MATH 2306
ORDINARY DIFFERENTIAL EQUATIONS

TIMOTHY E. FAVER

May 1, 2022

Contents

1. First-Order Differential Equations

We will not define precisely what a differential equation is for some time, as the most exact definition will make more sense after we see a number of examples, and nonexamples. For now, we will operate under the following terminology.

1.0.1 Undefinition.

An **ORDINARY DIFFERENTIAL EQUATION** *is an equation involving a function and one or more of its derivatives.*

The problem with this definition will be that most ambiguous word "involving." We will see some equations that certainly "involve" a function and its derivative and yet are not what we are culturally expected to define as a differential equation.

1.1. Direct integration.

Perhaps the simplest class of differential equations are those which involve only the first derivative of the unknown function and not the function itself.

1.1.1 Example.

Find all functions y such that

$$\frac{dy}{dx} = x. \tag{1.1.1}$$

Solution. We need to find all functions y whose derivative equals x. In other words, and in other notation, we need to find all functions y such that

$$y'(x) = x \tag{1.1.2}$$

for all x. If we assume that (1.1.2) holds, then y must be an antiderivative of x, and therefore

$$y(x) = \int x \ dx = \int x^1 \ dx = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$$

by the power rule. Here C is an arbitrary, fixed constant that is independent of x.

Conversely, we can check that

$$\frac{d}{dx}\left[\frac{x^2}{2} + C\right] = \frac{d}{dx}\left[\frac{x^2}{2}\right] + \frac{d}{dx}[C] = \frac{1}{2}\frac{d}{dx}[x^2] + 0 = \frac{1}{2}(2)x^{2-1} = x.$$

Thus the function $y(x) = x^2/2 + C$ satisfies (1.1.2). Moreover, the work with the antiderivative shows that *any* solution y to (1.1.2) must have the form $y(x) = x^2/2 + C$ for some constant C. ▲

We summarize the "direct integration" method of solving ODEs in the following algorithmic method and formal theorem.

1.1.2 Method: Solve $\frac{dy}{dx} = g(x)$ for a given function g.

All solutions have the form $y(x) = \int g(x)\ dx + C$, where C is a constant.

1.1.3 Theorem.

Suppose that g is continuous on the interval I, and let G be an antiderivative of g on I (i.e., $G'(x) = g(x)$ for all x in I.). Then a function f on I solves

$$f'(x) = g(x) \text{ for } x \text{ in } I$$

if and only if there is a constant C such that $f(x) = G(x) + C$ for all x in I.

Proof. ($\Longrightarrow$) Suppose that f satisfies $f'(x) = g(x)$ for all x in I. Then f is an antiderivative of g. Since G is also an antiderivative of g, calculus tells us that f and G differ by a constant. Thus there is a real number C such that $f(x) = G(x) + C$ for all x.

($\Longleftarrow$) Suppose that f has the form $f(x) = G(x) + C$, where C is a constant and $G' = g$. Then by the linearity of the derivative,

$$f'(x) = G'(x) + 0 = g(x).$$ ■

1.1.4 Remark.

One peril of the indefinite integral is the ambiguity of the "dummy" variable of integration. If g is a function, the symbols

$$\int g(x)\ dx, \qquad \int g(s)\ ds, \qquad \text{and} \qquad \int g(t)\ dt$$

all mean the same: they denote the set of all functions whose derivatives equal g. Thus we expect

$$\int g(x)\ dx = \int g(s)\ ds = \int g(t)\ dt.$$

If we try to define a specific function f by setting

$$f(x) = \int g(x)\ dx,$$

then we might also expect

$$f(x) = \int g(t)\ dt = f(t).\ (?!)$$

This sort of variable-switching can cause endless, and unnecessary, and avoidable, headaches. If and when we want to consider a particular antiderivative of a function g, we will frequently give it a name, like G (thus $G'(x) = g(x)$ for all x) that does not involve that most beautiful, and mystifying, of symbols, $\int$.

We constructed *infinitely* many solutions to the differential equation (1.1.1). This is wholly typical of differential equations: they usually have many solutions. If we impose additional requirements or data on the differential equation, then we can often winnow down to one particular solution.

This is where we finished on Monday, January 10, 2022 (Section 53).

1.1.5 Example.

Find all solutions to the INITIAL VALUE PROBLEM

$$\begin{cases} \dfrac{dy}{dx} = x \\ \\ y(0) = 0. \end{cases}$$

Solution. Now we are seeking a function y with two properties. First, we need $y'(x) = x$ for all x. Second, we need $y(0) = 0$.

We already know that the only functions y satisfying $y'(x) = x$ for all x have the form $y(x) = x^2/2 + C$ for some constant C. We want to choose C so that $y(0) = 0$. This demands

$$0 = y(0) = \frac{0^2}{2} + C = C.$$

Thus $C = 0$ and $y(x) = x^2/2$ solves the initial value problem. Also, this is the *only* solution to the initial value problem; all solutions are determined by the value of C, and only one value for C works here ($C = 0$). ▲

This is where we finished on Monday, January 10, 2022 (Section 54).

1.2. Explorations with exponentials.

Never underestimate the value of "fooling around" when learning something new. To review some essential calculus concepts, and to motivate some techniques and tricks for the future, we will "fool around" with several differential equations that are the least difficult ones that cannot be solved via direct integration.

1.2.1 Example.

Find all solutions to the differential equation $f' = f$.

Solution. We break our treatment of this (very simple) differential equation into a number of (very wordy) steps.

1. First, one might rewrite this problem as

"Find all functions f such that $f'(x) = f(x)$ for all x"

or

$$\text{"Find all } y \text{ such that } \frac{dy}{dx} = y.$$

Purely in words, with no symbols, we need to find all functions that differentiate back to themselves.

2. Without a calculus course behind us, we might not have a clue as to what functions could solve the differential equation, let alone how to be sure that we have found "all" solutions. (Although, without a calculus course, the words "derivatives" and "differentiate" probably will not make mathematical sense.) With a calculus course earlier in life, we might remember something useful: the function $f(x) := e^x$ satisfies $f'(x) = e^x$. Equivalently,

$$\frac{d}{dx}[e^x] = e^x.$$

That is, the exponential is its own derivative. And so one solution to $f' = f$ is $f(x) = e^x$.

3. Are there other solutions? Our immediate past experience with "direct integration" differential equations suggests that perhaps we should have an arbitrary constant in our answer. Perhaps we can "build" more solutions from this one solution $f(x) = e^x$? If we use the "linearity" of the derivative, we certainly can. Recall that if f is differentiable and C is a constant, then the function $g(x) := Cf(x)$ is differentiable, and $g'(x) = Cf'(x)$. In other words,

$$\frac{d}{dx}[Cf(x)] = Cf'(x) \quad \text{or} \quad (Cf(x))' = Cf'(x).$$

Returning to our specific solution $f(x) = e^x$, we let C be any constant and we define $g(x) = Ce^x$ to see that

$$g'(x) = \frac{d}{dx}[Ce^x] = C\frac{d}{dx}[e^x] = Ce^x = g(x).$$

Consequently, $g(x) = Ce^x$ also solves the differential equation. (A reasonable, but bad, idea would be to guess instead $g(x) = e^x + C$, which yields $g'(x) = e^x \neq g(x)$, unless $C = 0$.)

4. We now have an *infinite* family of solutions to the problem $f' = f$. For every real number, the map $f(x) = Ce^x$ is a solution. Can there be any other solutions? That is, can there exist a solution f to $f' = f$ such that there *does not* exist a real number C such that $f(x) = Ce^x$ for all x? More plainly, are there non-exponential solutions?

5. The answer is no, although the justification of this answer relies on a non-obvious trick. Suppose that f does satisfy $f' = f$. We want to force the existence of a real number C such that $f(x) = Ce^x$ for all x. We know that

$$f'(x) = f(x) \tag{1.2.1}$$

for all x. Then we can multiply both sides of (1.2.1) by e^{-x} to find

$$f'(x)e^{-x} = f(x)e^{-x}. \tag{1.2.2}$$

Why would we do this? Because it works. Because someone else told us to. Because we will revisit this later.

Now subtract the right side of (1.2.2) from the left to find

$$f'(x)e^{-x} - f(x)e^{-x} = 0. \tag{1.2.3}$$

The left side of (1.2.3) has a very special form. Recall that if g is another differentiable function, then the product rule tells us

$$\frac{d}{dx}[fg](x) = f'(x)g(x) + f(x)g'(x).$$

We rewrite (1.2.3) very slightly as

$$f'(x)e^{-x} + f(x)[-e^{-x}] = 0. \tag{1.2.4}$$

This, hopefully, looks very much like the product rule, if we recall that

$$\frac{d}{dx}[e^{-x}] = -e^{-x}$$

from the chain rule. Thus (1.2.4) is really

$$\frac{d}{dx}[f(x)e^{-x}] = 0. \tag{1.2.5}$$

We need yet another calculus fact: if h is a differentiable function and $h'(x) = 0$ for all x, then h is constant — there is a real number C such that $h(x) = C$ for all x. The statement (1.2.5) is really saying that if $h(x) = f(x)e^{-x}$, then h is constant. Thus there is a real number C such that

$$f(x)e^{-x} = C \tag{1.2.6}$$

for all x. We solve[1] for $f(x)$ as

$$f(x) = Ce^{x}.$$

Thus f must be a constant multiple of the exponential, and there are no other solutions to the differential equation. ▲

How should we pick the constant C? Without further "data" in our problem it is impossible to tell, but with additional data imposed, we may be directed to a particular value for C.

---

[1]If it helps with the algebra, rewrite (1.2.6) as

$$\frac{f(x)}{e^{x}} = C$$

and then multiply both sides by e^{x}.

1.2.2 Example.

Find all functions f satisfying both $f' = f$ and $f(0) = 1$.

Solution. We now know that if $f' = f$, then there is a constant C such that $f(x) = Ce^x$ for all x. We want $f(0) = 1$, so that demands $Ce^0 = 1$. Since $e^0 = 1$, we have $C = 1$, and thus $f(x) = e^x$. In particular, $f(x) = e^x$ is the only function satisfying both $f' = f$ and $f(0) = 1$. ▲

We see that while a differential equation may have infinitely many solutions, imposing an "initial" or "pointwise" condition on the solution can radically reduce the number of permissible solutions — and maybe even make the solution unique.

1.2.3 Example.

Let a be a fixed real number. Find all functions f satisfying $f' = af$.

Solution. The number a is a **PARAMETER** of our differential equation, and any solution that we find must somehow incorporate a. The equation $f' = f$, which we solved previously, is the special case $a = 1$.

Requiring $f' = af$ means that f and its derivative should be proportional, a word that might evoke, again, the exponential from our memory of calculus. If we guess $f(x) = Ce^{ax}$, where C is an arbitrary constant (and a is *not* arbitrary, but rather the same parameter that appears in our problem), we compute

$$f'(x) = \frac{d}{dx}[Ce^{ax}] = C\frac{d}{dx}[e^{ax}] = Ce^{ax}\frac{d}{dx}[ax] = Ce^{ax}a\frac{d}{dx}[x] = Ce^{ax}a = a(Ce^{ax}) = af(x).$$

Thus we have found an infinite family of solutions to our problem. We can check that any solution to $f' = af$ must have the form $f(x) = Ce^{ax}$ for some real number C using a trick similar to our solution of $f' = f$ (multiply both sides of $f'(x) = f(x)$ by e^{-ax} and work until the product rule appears), which we leave as an exercise. ▲

This is where we finished on Wednesday, January 12, 2022.

1.2.4 Example.

Find all functions f satisfying $f'(x) = xf(x)$ for all x.

Solution. We previously solved $f'(x) = f(x)$ and, more generally, $f'(x) = af(x)$, but now the coefficient on f is not constant. Nonetheless, since our previous solutions were $f(x) = Ce^x$ and $f(x) = Ce^{ax}$, we might be led to guess that here we should have a solution of the form $f(x) = Ce^{H(x)}$ for some function H. (The letter H is for "hope," as in "we hope that this is the case.") We can try to suss out what H is by evaluating our differential equation at this guess. We have

$$f'(x) = \frac{d}{dx}[Ce^{H(x)}] = C\frac{d}{dx}[e^{H(x)}] = Ce^{H(x)}H'(x).$$

Since we want $f'(x) = xf(x)$, this means that we need

$$Ce^{H(x)}H'(x) = x[Ce^{H(x)}].$$

We can divide $e^{H(x)}$ from both sides, since it is always the case that $e^{H(x)} > 0$. Thus we want

$$CH'(x) = Cx.$$

Next, we should eliminate C, and of course we can do so via division if $C \neq 0$. (If $C = 0$, then $f(x) = Ce^{H(x)} = 0 \cdot e^{H(x)} = 0$, and we can check that $f(x) = 0$ is a solution to the differential equation.) In the case $C \neq 0$, we find

$$H'(x) = x.$$

This is a separate, auxiliary differential equation for H, and it is one that we have solved before. We find that H must be the function

$$H(x) = \frac{x^2}{2} + K,$$

where K is a constant of integration. (We have already used C as a constant, so we are writing the constant of integration differently.)

Thus

$$f(x) = Ce^{H(x)} = Ce^{x^2/2+K} = Ce^K e^{x^2/2}.$$

Since K can be any real number, e^K can be any positive real number, and since C can be any real number, the product Ce^K can be any real number. So, we abuse notation and replace Ce^K by just C, where C is, once again, understood to be an arbitrary constant. Thus a family of solutions to $f'(x) = xf(x)$ is

$$f(x) = Ce^{x^2/2}.$$

Last, we need to establish that there are no other solutions. We can do this by assuming that $f'(x) = xf(x)$ and then multiplying both sides by $e^{-x^2/2}$ to obtain $f'(x)e^{-x^2/2} = (xe^{-x^2/2})f(x)$. This rearranges, once more, into the product rule. We leave this as an exercise. ▲

We have now solved the differential equation

$$f'(x) = h(x)f(x),$$

where h is a given function, in three different cases. Here is a summary.

