{per}},H^s_{\mathrm{per}})}\leq\sup_{k\in\mathbb{Z}}\frac{1}{(1+k^2)^{(r-s)/2}}|k\widetilde{\mu}'(\omega k)-k\widetilde{\mu}'(\grave{\omega}k)|$$

$$\leq\sup_{k\in\mathbb{Z}}\frac{\mathrm{Lip}(\widetilde{\mu}')|\omega k-\grave{\omega}k||k|}{(1+k^2)^{(r-s)/2}}$$

$$=\mathrm{Lip}(\widetilde{\mu}')|\omega-\grave{\omega}|\sup_{k\in\mathbb{Z}}\frac{k^2}{(1+k^2)^{(r-s)/2}}$$

$$\leq \operatorname{Lip}(\widetilde{\mu}')|\omega - \grave{\omega}| \sup_{k\in\mathbb{Z}} \frac{k^2}{1+k^2}$$

$$\leq \operatorname{Lip}(\widetilde{\mu}')|\omega - \grave{\omega}|. \qquad \blacksquare$$

**D.4. Operator conjugation.** Our goal in this section is to extend a Fourier multiplier defined on $H^r$ to map (a subspace of) $H_q^r$ into $H_q^r$. With these preliminaries underway, we can now state our major tool, which is a lemma by Beale [Bea91a] (Lemma 3), [Bea91b] (Lemma 5.1).

**D.4.1 Lemma (Beale).** *Let $q > 0$ and let $\widetilde{\mu}$ be meromorphic on the strip*

$$\overline{\Sigma}_q = \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| \leq q\}$$

*with the following properties.*

**(i)** *Let $\mathfrak{P}_\mu$ be the set of poles of $\widetilde{\mu}$ in $\overline{\Sigma}_q$. Suppose that $\widetilde{\mu}$ has a finite number of poles in $\overline{\Sigma}_q$, all of which are real and simple.*

**(ii)** *Let $s \geq 0$ and $C$, $z_0 > 0$ such that if $|z| \geq z_0$, then*

$$|\widetilde{\mu}(z)| \leq \frac{C}{|\operatorname{Re}(z)|^s}.$$

*Then for $r \geq 0$, set*

$$\mathfrak{D}_{\mu,q}^r := \left\{ f \in H_q^r \;\middle|\; z \in \mathfrak{P}_\mu \Longrightarrow \widehat{f}(z) = 0 \right\}$$

*and let $\mu$ be the Fourier multiplier with symbol $\widetilde{\mu}$. Then $\mu \in \mathbf{B}(\mathfrak{D}_{\mu,q}^r, H_q^{r+s})$ and*

$$\|\mu f\|_{r+s,q} \leq \left( \sup_{\substack{z\in\mathbb{C} \\ |\operatorname{Im}(z)|=q}} (1 + |\operatorname{Re}(z)|^2)^{s/2} |\widetilde{\mu}(z)| \right) \|f\|_{r,q}.$$

**D.4.2 Remark.** *For $f \in H_q^r$ we have $f \in \cap_{0\leq y<q} L_y^1$ by part (ii) of Proposition C.3.12, so $\widehat{f}(z)$ is defined pointwise by* (B.3.2) *for all $z \in \Sigma_q$. In particular, $\widehat{f}(z)$ is defined pointwise at all $z \in \mathfrak{P}_\mu$, since $\mathfrak{P}_\mu \subseteq \mathbb{R}$. This makes the definition of $\mathfrak{D}_{\mu,q}^r$ unambiguous.*

We apply Beale's lemma to prove the following.

**D.4.3 Proposition (Operator conjugation).** *Suppose that $\widetilde{\mu}$ satisfies the hypotheses of Beale's lemma on the closed strip $\overline{\Sigma}_{q_0}$ and that there is a constant $C > 0$ with*

$$|\widetilde{\mu}(k+iq) - \widetilde{\mu}(k)| \leq C|q| \tag{D.4.1}$$

*for all $|q| \leq q_0$. Let $\mu$ be the Fourier multiplier with symbol $\widetilde{\mu}$ and define*

$$\mu_q f := \cosh(q\cdot)\mu[\operatorname{sech}(q\cdot)f].$$

*Then $\mu_q \in \mathbf{B}(H^r)$ and*

$$\lim_{q \to 0^+} \|\mu_q - \mu\|_{\mathbf{B}(H^r)} = 0. \tag{D.4.2}$$

**Proof.** For convenience we abbreviate

$$c_q := \cosh(q\cdot) \quad \text{and} \quad s_q := \operatorname{sech}(q\cdot).$$

First we show that $\mu_q$ is a well-defined operator on $H^r$. Fix $f \in H^r$, so $s_q f \in H_q^r$. Beale's lemma tells us that $\mu$ maps $H_q^r$ to $H_q^r$ for $|q| \leq q_0$, so $\mu[s_q f] \in H_q^r$, and consequently $\mu_q f = c_q \mu[s_q f] \in H^r$, too.

For the limit, we compute, for $f \in H^r$ with $\|f\|_{H^r} \leq 1$,

$$\|c_q \mu[s_q f] - \mu f\|_{H^r}^2 = \int_{-\infty}^{\infty} (1+k^2)^r |\mathfrak{F}[c_q \mu[s_q f] - \mu[f]](k)|^2 \, dk.$$

By (B.3.5), we have

$$\mathfrak{F}[c_q \mu[s_q f]](k) = \frac{\mathfrak{F}[\mu[s_q f]](k+iq) + \mathfrak{F}[\mu[s_q f]](k-iq)}{2},$$

and since $\mu$ is a Fourier multiplier, this becomes

$$\mathfrak{F}[c_q \mu[s_q f]](k) = \frac{\widetilde{\mu}(k+iq)\widehat{s_q f}(k+iq) + \widetilde{\mu}(k-iq)\widehat{s_q f}(k-iq)}{2}.$$

Next, since $c_q s_q = 1$, we have

$$\widehat{f}(k) = \widehat{c_q s_q f}(k) = \frac{\widehat{s_q f}(k+iq) + \widehat{s_q f}(k-iq)}{2},$$

and so

$$\widehat{\mu f}(k) = \widetilde{\mu}(k)\widehat{f}(k) = \widetilde{\mu}(k)\frac{\widehat{s_q f}(k+iq) + \widehat{s_q f}(k-iq)}{2}.$$

Then

$$\begin{aligned}\mathfrak{F}[c_q\mu[s_qf]-\mu f](k) &= \frac{\widetilde{\mu}(k+iq)\widehat{s_qf}(k+iq)+\widetilde{\mu}(k-iq)\widehat{s_qf}(k-iq)}{2}\\ &\quad -\widetilde{\mu}(k)\frac{\widehat{s_qf}(k+iq)+\widehat{s_qf}(k-iq)}{2}\\ &= \left(\widetilde{\mu}(k+iq)-\widetilde{\mu}(k)\right)\frac{\widehat{s_qf}(k+iq)}{2}\\ &\quad +\left(\widetilde{\mu}(k-iq)-\widetilde{\mu}(k)\right)\frac{\widehat{s_qf}(k-iq)}{2}.\end{aligned}$$

By hypothesis on $\widetilde{\mu}$, we have

$$\left|\left(\widetilde{\mu}(k\pm iq)-\widetilde{\mu}(k)\right)\frac{\widehat{s_qf}(k\pm iq)}{2}\right| \leq C|q|\left|\widehat{s_qf}(k\pm iq)\right| = C|q||\mathfrak{F}[e^{\pm q\cdot}s_qf](k)|.$$

Now define the multiplication operator

$$T_q\colon H^r\to H^r\colon f\mapsto e^{q\cdot}s_qf.$$

To be clear, this is not a Fourier multiplier, and

$$\|T_q\|_{\mathbf{B}(H^r)}\leq C_r\left\|e^{q\cdot}s_q\right\|_{W^{r,\infty}}.$$

We conclude

$$\begin{aligned}\|c_q\mu[s_qf]-\mu[f]\|_{H^r}^2 &= \int_{-\infty}^{\infty}(1+k^2)^r|\mathfrak{F}[c_q\mu[s_qf]-\mu f](k)|^2\,dk\\ &= \int_{-\infty}^{\infty}(1+k^2)^r\left|\left(\widetilde{\mu}(k+iq)-\widetilde{\mu}(k)\right)\frac{\widehat{s_qf}(k+iq)}{2}\right|^2\,dk\\ &\quad +\int_{-\infty}^{\infty}(1+k^2)^r\left|\left(\widetilde{\mu}(k-iq)-\widetilde{\mu}(k)\right)\frac{\widehat{s_qf}(k-iq)}{2}\right|^2\,dk\\ &\leq C|q|\int_{-\infty}^{\infty}(1+k^2)^r|\widehat{T_qf}(k)|^2\,dk\\ &\quad +C|q|\int_{-\infty}^{\infty}(1+k^2)^r|\widehat{T_{-q}f}(k)|^2\,dk\end{aligned}$$

$$= C|q| \, \|T_q f\|_{H^r}^2 + C|q| \, \|T_{-q} f\|_{H^r}^2$$

$$\leq Cq \, \|T_q\|_{\mathbf{B}(H^r)}^2 + Cq \, \|T_{-q}\|_{\mathbf{B}(H^r)}^2 \text{ since } \|f\|_{H^r} \leq 1$$

$$\leq C_r |q|.$$

We used (D.4.5) to get the last inequality, and, taking $q \to 0$, this proves the limit (D.4.2). ∎

**D.4.4 Remark.** *The proof above could proceed just as well if we replaced the condition* (D.4.1) *with* $|\widetilde{\mu}(k + iq) - \widetilde{\mu}(k)| = \mathcal{O}(q^p)$ *uniformly in* $k$ *for some* $p > 0$. *We will have precisely* (D.4.1) *in our intended application below to the Friesecke-Pego operator* $\mathcal{A}$, *so we leave the statement as it is.*

**D.4.5 Lemma.** *For all* $r \geq 0$ *there is a constant* $C_r > 0$ *such that*

$$\sup_{|q| \leq 1} \|e^{q \cdot} s_q\|_{W^{r,\infty}} \leq C_r.$$

**Proof.** Set $f(x) = e^x$, $N(X) = X^{-1}$, and $g(x) = (e^x + e^{-x})/2$, so

$$\partial_x^r [e^x s_1(x)] = \partial_x^r [f(x) N(g(x))]$$

$$= \sum_{k=0}^r \binom{r}{k} \partial_x^{r-k} [f] \partial_X^k [N(g)]$$

$$= \sum_{k=0}^r \binom{r}{k} \partial_x^{r-k} [f] \sum_{j=1}^k \partial_X^j [N](g) \sum_{\boldsymbol{\sigma} \in \Sigma_j^k} C_{\boldsymbol{\sigma}} \prod_{\ell=1}^j \partial_x^{\sigma_j} [g]$$

$$= \sum_{k=0}^r \binom{r}{k} e^x \sum_{j=1}^k (-1)^j j! \left( \frac{e^x + e^{-x}}{2} \right)^{-(j+1)} \sum_{\boldsymbol{\sigma} \in \Sigma_j^k} C_{\boldsymbol{\sigma}} \prod_{\ell=1}^j \frac{e^x + (-1)^{\sigma_j} e^{-x}}{2}.$$

The $k$th term of this sum is bounded by

$$C_r \frac{e^x}{e^x + e^{-x}} \leq C_2.$$

∎

**D.5. The Friesecke-Pego operator.** Our goal in this section is to prove — from scratch — that the operator

$$\mathcal{A}f := f + \underbrace{\frac{2(\beta+\varkappa^3)}{\varkappa^2(\varkappa+1)}\varpi^0[\sigma f]}_{-\mathcal{K}[\sigma f]}, \qquad \varpi^0 = -c_\varkappa^2(1-\alpha_\varkappa\partial_X^2)^{-1},$$

is invertible on $E_q^r = \{f \in H_q^r \mid f \text{ is even}\}$. This operator arose in (4.2.2) when we constructed the nanopteron fixed point equations from Beale's ansatz. Our method follows exactly the proof in [FP99] for a differently scaled version of $\mathcal{A}$.

To see that $\mathcal{A}$ maps $E^r$ to $E^r$ for arbitrary $r \geq 1$, fix $f \in E^r$. Then $\sigma f \in E^r$ since $\sigma \in W^{r,\infty}$ is even, so $\varpi^0[\sigma f] \in E^r$ since $\varpi^0$ "smooths by two" and has an even symbol. Hence $\mathcal{K}[\sigma f] \in E^r$ and so also $\mathcal{A}f \in E^r$.

We will control the kernel of $\mathcal{A}$ with the following lemma.

**D.5.1 Lemma.** *Let $\lambda > 0$ and $\varsigma \in \mathcal{C}_0(\mathbb{R})$. Then the second-order differential equation*

$$\psi''(t) + \lambda(\varsigma(t)-1)\psi(t) = 0, \ \ t \in [0,\infty) \tag{D.5.1}$$

*does not have two linearly independent bounded solutions.*

**Proof.** Suppose instead that (D.5.1) has two bounded, linearly independent solutions $f_1$ and $f_2$. Since the coefficients of (D.5.1) are continuous, there exist scalars $\alpha_1, \alpha_2$ such that $f := \alpha_1 f_1 + \alpha_2 f_2$ solves

$$\begin{cases} f''(t) + \lambda(\varsigma(t)-1)f(t) = 0, & t \in [0,\infty) \\ f(t_0) = f'(t_0) = 1, \end{cases} \tag{D.5.2}$$

where by the hypothesis on $\varsigma$, we have chosen $t_0 \geq 0$ such that if $t \geq t_0$, then $|\varsigma(t)| < 1/2$.

Let

$$\mathcal{S} = \{t > t_0 \mid t_0 < s < t \Longrightarrow f(s) > 0\}.$$

Since $f(t_0) = 1$, by the continuity of $f$ the set $\mathcal{S}$ is nonempty, and so $T := \sup(\mathcal{S}) \in (t_0, \infty]$. We first show that $(t_0, T) \subseteq \mathcal{S}$ and then that $T = \infty$.

For the first claim, let $t \in (t_0, T)$. Then there must exist $t_1 \in \mathcal{S}$ such that $t < t_1$, as otherwise $T \le t$. So, $t \in (t_0, t_1)$, where $f(s) > 0$ for all $s \in (t_0, t_1)$. Hence $f(s) > 0$ for all $s \in (t_0, t] \subseteq (t_0, t_1)$, and so $t \in \mathcal{S}$. In particular, note that $f(t) > 0$ for all $t \in (t_0, T)$.