$h(x)$	Solution to $f'(x) = h(x)f(x)$
$h(x) = 1$	$f(x) = Ce^x$
$h(x) = a$	$f(x) = Ce^{ax}$
$h(x) = x$	$f(x) = Ce^{x^2/2}$

Perhaps we see a pattern emerging: the solutions always have the form $f(x) = Ce^{H(x)}$, where $H'(x) = h(x)$. That is, H is an antiderivative of h. These observations generalize in a substantial way, which we state formally. Before proceeding, though, we need to recall that if a function h is continuous on an interval I, then h has an antiderivative on I: there is a differentiable function H defined on I such that $H'(x) = h(x)$ for all x in I.

1.2.5 Theorem.

Let h be continuous on an interval I and let H be an antiderivative of h. Then a differentiable function f defined on I solves

$$f'(x) = h(x)f(x)$$

for all x in I if and only if there is a number C such that $f(x) = Ce^{H(x)}$ for all x in I.

Proof. ($\Longleftarrow$) Fix a real number C and let $f(x) = Ce^{H(x)}$. We need to show that $f'(x) = h(x)f(x)$. The chain rule tells us

$$f'(x) = \frac{d}{dx}[Ce^{H(x)}] = C\frac{d}{dx}[e^{H(x)}] = Ce^{H(x)}H'(x) = Ce^{H(x)}h(x) = h(x)[Ce^{H(x)}] = h(x)f(x).$$

($\Longrightarrow$) Suppose that f is a differentiable function defined on I and $f'(x) = h(x)f(x)$ for all x in I. We need to find a constant C such that $f(x) = Ce^{H(x)}$ for all x in I. We claim that

$$\frac{d}{dx}[f(x)e^{-H(x)}] = 0,$$

and that this forces the existence of such a C. We leave the details as an exercise but mention that it follows, broadly, the pattern of Step 5 in the solution of Example 1.2.1. ■

1.2.6 Method: Solve $f'(x) = h(x)f(x)$, where h is continuous.

All solutions have the form $f(x) = Ce^{H(x)}$, where $H'(x) = h(x)$. Take

$$H(x) = \int h(x)\ dx.$$

Omit the constant of integration.

1.2.7 Example.

(i) *Find all functions f defined on $(0, \infty)$ that satisfy*

$$f'(x) = \frac{f(x)}{x}. \tag{1.2.7}$$

(ii) *Find all functions f defined on $(0, \infty)$ that satisfy*

$$\begin{cases} f'(x) = \dfrac{f(x)}{x} \\ f(1) = 0. \end{cases}$$

Solution. (i) The differential equation (1.2.7) has the form $f'(x) = h(x)f(x)$, where $h(x) = 1/x$. Note that h is defined on the interval $(0, \infty)$ but not on $(-\infty, \infty)$. An antiderivative of h is

$$H(x) = \int \frac{1}{x}\ dx = \int \frac{dx}{x} = \ln(|x|).$$

Here we need only *one* antiderivative, and we may (shockingly) omit the constant of integration. Also, we are abusing notation (see Remark 1.1.4) by taking the independent variable of H and the dummy variable of integration both to be x. Life is a series of compromises.

We can simplify H slightly because we are working with $x > 0$: we have $H(x) = \ln(|x|) = \ln(x)$. Thus any solution to (1.2.7) has the form

$$f(x) = Ce^{H(x)} = Ce^{\ln(x)} = Cx.$$

We check our work:

$$f'(x) = \frac{d}{dx}[Cx] = C \quad \text{and} \quad \frac{f(x)}{x} = \frac{Cx}{x} = C.$$

Thus we have the equality in (1.2.7). Note also that we had no qualms about dividing by x above, since we are assuming that $x > 0$ all along.

(ii) We know that the solution f has the form $f(x) = Cx$, and we want $f(1) = 0$, thus $C \cdot 1 = 0$ and therefore $C = 0$. That is, $f(x) = 0 \cdot x = 0$ is the solution. ▲

This is where we finished on Friday, January 14, 2022.

1.3. Linear first-order differential equations.

1.3.1 Definition.

Let p and q be functions defined on the same interval I. A **LINEAR FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION** *is an equation of the form*

$$f'(x) + p(x)f(x) = g(x). \tag{1.3.1}$$

A **SOLUTION** *f to* (1.3.1) *is a differentiable function f defined on I that satisfies* (1.3.1) *for each x in I. The equation* (1.3.1) *is* **HOMOGENEOUS** *if $g(x) = 0$ for all x in I. Otherwise, the equation* (1.3.1) *is* **NONHOMOGENEOUS**.

We will sometimes suppress the "(x)" dependence in (1.3.1) and write instead

$$f' + p(x)f = g(x).$$

We will keep the "(x)" notation on p and g to emphasize that these functions are not necessarily constant. A linear homogeneous first-order ODE then has the form

$$f' + p(x)f = 0.$$

Often the interval I on which p and g, and later f, are defined will be implicit, i.e., not stated explicitly, but occasionally we will want to refer to it (as we did in Example 1.2.7).

1.3.2 Example.

The differential equation

$$f'(x) + 3xf(x) = e^x$$

is linear, first-order, and nonhomogeneous, while

$$f'(x) + 2f(x) = 0$$

is linear, first-order, and homogeneous.

Linear ODEs are important for at least three reasons.

- They appear in a variety of physical models, some of which we will meet later (like population growth and temperature distribution).

- They can be solved explicitly in the sense that we can develop a formula for every solution to (1.3.1).

- Their solutions exhibit rich theoretical behavior that can provide insight into solutions to much more complicated differential equations.

Toward the second point, we have already solved the homogeneous problem, in the slightly disguised form of Theorem 1.2.5.

1.3.3 Example.

Find all functions f satisfying

$$f'(x) + x\sin(x^2)f(x) = 0.$$

Solution. This equation is the same as

$$f'(x) = -x\sin(x^2)f(x).$$

Theorem 1.2.5 tells us that all solutions of this "new" equation have the form $f(x) = Ce^{H(x)}$, where $H'(x) = -x\sin(x^2)$. That is, we need to calculate

$$H(x) = \int x\sin(x^2)\ dx.$$

We substitute $u = x^2$ to find $du = 2x\ dx$, so $du/2 = x\ dx$ and therefore

$$\int x\sin(x^2)\ dx = \frac{1}{2}\int \sin(u)\ du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Since we only need one antiderivative, we will take $C = 0$. Then all solutions have the form

$$f(x) = C\exp\left(-\left[-\frac{1}{2}\cos(x^2)\right]\right) = C\exp\left(\frac{\cos(x^2)}{2}\right).$$

Here we are writing $\exp(X)$ instead of e^X. ▲

Here is the general result for homogeneous first-order linear equations.

1.3.4 Corollary.

Let p be continuous on the interval I and let P be an antiderivative of p. Then a differentiable function f defined on I solves

$$f'(x) + p(x)f(x) = 0 \tag{1.3.2}$$

for all x in I if and only if there is a constant C such that

$$f(x) = Ce^{-P(x)}.$$

Proof. It is a worthwhile exercise in calculus and algebra to check directly that

$$\frac{d}{dx}[Ce^{-P(x)}] + p(x)[Ce^{-P(x)}] = 0.$$

However, we can also rewrite (1.3.2) as

$$f'(x) = -p(x)f(x).$$

In the notation of Theorem 1.2.5, we would use $h(x) = -p(x)$. An antiderivative of h is $-P$, and so all solutions have the form $f(x) = Ce^{-P(x)}$, as claimed. ∎

1.3.5 Method: Solve $f'(x) + p(x)f(x) = 0$.

All solutions have the form $f(x) = Ce^{-P(x)}$, where $P'(x) = p(x)$. Take

$$P(x) = \int p(x)\ dx.$$

Omit the constant of integration.

Being able to solve the homogeneous problem $f' + p(x)f = 0$ does not immediately hand us a solution for the general nonhomogeneous problem (1.3.1). Instead, the solution procedure that we will develop relies on the following key, but perhaps wholly nonobvious, insights.

1. The left side of (1.3.1) is $f'(x)+p(x)f(x)$, and this looks vaguely like the product rule. Recall that if u is some function (some *u*nknown function at this moment), then

$$\frac{d}{dx}[f(x)u(x)] = f'(x)u(x) + f(x)u'(x).$$

There are two terms, one of which has a factor of $f'(x)$ and the other a factor of $f(x)$. In $f'(x) + p(x)f(x)$, there are two terms, and $f'(x)$ appears in one and $f(x)$ in the other. Is it possible to "convert" (1.3.1) into a problem in which the product rule appears?

2. Suppose that μ is a function such that $\mu(x) \neq 0$ for all x. Then f solves

$$f'(x) + p(x)f(x) = g(x)$$

if and only if f also solves

$$f'(x)\mu(x) + p(x)f(x)\mu(x) = g(x)\mu(x). \tag{1.3.3}$$

We have simply multiplied both sides of the original problem (1.3.1). That is, if (1.3.1) is true, then (1.3.3) must be true as well.

Conversely, if (1.3.3) is true, then because $\mu(x) \neq 0$ for all x, we can divide both sides of (1.3.3) by $\mu(x)$ to recover our original problem (1.3.1). It is culturally traditional to use the letter μ here, possibly to evoke "*mu*ltiply" (or, perhaps, "μltiply").

This is where we finished on Wednesday, January 19, 2022.

3. The expression

$$f'(x) + p(x)f(x) = f'(x) + f(x)p(x)$$

looks "almost" like the product rule involving the derivative of the product of f and another function. This expression is the sum of two terms, one of which has f' and the other of which has f.

The expression on the left of (1.3.3),

$$f'(x)\mu(x) + p(x)f(x)\mu(x) = f'(x)\mu(x) + f(x)[p(x)\mu(x)],$$

also looks like the product rule. Ideally, the product $p\mu$ should be the derivative of μ. That is, if μ satisfies

$$\mu'(x) = p(x)\mu(x),$$

then

$$f'(x)\mu(x) + f(x)[p(x)\mu(x)] = f'(x)\mu(x) + f(x)\mu'(x) = \frac{d}{dx}[f(x)\mu(x)].$$

4. So, suppose that we have a function μ with two properties: $\mu(x) \neq 0$ for all x and $\mu'(x) = p(x)\mu(x)$. Then our original problem $f'(x) + p(x)f(x) = g(x)$ is equivalent to

$$f'(x)\mu(x) + f(x)[p(x)\mu(x)] = \mu(x)g(x),$$

and this problem in turn is equivalent to

$$\frac{d}{dx}[f(x)\mu(x)] = \mu(x)g(x).$$

We can solve this third problem by direct integration:

$$f(x)\mu(x) = \int \mu(x)g(x)\, dx + C.$$

Here we are abusing notation (see, again, Remark 1.1.4) by using x for both the independent variable of f and μ and the dummy variable of integration on the right. Also, we are abusing $\int$ by letting $\int \mu(x)g(x)\ dx$ denote one particular antiderivative of the product μg, rather than a whole family. This is why we explicitly write the constant of integration C, as well.

Since $\mu(x) \neq 0$ for all x, we can solve for f:

$$f(x) = \frac{1}{\mu(x)} \int \mu(x)g(x)\ dx + \frac{C}{\mu(x)}. \tag{1.3.4}$$

5. Everything above seems to hinge on the existence of this function μ. Happily, existence is quite easy to establish. We first want μ to solve

$$\mu'(x) = p(x)\mu(x). \tag{1.3.5}$$

We know how to solve this equation, thanks to Theorem 1.2.5: take

$$\mu(x) = e^{P(x)},$$

where P is an antiderivative of p. (We might write $\mu(x) = e^{\int p(x)\ dx}$, again using $\int p(x)\ dx$ to denote one particular antiderivative of p. Also, Theorem 1.2.5 provides *all* solutions to (1.3.5); here we just need one, so there is no arbitrary constant in μ.)

Furthermore, we wanted $\mu(x) \neq 0$ for all x. Since μ is an exponential, we have something even stronger: $\mu(x) = e^{P(x)} > 0$ for all x. Thus we have satisfied the two conditions on μ from Step 4 above, and so we can proceed to solve the general problem $f' + p(x)f = g(x)$ via the formula in (1.3.4).

We will first apply the method sketched above to a concrete problem and then distill the general procedure.

1.3.6 Example.

Solve $f'(x) + 2xf(x) = e^{-x^2}$.

Solution. This equation has the form $f'(x)+p(x)f(x) = g(x)$ for $p(x) = 2x$ and $g(x) = e^{-x^2}$. We want to multiply the left side of the equation by a special function μ so that the resulting product equals the product rule derivative $(f\mu)'$. We take

$$\mu(x) = e^{\int p(x)\ dx} = e^{\int 2x\ dx} = e^{x^2}.$$

We do not bother including the constant of integration, since the method above only called for one particular antiderivative of p.

Then $f'(x) + 2xf(x) = e^{-x^2}$ is equivalent to

$$f'(x)e^{x^2} + 2xf(x)e^{x^2} = e^{-x^2}e^{x^2}.$$

Simplified and rearranged, this is

$$f'(x)e^{x^2} + f(x)[2xe^{x^2}] = 1.$$

The left side is

$$f'(x)e^{x^2} + f(x)[2xe^{x^2}] = \frac{d}{dx}[f(x)e^{x^2}].$$

Thus our original problem is the same as

$$\frac{d}{dx}[f(x)e^{x^2}] = 1.$$

We integrate both sides with respect to x and collect the constant of integration on the right side:

$$\int \frac{d}{dx}[f(x)e^{x^2}]\ dx = \int 1\ dx + C.$$

The integral and derivative on the left cancel each other out, and the right side is easy to integrate with the power rule. We find

$$f(x)e^{x^2} = x + C.$$

We solve for f:

$$f(x) = xe^{-x^2} + Ce^{-x^2}.$$

▲

1.3.7 Example.

Revisit Example 1.3.6 and match each calculation there with an operation on the general equation $f'(x) + p(x)f(x) = g(x)$, as sketched in Steps 1 through 5 above.

Solution. We work everything out in parallel. On the left we act on $f'(x)+p(x)f(x) = g(x)$ and on the right we solve the concrete problem $f'(x) + 2xf(x) = e^{-x^2}$.