For the second claim, let $t \in (t_0, T)$ and compute

$$f''(t) = \lambda(1 - \varsigma(t))f(t) \ge \lambda f(t) - \lambda\frac{f(t)}{2} = \lambda\frac{f(t)}{2} > 0.$$

Hence $f'$ is increasing on $(t_0, \infty)$, and so, using the initial conditions of (D.5.2)

$$f(t) = f(t_0) + \int_{t_0}^{t} f'(s)\ ds \ge f(t_0) + f'(t_0)(t - t_0) = 1 + t - t_0. \tag{D.5.3}$$

Then

$$f(T) = \lim_{t \to T^-} f(t) \ge \lim_{t \to T^-} 1 + t - t_0 = 1 + T - t_0 > 0.$$

By continuity, there is $\delta > 0$ such that $f(t) > 0$ for $t \in [T, T + \delta)$. Hence $T + \delta \in \mathcal{S}$, which is a contradiction unless $T = \infty$.

So, we have $\mathcal{S} = (t_0, \infty)$, and therefore $f(t) > 0$ for all $t \in (t_0, \infty)$. But then (D.5.3) holds for all $t > t_0$, and so $f$ is unbounded, a contradiction. ■

Now suppose $\mathcal{A}f = 0$, $f \in H^r$. That is,

$$f - \frac{2(\beta + \varkappa^3)}{\varkappa^2(\varkappa + 1)} c_\varkappa^2 (1 - \alpha_\varkappa \partial_X^2)^{-1}[\sigma f] = 0. \tag{D.5.4}$$

It is straightforward to rearrange this to

$$f'' + \frac{1}{\alpha_\varkappa}\left(\frac{2(\beta + \varkappa^3)}{\varkappa^2(\varkappa + 1)} c_\varkappa^2 \sigma - 1\right) f = 0, \tag{D.5.5}$$

and then Lemma (D.5.1) applies to show that (D.5.5) has at most one nontrivial bounded solution. Since (D.5.4) and (D.5.5) are equivalent, the dimension of the kernel of $\mathcal{A}$ (as an operator from $H^r$ to $H^r$) is at most one. Now we show the dimension is precisely one.

We know from (2.5.6) that $\sigma$ solves the KdV-type equation

$$\sigma - \frac{1}{2}\mathcal{K}[\sigma^2] = \sigma - c_\varkappa^2 \frac{\beta + \varkappa^3}{\varkappa^2(1 + \varkappa)}(1 - \alpha_\varkappa \partial_X^2)^{-1}[\sigma] = 0.$$

Then since derivatives commute with Fourier multipliers,

$$\begin{aligned}
\mathcal{A}\sigma' &= \sigma' - \frac{2(\beta+\varkappa^3)}{\varkappa^2(\varkappa+1)}\varpi^0[\sigma\sigma'] \\
&= \sigma' - \frac{\beta+\varkappa^3}{\varkappa^2(\varkappa+1)}c_\varkappa^2(1-\alpha_\varkappa\partial_X^2)^{-1}[2\sigma\sigma'] \\
&= \partial_X[\sigma] - \frac{\beta+\varkappa^3}{\varkappa^2(\varkappa+1)}c_\varkappa^2(1-\alpha_\varkappa\partial_X^2)^{-1}(\partial_X[\sigma^2]) \\
&= \partial_X\left[\sigma - \frac{1}{2}\mathcal{K}[\sigma^2]\right] \\
&= 0.
\end{aligned}$$

That is, $\ker(\mathcal{A}) = \operatorname{span}(\{\sigma'\})$. Since $\sigma$ is even, $\sigma'$ is odd, and so if we restrict $\mathcal{A}$ to $E^r$, then $\ker(\mathcal{A})$ is trivial. Finally, since $\mathcal{A} = \mathbb{1} - \mathcal{K}[\sigma\cdot]$, where $\mathcal{K}[\sigma\cdot]$ is compact by Proposition C.3.17, the Fredholm alternative guarantees that $\mathcal{A}\big|_{E^r}$ is also surjective, hence $\mathcal{A}\big|_{E^r}$ is invertible.

**D.5.2 Proposition.** *There exists $q_{\mathsf{FP}} > 0$ such that for each integer $r \geq 1$ and each $q \in [0, q_{\mathsf{FP}})$ the operator $\mathcal{A}$ maps $E_q^r$ bijectively onto $E_q^r$ and*

$$\sup_{0 \leq q < q_{\mathsf{FP}}} \left\{ \|\mathcal{A}\|_{\mathbf{B}(E_q^r)}, \left\|\mathcal{A}^{-1}\right\|_{\mathbf{B}(E_q^r)} \right\} < C_r < \infty. \tag{D.5.6}$$

**Proof.** We proved the case $q = 0$ above. Recall that the symbol of $\varpi^0$ is

$$\tilde{\varpi}^0(k) := \widetilde{\varpi^0}(k) = \frac{1}{1+\alpha_\varkappa k^2}, \tag{D.5.7}$$

where $\alpha_\varkappa > 0$. The function $\tilde{\varpi}^0$ has no singularities in the strip $\Sigma_{\sqrt{\alpha_\varkappa}}$, and clearly the other hypotheses of Beale's lemma (Lemma D.4.1) apply to it. Hence $\varpi^0$ maps $H_q^r$ into $H_q^r$ for $q \in (0, \sqrt{\alpha_\varkappa})$, and since $\tilde{\varpi}^0$ is even, $\varpi^0$ in fact maps $E_q^r$ into $E_q^r$. Next, $\sigma$ is even and $\sigma \in H_q^r$ for all $r \geq 0$ and all $q \in (0, 1/2\sqrt{\alpha_\varkappa})$ by definition of $\sigma$ in (2.5.8) and Lemma C.3.6. Then Lemma C.3.19 implies that multiplication by $\sigma$ maps $E_q^r$ to $E_q^r$, so the composition $\varpi^0(\sigma\cdot)$ also maps $E_q^r$ into $E_q^r$. Hence $\mathcal{K} \in \mathbf{B}(E_q^r)$ and so also

$\mathcal{A} \in \mathbf{B}(E_q^r)$.

It remains for us to show that $\mathcal{A}$ is bijective from $E_q^r$ to $E_q^r$. We provide two proofs.

**(i)** One easy way to do this is to note first that $\mathcal{A}$ has trivial kernel on $E_q^r$: $\mathcal{A}$ has trivial kernel on $E^r$, and $E_q^r \subseteq E^r$. Next, we note that Corollary C.3.17 implies that $\mathcal{K}$ is compact from $E_q^r$ to $E_q^r$. The Fredholm alternative applies (once again) to show that $\mathcal{A}$ is invertible on $E_q^r$. This approach does not, however, provide the uniform estimates in (D.5.6).

**(ii)** A longer proof illustrates the method of operator conjugation. Let

$$\mathcal{A}_q f := c_q \mathcal{A}(s_q f) = f - \mathcal{K}_q[\sigma f], \qquad \mathcal{K}_q[g] := c_q \mathcal{K}[s_q g],$$

with $c_q = \cosh(q\cdot)$ and $s_q = \operatorname{sech}(q\cdot)$. First we show that $\mathcal{A}_q \in \mathbf{B}(E^r)$. Beale's lemma tells us that if $f \in E^r$, then $s_q f \in E_q^r$, so $\mathcal{K}[\sigma s_q f] \in E_q^r$ as well, and then $c_q \mathcal{K}[\sigma s_q f] \in E^r$. So, $\mathcal{A}_q$ maps $E^r$ to $E^r$. For its boundedness, we only need to estimate

$$\|c_q \mathcal{K}[\sigma s_q f]\|_{E^r} = \|\mathcal{K}[\sigma s_q f]\|_{E_q^r} \leq \|\mathcal{K}\|_{\mathbf{B}(E_q^r)} \|\sigma s_q f\|_{E_q^r} \leq C_{r,q} \|\mathcal{K}\|_{\mathbf{B}(E_q^r)} \|\sigma\|_{W^{r,\infty}} \|f\|_{E^r}.$$

Next, we show that $\mathcal{A}$ is invertible on $E_q^r$ if and only if $\mathcal{A}_q$ is invertible on $E^r$. Given $f$, $g \in E_q^r$, we can write $f = s_q \widetilde{f}$ and $g = s_q \widetilde{g}$ for some $\widetilde{f}$, $\widetilde{g} \in E^r$. Then

$$\mathcal{A}f = \mathcal{A}g \iff c_q \mathcal{A}(s_q \widetilde{f}) = c_q \mathcal{A}(s_q \widetilde{g}) \iff \mathcal{A}_q \widetilde{f} = \mathcal{A}_q \widetilde{g}. \tag{D.5.8}$$

and

$$\mathcal{A}f = g \iff \mathcal{A}(s_q \widetilde{f}) = s_q \widetilde{g} \iff c_q \mathcal{A}(s_q \widetilde{f}) = \widetilde{g} \iff \mathcal{A}_q \widetilde{f} = \widetilde{g} \tag{D.5.9}$$

Since $f$ and $g$ (equivalently $\widetilde{f}$ and $\widetilde{g}$) are arbitrary, the problems of injectivity (D.5.8) and surjectivity (D.5.9) for $\mathcal{A}$ on $E_q^r$ are equivalent to establishing injectivity and surjectivity of $\mathcal{A}_q$ on $E^r$.

Now, we compute

$$\frac{1}{1+(k+iq)^2}-\frac{1}{1+k^2}=\underbrace{\frac{q^2(1-3k^2-q^2)}{(k^2+1)((k^2-q^2+1)^2+(2kq)^2}}_{\mathcal{R}_q(k)}-i\underbrace{\frac{2kq}{(k^2-q^2+1)^2+(2kq)^2}}_{\mathcal{I}_q(k)}.$$

If we restrict $q \in (0, 1/2)$, then

$$\left(\frac{3}{4}+k^2\right)^2 \leq (k^2-q^2+1)^2+(2kq)^2,$$

in which case

$$|\mathcal{R}_q(k)| \leq |q|\frac{|1-3k^2|}{(3/4+k^2)^2}+\frac{|q|}{4}\frac{1}{(3/4+k^2)^2} \leq 4|q|$$

and

$$|\mathcal{I}_q(k)| \leq |q|\left|\frac{2k}{(3/4+k^2)^2}\right| \leq 2|q|.$$

Rescaling $\mathcal{R}_q(k)$ and $\mathcal{I}_q(k)$ to $\mathcal{R}_q(\alpha_\varkappa k)$ and $\mathcal{I}_q(\alpha_\varkappa k)$, we see that $\widetilde{\varpi}^0(k)$ from (D.5.7) satisfies the hypothesis (D.4.1) for operator conjugation on the Fourier multiplier $\varpi^0$. It follows that

$$\|\mathcal{A}_q-\mathcal{A}\|_{\mathbf{B}(E^r)}=\|\mathcal{K}_q[\sigma\cdot]-\mathcal{K}[\sigma\cdot]\|_{\mathbf{B}(E^r)} \leq C_r\|\sigma\|_{W^{r,\infty}}\|\mathcal{K}_q-\mathcal{K}\|_{\mathbf{B}(E^r)} \to 0 \quad \text{(D.5.10)}$$

as $q \to 0$. The constant $C_r > 0$, which is independent of $q$, comes from Proposition C.3.12 (vii). Since $\mathcal{A}$ is invertible, this shows that for $q$ sufficiently small $\mathcal{A}_q$ is invertible as well.

For the estimates in (D.5.6), we take $f \in E^r_q$, so $c_q f \in E^r$, and compute

$$\begin{aligned}\|\mathcal{A}f\|_{E^r_q} &= \|c_q\mathcal{A}f\|_{E^r}\\ &= \|c_q\mathcal{A}(s_qc_qf)\|_{E^r}\\ &= \|\mathcal{A}_q(c_qf)\|_{E^r}\\ &\leq \|\mathcal{A}_q\|_{\mathbf{B}(E^r)}\|c_qf\|_{E^r}\end{aligned}$$

$$= \|\mathcal{A}_q\|_{\mathbf{B}(E^r)} \|f\|_{E^r_q} .$$

Hence

$$\|\mathcal{A}\|_{\mathbf{B}(E^r_q)} \leq \|\mathcal{A}_q\|_{\mathbf{B}(E^r)} \leq \|\mathcal{A}_q - \mathcal{A}\|_{\mathbf{B}(E^r)} + \|\mathcal{A}\|_{\mathbf{B}(E^r)} ,$$

and so taking $q$ sufficiently close to 0, we have the uniform bound on $\|\mathcal{A}\|_{\mathbf{B}(E^r_q)}$. Replacing $\mathcal{A}$ and $\mathcal{A}_q$ with $\mathcal{A}^{-1}$ and $\mathcal{A}_q^{-1}$ in the calculations above and using the convergence $\mathcal{A}_q^{-1} \to \mathcal{A}^{-1}$ in $\mathbf{B}(E^r)$, which follows from (D.5.10), we have the estimates on $\|\mathcal{A}^{-1}\|_{\mathbf{B}(E^r_q)}$. ■

## Appendix E. Composition Operators

In this appendix we prove a number of estimates that directly facilitate our treatment of the higher-order terms in the spring forces. All of these estimates treat composition operators: given a map $N$ and functions $f$ and $\grave{f}$, we need to control the norms of $N(f)$ and $N(f) - N(\grave{f})$ in a suitable function space. If our functions were always in $H^r$, this would be a well-understood problem [BM01]; to bound $N(f)$, we could rely on, for example, the proof of estimate (2.4) in [Mos66] or Proposition 3.9 in [Tay11], both of which state, roughly, that if $N \in \mathcal{C}^\infty(\mathbb{R})$ and $N(0) = 0$, then

$$\|N(f)\|_{H^r} \le C(1 + \|f\|_{H^r}),$$

where $C$ depends on $\|f\|_{L^\infty}$.

However, our functions will always be the superposition of a function $f \in H^r_q$ with a function $\varphi \in W^{r,\infty}$. Since $\partial_X^r[f]$ need not be bounded and $\varphi$ need not be square-integrable, the sum $f + \varphi$ belongs to neither of the spaces $H^r_q$ nor $W^{r,\infty}$. Moreover, the $W^{r,\infty}$-norm of $\varphi$ will depend delicately on the small parameter $\epsilon$, and we want careful, uniform estimates in $\epsilon$. So, we develop our composition operator estimates *ab ovo*.