$f'(x) + p(x)f(x) = g(x)$	$f'(x) + 2xf(x) = e^{-x^2}$
$f'(x)\mu(x) + p(x)f(x)\mu(x) = g(x)\mu(x)$	$f'(x)\mu(x) + 2xf(x)\mu(x) = \mu(x)e^{-x^2}$
$f'(x)\mu(x) + f(x)[p(x)\mu(x)] = \mu(x)g(x)$	$f'(x)\mu(x) + f(x)[2x\mu(x)] = \mu(x)e^{-x^2}$
Goal: $\mu'(x) = p(x)\mu(x)$	Goal: $\mu'(x) = 2x\mu(x)$
$\mu(x) = e^{P(x)},\ P(x) = \int p(x)\ dx$	$\mu(x) = e^{\int 2x\ dx} = e^{x^2}$
$f'(x)e^{P(x)} + f(x)[p(x)e^{P(x)}] = e^{P(x)}g(x)$	$f'(x)e^{x^2} + f(x)[2xe^{x^2}] = e^{x^2}e^{-x^2}$
$\frac{d}{dx}[f(x)e^{P(x)}] = e^{P(x)}g(x)$	$\frac{d}{dx}[f(x)e^{x^2}] = 1$
$\int \frac{d}{dx}[f(x)e^{P(x)}]\ dx = \int e^{P(x)}g(x)\ dx + C$	$\int \frac{d}{dx}[f(x)e^{x^2}]\ dx = \int 1\ dx + C$
$f(x)e^{P(x)} = \int e^{P(x)}g(x)\ dx + C$	$f(x)e^{x^2} = x + C$
$f(x) = Ce^{-P(x)} + e^{-P(x)}\int e^{P(x)}g(x)\ dx$	$f(x) = Ce^{-x^2} + e^{-x^2}x$

Arguably the most important steps were (1) figuring out what that special function μ should be and (2) multiplying through by μ to see the product rule on the left side. After that, the problem almost wrote itself with the direct integration. ▲

This "special function" μ has a very special name.

1.3.8 Definition.

An INTEGRATING FACTOR *for the differential equation $f'(x) + p(x)f(x) = g(x)$ is a function of the form $\mu(x) = e^{P(x)}$, where P is an antiderivative of p.*

1.3.9 Method: Solve $f'(x) + p(x)f(x) = g(x)$.

1. Find an integrating factor $\mu(x) = e^{\int p(x)\ dx}$. Omit the constant of integration in the antiderivative.

2. Multiply both sides of the differential equation by μ and recognize the left side as the product rule:

$$f'(x)\mu(x) + p(x)f(x)\mu(x) = g(x)\mu(x) \iff \frac{d}{dx}[f(x)\mu(x)] = g(x)\mu(x).$$

3. Integrate both sides and cancel the derivative on the left:

$$\begin{aligned}\frac{d}{dx}[f(x)\mu(x)] = g(x)\mu(x) &\iff \int \frac{d}{dx}[f(x)\mu(x)]\ dx = \int g(x)\mu(x)\ dx + C \\ &\iff f(x)\mu(x) = \int g(x)\mu(x)\ dx + C.\end{aligned}$$

4. Solve for f:

$$f(x) = \frac{1}{\mu(x)} \int g(x)\mu(x)\ dx + \frac{C}{\mu(x)}.$$

1.3.10 Example.

Find all functions f satisfying

$$\begin{cases} f'(x) + \dfrac{f(x)}{x} = x, & x > 0 \\ f(1) = 1. \end{cases}$$

Solution. This equation has the form $f'(x) + p(x)f(x) = g(x)$, where $p(x) = 1/x$ and $g(x) = x$. We calculate the integrating factor:

$$\mu(x) = e^{\int (1/x)\ dx} = e^{\ln(|x|)} = |x| = x,$$

since we are assuming $x > 0$ here.

We multiply both sides of our equation by $\mu(x) = x$ to find that f satisfies

$$f'(x)x + f(x) = x^2.$$

The left side really is a perfect derivative:

$$f'(x)x + f(x) = f'(x)x + f(x) \cdot 1 = \frac{d}{dx}[f(x)x],$$

and so f satisfies

$$\frac{d}{dx}[f(x)x] = x^2.$$

We integrate both sides to find

$$\int \frac{d}{dx}[f(x)x] \; dx = \int x^2 \; dx + C,$$

which gives

$$f(x)x = \frac{x^3}{3} + C.$$

Since $x > 0$, we may divide to find

$$f(x) = \frac{x^2}{3} + \frac{C}{x}.$$

To meet the initial condition $f(1) = 1$, we need

$$\frac{1^2}{3} + \frac{C}{1} = 1,$$

or

$$\frac{1}{3} + C = 1.$$

We solve for $C = 2/3$ and thus

$$f(x) = \frac{x^2}{3} + \frac{2}{3x}$$

solves the full initial value problem. ▲

This is where we finished on Friday, January 21, 2022.

1.3.11 Example.

Let r be any real number. Find all solutions to

$$f'(x) - rf(x) = e^{rx}.$$

Solution. This equation has the form $f'(x) + p(x)f(x) = g(x)$, where $p(x) = -r$ and $g(x) = e^{rx}$. (It is vitally important to include the minus sign on r in p.) The integrating factor is

$$\mu(x) = e^{\int -r \ dx} = e^{-rx}.$$

Multiplying both sides by μ, we find

$$f'(x)e^{-rx} - rf(x)e^{-rx} = e^{rx}e^{-rx}.$$

This simplifies to

$$f'(x)e^{-rx} + f(x)[-re^{-rx}] = 1.$$

Recognizing the product rule, we have

$$\frac{d}{dx}[f(x)e^{-rx}] = 1.$$

Integrating both sides, we find

$$f(x)e^{-rx} = \int 1 \ dx + C = x + C.$$

We solve for f:

$$f(x) = xe^{rx} + Ce^{rx}.$$

▲

1.3.12 Example.

Discuss the structure of solutions to

$$f'(x) + 3x^2 f(x) = e^{x^2 - x^3}.$$

Solution. This is a linear equation, and the coefficient on the f term is $p(x) = 3x^2$. The integrating factor is

$$\mu(x) = e^{\int 3x^2 \ dx} = e^{x^3}.$$

Multiplying both sides by μ, we find

$$f'(x)e^{x^3} + 3x^2 f(x)e^{x^3} = e^{x^2 - x^3}e^{x^3},$$

and this simplifies to

$$f'(x)e^{x^3} + f(x)[3x^2 e^{x^3}] = e^{x^2},$$

which in turn is equivalent to

$$\frac{d}{dx}[f(x)e^{x^3}] = e^{x^2}.$$

We integrate both sides to find

$$f(x)e^{x^3} = \int e^{x^2} \ dx + C. \tag{1.3.6}$$

Unfortunately, we cannot evaluate the integral on the right in terms of "elementary functions." The function $h(x) := e^{x^2}$ is continuous at each number x and consequently has an antiderivative defined on the whole real line; we just cannot find an "elementary" formula for this antiderivative. However, we recall the (perhaps less popular part of) the fundamental theorem of calculus, which tells us that the function

$$H(x) := \int_0^x e^{s^2}\,ds \tag{1.3.7}$$

is (1) defined for each real number x, since the integrand is continuous at all real numbers; (2) differentiable at all x; and (3) an antiderivative of h, since

$$H'(x) = \frac{d}{dx}\left[\int_0^x e^{s^2}\,ds\right] = e^{x^2}.$$

The symbol $\int e^{x^2}\,dx$ in (1.3.6) represents one particular antiderivative of $h(x) = e^{x^2}$, so we may as well replace it by the antiderivative H from (1.3.7). Thus our solution f satisfies

$$f(x)e^{x^3} = \int_0^x e^{s^2}\,ds + C,$$

and so

$$f(x) = e^{-x^3}\int_0^x e^{s^2}\,ds + Ce^{-x^3}.$$

This may not seem much better than (1.3.6), since we cannot evaluate the definite integral $\int_0^x e^{s^2}\,ds$ any more explicitly. However, numerical techniques could help us find approximate values to the integral for any given x. Additionally, we note that $H(0) = 0$; there was no need to take the lower limit of integration to be 0 — it could be any number — but a deft choice of lower limit can be helpful in solving initial value problems. Fundamentally, we should not be too concerned if we cannot evaluate an antiderivative explicitly, for we always have recourse to the definite integral; if h is continuous on an interval I, and x_0 is a point in I, then $H(x) := \int_{x_0}^x h(s)\,ds$ is an antiderivative of h on I. ▲

Sometimes a differential equation may be given in the form

$$r(x)f'(x) + p(x)f(x) = g(x). \tag{1.3.8}$$

At the values of x where $r(x) \neq 0$, this equation is equivalent to the linear problem

$$f'(x) + \frac{p(x)}{r(x)}f(x) = \frac{g(x)}{r(x)}, \tag{1.3.9}$$

and, in principle, we can solve (1.3.9) using the integrating factor method.

However, we must pay careful attention to the zeros (roots) of the function r. If g, p, and r are continuous, then the quotients p/r and g/r will only be continuous at those values of x for which $r(x) \neq 0$. Since continuity is a critical hypothesis to establish the existence of antiderivatives, and since (both abstractly and concretely) the integrating factor method hinges on the existence of antiderivatives, the location of the zeros of r, if they even exist, will determine the intervals on which solutions to (1.3.9) can be constructed.

1.3.13 Example.

Find all solutions to $xf'(x) + f(x) = xe^{x^2}$ *that are defined on* $(0, \infty)$.

Solution. This equation is not in the "standard form" $f'(x) + p(x)f(x) = g(x)$ because of the coefficient of x on $f(x)$. Since we are only interested in $x > 0$, we may divide by x to find that if f solves the given equation, then f also solves

$$f'(x) + \frac{f(x)}{x} = e^{x^2}.$$

We previously encountered the same left side in Example 1.3.10. As we did there, we take the integrating factor to be

$$\mu(x) = e^{\int (1/x)\ dx} = e^{\ln(|x|)} = |x| = x,$$

since we are assuming $x > 0$ here. Multiplying both sides by μ, we find

$$f'(x)x + f(x) = xe^{x^2},$$

which is exactly (and frustratingly) our original equation.

In other words, dividing by x and then beginning the integrating factor method took us back to where we started. However, perhaps doing so taught us that our original equation already had the form of the product rule:

$$xf'(x) + f(x) = \frac{d}{dx}[f(x)x].$$

Thus f satisfies

$$\frac{d}{dx}[f(x)x] = xe^{x^2},$$

and so

$$f(x)x = \int xe^{x^2}\ dx + C.$$

Substituting $u = x^2$ and $du = 2\ dx$, we have

$$\int xe^{x^2}\ dx = \int \frac{e^u}{2}\ du = \frac{e^u}{2} + C = \frac{e^{x^2}}{2} + C.$$

Thus

$$f(x)x = \frac{e^{x^2}}{2} + C,$$

and so

$$f(x) = \frac{e^{x^2}}{2x} + \frac{C}{x}.$$

It should be clear that this function f is not defined at $x = 0$. In fact, even if we take the very simple case of $C = 0$, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{x^2}}{2x} = \infty.$$

Curiously, if we *could* extend our solution to be defined at $x = 0$, the original differential equation would suggest very tame behavior. Evaluating the original ODE at $x = 0$, we find $0 \cdot f'(0) + f(0) = 0 \cdot e^{0^2}$. That is, we would expect $f(0) = 0$. The moral of this example is that we should not intuit behavior of f strictly from the problem $r(x)f'(x) + p(x)f(x) = g(x)$, which may not have solutions defined at all the values of x for which r is defined. ▲

This is where we finished on Monday, January 24, 2022.

We summarize all of our work above in the following theorem.

1.3.14 Theorem.

Suppose that p and g are continuous functions on the interval I. Let P be an antiderivative of p and B be an antiderivative of $e^{P(\cdot)}g$, i.e.,

$$P'(x) = p(x) \quad \text{and} \quad B'(x) = e^{P(x)}g(x). \tag{1.3.10}$$

Then all solutions f to

$$f'(x) + p(x)f(x) = g(x) \tag{1.3.11}$$

on the interval I have the form

$$f(x) = Ce^{-P(x)} + e^{-P(x)}B(x) \tag{1.3.12}$$

for some constant C. Conversely, any function of the form (1.3.12) *solves* (1.3.11).

Proof. To show that a function of the form (1.3.12) solves (1.3.11), one first needs to calculate $f'(x)$. This requires the identities, the product and chain rules, and some careful algebra; we leave this as an exercise.

To show that *all* solutions to (1.3.11) have the form (1.3.12), we use the ideas of Steps 1 through 5 above. Assume that f satisfies

$$f'(x) + p(x)f(x) = g(x).$$

Since p is continuous on I, there is an antiderivative P of p defined on I. Then

$$g(x)e^{P(x)} = f'(x)e^{P(x)} + f(x)[p(x)e^{P(x)}] = \frac{d}{dx}[f(x)e^{P(x)}].$$

Thus $fe^{P(\cdot)}$ is an antiderivative of $ge^{P(\cdot)}$, while B is another antiderivative of $ge^{P(\cdot)}$. Consequently, there is a constant C such that

$$f(x)e^{P(x)} = B(x) + C,$$

and so

$$f(x) = Ce^{-P(x)} + e^{-P(x)}B(x). \qquad \blacksquare$$

Here is how we interpret the structure (1.3.12) of solutions to the linear first-order problem (1.3.11).

1.3.15 Remark.

Theorem 1.3.14 tells us that every solution to $f'(x) + p(x)f(x) = g(x)$ has the form $f(x) = Ce^{-P(x)} + B(x)e^{-P(x)}$, where P is an antiderivative of p and B is a slightly more complicated function. Taking $C = 0$ here, we see that $f(x) = B(x)e^{-P(x)}$ is a "particular" solution to the nonhomogeneous problem $f'(x)+p(x)f(x) = g(x)$. Corollary 1.3.4 reminds us that taking $f(x) = e^{-P(x)}$ solves the homogeneous problem $f'(x) + p(x)f(x) = 0$. Thus the "general" solution to the nonhomogeneous problem is the sum of (1) a constant multiple of a (nontrivial!) solution to the homogeneous problem [$Ce^{-P(x)}$] and (2) a "variable-coefficient" multiple of the same solution to the homogeneous problem [$B(x)e^{-P(x)}$]. This structure will be very useful to keep in mind when we study equations with higher-order derivatives.