### E.1. A mapping estimate.

**E.1.1 Proposition.** *Let $r \in \mathbb{N}$, $N \in \mathcal{C}^r(\mathbb{R})$, and $C_\star$, $q > 0$. There exists an increasing map $\mathcal{M}\colon (0,\infty) \to (0,\infty)$ with the following property. Let $a$, $\epsilon \in (0,1)$. Suppose $f$, $g \in H^r_q$ and $\varphi \in W^{r,\infty}$ with $\left\|\partial_X^j[\varphi]\right\|_{L^\infty} \le C_\star \epsilon^{-j}$ for $j = 0, \ldots, r$. Then*

$$\|fN(\epsilon(g + a\varphi))\|_{r,q} \le \mathcal{M}[\|g\|_{r,q}](1 + |a|\epsilon^{1-r}) \|f\|_{r,q}. \tag{E.1.1}$$

*The function $\mathcal{M}$ depends on $N$, $r$, $q$, and $C_\star$, but not on $f$, $\epsilon$, or $\varphi$.*

**E.1.2 Remark.** *Each of the estimates that we prove in this appendix will include a nonnegative factor $\mathcal{M}$ that depends, by its (convoluted) construction, on the norms* *of one or more functions involved in the estimate. So, we can think of* $\mathcal{M}$ *as a map from* $\mathbb{R}_+^n$ *to* $\mathbb{R}_+$ *for some* $n \geq 1$*, where* $\mathbb{R}_+ = [0, \infty)$*. It turns out that* $\mathcal{M}$ *will always have the property that*

$$\sup_{\substack{\mathbf{v} \in \mathbb{R}_+^n \\ |\mathbf{v}| \leq r}} \mathcal{M}[\mathbf{v}] < \infty \tag{E.1.2}$$

*for any* $r > 0$*, where* $|\mathbf{v}| = \sum_{k=1}^n v_k$*. Then we may define a new map* $\mathcal{M}^\star \colon \mathbb{R}_+^n \to \mathbb{R}_+$ *by*

$$\mathcal{M}^\star[\mathbf{u}] := \sup_{\substack{\mathbf{v} \in \mathbb{R}_+^n \\ |\mathbf{v}| \leq |\mathbf{u}|}} \mathcal{M}[\mathbf{v}].$$

*This map* $\mathcal{M}^\star$ *enjoys three properties:*

**(i)** $\mathcal{M}^\star$ *is "radially increasing" in the sense that if* $|\mathbf{u}| \leq |\grave{\mathbf{u}}|$*, then* $\mathcal{M}^\star[\mathbf{u}] \leq \mathcal{M}^\star[\grave{\mathbf{u}}]$*. In particular, if* $n = 1$*, then* $\mathcal{M}$ *is increasing in the usual sense.*

**(ii)** *As a consequence of the first property,* $\mathcal{M}^\star$ *is locally bounded in the sense that*

$$\sup_{\mathbf{u} \in \mathbb{R}_+^n |\mathbf{u}| \leq r} \mathcal{M}^\star[\mathbf{u}] < \infty$$

*for any* $r > 0$*.*

**(iii)** *It is obvious from the definition of* $\mathcal{M}^\star$ *that* $\mathcal{M}[\mathbf{u}] \leq \mathcal{M}^\star[\mathbf{u}]$*.*

*The first two properties of* $\mathcal{M}^\star$ *above will be very useful for the nanopteron estimates. The third property allows us to replace the function* $\mathcal{M}$ *that we construct in a given proof by its relative* $\mathcal{M}^\star$*, and we will do so without further comment. So, for example, when we prove Proposition E.1.1, our proof will only demonstrate the property* (E.1.2)*.*

**E.1.3 Remark.** *In the proof of Proposition E.1.1 and the following proofs we will have a great many constants that depend more or less innocuously on different parameters. A constant* $C_r$ *depends only on* $r$*; a constant* $C_{r,q}$ *depends only on* $r$ *and* $q$*; and a constant* $C_{\star,r,q}$ *depends only on* $C_\star$*,* $r$*, and* $q$*. The value of these constants may change from line to line, but their dependence remains rests firmly and solely on* *their subscripts.*

Now we are ready to prove Proposition E.1.1.

**Proof.** We will construct three functions $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ in this proof as we progress toward the ultimate $\mathcal{M}$ that satisfies (E.1.1). To simplify notation, we will write $\mathcal{M}_0[g] = \mathcal{M}[\|g\|_{r,q}]$, etc. Since

$$\|fN(\epsilon(g+\varphi))\|_{r,q} = \|\cosh(q\cdot)fN(\epsilon(g+a\varphi))\|_{L^2} + \|\cosh(q\cdot)\partial_X^r[fN(\epsilon(g+a\varphi))]\|_{L^2}. \tag{E.1.3}$$

it suffices to verify that each $L^2$-norm in the sum above has an upper bound of the form given on the right side of (E.1.1).

First, let

$$\mathcal{I}(g) = \left\{ Y \in \mathbb{R} \;\middle|\; |Y| \le C_{r,q}\|g\|_{r,q} + C_\star \right\}.$$

Since

$$\begin{aligned}
\epsilon|g(X)+a\varphi(X)| &\le \|g\|_{L^\infty} + |a|\,\|\varphi\|_{L^\infty} \\
&\le C_{r,q}\|g\|_{r,q} + C_\star\epsilon^0 \\
&\le C_{r,q}\|g\|_{r,q} + C_\star,
\end{aligned}$$

we have

$$\left\|\partial_X^k[N](\epsilon(g+a\varphi))\right\|_{L^\infty} \le \|N\|_{W^{r,\infty}(\mathcal{I}(g))}.$$

Then the first term in (E.1.3) is easy to handle:

$$\begin{aligned}
\|\cosh(q\cdot)fN(\epsilon(g+a\varphi))\|_{L^2} &\le \|N(\epsilon(g+a\varphi))\|_{L^\infty}\|\cosh(q\cdot)f\|_{L^2} \\
&\le \|N\|_{W^{r,\infty}(\mathcal{I}(g))}\|\cosh(q\cdot)f\|_{r,q}.
\end{aligned}$$

Next, Leibniz's rule reduces the study of the second term in (E.1.3) to estimating terms of the form

$$\left\|\cosh(q\cdot)\partial_X^m[f]\partial_X^n[N(\epsilon(g+a\varphi))]\right\|_{L^2},$$

where $0 \le m, n \le r$ and $m + n \le r$. It turns out that estimating the $m = 0, n = r$ term is both the most complicated and the most instructive term, so we do it first. That is, we estimate

$$\left\|\cosh(q\cdot)f\partial_X^r[N(\epsilon(g+a\varphi))]\right\|_{L^2}.$$

We use Faá di Bruno's rule to expand

$$\partial_X^r[N(\epsilon(g+a\varphi))] = \sum_{k=1}^r \partial_X^k[N](\epsilon(g+a\varphi)) \sum_{\boldsymbol{\sigma}\in\Sigma_k^r} C_{\boldsymbol{\sigma}} \prod_{j=1}^k \partial_X^{\sigma_j}[\epsilon(g+a\varphi)].$$

A first pass then reduces our estimate to

$$\begin{aligned}
&\left\|\cosh(q\cdot)f\partial_X^r[N(\epsilon(g+a\varphi))]\right\|_{L^2} \\
&\qquad \le C_r \left\|N\right\|_{W^{r,\infty}(\mathcal{I}(g))} \sum_{k=1}^r \sum_{\boldsymbol{\sigma}\in\Sigma_k^r} \left\|\cosh(q\cdot)f\prod_{j=1}^k \epsilon\partial_X^{\sigma_j}[g+a\varphi]\right\|_{L^2}. \qquad \text{(E.1.4)}
\end{aligned}$$

When $k = 1$, we have $\Sigma_1^r = \{r\}$ by Remark A.2.1, and

$$\begin{aligned}
\left\|\cosh(q\cdot)f\epsilon\partial_X^r[g+a\varphi]\right\|_{L^2} &\le \left\|\cosh(q\cdot)f\right\|_{L^\infty}\left\|\partial_X^r[g]\right\|_{L^2} + \left\|\cosh(q\cdot)f\right\|_{L^2}\epsilon|a|\left\|\partial_X^r[\varphi]\right\|_{L^\infty} \\
&\le C_{r,q}\left\|f\right\|_{r,q}\left\|g\right\|_{r,q} + \left\|f\right\|_{r,q}|a|C_\star\epsilon^{1-r} \\
&\le \mathcal{M}_0[g]\left(1+|a|\epsilon^{1-r}\right)\left\|f\right\|_{r,q},
\end{aligned}$$

where we have set

$$\mathcal{M}_0[g] = \max\left\{C_{r,q}\left\|g\right\|_{r,q}, C_\star\right\}.$$

When $2 \le k \le r$, all of the factors in the product in (E.1.4) will be $L^\infty$ because the order of each derivative $\partial_X^{\sigma_j}[g]$ will be at most $r-1$. Then

$$\left\|\prod_{j=1}^{k}\epsilon\partial_X^{\sigma_j}[g+a\varphi]\right\|_{L^\infty} \le \prod_{j=1}^{k}\left(C_{r,q}\|g\|_{r,q}+\epsilon|a|\left\|\partial_X^{\sigma_j}[\varphi]\right\|_{L^\infty}\right)$$
$$\le C_{\star,r,q}\prod_{j=1}^{k}\left(\|g\|_{r,q}+|a|\epsilon^{1-\sigma_j}\right). \tag{E.1.5}$$

We use Lemma E.6.1 to write this product as

$$\prod_{j=1}^{k}\left(\|g\|_{r,q}+|a|\epsilon^{1-\sigma_j}\right) = \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^k\\ \alpha_j+\beta_j=1\ \forall j}}\prod_{j=1}^{k}\|g\|_{r,q}^{\alpha_j}\left(|a|\epsilon^{1-\sigma_j}\right)^{\beta_j}$$
$$= \|g\|_{r,q}^{k} + \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^k\\ \alpha_j+\beta_j=1\ \forall j\\ \boldsymbol{\beta}\neq\mathbf{0}}}\prod_{j=1}^{k}\|g\|_{r,q}^{\alpha_j}\left(|a|\epsilon^{1-\sigma_j}\right)^{\beta_j}. \tag{E.1.6}$$

We focus on the second term. First, because $0 \le \sum_{j=1}^{k}\alpha_j \le k$, we have

$$\prod_{j=1}^{k}\|g\|_{r,q}^{\alpha_j} = \|g\|_{r,q}^{\sum_{j=1}^{k}\alpha_j} \le \sum_{j=0}^{k}\|g\|_{r,q}^{j} =: \mathcal{M}_1[g]. \tag{E.1.7}$$

Next, because $|a| < 1$ and $\epsilon < 1$, we have

$$\prod_{j=1}^{k}\left(|a|\epsilon^{1-\sigma_j}\right)^{\beta_j} \le |a|\epsilon\prod_{j=1}^{k}\epsilon^{-\sigma_j\beta_j} = |a|\epsilon\epsilon^{-\sum_{j=1}^{k}\sigma_j\beta_j}.$$

Since $0 \le \sigma_j\beta_j \le \sigma_j$, this in turn becomes

$$|a|\epsilon\epsilon^{-\sum_{j=1}^{k}\sigma_j\beta_j} \le |a|\epsilon\epsilon^{-\sum_{j=1}^{k}\sigma_j} = |a|\epsilon\epsilon^{|\boldsymbol{\sigma}|} = |a|\epsilon^{1-r}. \tag{E.1.8}$$

Here we have used the stipulation $|\boldsymbol{\sigma}| = r$ from Remark A.2.1.

All together, (E.1.7) and (E.1.8) imply

$$\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^k\\ \alpha_j+\beta_j=1\ \forall j\\ \boldsymbol{\beta}\neq\mathbf{0}}}\prod_{j=1}^{k}\|g\|_{r,q}^{\alpha_j}\left(|a|\epsilon^{1-\sigma_j}\right)^{\beta_j} \le \mathcal{M}_1[g]\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^k\\ \alpha_j+\beta_j=1\ \forall j\\ \boldsymbol{\beta}\neq\mathbf{0}}}|a|\epsilon^{1-r} \le r^2\mathcal{M}_1[g]|a|\epsilon^{1-r},$$

and so (E.1.5) and (E.1.6) imply

$$\left\| \prod_{j=1}^{k} \epsilon \partial_X^{\sigma_j}[g + a\varphi] \right\|_{L^\infty} \leq \mathcal{M}_1[g] + r^2 \mathcal{M}_1[g]|a|\epsilon^{1-r} \leq \mathcal{M}_2[g]\left(1 + |a|\epsilon^{1-r}\right), \qquad \text{(E.1.9)}$$

where $\mathcal{M}_2[g] = r^2\mathcal{M}_1[g]$. Returning to (E.1.4), we find

$$\begin{aligned}\|\cosh(q\cdot) f \partial_X^r[N(\epsilon(g + a\varphi))]\|_{L^2} &\leq C_r \|N\|_{W^{r,\infty}(\mathcal{I}(g))} \mathcal{M}_0[g]\left(1 + |a|\epsilon^{1-r}\right)\|f\|_{r,q} \\ &\quad + C_r \|N\|_{W^{r,\infty}(\mathcal{I}(g))} \sum_{k=2}^{r} \sum_{\boldsymbol{\sigma} \in \Sigma_k^r} \|\cosh(q\cdot) f\|_{L^2} \mathcal{M}_2[g]\left(1 + |a|\epsilon^{1-r}\right) \\ &\leq \underbrace{C_r \|N\|_{W^{r,\infty}(\mathcal{I}(g))} \mathcal{M}_2[g]}_{\mathcal{M}[g]}\left(1 + |a|\epsilon^{1-r}\right)\|f\|_{r,q}.\end{aligned}$$

This estimate has the same form as the right side of (E.1.1), so we have completed our work on $\|\cosh(q\cdot) f \partial_X^r[N(\epsilon(g + a\varphi))]\|_{L^2}$. To treat the other terms

$$\|\cosh(q\cdot)\partial_X^m[f]\partial_X^n[N(\epsilon(g + a\varphi))]\|_{L^2}$$

where $0 \leq m, n \leq r, m + n \leq r$, and $(m, n) \neq (0, r)$, note that these strictures on $m$ and $n$ imply $n \leq r-1$. So, when we expand $\partial_X^n[N(\epsilon(g+a\varphi))]$ with Faà di Bruno's rule, all derivatives $\partial_X^j[g + \varphi]$ will be $L^\infty$, while of course $\cosh(q\cdot)\partial_X^m[f]$ will be $L^2$. Then we proceed exactly as we did above in the long treatment of the case $2 \leq k \leq r$. ■

**E.2. A Lipschitz estimate in $H_q^r$.**

**E.2.1 Proposition.** *Let $r \in \mathbb{N}$, $N \in \mathcal{C}^{r+1}(\mathbb{R})$, and $C_\star, q > 0$. There exists a radially increasing map $\mathcal{M} \colon \mathbb{R}_+^2 \to \mathbb{R}_+$ with the following property. Let $a$, $\epsilon \in (0, 1)$. Suppose $f$, $\grave{f} \in H_q^r$ and $\varphi \in W^{r,\infty}$ with $\left\|\partial_X^j[\varphi]\right\|_{L^\infty} \leq C_\star \epsilon^{-j}$ for $j = 0, \ldots, r$. Then*

$$\left\|N(\epsilon(f+a\varphi)) - N(\epsilon(\grave{f}+a\varphi))\right\|_{r,q} \leq \mathcal{M}[\|f\|_{r,q}, \|\grave{f}\|_{r,q}]\left(1 + |a|\epsilon^{1-r}\right)\left\|f - \grave{f}\right\|_{r,q}. \quad \text{(E.2.1)}$$

*The function $\mathcal{M}$ depends on $N$, $r$, $q$, and $C_\star$ but not on $f$, $\grave{f}$, $\epsilon$, or $\varphi$.*

**Proof.** Following our notation in the proof of Proposition E.1.1, we will build $\mathcal{M}$ out of several functions $\mathcal{M}_0, \ldots, \mathcal{M}_5$, and suppress norms within these functions, writing, for example,

$$\mathcal{M}_0[\|f\|_{r,q}, \|\grave{f}\|_{r,q}] = \mathcal{M}_0[f, \grave{f}] \quad \text{and} \quad \mathcal{M}_1[\|f\|_{r,q}] = \mathcal{M}_1[f].$$

We have

$$N(\epsilon(f + a\varphi)) - N(\epsilon(\grave{f} + a\varphi)) = \underbrace{\cosh(q\cdot)\left(N(\epsilon(f + a\varphi)) - N(\epsilon(\grave{f} + a\varphi))\right)}_{\Delta_1}$$

$$+ \underbrace{\cosh(q\cdot)\partial_X^r\left[N(\epsilon(f + a\varphi)) - N(\epsilon(\grave{f} + a\varphi))\right]}_{\Delta_2}.$$

Since

$$|\epsilon(f(X) + a\varphi(X))| \leq \|f\|_{L^\infty} + \|\varphi\|_{L^\infty} \leq C_{r,q}\|f\|_{r,q} + C_\star,$$

for any $0 \leq k \leq r$ we have

$$\left\|\partial_X^k[N](\epsilon(f + a\varphi))\right\|_{L^\infty} \leq \|N\|_{W^{r+1,\infty}(\mathcal{I}(f,\grave{f}))} =: \mathcal{M}_0[f, \grave{f}], \tag{E.2.2}$$

where

$$\mathcal{I}(f, \grave{f}) = \left\{Y \in \mathbb{R} \;\middle|\; |Y| \leq C_{r,q}\|f\|_{r,q} + C_{r,q}\|\grave{f}\|_{r,q} + 2C_\star\right\}.$$

If we replace $f$ with $\grave{f}$ in (E.2.2), the same estimate still holds. We will use this estimate frequently throughout the rest of the proof, starting with a bound on $\Delta_1^2$:

$$\begin{aligned}\|\Delta_1\|_{L^2}^2 &= \int_{-\infty}^{\infty} \cosh^2(qX)\left|N(\epsilon(f(X) + a\varphi(X))) - N(\epsilon(\grave{f}(X) + a\varphi(X)))\right|^2\, dX \\ &\leq \|\partial_X[N]\|_{L^\infty(\mathcal{I}(f,\grave{f}))}^2 \int_{-\infty}^{\infty} \cosh^2(qX)|f(X) - \grave{f}(X)|^2\, dX \\ &\leq \|N\|_{W^{r+1,\infty}(\mathcal{I}(f,\grave{f}))}^2 \left\|f - \grave{f}\right\|_{r,q}^2.\end{aligned} \tag{E.2.3}$$

So, $\|\Delta_1\|_{L^2}$ has the bound

$$\|\Delta_1\|_{L^2} \leq \mathcal{M}_0[f, \grave{f}]\left\|f - \grave{f}\right\|_{r,q}. \tag{E.2.4}$$

To estimate $\Delta_2$, we first rewrite

$$\begin{aligned}
\Delta_2 &= \sum_{k=1}^{r} \partial_X^k[N](\epsilon(f+a\varphi)) \sum_{\boldsymbol{\sigma}\in\Sigma_k^r} C_{\boldsymbol{\sigma}} \prod_{j=1}^{k} \partial_X^{\sigma_j}[\epsilon(f+a\varphi)] \\
&\quad - \sum_{k=1}^{r} \partial_X^k[N](\epsilon(\grave{f}+a\varphi)) \sum_{\boldsymbol{\sigma}\in\Sigma_k^r} C_{\boldsymbol{\sigma}} \prod_{j=1}^{k} \partial_X^{\sigma_j}[\epsilon(\grave{f}+a\varphi)] \\
&\quad = \sum_{k=1}^{r} \underbrace{\left(\partial_X^k[N](\epsilon(f+a\varphi)) - \partial_X^k[N](\epsilon(\grave{f}+a\varphi))\right)}_{\Delta_3(k)} \sum_{\boldsymbol{\sigma}\in\Sigma_k^r} \underbrace{\prod_{j=1}^{k} \partial_X^{\sigma_j}[\epsilon(f+a\varphi)]}_{\Pi(\boldsymbol{\sigma})} \\
&\quad + \sum_{k=1}^{r} \partial_X^k[N](\epsilon(\grave{f}+a\varphi)) \sum_{\boldsymbol{\sigma}\in\Sigma_k^r} \underbrace{\left(\prod_{j=1}^{k} \partial_X^{\sigma_j}[\epsilon(f+a\varphi)] - \prod_{j=1}^{k} \partial_X^{\sigma_j}[\epsilon(\grave{f}+a\varphi)]\right)}_{\Delta_4(\boldsymbol{\sigma})}.
\end{aligned}$$

So, now we need to estimate

$$\|\Delta_2\|_{L^2} \le C_r \sum_{k=1}^{r} \sum_{\boldsymbol{\sigma}\in\Sigma_k^r} \underbrace{\|\cosh(q\cdot)\Delta_3(k)\Pi(\boldsymbol{\sigma})\|_{L^2}}_{\mathrm{T}_1(\boldsymbol{\sigma},k)} + \underbrace{\left\|\cosh(q\cdot)\partial_X^k[N](\epsilon(f+a\varphi))\Delta_4(\boldsymbol{\sigma})\right\|_{L^2}}_{\mathrm{T}_2(\boldsymbol{\sigma},k)}. \tag{E.2.5}$$

When $k=1$, $\boldsymbol{\sigma}=\{r\}$, and so

$$\begin{aligned}
\mathrm{T}_1(\{r\},1) &= \left\| \cosh(q\cdot)\left(\partial_X[N](\epsilon(f+a\varphi)) - \partial_X[N](\epsilon(f+a\varphi))\right) \partial_X^r[\epsilon(f+a\varphi)] \right\|_{L^2} \\
&\le \left\| \left(\partial_X[N](\epsilon(f+a\varphi)) - \partial_X[N](\epsilon(f+a\varphi))\right) \cosh(q\cdot)\partial_X^r[f] \right\|_{L^2} \\
&+ \left\| \cosh(q\cdot)\left(\partial_X[N](\epsilon(f+a\varphi)) - \partial_X[N](\epsilon(f+a\varphi))\right) \partial_X^r[\varphi] \right\|_{L^2} \\
&\le \left\|\partial_X[N](\epsilon(f+a\varphi)) - \partial_X[N](\epsilon(f+a\varphi))\right\|_{L^\infty} \|\cosh(q\cdot)\partial_X^r[f]\|_{L^2} \\
&+ \left\| \cosh(q\cdot)\left(\partial_X[N](\epsilon(f+a\varphi)) - \partial_X[N](\epsilon(f+a\varphi))\right) \right\|_{L^2} \|\partial_X^r[\varphi]\|_{L^\infty}
\end{aligned}$$

$$\leq \|N\|_{W^{r+1,\infty}(\mathcal{I}(f,\grave{f}))}\|f-\grave{f}\|_{r,q}\|f\|_{r,q}$$
$$+\|N\|_{W^{r+1,\infty}(\mathcal{I}(f,\grave{f}))}\|f-\grave{f}\|_{r,q}C_\star\epsilon^{-r}$$

$$\leq \mathcal{M}_0[f,\grave{f}](1+|a|\epsilon^{1-r})\|f-\grave{f}\|_{r,q}.$$

Similarly, $\mathrm{T}_2(\{r\},1)$ is bounded by

$$\|\cosh(q\cdot)\partial_X[N](\epsilon(f+a\varphi))\Delta_4(r)\|_{L^2}$$

$$= \|\cosh(q\cdot)\partial_X[N](\epsilon(f+a\varphi))\left(\partial_X^r[\epsilon(f+a\varphi)]-\partial_X^r[\epsilon(f+a\varphi)]\right)\|_{L^2}$$

$$\leq \|\partial_X[N](\epsilon(f+a\varphi))\|_{L^\infty}\|\cosh(q\cdot)\partial_X[f-\grave{f}]\|_{L^2}$$

$$\leq \mathcal{M}_0[f,\grave{f}](1+|a|\epsilon^{1-r})\|f-\grave{f}\|_{r,q}.$$

Now let $2\leq k\leq r$. Since all of the derivatives $\partial_X^{\sigma_j}[f]$ are at most order $r-1$ when $\boldsymbol{\sigma}\in\Sigma_k^r$, they are $L^\infty$. So we estimate

$$\mathrm{T}_1(\boldsymbol{\sigma},k)\leq \|\cosh(q\cdot)\left(\partial_X^k[N](\epsilon(f+a\varphi))-\partial_X^k[N](\epsilon(f+a\varphi))\right)\|_{L^2}$$
$$\times\prod_{j=1}^k\|\partial_X^{\sigma_j}[\epsilon(f+a\varphi)]\|_{L^\infty}$$

We bound the $L^2$ factor above using (E.2.2):

$$\|\cosh(q\cdot)\left(\partial_X^k[N](\epsilon(f+a\varphi))-\partial_X^k[N](\epsilon(f+a\varphi))\right)\|_{L^2}\leq \mathcal{M}_0[f,\grave{f}]\|f-\grave{f}\|_{r,q}.$$

And we bound the product using *exactly* the same reasoning that led to the estimate (E.1.9) in the proof of Proposition E.1.1. That is, we find

$$\prod_{j=1}^k\|\partial_X^{\sigma_j}[\epsilon(f+a\varphi)]\|_{L^\infty}\leq \mathcal{M}_1[f]\left(1+|a|\epsilon^{1-r}\right),$$

where $\mathcal{M}_1\colon\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function that depends on $r$, $q$, and $C_\star$, but not on $a$, $\epsilon$, $\varphi$, or $f$. Thus

$$\mathrm{T}_1(\boldsymbol{\sigma},k)\leq \mathcal{M}_0[f,\grave{f}]\mathcal{M}_1[f]\left(1+|a|\epsilon^{1-r}\right)\|f-\grave{f}\|_{r,q} \tag{E.2.6}$$

for $2 \leq k \leq r$ and $\boldsymbol{\sigma} \in \Sigma_k^r$.

Next, a first glance at $\mathrm{T}_2(\boldsymbol{\sigma}, k)$ shows

$$\mathrm{T}_2(\boldsymbol{\sigma}, k) \leq \mathcal{M}_0[f, \grave{f}] \, \|\Delta_4(\boldsymbol{\sigma})\|_{L^2} \, .$$

We use Lemma E.6.2 to rewrite

$$\begin{aligned}
\Delta_4(\boldsymbol{\sigma}) &= \prod_{j=1}^{k} \partial_X^{\sigma_j}[\epsilon(f + a\varphi)] - \prod_{j=1}^{k} \partial_X^{\sigma_j}[\epsilon(f + a\varphi)] \\
&= \sum_{j=1}^{k} \underbrace{\left(\partial_X^{\sigma_j}[\epsilon(f + a\varphi)] - \partial_X^{\sigma_j}[\epsilon(f + a\varphi)]\right)}_{\Delta_5(\sigma_j)} \underbrace{\prod_{\ell=1}^{j-1} \partial_X^{\sigma_\ell}[\epsilon(f + a\varphi)]}_{\Pi_{j-1}(\boldsymbol{\sigma})} \underbrace{\prod_{\ell=j+1}^{k} \partial_X^{\sigma_\ell}[\epsilon(f + a\varphi)]}_{\Pi^{j+1}(\boldsymbol{\sigma})} .
\end{aligned} \tag{E.2.7}$$

We easily estimate

$$\|\cosh(q\cdot)\Delta_5(\sigma_j)\|_{L^2} \leq \mathcal{M}_0[f, \grave{f}] \big\| f - \grave{f} \big\|_{r,q}.$$

The products $\Pi_{j-1}(\boldsymbol{\sigma})$ and $\Pi^{j+1}(\boldsymbol{\sigma})$ require a little more care. Since the derivatives $\partial_X^{\sigma_j}[f]$ and $\partial_X^{j}[\grave{f}]$ are still order at most $r-1$, these products are $L^\infty$. More precisely, we can carefully replicate the steps that led to the estimate (E.1.9) to find

$$\|\Pi_{j-1}(\boldsymbol{\sigma})\|_{L^\infty} \leq \mathcal{M}_2[f] \left(1 + |a|\epsilon^{1 - \sum_{\ell=1}^{j-1} \sigma_\ell}\right)$$

and

$$\big\|\Pi^{j+1}(\boldsymbol{\sigma})\big\|_{L^\infty} \leq \mathcal{M}_3[\grave{f}] \left(1 + |a|\epsilon^{1 - \sum_{\ell=j+1}^{k} \sigma_j}\right) .$$

Multiplying these estimates and using, as always, the assumptions $0 < \epsilon < 1$ and $|a| < 1$ and the relation $r = |\boldsymbol{\sigma}| = \sum_{j=1}^{k} \sigma_j$, we find

$$\big\|\Pi_{j-1}(\boldsymbol{\sigma})\Pi^{j+1}(\boldsymbol{\sigma})\big\|_{L^\infty} \leq \mathcal{M}_4[f, \grave{f}] \left(1 + |a|\epsilon^{1-r}\right) . \tag{E.2.8}$$

Then (E.2.7) and (E.2.8) together yield

$$\mathrm{T}_2(\boldsymbol{\sigma}, k) \leq r\mathcal{M}_0[f, \grave{f}]^2 \mathcal{M}_3[f, \grave{f}] \left(1 + |a|\epsilon^{1-r}\right) \big\| f - \grave{f} \big\|_{r,q}. \tag{E.2.9}$$

Now that we have bounded both $\mathrm{T}_1(\boldsymbol{\sigma}, k)$ and $\mathrm{T}_2(\boldsymbol{\sigma}_k, k)$ for $k = 2, \ldots, r$ and $\boldsymbol{\sigma} \in \Sigma_k^r$, we can use (E.2.6) and (E.2.9) to our estimate (E.2.5) for $\Delta_2$ and conclude

$$\|\Delta_2\|_{L^2} \leq \mathcal{M}_5[f, \grave{f}]\left(1+|a|\epsilon^{1-r}\right)\left\|f-\grave{f}\right\|_{r,q}.$$

for some continuous map $\mathcal{M}_5 \colon \mathbb{R}_+^2 \to \mathbb{R}_+$ that is independent of $\epsilon$, $|a|$, and $\varphi$ (but dependent on $r$ and $q$). This, together with (E.2.4), gives (E.2.1). ■

By taking $\varphi = 0$ and $\epsilon = a = 1/2$ in Proposition E.2.1, we obtain the following simpler estimate.