All of the initial value problems that we have so far considered have had unique solutions. This is no accident.

1.3.16 Corollary.

Suppose that p and g are continuous functions on the interval I. Let x_0 be a point in I and let y_0 be any real number. Then the initial value problem

$$\begin{cases} f'(x) + p(x)f(x) = g(x) \\ f(x_0) = y_0 \end{cases} \tag{1.3.13}$$

has a unique solution that is defined on I.

Proof. First we establish the existence of such a solution. Theorem 1.3.14 tells us that any solution to $f'(x) + p(x)f(x) = g(x)$ has the form $f(x) = Ce^{-P(x)} + B(x)e^{-P(x)}$, where $P'(x) = p(x)$ and $B'(x) = e^{P(x)}g(x)$. We are free to use any of the (infinitely many possible) antiderivatives P and B here. We will pick them so that $P(x_0) = B(x_0) = 0$. Then we will have

$$f(x_0) = Ce^{-P(x_0)} + B(x_0)e^{-P(x_0)} = Ce^0 + 0 \cdot e^0 = C,$$

and so we should take $C = y_0$.

Here is how we choose the antiderivatives. Put

$$P(x) := \int_{x_0}^{x} p(s)\ ds \quad \text{and} \quad B(x) := \int_{x_0}^{x} e^{P(s)}g(s)\ ds = \int_{x_0}^{x} e^{\int_{x_0}^{s} p(t)\ dt}g(s)\ ds. \tag{1.3.14}$$

Since $\int_{x_0}^{x_0} h(s)\ ds = 0$ for any function h, it follows that $P(x_0) = B(x_0) = 0$, as desired. Thus a solution to the initial value problem is

$$f(x) = y_0e^{-P(x)} + B(x)e^{-P(x)} \tag{1.3.15}$$

with P and B defined in (1.3.14).

To establish the uniqueness of the solution, suppose that f_1 is also a solution to (1.3.13). Define $h(x) := f(x) - f_1(x)$, where f is defined in (1.3.15). We leave it as an exercise to verify that h solves the homogeneous initial value problem

$$\begin{cases} h'(x) + p(x)h(x) = 0 \\ h(x_0) = 0 \end{cases} \tag{1.3.16}$$

and, moreover, that the only solution to (1.3.16) is the zero function. That is, $h(x) = 0$ for all x, and so $f(x) = f_1(x)$ for all x. ■

1.4. Nonlinear first-order equations I: separable equations.

1.4.1 Undefinition.

An equation involving a function and its first derivative that cannot be written as a linear equation (Definition 1.3.1) is **NONLINEAR**.

Typically nonlinear equations involve "powers" on the unknown function or "composition" of the unknown function with some other, given function. For example,

$$f'(x) + [f(x)]^2 = 0$$

cannot be written in the form $f'(x) + p(x)f(x) = g(x)$ for some functions p and g. There is simply no way to do this because of the squared term.

The outcomes for a nonlinear differential equation — whether or not solutions exist, just how unique those solutions are — can be fairly vast. We will study a few families of nonlinear equations that arise rather naturally and frequently in applications.

1.4.2 Definition.

Let p and q be functions. A **SEPARABLE FIRST-ORDER DIFFERENTIAL EQUATION** *is an equation of the form*

$$f'(x) = p(x)q(f(x)). \tag{1.4.1}$$

In Leibniz notation, this reads

$$\frac{dy}{dx} = p(x)q(y), \tag{1.4.2}$$

where on the right y is a function of x.

Differential equations like (1.4.1) and (1.4.2) are called "separable" because the product structure of their right sides "separates" the dependence on x from the dependence on y. It may not be clear where to start in solving a separable equation, but the following example suggests searching for some easy solutions.

1.4.3 Example.

Verify[2] *that $y = 0$ solves*

$$\frac{dy}{dx} = -2xy^2.$$

Solution. Since $y = 0$ is constant, we have

$$\frac{dy}{dx} = \frac{d}{dx}[0] = 0.$$

Also, we calculate

$$-2xy^2 = -2x \cdot 0^2 = 0$$

for all x. Thus

$$\frac{dy}{dx} = 0 = -2xy^2$$

for all x, and so $y = 0$ solves the equation. ▲

Here is the generalization of this example.

1.4.4 Theorem.

Suppose that p and q are functions and y_0 is a real number such that $q(y_0) = 0$. Then the constant function $y = y_0$ solves the separable equation

$$\frac{dy}{dx} = p(x)q(y).$$

A constant solution to this equation is an **EQUILIBRIUM SOLUTION** *for the equation.*

This is where we finished on Wednesday, January 26, 2022.

Proof. Since $y = y_0$ is constant, we have

$$\frac{dy}{dx} = \frac{d}{dx}[y_0] = 0.$$

And since $q(y_0) = 0$, we also have

$$p(x)q(y) = p(x)q(y_0) = p(x) \cdot 0 = 0$$

for all x. Thus

$$\frac{dy}{dx} = 0 = p(x)q(y)$$

for all x, and so $y = y_0$ satisfies the equation. ■

Of course, we should try to find more dynamic solutions than just the equilibrium ones.

[2] One might wish to keep in mind a distinction between the commands "Verify/check that y solves an equation" and "Solve the equation." For example, to verify that $y = 2$ solves $2y + 1 = 5$, one simply computes $(2 \cdot 2) + 1 = 4 + 1 = 5$. To solve the equation $2y + 1 = 5$, one first subtracts 1 from both sides to obtain $2y = 4$ and then divides by 2 to conclude $y = 2$. Using calculus and algebra, we should be able to check that a given function solves a differential equation even if we have no clue how to generate the solution in the first place.

1.4.5 Example.

Find solutions $y \neq 0$ *to*

$$\frac{dy}{dx} = -2xy^2.$$

Solution. 1. We work backwards: suppose that y is a solution. What can we discover about y? A natural idea is to try to integrate both sides:

$$\int \frac{dy}{dx}\ dx = \int -2xy^2\ dx,$$

but this only leads to

$$y = -2\int xy^2\ dx + C.$$

Since y should depend on x, we cannot factor y^2 out of the indefinite integral, and there is no hope of solving further for y.

2. However, since we are looking for nonzero solutions, let us also suppose that $y(x) \neq 0$, at least for all x in some interval I. Then we can divide by y^2:

$$\frac{dy}{dx}\frac{1}{y^2} = -2x. \tag{1.4.3}$$

Here is a key observation: the left side is the product of a function of x (namely, $1/y(x)^2$) and a derivative with respect to x. This is exactly what the chain rule says: if G is a function and y is a function of x, then

$$\frac{d}{dx}[G(y)] = G'(y)\frac{dy}{dx}.$$

3. Suppose that we have a function G such that

$$G'(y) = \frac{1}{y^2}. \tag{1.4.4}$$

Then

$$\frac{d}{dx}[G(y)] = G'(y)\frac{dy}{dx} = \frac{1}{y^2}\frac{dy}{dx}.$$

This allows us to rewrite (1.4.3) as

$$\frac{d}{dx}[G(y)] = -2x. \tag{1.4.5}$$

4. We integrate both sides of (1.4.5) with respect to x to find

$$\int \frac{d}{dx}[G(y)]\ dx = \int -2x\ dx + C$$

and thus

$$G(y) = -x^2 + C. \tag{1.4.6}$$

This is an *implicit* equation for y as a function of x. If we know more about G, perhaps it will be possible to solve for y explicitly as a function of x.

5. Recall from (1.4.4) that we want $G'(y) = y^{-2}$, and so

$$G(y) = \int G'(y)\ dy + K = \int y^{-2}\ dy + K = -y^{-1} + K,$$

where K is a constant of integration. (We are writing K, not C, here, since we have already used C, but we will soon absorb K into C anyway.)

Thus, per (1.4.6), we want y to satisfy

$$-y^{-1} + K = -x^2 + C.$$

We first combine both arbitrary constants K and C into the same constant C:

$$-y^{-1} = -x^2 + C.$$

Then we multiply both sides by -1 and absorb the -1 factor into C:

$$y^{-1} = x^2 + C.$$

We conclude

$$y = (x^2 + C)^{-1}.$$

6. This has been a long process, so we check that $y = (x^2+C)^{-1}$ solves the original equation:

$$\frac{dy}{dx} = \frac{d}{dx}[(x^2 + C)^{-1}] = -(x^2 + C)\frac{d}{dx}[x^2 + C] = -(x^2 + C)(2x),$$

while

$$-2xy^2 = -2x[(x^2 + C)^{-1}]^2 = -2x(x^2 + C)^{-2}.$$

This y therefore works. ▲

1.4.6 Example.

Solve each initial value problem.

(i) $\begin{cases} \dfrac{dy}{dx} = -2xy^2 \\ y(0) = 0 \end{cases}$

(ii) $\begin{cases} \dfrac{dy}{dx} = -2xy^2 \\ y(0) = 1 \end{cases}$

(iii) $\begin{cases} \dfrac{dy}{dx} = -2xy^2 \\ y(0) = -1 \end{cases}$

Solution. (i) The zero solution $y(x) = 0$ certainly works. The other solutions do not: if $y(x) = (x^2 + C)^{-1}$ satisfies $y(0) = 0$, then $C^{-1} = 0$. This is impossible; no real number C satisfies $C^{-1} = 0$.

(ii) The zero solution $y(x) = 0$ definitely does not work. Let us try $y(x) = (x^2 + C)^{-1}$. We need

$$1 = y(0) = (0^2 + C)^{-1} = C^{-1},$$

and so $C = 1$. Thus

$$y(x) = (x^2 + 1)^{-1} = \frac{1}{x^2 + 1}$$

solves the initial value problem.

(iii) Again, the zero solution $y(x) = 0$ cannot work. We try $y(x) = (x^2 + C)^{-1}$ and find

$$-1 = y(0) = C^{-1}$$

and thus $C = -1$. Hence

$$y(x) = (x^2 - 1)^{-1} = \frac{1}{x^2 - 1}$$

solves the initial value problem. ▲

1.4.7 Remark.

In Example 1.4.5, we found two kinds of solutions to

$$\frac{dy}{dx} = -2xy^2,$$

the constant solution $y(x) = 0$ and the infinite family of nonconstant functions $y(x) = (x^2 + C)^{-1}$. There is no choice of C such that $(x^2 + C)^{-1} = 0$ for all x, and so these two groups of solutions are distinct.

Moreover, we saw in Example 1.4.6 that depending on the choice of C, the solution may or may not be defined for all x. The solution $y(x) = (x^2 + 1)^{-1}$ is defined for all x, while $y(x) = (x^2 - 1)^{-1}$ is not defined at $x = \pm 1$. This contrasts with the solutions to the first-order linear problem: if p and g are defined on the interval I, then all solutions to $f'(x) + p(x)f'(x) = g(x)$ are defined on I (and maybe on a larger interval). In particular, all solutions are defined on the same interval. In the nonlinear problem of Examples 1.4.5-1.4.6, not all solutions are defined on the same interval. This is one illustration of the many ways that nonlinear problems can diverge from linear ones.

1.4.8 Method: Solve $\frac{dy}{dx} = p(x)q(y)$

1. Check for equilibrium solutions: solve $q(y) = 0$ if possible. The roots (zeros) of q, if q has any, are the equilibrium solutions.

2. Assume $q(y) \neq 0$ and rewrite the problem as

$$\frac{1}{q(y)}\frac{dy}{dx} = p(x).$$

3. Integrate both sides with respect to x:

$$\int \left(\frac{1}{q(y)} \frac{dy}{dx} \right) \, dx = \int p(x) \, dx + C.$$

Simplify $\int p(x) \, dx$ as much as possible.

4. Change variables in the integral on the left:

$$\int \left(\frac{1}{q(y)} \frac{dy}{dx} \right) \, dx = \int \frac{1}{q(y)} \, dy.$$

Simplify this integral as much as possible.

5. Obtain an implicit equation for y as a function of x:

$$\int \frac{1}{q(y)} \, dy = \int p(x) \, dx + C. \tag{1.4.7}$$

Try to solve for y explicitly as a function of x. Keep track of multiple distinct solutions.

This is where we finished on Friday, January 28, 2022.

1.4.9 Example.

Find all solutions to

$$\frac{dy}{dx} = \sin(x)(y-1)^3.$$

Solution. We follow exactly the steps of Method 1.4.8.

1. The differential equation has the form

$$\frac{dy}{dx} = p(x)q(y), \qquad \text{where} \qquad p(x) = \sin(x) \qquad \text{and} \qquad q(y) = (y-1)^2.$$

We solve $q(y) = 0$, which is $(y-1)^3 = 0$. This implies $y - 1 = 0$ and thus $y = 1$. That is, $y(x) = 1$ is an (and the only) equilibrium solution.

2. We assume $(y-1)^3 \neq 0$ and rewrite the problem as

$$\frac{1}{(y-1)^3} \frac{dy}{dx} = \sin(x).$$

3. We integrate both sides with respect to x:

$$\int \left(\frac{1}{(y-1)^3} \frac{dy}{dx} \right) \, dx = \int \sin(x) \, dx + C.$$

We evaluate

$$\int \sin(x)\ dx + C = -\cos(x) + C.$$

4. We evaluate

$$\int \left(\frac{1}{(y-1)^3} \frac{dy}{dx} \right)\ dx = \int \frac{1}{(y-1)^3}\ dy = \int (y-1)^{-3}\ dy = -\frac{(y-1)^{-2}}{2}.$$

5. We have therefore found

$$-\frac{(y-1)^{-2}}{2} = -\cos(x) + C. \tag{1.4.8}$$

Now we solve for y. First we have

$$(y-1)^2 = \frac{1}{2\cos(x) + C}.$$

We need to undo a square, so we will want to keep track of the positive and negative roots:

$$y - 1 = \pm \left(\frac{1}{2\cos(x) + C} \right)^{1/2}$$

and so there are two families of solutions:

$$y = \left(\frac{1}{2\cos(x) + C} \right)^{1/2} + 1 \quad \text{and} \quad y = -\left(\frac{1}{2\cos(x) + C} \right)^{1/2} + 1.$$

For the square root to be defined, we really need $2\cos(x) + C \geq 0$, and for the quotient to be defined, we really need $2\cos(x) + C \neq 0$. Thus our solutions only make sense if $2\cos(x) + C > 0$. There are a variety of ways to achieve this, depending on what value of C we choose. For example, since $-1 \leq \cos(x) \leq 1$ for all x, if we take $C > 2$, we will have $2\cos(x) + C > 0$, and all will be well. ▲

In Examples 1.4.5, 1.4.6, and 1.4.9, we derived nonconstant solutions to our separable problems by first *assuming* the existence of a solution to

$$\frac{dy}{dx} = p(x)q(y)$$

and then *forcing* x and y to satisfy an IMPLICIT relation of the form

$$F(x, y) = C$$

for some constant C. For example, if we put

$$F(x, y) := -\frac{(y-1)^{-2}}{2} + \cos(x),$$

then any solution y to

$$\frac{dy}{dx} = \sin(x)(y-1)^3$$

must satisfy $F(x, y) = C$ for some real number C, per (1.4.8). More precisely, the function y, which depends on x, satisfies

$$F(x, y(x)) = C$$

for all x in the domain of y. This is really just another way of rewriting the final conclusion (1.4.7) in Step 5 of Method 1.4.8.

We may or may not be successful in solving $F(x, y) = C$ "explicitly" for y as a "formula" depending on x. Nonetheless, since the expression $F(x, y) = C$ no longer involves derivatives, solving $F(x, y) = C$ is usually an "easier" problem than our original ODE.

1.4.10 Example.

(i) *Determine an expression $F(x, y)$ that depends on the variables x and y such that any solution y to*

$$\frac{dy}{dx} = \frac{3x^2}{2(y-1)}$$

satisfies $F(x, y) = C$ for some constant C.

(ii) *Solve the initial value problem*

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2}{2(y-1)} \\ y(0) = -1. \end{cases}$$

Solution. (i) This equation has the form

$$\frac{dy}{dx} = p(x)q(y), \qquad \text{where} \qquad p(x) = 3x^2 \qquad \text{and} \qquad q(y) = \frac{1}{2(y-1)}.$$

Consequently, $q(y) \neq 0$ for all $y \neq 1$, and q is not defined at $y = 1$. Thus there are no equilibrium solutions to consider.

We separate variables to find the nonconstant solutions:

$$2(y-1)\frac{dy}{dx} = 3x^2$$

and integrate:

$$\int 2(y-1)\frac{dy}{dx}\ dx = \int 3x^2\ dx + C = x^3 + C.$$

We simplify the y-integral:

$$\int 2(y-1)\frac{dy}{dx}\ dx = 2\int (y-1)\ dy = y^2 - 2y.$$

Then we want y to satisfy

$$y^2 - 2y = x^3 + C.$$

In other words, y satisfies the implicit relation

$$\underbrace{y^2 - 2y - x^3}_{F(x,y)} = C. \tag{1.4.9}$$

This is where we finished on Monday, January 31, 2022.

(ii) We know that any solution to the differential equation satisfies $F(x, y) = C$ for some constant C. That is, $F(x, y(x)) = C$ for all x at which y is defined. Rearranged, this reads

$$y^2 - 2y - (x^3 + C) = 0.$$

This is really a quadratic equation in y with an x-dependent coefficient. In general, we know that the solutions to

$$ay^2 + by + c = 0$$

are

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and here

$$a = 1, \qquad b = -2, \qquad \text{and} \qquad c = -(x^3 + C).$$

Thus the *function* y satisfies

$$y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)(x^3 + C)}}{2(1)} = 1 \pm \sqrt{x^3 + C}.$$

We therefore find two "branches" of solutions to the original differential equation:

$$y_+(x) = 1 + \sqrt{x^3 + C} \quad \text{and} \quad y_-(x) = 1 - \sqrt{x^3 + C}. \tag{1.4.10}$$

One of these branches will, hopefully, give us the solution to the initial value problem. We try

$$-1 = y_+(0) = 1 + \sqrt{0^3 + C} = 1 + \sqrt{C}$$

to find that if $y_+(0) = 1$, then C must satisfy $\sqrt{C} = -2$. This is impossible, since $\sqrt{C} \geq 0$ for all $C \geq 0$. (Another way to look at this is to note that $\sqrt{x^3 + C} \geq 0$ for all x and C satisfying $x^3 + C \geq 0$, and thus $y_+(x) \geq 0$ for all x at which y_+ is defined.)

We try the other branch: to have $y_-(0) = -1$, we need

$$-1 = y_-(0) = 1 - \sqrt{0^3 + C} = 1 - \sqrt{C},$$

which rearranges to $\sqrt{C} = 2$. The only value of C that works is $C = 4$. Thus the solution to the initial value problem is

$$y(x) = 1 - \sqrt{x^3 + 4}.$$

This, incidentally, is defined only for $x^3 + 4 \geq 0$, which is to say, for $x \geq (-4)^{1/3}$. ▲

1.4.11 Remark.

The implicit relation $F(x, y) = C$ *that a solution to a separable problem satisfies is not exactly unique. In Example 1.4.10, we had to integrate* $\int 2(y-1)\ dy$. *If we substitute* $u = y - 1$, *we find*

$$\int 2(y-1)\ dy = (y-1)^2.$$

As usual, we are omitting a constant of integration. Then the implicit relation that y *satisfies is*

$$(y-1)^2 - x^3 = C. \tag{1.4.11}$$

If we expand the square and rearrange, we find

$$F(x, y) + 1 = C,$$

where F *was defined in* (1.4.9). *Nonetheless, when we dive into the algebra and solve for* y, *we still find* $y = y_{\pm}$ *as in* (1.4.10). *Indeed, in this case it is probably easier to obtain* $(y-1)^2 = x^3 + C$ *from* (1.4.11) *and then take square roots. The point is that two different approaches can still yield the same correct explicit solution(s).*

This is where we finished on Wednesday, February 2, 2022.

1.5. Elementary modeling problems.

We will study two kinds of elementary models: temperature flow models and population growth models. Both problems will lead to first-order differential equations that can be solved via integrating factors and/or separation of variables. Our purpose here is to present universally accessible models (we all experience temperature, we are all more or less concerned with making or not making babies) that will demand and reinforce some of our symbolic techniques.

Broadly, any experience of mathematical modeling involves the following steps.

1. Contemplate some "physical" or "real-world" situation.

2. Convert (abstractify) that situation into mathematics. The mathematics should be both *physically reasonable* and *theoretically tractable*. We do not want a mathematical problem that does not reflect the "real world," nor do we want an all-encompassing model that is impossible to analyze.

3. Study the mathematical model. For us, that will mean solving a differential equation.

4. Interpret the solution. Does it broadly behave the way we physically expect? Revise the model of Step 2 as needed and repeat.

1.5.1. Temperature I: an object in an environment of constant temperature.

1.5.1 Undefinition.

The TEMPERATURE *of an object is the number that comes out when the thermometer goes in.*

1.5.2 Example.

Consider the following two situations.

1. *You take a stick of butter out of your freezer. The stick is frozen solid. You place the stick on your kitchen countertop and walk away. Your kitchen is kept at a constant temperature, say 70°F. What happens to the butter?*

2. *You preheat a saucepan on your stovetop to some hot temperature. The surrounding environment in the kitchen remains at a constant temperature. You take a frozen stick of butter out of the freezer and place it in the saucepan. What happens to the butter?*

Solution. (i) The butter should warm up and soften and reach, more or less, a temperature of 70°F. Once the butter's temperature has reached 70°F, it will not change. The butter should definitely not get colder by sitting out on the counter, nor should it suddenly become engulfed in flames and get much hotter than the ambient kitchen.

(ii) Again, the butter should get warmer. This time, the butter probably should melt entirely rather than just soften. Moreover, it is likely that the butter will get warmer *faster* in the heated saucepan than it would sitting on the countertop. In other words, it should take less time for the butter to increase in temperature by, say, 10°F when it is in the saucepan then when it is on the countertop. ▲

Here are some principles of temperature that, hopefully, the stories of the delicious butter illustrate.

1.5.3 Nontheorem.

An object is placed in an environment of constant temperature. The differences between the object and the environment are so significant that the object cannot change this constant temperature of the environment. No internal property of the object causes its temperature to change.

(T1) *Over time (possibly a long time) the object's temperature will roughly equal that of the environment.*

(T2) *If the object's temperature ever equals the environment's temperature, then the object's temperature will not change any more.*

(T3) *An object will change temperature faster if the difference between its temperature and the environment's is large. More generally, the rate at which the object's temperature changes should be proportional to the difference between its temperature and the environ-*

ment's temperature.

(T4) *If the temperature of the surrounding environment is greater than the object's initial temperature, the temperature of the object will increase. If the temperature of the surrounding environment is less than the object's initial temperature, the temperature of the object will decrease.*

Let $f(t)$ be the temperature of the object at time t and let U_* be the constant temperature of the surrounding environment. We will not be bothered with assigning units to time and temperature, but we will assume that time is always nonnegative: $t \geq 0$. Principle T1 suggests that

$$\lim_{t \to \infty} f(t) = U_*,$$

while Principle T2 suggests that if $f(t_*) = U_*$ for some time $t_* \geq 0$, then $f(t) = U_*$ for all time $t \geq t_*$.

Can we say anything about the temperatures $f(t)$ for "smaller" times than ∞ and when we know $f(t) \neq U_*$? The difference between the object's temperature and the environment's is $f(t) - U_*$. The rate of change of the object's temperature is, of course, $f'(t)$. Two time-dependent quantities A and B are proportional if there is a constant k such that $A(t) = kB(t)$ for all t; the constant k should not depend on time t. Thus Principle T3 above leads us to expect there to be a constant k such that

$$f'(t) = k[f(t) - U_*]$$

for all t. This constant k will somehow depend[3] on the material and the environment; if we change either, k should change. (The k that works for butter in your kitchen may not work for applesauce in your basement.)

Suppose that we have an initial temperature for our object: $f(0) = y_0$. Then temperature must satisfy the initial value problem

$$\begin{cases} f'(t) = k[f(t) - U_*] \\ f(0) = y_0. \end{cases} \tag{1.5.1}$$

This is **Newton's law of cooling**, and it is both a linear first-order problem[4] and a separable problem[5]. However we solve it, we find

$$f(t) = (y_0 - U_*)e^{kt} + U_*.$$

[3]Incidentally, although we just said that we will not care about units, the units of $f'(t)$ should be something like temperature degrees per time unit, e.g., °F/s. Since $f(t)$ and U_* are measured in temperature degrees, k should be measured in 1/time.

[4]Rewrite it as $f'(t) - kf(t) = -kU_*$. This has the familiar linear structure with $p(t) = -k$ and $g(t) = -kU_*$.

[5]Write y instead of f and find $\dfrac{dy}{dt} = k(y - U_*)$. This has the familiar separable structure with $p(t) = k$ and $q(y) = y - U_*$; we could also take $p(t) = 1$ and $q(y) = k(y - U_*)$. Either way, note that $y = U_*$ is the only equilibrium solution.

We have not said anything about the constant k. Since we want

$$\lim_{t\to\infty} f(t) = U_*,$$

we need

$$\lim_{t\to\infty} (y_0 - U_*)e^{kt} = 0.$$

If $y_0 \neq U_*$, we really need

$$\lim_{t\to\infty} e^{kt} = 0.$$

This happens if $k < 0$, but the limit above is ∞ if $k > 0$. (And if $k = 0$, then nothing interesting happens; in that case, f is constant, and $f(0) = y_0$ forces $f(t) = y_0$ for all t.)

In fact, we could have predicted the requirement $k < 0$ earlier using Principle T4. We know that f should satisfy

$$f'(t) = k[f(t) - U_*].$$

If for some t_* we have $f(t_*) = U_*$, then $f(t) = U_*$ for all $t \geq t_*$, and the problem is no longer interesting. So, either $f(t) < U_*$ for all t or $f(t) > U_*$ for all t. First uppose $f(t) < U_*$. Principle T4 suggests that the object's temperature is increasing: $f'(t) > 0$. Since $f(t) - U_* < 0$, we want $k < 0$, too. If, however, $f(t) > U_*$, then the object's temperature should be decreasing: $f'(t) < 0$. Since $f(t) - U_* > 0$ in this case, we want $k < 0$.

Thus we conclude that the constant k should always be negative, and so from now on we will replace k by $-\kappa$, where $\kappa > 0$.

This is where we finished on Monday, February 7, 2022.

1.5.4 Nontheorem.

Suppose that an object of initial temperature y_0 is placed in an environment of constant temperature U_*. There is a constant $\kappa > 0$ such that for all time $t \geq 0$, the temperature f satisfies the initial value problem

$$\begin{cases} f'(t) = -\kappa[f(t) - U_*] \\ f(0) = y_0, \end{cases}$$

and so f is

$$f(t) = (y_0 - U_*)e^{-\kappa t} + U_*. \tag{1.5.2}$$

Not only does this temperature model respect our physical intuitions from above, it also rigorously predicts the following "comparison" situation.

1.5.5 Example.

You make a cup of coffee to drink after a Zoom meeting. During your meeting, the cup stays in your kitchen, where the temperature is constant. You want your coffee to be as hot as possible when you drink it. Do you add the milk right after you make the coffee or *wait until after your meeting?*

Solution. Experience probably suggests that we should add the milk after the meeting. Mathematically, this is a tale of two initial value problems. Suppose that the fresh-out-of-the-pot coffee has the temperature y_0 and that your kitchen's temperature is U_*. Then there is a constant κ (which probably depends on your brand of coffee, your mug, and your kitchen, but definitely not time) such that your coffee's temperature, without adding the milk, is given by (1.5.2).

However, suppose you add the milk right after making the coffee. Then your coffee's initial temperature is $y_0 - m_0$, where m_0 is (probably) the temperature of the milk. Then your coffee's temperature f satisfies

$$\begin{cases} f'(t) = -\kappa[f(t) - U_*] \\ f(0) = y_0 - m_0, \end{cases}$$

at least if we assume that the amount of milk added is small enough not to affect the material constant κ from the no-milk situation above. The solution to this initial-value problem is

$$\widetilde{f}(t) = (y_0 - m_0 - U_*)e^{-\kappa t} + U_*;$$

just replace y_0 with $y_0 - m_0$ in (the nonproof of) Nontheorem 1.5.4. Since $y_0 - m_0 - U_* < y_0 - m_0$, we have $\widetilde{f}(t) < f(t)$ for all t. ▲

Our model also allows for a quick "experimental" prediction of temperature.