**E.2.2 Corollary.** *Let $N \in \mathcal{C}^{r+1}(\mathbb{R})$. There exists a radially increasing map $\mathcal{M} \colon \mathbb{R}_+^4 \to \mathbb{R}_+$ such that*

$$\left\|N(f)-N(\grave{f})\right\|_{r,q} \leq \mathcal{M}[\|f\|_{r,q}, \left\|\grave{f}\right\|_{r,q}, \|\varphi\|_{W^{r,\infty}}, \|\grave{\varphi}\|_{W^{r,\infty}}]\left\|f-\grave{f}\right\|_{r,q}$$

*for all $f, \grave{f} \in H_q^r$.*

## E.3. A Lipschitz estimate in $H_q^1$.

**E.3.1 Proposition.** *Let $N \in \mathcal{C}^1(\mathbb{R})$. There exists a radially increasing map $\mathbb{R}_+^4 \to \mathbb{R}_+$ such that*

$$\left\|(f-\grave{f})(g+\varphi)N(h+\grave{\varphi})\right\|_{1,q} \leq \mathcal{M}[\|g\|_{1,q}, \|h\|_{1,q}, \|\varphi\|_{W^{1,\infty}}, \|\grave{\varphi}\|_{W^{1,\infty}}]\left\|f-\grave{f}\right\|_{1,q}$$

*for all $f$, $\grave{f}$, $g$, $h \in H_q^1$ and $\varphi$, $\grave{\varphi} \in W^{1,\infty}$.*

**Proof.** This is a straightforward calculation that requires only one pass with the chain rule, so we omit the details. ■

**E.4. Lipschitz estimates in $W^{1,\infty}$.** The estimates in this section are much simpler than the preceding $H_q^r$ mapping and Lipschitz estimates because we work only with $r = 1$ and so do not need to keep careful track of powers of $\epsilon$. However, we do need a "decay borrowing" product estimate, which we take from Lemma A.2 in [FW18]: for $r \geq 0$ and $q > 0$, we have

$$\|fg\|_{r,q/2} \leq C_r\|f\|_{r,q}\left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)g\right\|_{W^{r,\infty}}. \tag{E.4.1}$$

**E.4.1 Proposition.** *Let $N \in \mathcal{C}^2(\mathbb{R})$. There exists a radially increasing map $\mathcal{M}\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ such that*

$$\|f \cdot (N(g+\varphi) - N(g+\grave{\varphi}))\|_{1,q/2} \leq \mathcal{M}[g,\varphi,\grave{\varphi}] \left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(\varphi - \grave{\varphi})\right\|_{W^{1,\infty}} \|f\|_{1,q} \tag{E.4.2}$$

*for all $f$, $g \in H_q^1$ and $\varphi$, $\grave{\varphi} \in W^{1,\infty}$.*

**Proof.** We have

$$\|f \cdot (N(g+\varphi) - N(g+\grave{\varphi}))\|_{1,q/2} = \underbrace{\left\|\cosh\left(\frac{q}{2}\cdot\right) f \cdot (N(g+\varphi) - N(g+\grave{\varphi}))\right\|_{L^2}}_{\Delta_1}$$

$$+ \underbrace{\left\|\cosh\left(\frac{q}{2}\cdot\right) \partial_X[f \cdot (N(g+\varphi) - N(g+\grave{\varphi}))]\right\|_{L^2}}_{\Delta_2}.$$

The first estimate for $\Delta_1^2$ is similar to that for $\Delta_1^2$ in the proof of Proposition E.2.1, i.e., a direct calculation with the integral yields

$$\Delta_1 \leq \|N\|_{W^{2,\infty}(\mathcal{I}(g,\varphi,\grave{\varphi}))} \left\|\cosh\left(\frac{q}{2}\cdot\right) f(\varphi - \grave{\varphi})\right\|_{L^2}.$$

Next, we use the decay-borrowing estimate (E.4.1):

$$\begin{aligned}
\Delta_1 &\leq \|N\|_{W^{2,\infty}(\mathcal{I}(g,\varphi,\grave{\varphi}))} \|f(\varphi - \grave{\varphi})\|_{0,q/2} \\
&\leq C \|N\|_{W^{2,\infty}(\mathcal{I}(g,\varphi,\grave{\varphi}))} \|f\|_{0,q} \left\|\operatorname{sech}\left(\left|q - \frac{q}{2}\right| \cdot\right)(\varphi - \grave{\varphi})\right\|_{W^{0,\infty}} \\
&\leq C \|N\|_{W^{2,\infty}(\mathcal{I}(g,\varphi,\grave{\varphi}))} \|f\|_{1,q} \left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(\varphi - \grave{\varphi})\right\|_{W^{1,\infty}}.
\end{aligned}$$

This is exactly the kind of estimate we want for $\Delta_1$, and so we move on to $\Delta_2$. Since we only ever take one derivative in this proof, we write $N' = \partial_X[N]$, etc. We have

$$\Delta_2 \leq \underbrace{\left\|\cosh\left(\frac{q}{2}\cdot\right) f' \cdot (N(g+\varphi) - N(g+\grave{\varphi}))\right\|_{L^2}}_{\Delta_3}$$

$$+\underbrace{\left\|\cosh\left(\frac{q}{2}\cdot\right)f\cdot\left(N'(g+\varphi)(g'+\varphi')-N'(g+\grave{\varphi})(g'+\grave{\varphi}')\right)\right\|_{L^2}}_{\Delta_4}.$$

We get an estimate on $\Delta_3$ of the form in (E.4.2) by working directly with the integral, and we have

$$\Delta_4\leq\underbrace{\left\|\cosh\left(\frac{q}{2}\cdot\right)fN'(g+\varphi)(\varphi'-\grave{\varphi}')\right\|_{L^2}}_{\Delta_5}$$

$$+\underbrace{\left\|\cosh\left(\frac{q}{2}\cdot\right)f\cdot(N'(g+\varphi)-N'(g+\grave{\varphi}))g'\right\|_{L^2}}_{\Delta_6}$$

$$+\underbrace{\left\|\cosh\left(\frac{q}{2}\cdot\right)f\cdot(N'(g+\varphi)-N'(g+\grave{\varphi}))\grave{\varphi}'\right\|_{L^2}}_{\Delta_7}.$$

For $\Delta_5$, we have

$$\begin{aligned}\Delta_5&=\|fN'(g+\varphi)(\varphi'-\grave{\varphi}')\|_{0,q/2}\\&\leq\|N'(g+\varphi)\|_{L^\infty}\|f(\varphi'-\grave{\varphi}')\|_{0,q/2}\\&\leq C\|N\|_{W^{2,\infty}(\mathcal{I}(g,\varphi,\grave{\varphi}))}\|f\|_{0,q}\left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(\varphi'-\grave{\varphi}')\right\|_{W^{1,\infty}}.\end{aligned}$$

For $\Delta_6$, we have

$$\begin{aligned}\Delta_6&\leq\|\cosh(q\cdot)f\|_{L^\infty}\|(N'(g+\varphi)-N'(g+\grave{\varphi}))g'\|_{L^2}\\&=\|\cosh(q\cdot)f\|_{L^\infty}\|(N'(g+\varphi)-N'(g+\grave{\varphi}))g'\|_{0,0}\\&\leq C_q\|f\|_{1,q}\|g'\|_{0,q/2}\left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(N'(g+\varphi)-N'(g+\grave{\varphi}))\right\|_{W^{0,\infty}}\\&\leq C_q\|f\|_{1,q}\|g\|_{1,q}\|N\|_{W^{2,\infty}(\mathcal{I}(g,\varphi,\grave{\varphi}))}\left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(\varphi-\grave{\varphi})\right\|_{W^{1,\infty}}.\end{aligned}$$

For the last Lipschitz estimate on $N'$, we just work pointwise to get the estimate in $W^{0,\infty} = L^\infty$.

Finally, for $\Delta_7$, we use the same techniques as above to produce

$$\begin{aligned}\Delta_7 &\leq \|\dot{\varphi}'\|_{L^\infty} \|f(N'(g+\varphi) - N'(g+\dot{\varphi}))\|_{0,q/2} \\ &\leq \|\dot{\varphi}\|_{W^{1,\infty}} \|f\|_{0,q} \left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(N'(g+\varphi) - N'(g+\dot{\varphi}))\right\|_{W^{0,\infty}} \\ &\leq \|\dot{\varphi}\|_{W^{1,\infty}} \|f\|_{1,q} \|N\|_{W^{2,\infty}(\mathcal{I}(g,\varphi,\dot{\varphi}))} \left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(\varphi - \dot{\varphi})\right\|_{W^{1,\infty}}.\end{aligned}$$

Combining the estimates on $\Delta_1$ through $\Delta_7$, we have (E.4.2). ■

**E.4.2 Proposition.** *Let $N \in \mathcal{C}^3(\mathbb{R})$. There exists a radially increasing map $\mathcal{M}\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ such that*

$$\begin{aligned}&\|(N(f+\varphi) - N(\varphi)) - (N(f+\dot{\varphi}) - N(\dot{\varphi}))\|_{1,q/2} \\ &\qquad\qquad \leq \mathcal{M}[f,\varphi,\dot{\varphi}] \left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(\varphi - \dot{\varphi})\right\|_{W^{1,\infty}} \|f\|_{1,q}.\end{aligned}$$

**Proof.** We use the fundamental theorem of calculus twice to rewrite

$$(N(f+\varphi)-N(\varphi))-(N(f+\dot{\varphi})-N(\dot{\varphi})) = f(\varphi-\dot{\varphi}) \int_0^1 \int_0^1 N''(tf+\dot{\varphi}+s(\varphi-\dot{\varphi}))\, ds\, dt.$$

The necessary estimates are then similar to those in Proposition E.4.1. Of these estimates, arguably the most complicated involves controlling the $L^2$-norm of

$$\cosh\left(\frac{q}{2}\cdot\right) f(\varphi - \dot{\varphi})\partial_X \left[\int_0^1 \int_0^1 N''(tf + \dot{\varphi} + s(\varphi - \dot{\varphi}))\, ds\, dt\right]. \tag{E.4.3}$$

After differentiating under the integral in (E.4.3), the terms that we need to bound are by now routine; we remove a number of factors in the $L^\infty$-norm and then use decay borrowing on the rest. For example, one of these terms is

$$\Delta_{\text{hard}} = \cosh\left(\frac{q}{2}\cdot\right) f(\varphi - \grave{\varphi})(\varphi' - \grave{\varphi}') \underbrace{\int_0^1 \int_0^1 |N'''(tf + \grave{\varphi} + s(\varphi - \grave{\varphi}))| \, ds \, dt}_{\mathcal{I}},$$

and we have

$$\begin{aligned}\|\Delta_{\text{hard}}\|_{L^2} &\le \|\varphi - \grave{\varphi}\|_{L^\infty} \|\mathcal{I}\|_{L^\infty} \left\|\cosh\left(\frac{q}{2}\cdot\right) f(\varphi' - \grave{\varphi}')\right\|_{L^2} \\ &\le (\|\varphi\|_{L^\infty} + \|\grave{\varphi}\|_{L^\infty}) \|\mathcal{I}\|_{L^\infty} \|f\|_{1,q} \left\|\operatorname{sech}\left(\frac{q}{2}\cdot\right)(\varphi' - \grave{\varphi}')\right\|_{W^{0,\infty}}.\end{aligned}$$

We omit the other details, as they are by now routine. ■

**E.5. Estimates in $H^r_{\text{per}}$.** All of our estimates so far have involved the space $H^r_q$. That is, we have proved estimates for functions that are square-integrable on all of $\mathbb{R}$. However, none of our proofs relied in an essential way on the domain of integration being $\mathbb{R}$, and so we can replace $\mathbb{R}$ with $[0, 2\pi]$ and find that our proofs are still valid for the space $H^r_{\text{per}}$. Specifically, by taking $q = a = 0$, $\varphi = 0$, and $\epsilon = 1/2$, we can rerun the proofs of Propositions E.1.1 and E.2.1 in $H^r_{\text{per}}$ to obtain the following.

**E.5.1 Proposition.** *Let $r \ge 1$ and $N \in \mathcal{C}^r([0, 2\pi])$. There exists an increasing map $\mathcal{M}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|N(f)\|_{H^r_{\text{per}}} \le \mathcal{M}[\|f\|_{H^r_{\text{per}}}]$$

*for all $f \in H^r_{\text{per}}$.*

**E.5.2 Proposition.** *Let $r \ge 1$ and $N \in \mathcal{C}^{r+1}([0, 2\pi])$. There exists a radially increasing map $\mathcal{M}\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ such that*

$$\|N(f) - N(\grave{f})\|_{H^r_{\text{per}}} \le \mathcal{M}[\|f\|_{H^r_{\text{per}}}, \|\grave{f}\|_{H^r_{\text{per}}}]\|f - \grave{f}\|_{H^r_{\text{per}}}.$$

**E.6. Auxiliary identities for sums and products.**

**E.6.1 Lemma.** *Let $\{A_j\}_{j=1}^r, \{B_j\}_{j=1}^r \subseteq \mathbb{R}$. Then*

$$\prod_{j=1}^{r}(A_j+B_j)=\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^r\\ \alpha_j+\beta_j=1,\ j=1,\ldots,r}}\prod_{i=1}^{r}A_i^{\alpha_i}B_i^{\beta_i}$$

**Proof.** We induct on $r$. If $r=1$, then

$$\sum_{\substack{\alpha_1,\beta_1\in\{0,1\}\\ \alpha_1+\beta_1=1}}A_1^{\alpha_1}B_1^{\alpha_1}=A_1^1B_1^0+A_1^0B_1^1=A_1+B_1.$$

Suppose the formula holds for some $r\geq 1$. Then

$$\prod_{j=1}^{r+1}(A_j+B_j)=(A_{r+1}+B_{r+1})\prod_{j=1}^{r}(A_j+B_j)^r$$

$$=(A_{r+1}+B_{r+1})\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^r\\ \alpha_j+\beta_j=1,\ j=1,\ldots,r}}\prod_{i=1}^{r}A_i^{\alpha_i}B_i^{\beta_i}$$

$$=\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^r\\ \alpha_j+\beta_j=1,\ j=1,\ldots,r}}\prod_{i=1}^{r}A_i^{\alpha_i}B_i^{\beta_i}A_{r+1}+\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^r\\ \alpha_j+\beta_j=1,\ j=1,\ldots,r}}\prod_{i=1}^{r}A_i^{\alpha_i}B_i^{\beta_i}B_{r+1}$$

$$=\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^{r+1}\\ \alpha_j+\beta_j=1,\ j=1,\ldots,r+1}}\prod_{i=1}^{r+1}A_i^{\alpha_i}B_i^{\beta_i}.$$

■

**E.6.2 Lemma.** *Let* $z_1,\ldots,z_n,\grave{z}_1,\ldots,\grave{z}_n\in\mathbb{C}$. *Then*

$$\prod_{k=1}^{n}z_k-\prod_{k=1}^{n}\grave{z}_k=\sum_{k=1}^{n}(z_k-\grave{z}_k)\prod_{j=1}^{k-1}\grave{z}_j\prod_{j=k+1}^{n}z_j.$$

**Proof.** When $n=1$ it is obvious, so assume it holds for some $n$ and consider the $n+1$ case:

$$\prod_{k=1}^{n+1}z_k-\prod_{k=1}^{n+1}\grave{z}_k=z_{n+1}\prod_{k=1}^{n}z_k-\grave{z}_{n+1}\prod_{k=1}^{n}\grave{z}_k$$

$$=z_{n+1}\prod_{k=1}^{n}z_k-z_{n+1}\prod_{k=1}^{n}\grave{z}_k+z_{n+1}\prod_{k=1}^{n}\grave{z}_k-\grave{z}_{k+1}\prod_{k=1}^{n}\grave{z}_k$$

$$
\begin{aligned}
&= z_{n+1}\left(\prod_{k=1}^{n} z_k - \prod_{k=1}^{n} \grave{z}_k\right) + (z_{n+1} - \grave{z}_{n+1})\prod_{k=1}^{n} \grave{z}_k \\
&= z_{n+1}\sum_{k=1}^{n}(z_k - \grave{z}_k)\left(\prod_{j=1}^{k-1} \grave{z}_j\right)\left(\prod_{j=k+1}^{n} z_j\right) + (z_{n+1} - \grave{z}_{n+1})\prod_{k=1}^{n} \grave{z}_k \\
&= \sum_{k=1}^{n}(z_k - \grave{z}_k)\left(\prod_{j=1}^{k-1} \grave{z}_j\right)\left(\prod_{j=k+1}^{n+1} z_j\right) + (z_{n+1} - \grave{z}_{n+1})\prod_{k=1}^{n} \grave{z}_k \\
&= \sum_{k=1}^{n+1}(z_k - \grave{z}_k)\left(\prod_{j=1}^{k-1} \grave{z}_k\right)\left(\prod_{j=k+1}^{n+1} z_j\right). \qquad \blacksquare
\end{aligned}
$$

## Appendix F. Assorted Proofs

**F.1. Proof of Theorem 1.4.1.** The traveling wave ansatz (2.1.1) with wave speed $c = c_\epsilon = \sqrt{c_\varkappa^2 + \epsilon^2}$ gave

$$r_j(t) = \begin{cases} p_1(j - c_\epsilon t), & j \text{ is odd} \\ p_2(j - c_\epsilon t), & j \text{ is even,} \end{cases}$$

and the change of variables (2.2.10) and the long wave scaling (2.4.1) converted $\mathbf{p} = (p_1, p_2)$ into

$$\mathbf{p}(x) = (J\mathbf{h})(x) = \epsilon^2(J\boldsymbol{\theta}(\epsilon\cdot))(x) = \epsilon^2(J^\epsilon\boldsymbol{\theta})(\epsilon x).$$

With $\boldsymbol{\theta} = \boldsymbol{\sigma} + a_\epsilon\boldsymbol{\varphi}_\epsilon^{a_\epsilon} + \boldsymbol{\eta}_\epsilon$ from Beale's ansatz (4.1.1) and Theorem 4.3.2, we find

$$\mathbf{p}(x) = \epsilon^2(J^\epsilon\boldsymbol{\sigma})(\epsilon x) + \epsilon^2 a_\epsilon(J^\epsilon\boldsymbol{\varphi}_\epsilon^{a_\epsilon})(\epsilon x) + \epsilon^2(J^\epsilon\boldsymbol{\eta}_\epsilon)(\epsilon x).$$

We now want to isolate what will be the lowest order term in $\epsilon$. With $J^0$ defined in (2.5.3), we have

$$\mathbf{p}(x) = \epsilon^2(J^0\boldsymbol{\sigma})(\epsilon x) + \epsilon^2 a_\epsilon(J^\epsilon\boldsymbol{\varphi}_\epsilon^{a_\epsilon})(\epsilon x) + \epsilon^2\left((J^\epsilon\boldsymbol{\eta}_\epsilon)(\epsilon x) + ((J^\epsilon - J^0)\boldsymbol{\sigma})(\epsilon x)\right),$$

where

$$\epsilon^2\left\|a_\epsilon(J^\epsilon\boldsymbol{\varphi}_\epsilon^{a_\epsilon})(\epsilon\cdot)\right\|_{W^{r,\infty}} \le \epsilon^2(C_r\epsilon^r)\epsilon^r\left\|J^\epsilon\boldsymbol{\varphi}_\epsilon^{a_\epsilon}\right\|_{W^{r,\infty}} \le C_r\epsilon^{r+2} \tag{F.1.1}$$

and

$$\epsilon^2\left\|J^\epsilon\boldsymbol{\eta}_\epsilon + (J^\epsilon - J^0)\boldsymbol{\sigma}\right\|_{r,q_\star} \le C_r\epsilon^3. \tag{F.1.2}$$

by the estimates in Theorem 4.3.2 and (G.2.7). We abbreviate

$$\begin{pmatrix} p_1^\epsilon(x) \\ p_2^\epsilon(x) \end{pmatrix} := \epsilon^2 a_\epsilon(J^\epsilon\boldsymbol{\varphi}_\epsilon^{a_\epsilon})(\epsilon x) \quad \text{and} \quad \begin{pmatrix} v_1^\epsilon(x) \\ v_2^\epsilon(x) \end{pmatrix} := \epsilon^2\left((J^\epsilon\boldsymbol{\eta}_\epsilon)(x) + ((J^\epsilon - J^0)\boldsymbol{\sigma})(x)\right).$$

We get the estimates for $p_j$ and $v_j$ from (F.1.1) and (F.1.2). For the period of $p_j$, observe that by Theorem 3.1.1

$$(J^\epsilon\boldsymbol{\varphi}_\epsilon^{a_\epsilon})(\epsilon x) = (J^\epsilon\boldsymbol{\nu}(\omega_\epsilon^{a_\epsilon}\cdot))(\epsilon x) + (J^\epsilon\boldsymbol{\psi}_\epsilon^{a_\epsilon}(\omega_\epsilon^{a_\epsilon}\cdot))(\epsilon x) = (J^\epsilon\boldsymbol{\nu})(\epsilon\omega_\epsilon^{a_\epsilon}x) + (J^\epsilon\boldsymbol{\psi}_\epsilon^{a_\epsilon})(\epsilon\omega_\epsilon^{a_\epsilon}x),$$

where $\epsilon\omega_\epsilon^{a_\epsilon}$ is uniformly bounded in $\epsilon$ and $\boldsymbol{\nu}$ and $\boldsymbol{\psi}_\epsilon^{a_\epsilon}$ are $2\pi$-periodic.

Finally,

$$J^0\boldsymbol{\sigma} = \begin{bmatrix} 1/\varkappa & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} \sigma \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma/\varkappa \\ \sigma \end{pmatrix},$$

and so for $j$ odd we have

$$r_j(t) = \epsilon^2 \frac{1}{\varkappa}\sigma(\epsilon(j - c_\epsilon t)) + v_1^\epsilon(\epsilon(j - c_\epsilon t)) + p_1^\epsilon(j - c_\epsilon t),$$

while for $j$ even,

$$r_j(t) = \epsilon^2\sigma(\epsilon(j - c_\epsilon t)) + v_2^\epsilon(\epsilon(j - c_\epsilon t)) + p_2^\epsilon(j - c_\epsilon t).$$

**F.2. Consistency of Theorems 1.4.2 and 1.4.1 with [GMWZ14].** We stated the consistency of Theorem 1.4.2 with the homogenization-based estimates of [GMWZ14] as Remark 3.3 in [FW18]. Here we discuss in greater detail the spring dimer estimates from Theorem 1.4.1.

Making the traveling wave ansatz

$$U_\pm(X,T) = V\left(X \pm \frac{T}{2c_\varkappa}\right)$$

rearranging, and integrating, (1.5.2) in the case $w = 1$ becomes

$$\alpha_\varkappa V'' - V + c_\varkappa^2 \frac{\beta + \varkappa^3}{\varkappa^2(1+\varkappa)} V^2 = 0,$$

which is precisely the ordinary differential equation (2.5.6) that we derived in our study of the formal long wave limit at $\epsilon = 0$. Hence $V = \sigma$ as defined in (2.5.8) and

$$U_\pm(X,T) = \sigma\left(X \pm \frac{T}{2c_\varkappa}\right),$$

and so in the approximation (1.5.1) for $r_j(t)$ we have

$$U_-(\epsilon(j - c_\varkappa t), \epsilon^3 t) = \sigma\left(\epsilon(j - c_\varkappa t) - \frac{\epsilon^3 t}{2c_\varkappa}\right) = \sigma\left(\epsilon\left(j - \left(c_\varkappa + \frac{1}{2c_\varkappa}\epsilon^2\right)t\right)\right).$$

Since

$$c_\epsilon = \sqrt{c_\varkappa^2 + \epsilon^2} = c_\varkappa + \frac{1}{2c_\varkappa}\epsilon^2 + \mathcal{O}(\epsilon^4)$$

and $\sigma' \in L^\infty$, we have (for $|t| \le T_0$)

$$\sigma\left(\epsilon\left(j-\left(c_{\varkappa}+\frac{1}{2c_{\varkappa}}\epsilon^2\right)t\right)\right)=\sigma(\epsilon(j-c_\epsilon t))+\mathcal{O}(\epsilon^5).$$

Last, since $\sigma \in L^\infty$, we have

$$\epsilon^2 U_+(\epsilon(j+c_{\varkappa}t),\epsilon^3 t)=\epsilon^2\sigma\left(\epsilon(j+c_{\varkappa}t)+\frac{\epsilon^3 t}{2c_{\varkappa}}\right)=\mathcal{O}(\epsilon^2).$$

All together, (1.5.1) becomes

$$r_j(t)=\frac{\epsilon^2}{\mathrm{K}_j}\sigma(\epsilon(j-c_\epsilon t))+\mathcal{O}(\epsilon^2),$$

which agrees with the expression for $r_j(t)$ in Theorem 1.4.1 and the subsequent estimates.

**F.3. Proof of Proposition 2.2.1.**

**(i)** *Analyticity of* $\widetilde{\varrho}$ *and* $\widetilde{\lambda}_\pm$ *on a strip.* Assume $w > 1$. We use the definition of $\cos(z)$, $z \in \mathbb{C}$, to compute, for $\tau \in \mathbb{R}$,

$$\begin{aligned}\mathrm{Re}((1+w)^2(1-\varkappa)^2+4\varkappa((1-w)^2+4w\cos^2(k+i\tau)))\\ =(1+w)^2(\varkappa^2+1)+2(w-1)^2\varkappa+8\varkappa w\cosh(2\tau)\cos(2k)\\ \geq(1+w)^2(\varkappa^2+1)+2(w-1)^2\varkappa-8\varkappa w\cosh(2\tau)=:f(\tau).\end{aligned}$$

We have

$$f(0)=(1+w)^2\varkappa^2+2(w^2-6w+1)\varkappa+(1+w)^2=:g(\varkappa).$$

That is, $g$ is quadratic in $\varkappa$, and the discriminant of $g$ is

$$-64w(w-1)^2$$

This is negative for all $w \neq 1$, so $g$ has constant sign. Since we are assuming $w > 1$ and the coefficient on $\varkappa^2$ is $(1+w)^2 > 0$, we conclude that $g$ is always positive.

Consequently, $f(0) > 0$, and so there is $\tau_1 > 0$ such that $f(\tau) > 0$ for all $|\tau| \leq \tau_1$. Thus $\mathrm{Re}((1+w)^2(1-\varkappa)^2+4\varkappa((1-w)^2+4w\cos^2(k+i\tau))) > 0$ for all $w > 1$, $\varkappa \neq 0$, $k \in \mathbb{R}$, and $|\tau| \leq \tau_1$. Taking the branch cut of the square root to be the negative real axis, we see that $\widetilde{\varrho}(z)$ is defined (and analytic) for all $z \in \overline{\Sigma}_{\tau_1}$. In particular, if $z$ is real (i.e, $z = k \in \mathbb{R}$ with $\tau = 0$ above), we have $\mathrm{Re}((1+w)^2(1-\varkappa)^2 + 4\varkappa((1-w)^2 + 4w\cos^2(k))) > 0$, and so $\widetilde{\varrho}(k)$ is real-valued.

By symmetry in $\varkappa$ and $w$, the same results hold if $\varkappa > 1$ and $w > 0$. We also see that $\widetilde{\lambda}_\pm$ are analytic on the same strip $\overline{\Sigma}_{\tau_1}$.

*Analyticity of $\widetilde{v}_\pm$ on a strip in the case $w \neq 1$.* This depends on the zeros of $f(z) = we^{iz} + e^{-iz}$ on a strip. First, we compute

$$f(k+i\tau) = we^{i(k+\tau)} + e^{-i(k+i\tau)} = (we^{-\tau} + e^{\tau})\cos(k) + i(we^{-\tau} - e^{\tau})\sin(k).$$

Then $f(k+i\tau) = 0$ if and only if both

$$\mathrm{Re}[f(k+i\tau)] = (we^{-\tau} + e^{\tau})\cos(k) = 0 \tag{F.3.1}$$

and

$$\mathrm{Im}[f(k+i\tau)] = (we^{-\tau} - e^{\tau})\sin(k). \tag{F.3.2}$$

Since $we^{-\tau} + e^{\tau} > 0$ for all $w > 0$ and $\tau \in \mathbb{R}$, we have (F.3.1) if and only if

$$k = \frac{2j+1}{2}\pi =: k_j \tag{F.3.3}$$

for some $j \in \mathbb{Z}$. In that case, (F.3.2) holds if and only if $we^{-\tau} - e^{\tau} = 0$. We can compute directly that this equation has the unique solution

$$\tau_w := \frac{1}{2}\ln(w).$$

Since $w \neq 1$, we have $\tau_w \neq 0$. We set $\tau_2 := |\tau_w|/2$ and conclude that if $z \in \overline{\Sigma}_{\tau_w}$, then $\mathrm{Im}(f(z)) \neq 0$, hence $f(z) \neq 0$. Consequently, we may divide by $f$ to obtain $\widetilde{\mathbf{v}}_\pm$ as analytic.

*Analyticity of $\widetilde{v}_\pm$ on a strip in the case $w = 1$.* We are using the definitions of $\widetilde{v}_\pm$ from (2.2.2). With $k_j$ from (F.3.3), we easily compute

$$2 - \widetilde{\lambda}_-(k_j) = 2\varkappa - \widetilde{\lambda}_+(k_j) = 0.$$

Since the zeros of $\cos(\cdot)$ are simple, we conclude that $\widetilde{v}_\pm$ have removable singularities at $k = k_j$. Consequently, they are analytic on the strip $\overline{\Sigma}_{\tau_1}$ from above.

*Analyticity of the eigenvector scalings $\widetilde{\gamma}_\pm$ from* (2.7.1). This follows from our proof in part (ii) that $we^{iz} + e^{-iz} \neq 0$ for $z \in \overline{\Sigma}_{\tau_2}$.