1.5.6 Example.

Explain why if we know the temperature of the object at two distinct points in time, then we know it for all time.

Solution. Suppose that one of these points in time is time $t = 0$, so we know $f(0) = y_0$. If $y_0 = U_*$, then $f(t) = U_*$ for all t and there is nothing further to discuss. So, let us assume $y_0 \neq U_*$.

Suppose as well that $f(t_1) = y_1$, where $t_1 > 0$. From (1.5.2), we have

$$y_1 = f(t_1) = (y_0 - U_*)e^{-\kappa t_1} + U_*.$$

Since $y_0 - U_* \neq 0$, we have

$$\frac{y_1 - U_*}{y_0 - U_*} = e^{-\kappa t_1}.$$

Incidentally, this presumes that both $y_1 - U_*$ and $y_0 - U_*$ are positive or both are negative; we leave the details as an exercise. Thus

$$-\kappa t_1 = \ln\left(\frac{y_1 - U_*}{y_0 - U_*}\right),$$

and so

$$\kappa = -\frac{1}{t_1}\ln\left(\frac{y_1 - U_*}{y_0 - U_*}\right) = \ln\left(\left[\frac{y_1 - U_*}{y_0 - U_*}\right]^{-1/t_1}\right).$$

This allows us to cast the formula (1.5.2) for f purely in terms of the five parameters y_0, y_1, y_0, U_*, and t_1, all of which are, in principle, experimentally verifiable. Using properties of exponentials and natural logarithms, we could also simplify this formula for f considerably; we leave the details as an exercise. ▲

Here is a possible deficiency of our model: while we certainly have

$$\lim_{t\to\infty}[(y_0 - U_*)e^{-\kappa t} + U_*] = U_*,$$

so that over long times the object's temperature is, roughly, the environment's, the model does not allow $f(t) = U_*$ for some $t > 0$ unless $f(0) = U_*$. That is, the object's temperature only equals the environment's if the object starts out at the same temperature as the environment.

1.5.7 Example.

Show that if f is defined by (1.5.2) *and $y_0 \neq U_*$, then $f(t) \neq U_*$ for all $t \geq 0$.*

Solution. Suppose to the contrary that we do have $f(t) = U_*$ for some time $t \geq 0$. Then

$$U_* = (y_0 - U_*)e^{-\kappa t} + U_*,$$

and so

$$(y_0 - U_*)e^{-\kappa t} = 0.$$

Since $e^{-\kappa t} > 0$ for all t, we must have $y_0 - U_* = 0$, thus $y_0 = U_*$. This contradicts the assumption $y_0 \neq U_*$. ▲

This is where we finished on Wednesday, February 9, 2022.

1.5.2. Temperature II: an object in an environment of variable temperature.

Up to now we have assumed that the environment's temperature is constant; it is quite reasonable, however, that the environment's temperature should not be constant. We still assume, though, that the object cannot influence the environment's temperature. Say that the environment's temperature is now $U(t)$ at time t. All of the reasoning above suggests that the object's temperature satisfies

$$\begin{cases} f'(t) = -\kappa[f(t) - U(t)] \\ f(0) = y_0. \end{cases} \tag{1.5.3}$$

The integrating factor method[6] tells us that

$$f(t) = y_0 e^{-\kappa t} + \kappa e^{-\kappa t} \int_0^t e^{\kappa s} U(s)\ ds; \tag{1.5.4}$$

we leave the full verification as an exercise.

If the environment's temperature is not constant, there is no reason to expect that the object's temperature should asymptote to some constant value, either. At the physical level, if the environment's temperature fluctuates, say, periodically, then the object's temperature too should increase and decrease with some periodicity. At the mathematical level, without further knowledge of the function U, there is no reason to expect that that the limit

$$\lim_{t \to \infty} e^{-\kappa t} \int_0^t e^{\kappa s} U(s)\ ds$$

should even exist.

1.5.8 Example.

Over a 24-hour period, an environment has an average temperature A. Suppose that the temperature is lowest at midnight and highest at noon. For simplicity, assume that these extremes are "symmetric around the average temperature": the lowest temperature is $A - r$ and the highest is $A + r$.

Show that if we start measuring time in hours, with $t = 0$ corresponding to midnight, then modeling the environment's temperature by

$$U(t) = A - r\cos\left(\frac{\pi t}{12}\right) \tag{1.5.5}$$

has all of these features.

Solution. Since $\cos(\cdot)$ is 2π-periodic, we check[7] that

$$U(t+24) = A - r\cos\left(\frac{\pi(t+24)}{12}\right) = A - r\cos\left(\frac{\pi t}{12} + 2\pi\right) = A - r\cos\left(\frac{\pi t}{12}\right) = U(t).$$

Thus U is 24-periodic.

We calculate the midnight temperature to be

$$U(0) = A - r\cos(0) = A - r$$

[6]This problem is not separable, as it reads

$$\frac{dy}{dt} = -\kappa y + \kappa U(t),$$

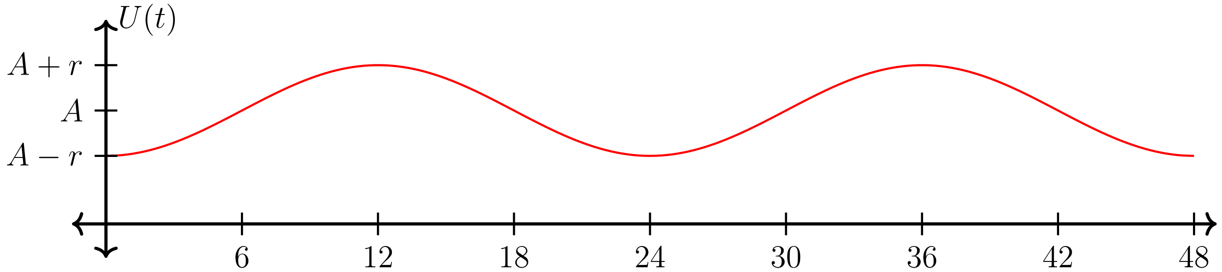
and the right side cannot be written as the product $p(t)q(y)$ if U is not constant.

[7]If a function f is P-**PERIODIC**, in the sense that $f(x+P) = f(x)$ for all x, and if $\omega \neq 0$, then $g(x) := f(\omega x)$ is P/ω-periodic. That is, $g(x + P/\omega) = g(x)$ for all x.

and the noon temperature to be

$$U(12) = A - r\cos(\pi) = A - r(-1) = A + r.$$

To see that these are the minimum and maximum temperatures, respectively, over a 24-hour period, we would need to determine the extrema of U on the interval $[0, 24]$. We leave this as an exercise.



Finally, the graph above suggests that A is the "average" value of U, and the rigorous definition of average value[8] leads us to calculate

$$\int_0^{24} U(t)\,dt = A,$$

a calculation that, yet again, we leave as an exercise. ▲

1.5.9 Example.

A cool rock of temperature y_0 is removed from a cool creek bed (at the stroke of midnight) and placed on a dry patch of the surrounding woods. Suppose the temperature $U(t)$ of this part of the woods is given by (1.5.5).

[8]The average value of an integrable function f on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(t)\,dt.$$

(i) *What will the temperature of the rock be?*

(ii) *Explain (in mathematical language) how the temperature of the rock behaves over long times.*

(iii) *Explain (in nonmathematical language) how the temperature of the rock behaves over long times.*

Solution. **(i)** Using the formula (1.5.4), the rock's temperature is

$$f(t) = y_0 e^{-\kappa t} + \kappa e^{-\kappa t} \int_0^t e^{\kappa s} \left[A - r\cos\left(\frac{\pi s}{12}\right)\right] \, ds$$

for some constant $\kappa > 0$. We can evaluate this integral with integration by parts ($u = A - r\cos(\pi s/12)$ and $dv = e^{\kappa s}$). After a fair amount of work, which we cheerfully omit, we find

$$f(t) = \left(y_0 - A + \frac{r}{1+\phi^2} \right) e^{-\kappa t} + A - \frac{r}{\sqrt{1+\phi^2}} \cos\left(\frac{\pi t}{12} - \psi\right), \tag{1.5.6}$$

where

$$\phi := \frac{\pi}{12\kappa} \quad \text{and} \quad \psi := \arctan(\phi).$$

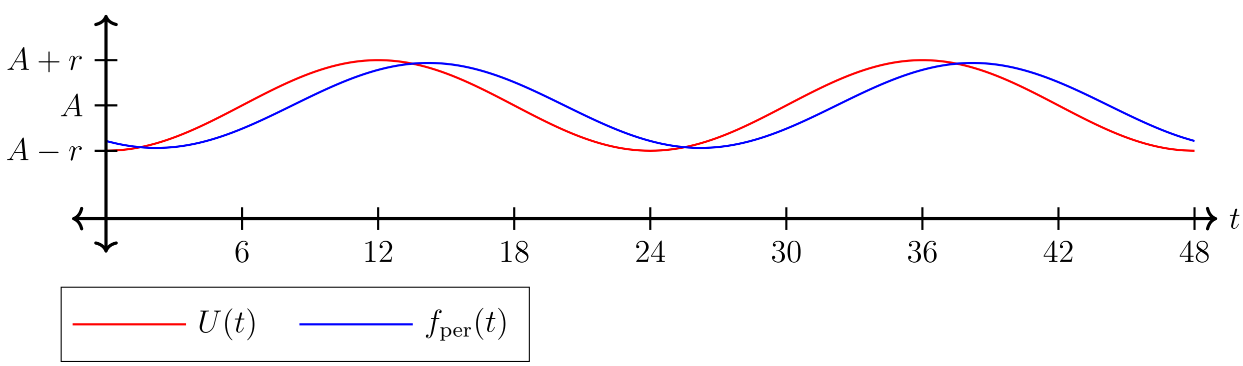
(ii) Since $\lim_{t\to\infty} e^{-\kappa t} = 0$, over long times the first term in (1.5.6) becomes negligible, and the real contribution to f comes from its "periodic" part, which we call

$$f_{\text{per}}(t) := A - \frac{r}{\sqrt{1+\phi^2}} \cos\left(\frac{\pi t}{12} - \psi\right).$$

This function f_{per} closely resembles the environmental temperature U, except for the denominator of $\sqrt{1+\phi^2}$ and the "phase shift" of ψ. Since $\kappa > 0$, we have $\phi > 0$, too, and therefore $1 + \phi^2 > 1$. Thus $1 < \sqrt{1+\phi^2}$, and so

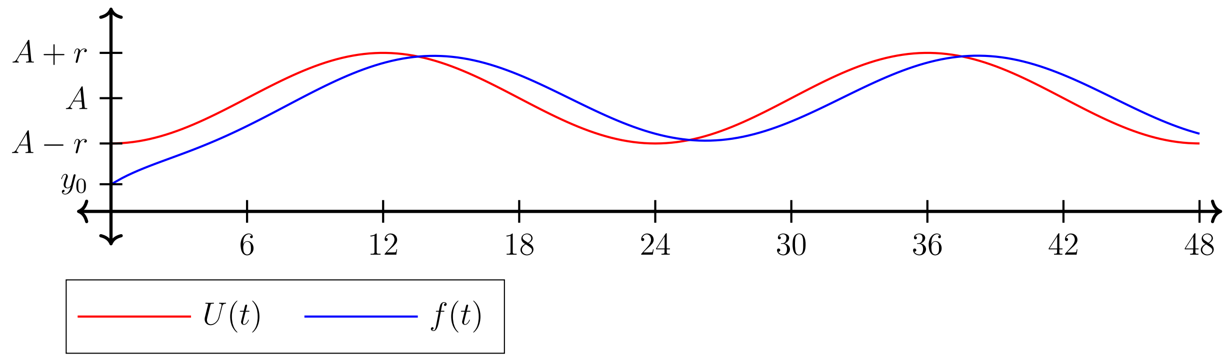
$$\frac{r}{\sqrt{1+\phi^2}} < r.$$

Thus the cosine term in f_{per} is slightly smaller than the cosine term in U. Next, since $\phi > 0$, we have $\psi = \arctan(\phi) > 0$, and so the graph of the cosine term in f_{per} is shifted slightly to the right of the graph of the cosine term in U. Here is a graph of U and f_{per} together, where we clearly see the "phase shift" in f_{per} compared to U.



(iii) We might expect that the temperature of the rock should "lag behind" the temperature of the environment. As the day dawns, first the environment heats up, and then some of that heat transfers to the rock.

We sketch below the graphs of the environment's temperature U and the rock's temperature f; we used a computer system to produce these graphs. We see that the extreme values of f occur slightly later in time than the extreme values of U.



The "delay" in the rock's heating corresponds to the phase shift ψ in the cosine term of f_{per}. The rock will not get quite as hot as the environment, perhaps because by the time the rock has warmed up, the environment is starting to cool, and this corresponds to the coefficient $r(1+\phi^2)^{-1/2} < r$ on the cosine term in f_{per}. ▲

Finally, we may want to consider the situation in which the object has some internal heat source/sink. For example, suppose the object is a building. In addition to the effect of the surrounding environment's temperature $U(t)$, the building may have an internal heating or cooling system, which contributes $H(t)$ temperature units per time unit. The inhabitants of the building could also affect its temperature; we lump their contribution into this function H. Then the building's rate of temperature change is no longer directly proportional to the difference between its internal temperature and the outside. Rather, a good model for building temperature f is

$$f'(t) = -\kappa[f(t) - U(t)] + H(t).$$

This is still a linear first-order equation for f, and so the integrating factor method still applies. A natural extension of this problem is a building with multiple rooms whose temperatures influence each other; such a situation would require a *system* of differential equations, one for each room.

1.5.3. Population growth I: exponential growth.

We consider two models for population growth. Ordinarily we think of a population as *integer*-valued: 30 students in a class, 6 million people in the Atlanta metropolitan area. If our population is very large, then it may make sense to count it using noninteger values: 5.85 million people, 61.5 thousand frisky rabbits, etc. For this reason we will model populations by *continuous* functions that may take noninteger values, rather than functions that take only staggered, "discrete" values.