*Boundedness of $\widetilde{\varrho}$, $\widetilde{\lambda}_\pm$, $\widetilde{v}_\pm$, and $\widetilde{\gamma}_\pm$ on strips.* Set $q_0 = \min\{\tau_1, \tau_2\}$. By definition, each of these functions has the form of a function $f$ analytic on $\overline{\Sigma}_{q_0}$ with the property that $f(k + iy) = f(k + 2\pi + iy)$ for all $k$, $y \in \mathbb{R}$. Consequently,

$$\sup_{z \in \overline{\Sigma}_{q_0}} |f(z)| = \sup_{\substack{|k| \leq 2\pi \\ |y| \leq q_0}} |f(z)| < \infty,$$

by the compactness of $[0, 2\pi] \times [-q_0, q_0]$.

**(ii)** This is a direct computation.

**(iii)** These inequalities follow directly from the implication

$$\cos^2\left(\frac{\pi}{2}\right) = 0 \leq \cos^2(k) \leq 1 = \cos^2(0) \Longrightarrow \widetilde{\varrho}\left(\frac{\pi}{2}\right) \leq \widetilde{\varrho}(k) \leq \widetilde{\varrho}(0).$$

**(iv)** This is a direct computation.

**(v)** We have

$$|\widetilde{\lambda}_\pm'(k)| = \left|\frac{\widetilde{\varrho}'(k)}{2}\right| = \frac{8\varkappa w |\sin(k)\cos(k)|}{\widetilde{\varrho}(k)}. \tag{F.3.4}$$

*Proof of the inequality $|\widetilde{\lambda}_\pm'(k)| \leq 2\varkappa/(1+\varkappa)$.* This is equivalent to establishing

$$4w^2\cos^4(k) + 4w\left(\frac{4\varkappa}{(1+\varkappa)^2} - w\right)\cos^2(k) \\ + \frac{1}{(1+\varkappa)^2}\left[(1+w)^2(1-\varkappa)^2 + 4\varkappa(1-w)^2\right] \geq 0.$$

Taking $x = \cos^2(k)$, we just need to show

$$f(x) := 4w^2x^2 + 4w\left(\frac{4\varkappa}{(1+\varkappa)^2} - w\right)x \\ + \frac{1}{(1+\varkappa)^2}\left[(1+w)^2(1-\varkappa)^2 + 4\varkappa(1-w)^2\right] \geq 0$$

on $[0,1]$. In fact, this quadratic turns out to be nonnegative on all of $\mathbb{R}$. Its discriminant is

$$-\frac{16w(\varkappa-1)^2}{(1+\varkappa)^4}\left[2(1+\varkappa)^2w+\varkappa^2+6\varkappa+1\right]$$

and so the discriminant is always nonpositive, as we require $\varkappa$, $w>0$. Since the coefficient on $x^2$ in $f$ is $4w^2>0$, $f$ is always nonnegative.

*Proof of the inequality* $|\widetilde{\lambda}'_\pm(k)|\le 2w/(1+w)$. This follows from the previous inequality by the symmetry of $\widetilde{\lambda}'_\pm(k)$ in $\varkappa$ and $w$.

*Proof of the inequality* $\widetilde{\lambda}'_\pm(k)|\le 2c_\star^2|k|$. We rewrite (F.3.4) as

$$|\widetilde{\lambda}'_\pm(k)|=\frac{8\varkappa w}{(1+w)(1+\varkappa)}|\sin(k)|M(k), \tag{F.3.5}$$

with

$$M(k):=\frac{|\cos(k)|}{\sqrt{\dfrac{(1-\varkappa)^2}{(1+\varkappa)^2}+\dfrac{4\varkappa}{(1+w)^2(1+\varkappa)^2}((1-w)^2+4w\cos^2(k))}}.$$

We claim

$$0\le M(k)\le 1. \tag{F.3.6}$$

Then (F.3.5) implies both

$$|\widetilde{\lambda}'_\pm(k)|\le\frac{8\varkappa w}{(1+w)(1+\varkappa)}\cdot 1=2c_\star^2$$

and, using the inequality $|\sin(k)|\le|k|$,

$$|\widetilde{\lambda}'_\pm(k)|\le 2c_\star|k|.$$

Now we prove (F.3.6). First, we rewrite

$$M(k)=\frac{|1-\sin^2(k)|}{\sqrt{\dfrac{(1-\varkappa)^2}{(1+\varkappa)^2}+\dfrac{4\varkappa}{(1+w)^2(1+\varkappa)^2}((1+w)^2-4w\sin^2(k))}}.$$

Since $\sin^2(\cdot)$ maps $\mathbb{R}$ onto $[0,1]$, it then suffices to prove

$$0 \leq \sup_{0\leq s\leq 1} \frac{|1-s|}{\sqrt{\frac{(1-\varkappa)^2}{(1+\varkappa)^2} + \frac{4\varkappa}{(1+w)^2(1+\varkappa)^2}((1+w)^2 - 4ws)}} \leq 1. \tag{F.3.7}$$

We have

$$\frac{(1-\varkappa)^2}{(1+\varkappa)^2} + \frac{4\varkappa}{(1+w)^2(1+\varkappa)^2}((1+w)^2 - 4ws) = 1 - \frac{16\varkappa w}{(1+w)^2(1+\varkappa)^2} s.$$

Now observe that for any $0 \leq r < 1$, we have

$$\sup_{0\leq s\leq 1} \frac{1-s}{1-rs} \leq 1. \tag{F.3.8}$$

Indeed, we have

$$0 \leq rs < s \leq 1,$$

and so

$$0 \leq 1-s < 1-rs,$$

which implies (F.3.8). So, if we can establish

$$\frac{16\varkappa w}{(1+w)^2(1+\varkappa)^2} < 1, \tag{F.3.9}$$

then we can invoke (F.3.8) to conclude (F.3.7).

We consider three cases.

*Case 1.* $w > 1$ and $\varkappa = 1$. Here (F.3.9) reduces to

$$\frac{4w}{(1+w)^2} < 1,$$

and this is equivalent to

$$0 < (1+w)^2 - 4w = (w-1)^2.$$

Since $w > 1$, the inequality above holds.

*Case 2.* $w > 1$ and $\varkappa > 0$ but $\varkappa \neq 1$. After cross-multiplying, (F.3.9) is equivalent to

$$\underbrace{(1+w)^2(1+\varkappa)^2 - 16\varkappa w}_{f(w,\varkappa)} > 0.$$

Expanding $f(w,\varkappa)$ as a quadratic in $w$, we have

$$f(w,\varkappa)=(\varkappa+1)^2+2(\varkappa^2-6\varkappa+1)w+(\varkappa+1)^2.$$

Clearly

$$f(0,\varkappa)=(\varkappa+1)^2>0,$$

and the discriminant of $f(\cdot,\varkappa)$ is

$$\Delta(\varkappa):=4(\varkappa^2-6\varkappa+1)^2-4(\varkappa+1)^4=-64\varkappa(\varkappa-1)^2.$$

Since $\varkappa\neq-1$, $\Delta(\varkappa)<0$, so $f(w,\varkappa)\neq0$ and therefore $f(w,\varkappa)$ is always positive.

*Case 3.* $\varkappa>1$ and $w>0$. Since (F.3.9) is symmetric in $\varkappa$ and $w$, this follows from the previous two cases.

**(vi)** We combine the hypothesis $c^2>c_\star^2$, the second inequality in (i), and the fundamental theorem of calculus to produce

$$c^2k^2-\widetilde{\lambda}_-(k)=\int_0^k\left(2c^2s-\widetilde{\lambda}'_-(s)\right)\,ds>\int_0^k\left(2c_\star^2s-\widetilde{\lambda}'_-(s)\right)\,ds\geq0.$$

**(vii)** Let

$$k_1(c):=\frac{\sqrt{\widetilde{\lambda}_+(\pi/2)}}{c}\qquad\text{and}\qquad k_2(c):=\frac{\sqrt{(1+w)(1+\varkappa)}}{c}.$$

If $k<k_1(c)$, then

$$\widetilde{\xi}_c(k)=-c^2k^2+\widetilde{\lambda}_+(k)>-\widetilde{\lambda}_+\left(\frac{\pi}{2}\right)+\widetilde{\lambda}_+(k)\geq0,$$

and if $k>k_2(c)$, then

$$\widetilde{\xi}_c(k)<-(1+w)(1+\varkappa)+\widetilde{\lambda}_+(k)\leq0.$$

The intermediate value theorem then guarantees the existence of some $\Omega_c\in[k_1(c),k_2(c)]$ such that $\widetilde{\xi}_c(\Omega_c)=0$. Moreover, the estimates above show that we can have $\widetilde{\xi}_c(k)=0$ only for $k\in[k_1(c),k_2(c)]$.

We now show $2c^2k-\widetilde{\lambda}'_+(k)>0$ for $k>k_1(c)$, so that $\widetilde{\xi}'_c(k)<0$ for $k>k_1(c)$ and therefore this root $\Omega_c$ is unique. We prove this in the case $w>1$, the result for $\varkappa>1$

arising by symmetry. Then $(1+w)/2 > 1$ and

$$\widetilde{\lambda}_+\left(\frac{\pi}{2}\right) = \frac{(1+w)(1+\varkappa)}{2} + \frac{1}{2}\widetilde{\varrho}\left(\frac{\pi}{2}\right) \geq \frac{(1+w)(1+\varkappa)}{2} > 1+\varkappa,$$

and we also recall

$$|\widetilde{\lambda}_+'(k)| \leq \frac{4\varkappa}{1+\varkappa}.$$

Let $c_- = 9/10$. Then if $k > k_1(c)$ and $c > c_-$, we have

$$\begin{aligned}
2c^2k - \widetilde{\lambda}_+'(k) &\geq 2c^2k_1(c) - \frac{4\varkappa}{1+\varkappa} \\
&\geq 2c\sqrt{\frac{(1+w)(1+\varkappa)}{2}} - \frac{4\varkappa}{1+\varkappa} \\
&\geq 2c\sqrt{1+\varkappa} - \frac{4\varkappa}{1+\varkappa} \\
&\geq 2\left(\frac{9}{10}\right)\sqrt{1+\varkappa} - \frac{4\varkappa}{1+\varkappa}.
\end{aligned}$$

Set

$$f(\varkappa) := \frac{9}{5}\sqrt{1+\varkappa} - \frac{4\varkappa}{1+\varkappa} - \frac{1}{4}.$$

We claim $f(\varkappa) > 0$ for all $\varkappa > 0$. This will establish (2.2.9) with $b_0 = 1/4$.

First, $f(0) = 31/20 > 1 > 1/4$. Next,

$$f'(\varkappa) = \frac{9(\varkappa+1)^2 - 40\sqrt{\varkappa+1}}{10(\varkappa+1)^{5/2}}.$$

The only critical points of $f$ on $(0,\infty)$, then, are the values of $\varkappa$ such that

$$9(\varkappa+1)^2 - 40\sqrt{\varkappa+1} = 0.$$

This is equivalent to $\varkappa = -1$, which we rule out, or

$$\varkappa = \left(\frac{40}{9}\right)^{2/3} - 1 =: \varkappa_0.$$

We test the critical point $\varkappa_0$:

$$f''(\varkappa) = \frac{160\sqrt{\varkappa+1} - 9(\varkappa+1)^2}{(\varkappa+1)^{7/2}} \quad \text{and} \quad f''(\varkappa_0) = 80\sqrt[3]{15} > 0.$$

Hence $f$ has a local minimum at $\varkappa = \varkappa_0$, and this local minimum is in fact the global minimum of $f$ on $(0, \infty)$. Last,

$$f(\varkappa_0) = \frac{9}{5}(15)^{1/3} - \frac{17}{4} > 0.$$

This concludes the proof in the case $w > 1$. We take

$$c_- := \frac{9}{10} \quad \text{and} \quad b_0 := \frac{9}{5}(15)^{1/3} - \frac{17}{4}.$$

If $\varkappa > 1$, then we use instead the inequalities

$$\widetilde{\lambda}_+\left(\frac{\pi}{2}\right) > 1 + w \quad \text{and} \quad |\widetilde{\lambda}_+'(k)| \le \frac{4w}{1+w}$$

and replace $f(\varkappa)$ with the identically defined $f(w)$ to obtain the same conclusion.

**F.4. Proof of Lemma 3.4.1.** Let

$$M = \max\left\{\max_{0 \le s \le 1} \mathcal{M}_{\mathrm{map}}[s], \max_{0 \le s, \grave{s} \le 1} \mathcal{M}_{\mathrm{lip}}[s, \grave{s}], \max_{0 \le s \le 1} \mathcal{M}_{\mathrm{max}}[s]\right\}.$$

Set

$$r_0 = \min\left\{\frac{1}{6M}, 1\right\} \quad \text{and} \quad a_0 = \min\left\{\frac{r_0}{2M}, a_1, \frac{1}{6M}\right\}.$$

Observe that

$$r_0 \le \frac{1}{2M} \Longrightarrow r_0^2 \le \frac{r_0}{2M} \Longrightarrow Mr_0^2 \le \frac{r_0}{2}.$$

Then for $x \in \mathfrak{B}(r_0)$ and $|a| \le a_0$, we have

$$\begin{aligned} \|F_\epsilon(x,a)\| &\le \mathcal{M}_{\mathrm{map}}[\|x\|]\left(|a| + \|x\|^2\right) \\ &\le M(a_0 + r_0^2) \\ &\le M\left(\frac{r_0}{2M}\right) + Mr_0^2 \\ &\le \frac{r_0}{2} + \frac{r_0}{2} \\ &= r_0. \end{aligned} \tag{F.4.1}$$

Next, for $x$, $\grave{x} \in \mathfrak{B}(r_0)$ and $|a| \le a_0$, we compute

$$\begin{aligned}
\|F_\epsilon(x,a) - F_\epsilon(\grave{x},\grave{a})\| &\le \mathcal{M}_{\text{lip}}[\|x\|,\|\grave{x}\|]\left(|a| + \|x\| + \|\grave{x}\|\right)\|x-\grave{x}\| \\
&\le M(|a| + 2r_0)\|x-\grave{x}\| \\
&\le M\left(\frac{1}{6M} + \frac{2}{6M}\right)\|x-\grave{x}\| \\
&= \frac{1}{2}\|x-\grave{x}\|.
\end{aligned} \tag{F.4.2}$$

Together, (F.4.1) and (F.4.2) imply that each map $F_\epsilon(\cdot,a)\colon \mathfrak{B}(r_0) \to \mathfrak{B}(r_0)$ is a contraction on $\mathfrak{B}(r_0)$, which means there exists a unique $x_\epsilon^a \in \mathfrak{B}(r_0)$ such that $F_\epsilon(x_\epsilon^a, a) = x_\epsilon^a$.

Finally, we have

$$\begin{aligned}
\left\|x_\epsilon^a - x_\epsilon^{\grave{a}}\right\| &= \left\|F_\epsilon(x_\epsilon^a,a) - F_\epsilon(x_\epsilon^{\grave{a}},\grave{a})\right\| \\
&\le \|F_\epsilon(x_\epsilon^a,a) - F_\epsilon(x_\epsilon^a,\grave{a})\| + \left\|F_\epsilon(x_\epsilon^a,\grave{a}) - F_\epsilon(x_\epsilon^{\grave{a}},\grave{a})\right\| \\
&\le \mathcal{M}_{\max}[\|x_\epsilon^a\|]|a-\grave{a}| + \mathcal{M}_{\text{lip}}[\|x_\epsilon^a\|, \left\|x_\epsilon^{\grave{a}}\right\|]|\grave{a}|\left\|x_\epsilon^a - x_\epsilon^{\grave{a}}\right\| \\
&\le M|a-\grave{a}| + Ma_0\left\|x_\epsilon^a - x_\epsilon^{\grave{a}}\right\|.
\end{aligned}$$

Since

$$Ma_0 \le M\left(\frac{1}{2M}\right) = \frac{1}{2},$$

we can rearrange this last inequality to

$$\frac{1}{2}\left\|x_\epsilon^a - x_\epsilon^{\grave{a}}\right\| \le M|a-\grave{a}|,$$

and thus

$$\left\|x_\epsilon^a - x_\epsilon^{\grave{a}}\right\| \le 2M|a-\grave{a}|.$$

This proves (3.4.5).

**F.5. The traveling wave problem in the case $\beta_2 = 0$.** Recall that in our rescaling (1.3.6) we assumed $\beta_2 \neq 0$. If $\beta_2 = 0$, then we must have $\beta_1 \neq 0$, and we redefine $a_2$ from (1.3.5) as

$$a_2 := \frac{\varkappa_2}{\beta_1}.$$

We can then continue with the nondimensionalization as before and obtain a slightly different version of (1.3.7); then we proceed to write this system in terms of relative displacements like (1.3.8) and finally make the traveling wave ansatz (2.1.1). We obtain a system almost identical to (2.1.2):

$$c^2\overline{\mathbf{p}}'' + L[\varkappa, w]\overline{\mathbf{p}} + L[1, w]\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\overline{\mathbf{p}}^{.2} + L[1, w](\overline{\mathbf{p}}^{.3}.N(\overline{\mathbf{p}})) = 0. \qquad \text{(F.5.1)}$$

Here we are writing $\overline{\mathbf{p}}$ instead of $\mathbf{p}$ for the column vector of traveling wave profiles to emphasize that (F.5.1) corresponds to a materially distinct lattice from the one that yielded (2.1.2). Rewriting (F.5.1) to expose a factor of $L[\varkappa, w]$ in the nonlinear terms as in (2.1.4), we find

$$c^2\overline{\mathbf{p}}'' + L[\varkappa, w]\overline{\mathbf{p}} + L[\varkappa, w]\begin{bmatrix}1/\varkappa & 0\\ 0 & 0\end{bmatrix}\overline{\mathbf{p}}^{.2} + L[\varkappa, w]M_{1/\varkappa}(\overline{\mathbf{p}}^{.3}.N(\overline{\mathbf{p}})) = 0. \qquad \text{(F.5.2)}$$

Now we rescale $\overline{\mathbf{p}}(x) = \varkappa\mathbf{q}(x)$, so (F.5.2) becomes

$$c^2\mathbf{q}'' + L[\varkappa, w]\mathbf{q} + L[\varkappa, w]\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathbf{q}^{.2} + L[\varkappa, w]\begin{bmatrix}\varkappa^2 & 0\\ 0 & \varkappa\end{bmatrix}(\mathbf{q}^{.3}.N(\varkappa\mathbf{q})) = 0. \qquad \text{(F.5.3)}$$

We recall from (1.3.4) that we defined $\beta = 0$ when $\beta_2 = 0$, and so, to second order in $\mathbf{q}$, (F.5.3) is

$$c^2\mathbf{q}'' + L[\varkappa, w]\mathbf{q} + L[\varkappa, w]\begin{bmatrix}1 & 0\\ 0 & \beta/\varkappa\end{bmatrix}\mathbf{q}^{.2}. \qquad \text{(F.5.4)}$$

This agrees with (2.1.4) to second order as well. The formal long wave limit in Section 2.5 depended only on the linear and quadratic terms of (2.1.4), and likewise the techniques for the existence of periodic solutions in Chapter 3 and the nanopteron solutions in Chapter 4 relied ultimately on only the linear and quadratic terms of the traveling wave equation having the explicit form (F.5.4). The manipulations of and higher-order estimates for the "higher-order terms" in these two chapters carry over in the same way to the rescaled nonlinearity in (F.5.3).

## Appendix G. Operator Estimates from [FW18]

In this appendix we present two collections of results that we proved earlier in [FW18]. The set-up there is slightly different, of course; for example, the eigenvector operators $J$ and $J_1$ are not the same, and the "symmetry" in the mass dimer problem analogous to the "even × even" symmetry in Lemma 2.7.1 is "even × odd." But the proofs, happily, do not depend on these superficial differences, and so we just provide the long litany of estimates and other properties below.

### G.1. Estimates for the periodic problem.

**G.1.1 Proposition.** *There exists $\epsilon_{\mathrm{per}} \in (0,1)$ with the following properties.*

**(i)** *For $\epsilon \in (0, \epsilon_{\mathrm{per}})$ and $t \in \mathbb{R}$, there is a function $\mathcal{R}_\epsilon$ such that*

$$\widetilde{\xi}_{c_\epsilon}(\epsilon\omega_\epsilon + t) = \underbrace{\widetilde{\xi}'_{c_\epsilon}(\epsilon\omega_\epsilon)}_{\Upsilon_\epsilon} t + t^2\mathcal{R}_\epsilon(t). \tag{G.1.1}$$

*The constant $b_0 > 0$ from (2.2.9) also satisfies the estimate*

$$|\Upsilon_\epsilon| \geq b_0 \tag{G.1.2}$$

*for all $\epsilon \in (0, \epsilon_{\mathrm{per}})$.*

**(ii)** *For $|t| \leq 1$ and $\epsilon \in (0, \epsilon_{\mathrm{per}})$, the multiplier $\xi^{\epsilon,t}$ maps $E^{r+2}_{\mathrm{per}}$ bijectively onto the space*

$$\left\{\psi \in E^r_{\mathrm{per}} \;\middle|\; \widehat{\psi}(1) = 0\right\}.$$

*In particular, if $\Pi_2$ is the multiplier defined in Appendix 3.3 with symbol $\widetilde{\Pi}_2(k) = 1 - \delta_{|k|,1}$, then $\xi^{\epsilon,t}$ is invertible on the range of $\Pi_2$.*

**(iii)** *There exists $C_{\mathrm{map}} > 0$ such that*

$$\sup_{\substack{0<\epsilon<\epsilon_{\mathrm{per}} \\ |t|\leq 1 \\ r\in\mathbb{R}}} \|\mathcal{L}_1^\epsilon(t)\|_{\mathbf{B}(\mathcal{W}^r,\mathcal{W}^{r+2})} \leq C_{\mathrm{map}}, \tag{G.1.3}$$

$$\sup_{\substack{0<\epsilon<\epsilon_{\mathrm{per}} \\ |t|\le 1 \\ r\in\mathbb{R} \\ j=2,3}} \left\|\mathcal{L}_j^{\epsilon}(t)\right\|_{\mathbf{B}(\mathcal{W}^r)} \le C_{\mathrm{map}}, \tag{G.1.4}$$

*and*

$$\sup_{\substack{0<\epsilon<\epsilon_{\mathrm{per}} \\ t\in\mathbb{R}}} |\mathcal{R}_\epsilon(t)| \le C_{\mathrm{map}}. \tag{G.1.5}$$

**(iv)** *There exists* $C_{\mathrm{lip}} > 0$ *such that*

$$\sup_{\substack{0<\epsilon<\epsilon_{\mathrm{per}} \\ r\in\mathbb{R}}} \left\|\mathcal{L}_1^{\epsilon}(t) - \mathcal{L}_1^{\epsilon}(\grave{t})\right\|_{\mathbf{B}(\mathcal{W}^r)} \le C_{\mathrm{lip}}|t-\grave{t}|, \tag{G.1.6}$$

$$\sup_{\substack{0<\epsilon<\epsilon_{\mathrm{per}} \\ r\in\mathbb{R} \\ j=2,3,4}} \left\|\mathcal{L}_j^{\epsilon}(t) - \mathcal{L}_j^{\epsilon}(\grave{t})\right\|_{\mathbf{B}(\mathcal{W}^r,\mathcal{W}^{r-1})} \le C_{\mathrm{lip}}|t-\grave{t}|, \tag{G.1.7}$$

*and*

$$\sup_{0<\epsilon<\epsilon_{\mathrm{per}}} |\mathcal{R}_\epsilon(t) - \mathcal{R}_\epsilon(\grave{t})| \le C_{\mathrm{lip}}|t-\grave{t}|. \tag{G.1.8}$$

*for any* $|t|$, $|\grave{t}| \le 1$.

## G.2. Estimates for the nanopteron equations.

**G.2.1 Proposition.** *There exists* $q_\star \in (0, 1/2\sqrt{\alpha_\varkappa})$ *and* $\overline{\epsilon} \in (0, \epsilon_{\mathrm{per}})$ *such that the following hold.*

**(i)** *For all* $r \ge 0$*, there exists* $C_r > 0$ *such that for all* $\epsilon \in (0, \overline{\epsilon})$ *and all* $a$*,* $\grave{a} \in [-a_{\mathrm{per}}, a_{\mathrm{per}}]$*, the periodic solutions* $\boldsymbol{\varphi}_\epsilon^a$ *defined in Theorem 3.1.1 satisfy*

$$\|\boldsymbol{\varphi}_\epsilon^a\|_{W^{r,\infty}} + \|J^\epsilon \boldsymbol{\varphi}_\epsilon^a\|_{W^{r,\infty}} \le C_r \epsilon^{-r} \tag{G.2.1}$$

*and*

$$|\partial_X^r [J^\epsilon(\boldsymbol{\varphi}_\epsilon^a - \boldsymbol{\varphi}_\epsilon^{\grave{a}})](X)| \le C_r \epsilon^{-r} |a - \grave{a}|(1+|X|),\ X \in \mathbb{R}. \tag{G.2.2}$$

**(ii)** *There exists* $C > 0$ *such that for all* $r \ge 0$*,* $q > 0$*,* $\epsilon \in (0, \overline{\epsilon})$*, and and* $f \in H_q^r$*, the operator* $\iota_\epsilon$ *defined in* (4.2.4) *satisfies*

$$|\iota_\epsilon[f]| \le \frac{C\epsilon^r}{\sqrt{q}} \|f\|_{r,q}. \tag{G.2.3}$$

**(iii)** *There exists $C > 0$ such that the quantity $\upsilon_\epsilon$, defined in (4.2.6), satisfies*

$$|\upsilon_\epsilon| \geq C \tag{G.2.4}$$

*for all $\epsilon \in (0, \overline{\epsilon})$.*

**(iv)** *The operator $\mathcal{T}_\epsilon$ defined in (2.6.1) has the following properties.*

- *Let $r \geq 0$ and $q \in [0, q_\star]$. Given $g \in H^r_q$, there exists $f \in H^{r+2}_q$ such that $\mathcal{T}_\epsilon f = g$ if and only if $\widehat{g}(\pm\omega_\epsilon) = 0$, in which case $f$ is unique;*
- *Let $r \geq 0$, $q \in [0, q_\star]$, and $\epsilon \in (0, \overline{\epsilon})$. For all $g \in E^r_q$, there exists a unique $f \in E^{r+2}_q$ such that*

$$\mathcal{T}_\epsilon f = g - \frac{1}{\upsilon_\epsilon} \iota_\epsilon[g] \chi_\epsilon. \tag{G.2.5}$$

  *As in (4.2.7), we set $f := \mathcal{P}_\epsilon g$. Equivalently,*

$$\mathcal{P}_\epsilon g := \mathcal{T}_\epsilon^{-1} \left( g - \frac{1}{\upsilon_\epsilon} \iota_\epsilon[g] \chi_\epsilon \right).$$

- *For each $q \in [0, q_\star]$, there exists $C_q > 0$ such that*

$$\|\mathcal{P}_\epsilon g\|_{r+j,q} \leq \frac{C_q}{\epsilon^{j+1}} \|g\|_{r,q}, \ j = 0, 1, 2 \tag{G.2.6}$$

  *for all $r \geq 0$ and $\epsilon \in (0, \overline{\epsilon})$.*

**(v)** *There exists $C > 0$ such that for all $r \geq 0$, $\epsilon \in (0, \overline{\epsilon})$, $q \in (0, q_\star]$, and $\mathbf{h} \in H^r_q \times H^r_q$*

$$\left\|(J^\epsilon - J^0)\mathbf{h}\right\|_{r,q} \leq C\epsilon \|\mathbf{h}\|_{r+1,q}. \tag{G.2.7}$$

**(vi)** *There exists $C > 0$ such that for all $\epsilon \in (0, \overline{\epsilon})$, $q \in (0, q_\star]$, $r \geq 0$, and $f \in H^r_q$, the Fourier multiplier $\varpi^\epsilon$, whose symbol $\widetilde{\varpi}^\epsilon$ is given in (2.4.3), satisfies*

$$\|\varpi^\epsilon f\|_{r+2,q} \leq C \|f\|_{r,q}. \tag{G.2.8}$$

# Appendix H. Notation

a.e. = almost everywhere with respect to Lebesgue measure on $\mathbb{R}$

$\mathbf{B}(\mathcal{X},\mathcal{Y})$ = space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$

$\mathbf{B}(\mathcal{X}) = \mathbf{B}(\mathcal{X},\mathcal{X})$

$\mathcal{C}(\mathcal{X},\mathcal{Y})$ = space of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$

$\mathcal{C}(\mathcal{X}) = \mathcal{C}(\mathcal{X},\mathcal{X})$

$\mathcal{C}_0(\mathbb{R}) = \left\{ f \in \mathcal{C}(\mathbb{R}) \;\middle|\; \lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0 \right\}$

$\partial_X[f]$ = strong or weak derivative of $f = f(X)$

$\mathfrak{F}[f] = \widehat{f}$ = Fourier transform of a function $f$

$H^r$ = Sobolev space of order $r$ on $\mathbb{R}$

$H^r_{\text{per}}$ = Sobolev space of order $r$ of $2\pi$-periodic functions

$H^r_q$ = Sobolev space of functions that decay like $e^{q|x|}$ at infinity

$\mathbb{1}$ = identity operator

$\mathbf{i} = (1,0)$

$\mathbf{j} = (0,1)$

$L^2 = L^2(\mathbb{R})$ = square-integrable functions on $\mathbb{R}$

$L^\infty = L^\infty(\mathbb{R})$ = essentially bounded functions on $\mathbb{R}$

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