First suppose that a population grows at a rate directly proportional to its current population. The more members there are, the faster the population grows. Let $y(t)$ denote the population at time t, with $t \geq 0$. Then by the assumption of direct proportionality, there is a constant a such that

$$y'(t) = ay(t).$$

We solved this differential equation all the way back in Example 1.2.3, and we have

$$y(t) = Ce^{at}$$

for some constant C. (We could also use separation of variables or an integrating factor.)

However, we do not quite want C to be any real number, and especially not a negative number. The initial population is

$$y(0) = Ce^{a \cdot 0} = Ce^0 = C.$$

Thus we have an immediate physical constraint on this otherwise arbitrary constant C. Here is what we have shown.

1.5.10 Nontheorem.

Suppose that a population grows with rate a directly proportional to its current population. If the population at time $t = 0$ is y_0, then the population at any time $t \geq 0$ is $y(t) = y_0 e^{at}$.

Our model and our conclusion are unrealistic for several reasons. First, the model does not take into account any factor that impedes population growth: limited resources, predators, disease, inter/intra-population conflict. Second, we have the long-time behavior

$$\lim_{t \to \infty} y(t) = y_0 \lim_{t \to \infty} e^{at} = \begin{cases} \infty, & a > 0 \\ 0, & a < 0. \end{cases}$$

Here we are assuming $a \neq 0$, as otherwise the population is the constant $y(t) = y_0$. Thus the population either explodes without bound or ultimately dies away, and neither of these are realistic scenarios. However, over shorter times (perhaps on the order of a few centuries for a nation's population), this exponential growth model is realistic and compatible with experimental/genealogical data.

This is where we finished on Friday, February 11, 2022.

1.5.4. Population growth II: logistic growth.

Suppose that we no longer demand that the growth rate of a population is directly proportional to the current population size. Instead, we just assume that the growth rate depends *in some way* on the current population. This is a very broad assumption, and mathematically we express it by saying

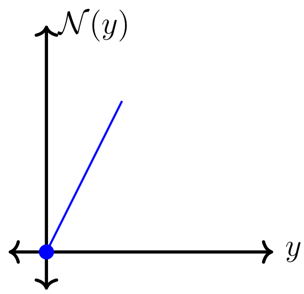
$$y'(t) = \mathcal{N}(y(t)), \tag{1.5.7}$$

where $y(t)$ is still the population at time t and $\mathcal{N}$ is some function. We will denote the independent variable of $\mathcal{N}$ by y. The symbol y is now playing dual roles, sometimes as the dependent variable of t, sometimes as the independent variable of $\mathcal{N}$.

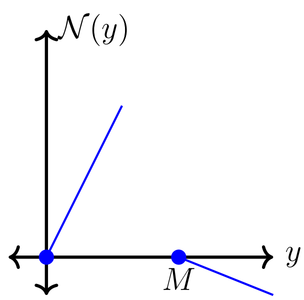
For exponential growth we took $\mathcal{N}(y) = ay$, where a is some constant. That is, $\mathcal{N}$ was linear. What sort of function $\mathcal{N}$ models a more nuanced sort of population growth? For a more nuanced model, we should look for a more complicated $\mathcal{N}$.

Here are two guiding assumptions on the growth rate of the population and the translation of those assumptions into "qualitative" features of $\mathcal{N}$.

1. *When the population is small, its growth rate is proportional to the existing population.* (This is exactly the assumption of exponential growth, but here it is only valid for a suitably *small* population.) In symbols, if $y \approx 0$, then $y' \approx ry$ for some constant r, and so we want $\mathcal{N}(y) \approx ry$ for $y \approx 0$.



2. *When the population is too large (for its environment, for existing resources, etc.), it should decrease.* This will prevent the unbounded sort of growth inherent to exponential growth. In symbols, there needs to be some population size $M > 0$ such that if $y(t) > M$ at time t, then $y'(t) < 0$. For $\mathcal{N}$, this means that if $y > M$, then $\mathcal{N} < 0$.



In summary, we want $\mathcal{N}$ to satisfy

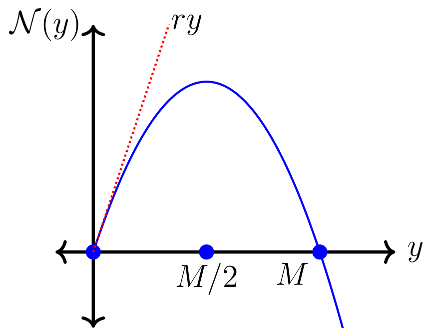
$$\mathcal{N}(0) = \mathcal{N}(M) = 0, \qquad y \approx 0 \Longrightarrow \mathcal{N}(y) \approx ry, \quad \text{and} \quad y > M \Longrightarrow \mathcal{N}(y) < 0.$$

There are many, many functions $\mathcal{N}$ that have this sort of behavior.

After some experimentation, a historically reasonable and popular choice is to set

$$\mathcal{N}(y) := ry\left(1 - \frac{y}{M}\right)$$

for some fixed positive constants r and M. Here is the graph of $\mathcal{N}$, which is a parabola.



When y is close to 0, we have $y^2 \approx 0$, and so

$$\mathcal{N}(y) = ry - \frac{ry^2}{M} \approx ry.$$

So, for y close to 0, $\mathcal{N}$ "grows linearly" like ry. The function $\mathcal{N}$ has a (global) maximum at $y = M/2$, and $\mathcal{N}(y) < 0$ for $y > M$.

Our population model y now satisfies

$$y' = ry\left(1 - \frac{y}{M}\right). \tag{1.5.8}$$

This is a separable differential equation, which we will solve shortly. It is called the **LOGISTIC EQUATION**.

We can predict many features of solutions to the logistic equation without using any formulas at all.

1.5.11 Example.

Suppose that y satisfies the initial value problem

$$\begin{cases} y' = ry\left(1 - \dfrac{y}{M}\right) \\ y(0) = y_0, \end{cases}$$

where $y_0 \geq 0$. What can you say about the shape of the graph of y and, if it exists, its limit at ∞?

Solution. We consider a number of cases on y_0, motivated by the observation that the equilibrium solutions of the underlying differential equation are $y = 0$ and $y = M$. Throughout, we will draw conclusions from both the formula and the graph of $\mathcal{N}$ above.

1. $y_0 = 0$. One solution in this case is $y(t) = 0$ for all t. (We do not yet have the technical tools to claim that the *only* solution in this case is $y = 0$.) Then the population never grows at all, which is probably unrealistic and definitely boring.

2. $y_0 = M$. One solution in this case is $y(t) = M$ for all t. Again, the population never grows (or decreases) at all, which is still unlikely and still boring.

3. $0 < y_0 < M$. Then $y'(0) = \mathcal{N}(y_0) > 0$, so the population is increasing at time $t = 0$. As long as $y(t) < M$, we will have $\mathcal{N}(y(t)) > 0$, so the population will continue to increase, provided that it does not hit the value of M. However, as y approaches M, the values of $\mathcal{N}(y)$ approach 0:

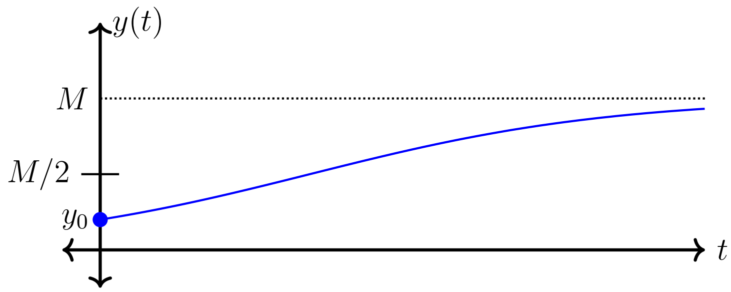
$$\lim_{y \to M^-} \mathcal{N}(y) = 0.$$

Thus as the population approaches the value of M "from below," its growth rate slows. We might conjecture[9] that the population's graph then "flattens out" just below M:

$$\lim_{t \to \infty} y(t) = M.$$

To be clear, we have certainly not proved that $y(t) \neq M$ for all t or $y(t) < M$ for all t.

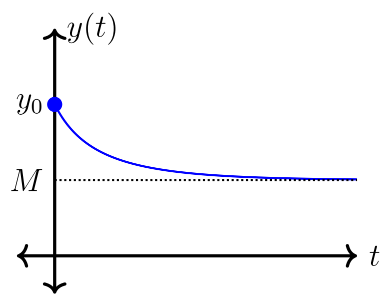
Taking into account these considerations, here is a possible graph of y.



4. $M < y_0$. Then $y'(0) = \mathcal{N}(y_0) < 0$, so the population is decreasing at time $t = 0$. As long as $y(t) > M$, we will have $\mathcal{N}(y(t)) < 0$, so the population will continue to decrease, provided that it does not hit the value of M. Again, as y decreases toward M, the values of $\mathcal{N}(y)$ approach 0:

$$\lim_{y \to M^+} \mathcal{N}(y) = 0.$$

Thus the rate of decrease of the population slows, and we might conjecture[10] that the population's graph asymptotes to M "from above."



[9] Here is a sketch of a deeper analysis of this conjecture, just to show the power of studying the differential equation $y' = \mathcal{N}(y)$ without further consideration of a formula for y. Recall that the sign of y'' determines the concavity of y. We are assuming $0 < y < M$, so $\mathcal{N}(y) > 0$. Since $y' = \mathcal{N}(y)$, we have $y'' = \mathcal{N}'(y)\mathcal{N}(y)$. Hence $y'' > 0$ if $\mathcal{N}'(y) > 0$, and $y'' < 0$ if $\mathcal{N}'(y) < 0$. Since $\mathcal{N}'(y) = r(1 - 2y/M)$, we have $\mathcal{N}'(y) > 0$ if $y > M/2$ and $\mathcal{N}'(y) < 0$ if $y < M/2$. Thus y is concave up provided that $0 < y < M/2$ and concave down when $M/2 < y$. In particular, if $y(t) \to M^-$ as $t \to \infty$, it follows that the graph of y is eventually concave down.

[10] More precisely, following the concavity reasoning of Footnote 9, we have $y''(t) < 0$ whenever $y(t) > M/2$. Consequently, if $y(t) > M$ for all t, then y is concave down for all time.

Again, this is largely a conjecture — a reasonable one, but one that we should back up with actual solutions (the existence of which we have not yet determined!) ▲

The value of the analysis in the preceding example is that it may be possible to predict features of solutions to a general differential equation of the form $y' = \mathcal{N}(y)$ based on knowledge of the initial condition of y and the function $\mathcal{N}$ but *not* using a formula for the solution. Indeed, in many cases a formula is difficult or impossible to construct, and even if it exists, its opaque structure may occlude any obvious observations. However, we can certainly solve the logistic equation. We do so below with $r = M = 1$, for simplicity.

1.5.12 Example.

Solve the separable problem

$$\begin{cases} \dfrac{dy}{dt} = y(1-y) \\ y(0) = y_0. \end{cases}$$

This is where we finished on Monday, February 14, 2022.

Solution. The algebraic steps in this problem are delicate and well worth understanding completely.

This is a separable problem, with $p(t) = 1$ and $q(y) = y(1-y) = \mathcal{N}(y)$. The equilibrium solutions are $y = 0$, 1, and otherwise we separate variables to find

$$\frac{1}{y(1-y)}\frac{dy}{dt} = 1.$$

We integrate with respect to t to find (after changing variables on the left)

$$\int \frac{1}{y(1-y)}\,dy = \int 1\,dt + C = t + C.$$

The y-integral calls for the dreaded method of partial fractions. Since[11]

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y},$$

we have

$$\int \frac{1}{y(1-y)}\,dy = \int \frac{1}{y}\,dy + \int \frac{1}{1-y}\,dy = \ln(|y|) - \ln(|1-y|) = \ln\left(\left|\frac{y}{1-y}\right|\right).$$

[11]Guess

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

Then

$$1 = A(1-y) + By = A - Ay + By = (B-A)y + A.$$

Each side is a polynomial in y, so the coefficients on corresponding powers must be equal. We have $A = 1$ and $B - A = 0$, so $B = A = 1$.

Here we substituted $u = 1 - y$ in the second integral.

Thus

$$\ln\left(\left|\frac{y}{1-y}\right|\right) = t + C,$$

and so exponentiating we find

$$\left|\frac{y}{1-y}\right| = e^C e^t.$$

Eliminating the absolute values, we have

$$\frac{y}{1-y} = e^C e^t \quad \text{or} \quad \frac{y}{1-y} = -e^C e^t.$$

Since e^C is an arbitrary positive number, we conclude

$$\frac{y}{1-y} = Ce^t \tag{1.5.9}$$

for some (other) arbitrary real number C. It may appear that we need $C \neq 0$, but taking $C = 0$ means $y/(1-y) = 0$, thus $y = 0$, and this is an equilibrium solution. So, it is legitimate to say that C can be any real number.

Now we must solve for y. Multiply by $1 - y$ to find

$$y = Ce^t(1-y) = Ce^t - yCe^t,$$

thus

$$y + Ce^t y = Ce^t, \tag{1.5.10}$$

and so[12]

$$y = \frac{Ce^t}{1 + Ce^t}. \tag{1.5.11}$$

Taking $t = 0$, we have

$$y_0 = y(0) = \frac{Ce^0}{1 + Ce^0} = \frac{C}{1+C}.$$

We could solve this algebraically for C in terms of y_0, but we would be repeating ourselves. Instead, use (1.5.9) to calculate

$$C = Ce^0 = \frac{y(0)}{1 - y(0)} = \frac{y_0}{1 - y_0}. \tag{1.5.12}$$

Incidentally, this presumes $y_0 \neq 1$; if $y_0 = 1$, then the equilibrium solution $y(t) = 1$ solves this initial value problem. Thus

$$y(t) = \frac{y_0 e^t}{(1 - y_0)\left(1 + \frac{y_0 e^t}{(1 - y_0)}\right)}. \tag{1.5.13}$$

[12] Depending on the choice of C and t, it is possible to have $1 + Ce^t = 0$, in which case our division by $1 + Ce^t$ to convert (1.5.10) into (1.5.11) could be questionable. Namely, we would need $e^t = -1/C$. This can only happen if $C < 0$. Below in (1.5.12) we will see that for our initial value problem, we have $C < 0$ if $y_0 > 1$. However, it can be shown that our ultimate solution (1.5.14) is defined for all $t \geq 0$ regardless of the value of y_0.

It will pay off later to simplify this algebraically now:

$$y(t) = \frac{y_0 e^t}{1 - y_0 + y_0 e^t} = \frac{y_0}{y_0 + (1 - y_0)e^{-t}}. \tag{1.5.14}$$

▲

We claim that the solution to the full logistic initial value problem

$$\begin{cases} y' = ry\left(1 - \frac{y}{M}\right) \\ y(0) = y_0 \end{cases}$$

is

$$y(t) = \frac{My_0}{y_0 + (M - y_0)e^{-rt}}.$$

and, moreover, that this solution can be deduced from Example 1.5.12. We defer the clever algebraic details to the exercises. However, it is a straightforward bit of arithmetic to show, indeed, $y(0) = y_0$, and since

$$\lim_{t\to\infty} e^{-rt} = 0,$$

provided that $r > 0$, we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{My_0}{y_0 + (M - y_0)e^{-rt}} = \frac{My_0}{y_0 + (M - y_0)\lim_{t\to\infty} e^{-rt}} = \frac{My_0}{y_0 + 0} = M.$$

Thus, over very long times, the population hovers around the carrying capacity, as we expected. A lengthier excursion into derivatives could also check the claims of Example 1.5.11 about the concavity of y, depending on the value of y_0.

1.6. The true definition of a differential equation.

We began our study of differential equations with the well-intentioned but vague Undefinition 1.0.1: a differential equation is an equation involving a function and one or more of its derivatives. We have now learned how to solve two broad classes of differential equations: linear equations, which have the form

$$f'(x) + p(x)f(x) = g(x) \tag{1.6.1}$$

for given functions p and g, and separable equations, which have the form

$$f'(x) = p(x)q(f(x)) \tag{1.6.2}$$

for given functions p and q. With a little reorganization, we can rewrite both problems in the same very general form, a form suggested by the abstract equation (1.5.7) that led to the logistic equation. It is this general form that will be the key to stating precisely what a differential equation is.

1.6.1 Undefinition.

Let I and J be intervals of real numbers. A **FUNCTION $\mathcal{N}$ OF TWO VARIABLES** *defined for x in I and y in J is a rule that assigns to each number x in I and y in J a third, unique number $\mathcal{N}(x, y)$.*

1.6.2 Example.

(i) *Setting $\mathcal{N}_1(x, y) := xy$ gives a function defined for all real numbers x and y.*

(ii) *Taking $\mathcal{N}_2(x, y) := xy^{-1}$ yields a function defined for all real numbers x and all $y \neq 0$.*

(iii) *Let p and g be functions defined on the same interval I. Then*

$$\mathcal{N}_3(x, y) := g(x) - p(x)y$$

is defined for all x in I and all real numbers y. In particular, the linear differential equation (1.6.1) *is equivalent to*

$$f'(x) = \mathcal{N}_3(x, f(x)).$$

(iv) *Let p be a function defined on I and q be a function defined on J. Then*

$$\mathcal{N}_4(x, y) := p(x)q(y)$$

is a function defined for all x in I and y in J. In particular, the linear differential equation (1.6.2) *is equivalent to*

$$f'(x) = \mathcal{N}_4(x, f(x)).$$

(v) *Let r and M be fixed positive numbers and define*

$$\mathcal{N}_5(x, y) := ry\left(1 - \frac{y}{M}\right).$$

Note that $\mathcal{N}_5$ is really independent of x. Then the logistic equation (1.5.8) *is the same as $f'(x) = \mathcal{N}_5(x, f(x))$.*

This is where we finished on Wednesday, February 16, 2022.

1.6.3 Example.

(i) *Let $\mathcal{N}_1(x, y) := e^{-x^2} - 2xy$. Then*

$$\mathcal{N}_1(0, 0) = 1, \qquad \mathcal{N}_1(1, e^{-1}/2) = 0, \qquad \text{and} \qquad \mathcal{N}_1(x, f(x)) = e^{-x^2} - 2xf(x).$$

The differential equation in Example 1.3.6 has the structure $f'(x) = \mathcal{N}_1(x, f(x))$.

(ii) *Let $\mathcal{N}_2(x, y) := -2xy^2$. Then $\mathcal{N}_2(1, 2) = -8$, while $\mathcal{N}_2(2, 1) = -4$. The differential equation in Example 1.4.5 has the structure $y'(x) = \mathcal{N}_2(x, y(x))$.*

Motivated by the common structures of the (few) differential equations that we have studied so far, we make the following sweeping definition.

1.6.4 Definition.

Let $\mathcal{N}$ be a function of the two variables x and y, defined for x in an interval I and y in an interval J. A FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION **(ODE)** *is an equation of the form*

$$f'(x) = \mathcal{N}(x, f(x)). \tag{1.6.3}$$

A SOLUTION *to* (1.6.3) *is a differentiable function such that*

(i) *f is defined on an interval I_0 that is contained in I;*

(ii) *$f(x)$ is in J for all x in I_0;*

(iii) *The equality* (1.6.3) *is true for each x in I_0;*

(iv) *f' is continuous on I_0.*

Now let x_0 be a point in I and y_0 be a point in J. A FIRST-ORDER INITIAL VALUE PROBLEM **(IVP)** *is a problem of the form*

$$\begin{cases} f'(x) = \mathcal{N}(x, f(x)) \\ f(x_0) = y_0. \end{cases} \tag{1.6.4}$$

A solution f to (1.6.4) *is a function satisfying conditions (i), (ii), (iii), and (iv) above, along with the final requirements*

(v) *$f(x_0) = y_0$;*

(vi) *I_0 has the form $I_0 = (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$, with $\delta = \infty$ allowed.*

The initial value problem (1.6.4) *is* WELL-POSED[13] WITH RESPECT TO I AND J *if it has a unique solution for each x_0 in I and y_0 in J.*

The careful observer may question why, or if, certain aspects of the definition above are necessary.

[13]This is an incomplete definition of the concept of "well-posedness." A fuller definition would take into account how the solutions depend on the initial conditions. Namely, if two initial conditions are "close," then the corresponding solutions should be "close." This is a property called "continuous dependence on the initial conditions," and of course it hinges on the right definition of "close." (If you shoot a rocket from one position and then fire the same kind of rocket from two inches over to the right, your rockets probably should have very similar trajectories.)

1.6.5 Example.

(i) *Why do we demand that solutions to differential equations be defined on intervals? For example, the function*

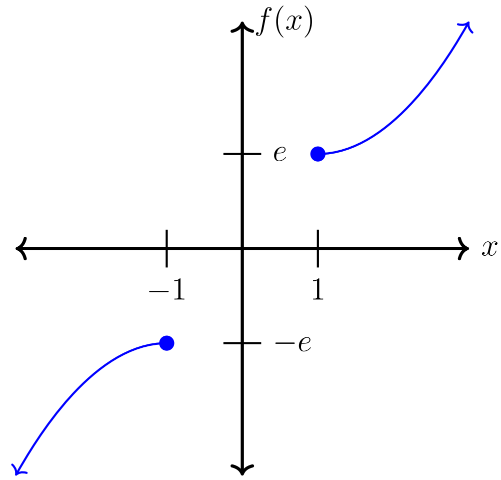
$$f(x) := \begin{cases} -e^x, & x \le -1 \\ e^x, & x \ge 1 \end{cases} \tag{1.6.5}$$

certainly satisfies $f'(x) = f(x)$ *for all numbers* $x < 1$ *and* $x > 1$*. However, we would not say that* f *is a solution to the differential equation* $f' = f$*.*

(ii) *Why do we demand that solutions to differential equations have continuous derivatives?*

(iii) *Why do we demand that first-order differential equations be written in the structure* $f'(x) = \mathcal{N}(x, f(x))$*?*

Solution. **(i)** If we are using a differential equation to model some "physical process," typically we expect that the solution should be defined over an unbroken interval. If the differential equation models a time-dependent process, like population growth, it would probably be very strange to care about modeling, say, the years between 1977 and 1983, and then, separately, the years between 1999 and 2005, with both ranges $[1977, 1983]$ and $[1999, 2005]$ enjoying the same model equation. Or perhaps our differential equation takes place over a spatial domain; it is unlikely that there would be a physical gap in the domain and that the same model would work over both parts of the domain. Here is the graph of the function f defined in (1.6.5), and we can very clearly see the "gap" in its domain.



At the mathematical level, working on intervals is quite nice. We integrate over intervals (which is the theoretical heart of separation of variables) and we draw conclusions from having derivatives valid over intervals (the integrating factor method). The structure of abstract existence arguments for proving the existence of solutions to initial value problems like (1.6.4) also tends to hinge on an interval basis for the problem.

(ii) At the physical level, a discontinuous first derivative suggests that the rate of change of the modeled phenomenon abruptly changes. Our experience suggests that this is not typically the case; if we are studying, say, the motion of a particle along a line or track, the particle does not suddenly teleport around (its position is continuous), nor does it abruptly change speeds (its velocity is continuous).

At the mathematical level, imposing a continuity requirement on the derivative certainly helps the abstract proof structure. It is also somewhat contrived to construct a function that is continuous and differentiable but whose derivative is continuous[14], and such functions do not "naturally" occur with much frequency.

(iii) This is primarily a mathematical, not physical, decision. The structure of the problem $f'(x) = \mathcal{N}(x, f(x))$ lends itself well to abstract existence/uniqueness arguments. There are plenty of worthwhile models that both involve derivatives and do not have the form $f'(x) = \mathcal{N}(x, f(x))$. ▲

1.6.6 Example.

Consider a population whose growth rate is proportional to the existing population. For simplicity, measure time in years. Previously we said that a model for this population is $y'(t) = ay(t)$.

However, suppose now that there is a time delay between birth and being able to reproduce (which, realistically, happens in most populations). Specifically, suppose that only members of the population that are at least d *years old can reproduce. Then the population's growth rate at time* t *really should be proportional not to the whole population at time* t *but rather to the population at* $t - d$*, for it is the population at time* $t - d$ *that will be* d *years old at time* t*. Thus a revised model could be*

$$y'(t) = ay(t - d). \tag{1.6.6}$$

The presence of the **DELAY** *of* $-d$ *prevents this equation from being a differential equation; we cannot find a function* $\mathcal{N}$ *of two variables such that* (1.6.6) *is equivalent to* $y' = \mathcal{N}(t, y)$*. After all, if* $\mathcal{N}$ *is just acting on the two real variables* t *and* y*, how could* $\mathcal{N}$ *introduce the "shift" by* $-d$*?*

Do all differential equations have solutions? And if there is a solution, is there only one? Our experience with linear first-order equations and separable equations suggests that the first question has an affirmative answer. Our experience with constants of integration and "free parameters" suggests that the second does not *always* have an affirmative answer, but our experience with initial value problems suggests that imposing an initial condition *may*

[14]Everyone's instinct is probably to turn the absolute value, but that does not pan out here — $f(x) := \int_0^x |s| \, ds$ is continuous and differentiable, and $f'(x) = |x|$ is continuous on $(-\infty, \infty)$ and just not differentiable at $x = 0$. Instead, taking

$$f(x) := \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

gives a function that is continuous and differentiable, yet whose derivative is not continuous (at $x = 0$, unsurprisingly).

tip the scales to the affirmative.

The reality is a bit more complicated than the special problems that we have studied; it is the structure of $\mathcal{N}$ greatly determines existence and uniqueness properties of solutions to (1.6.3) and (1.6.4). The most useful general conditions on $\mathcal{N}$ require some knowledge of multivariable calculus, which we are hesitant to presume. Instead, here are two kinds of $\mathcal{N}$ for which the corresponding initial value problem (1.6.4) does have a unique solution, i.e., for which it is well-posed.

1.6.7 Theorem.

Let I and J be open intervals and let $\mathcal{N}$ be a function defined for x in I and y in J. Then the initial value problem

$$\begin{cases} f'(x) = \mathcal{N}(x, f(x)) \\ f(x_0) = y_0 \end{cases}$$

is well-posed on I and J if $\mathcal{N}$ has either of the forms below.

(i) *Suppose that $p_0, p_1, \ldots, p_n$ are continuous functions defined on I and take $J = (-\infty, \infty)$. Define*

$$\mathcal{N}(x, y) := \sum_{k=0}^{n} p_k(x)y^k = p_0(x) + p_1(x)y + p_2(x)y^2 + \cdots + p_n(x)y^n.$$

(ii) *Suppose that p is continuous on I and q is continuous and differentiable on J. Suppose further that q' is continuous on J. Define*

$$\mathcal{N}(x, y) := p(x)q(y).$$

Nonproof. The true proof is rather beyond the scope of our course, but we can make some enlightening comments.

The function $\mathcal{N}$ defined in part (i) is essentially a polynomial in the variable y with x-dependent coefficient. We have actually proved part (i) in the very special case of $n = 1$. Here $\mathcal{N}(x, y) = p_0(x) + p_1(x)y$, and so the problem $f'(x) = \mathcal{N}(x, f(x))$ really is the first-order linear equation $f'(x) - p_1(x)f(x) = p_0(x)$.

In part (i), if $p_0 = p_1 = \cdots = p_n$, then the problem is really separable. Say that $p_k(x) = a_k p(x)$ for all x and each k, where a_k is a real number; then $\mathcal{N}(x, y) = p(x)q(y)$, where $q(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n$. Integrating the rational function $1/q(y)$, however, may not be pleasant.

The form of $\mathcal{N}$ in part (ii) is, of course, also separable; while we have a method for handling separable equations, there are no guarantees that we will be able to solve the implicit equation that results, or that we can solve it uniquely. This abstract theorem guarantees that we actually always can! ■

1.6.8 Example.

Explain why each initial value problem has a unique solution.

(i) $\begin{cases} y' = x^2y + xy^2 \\ y(0) = 0 \end{cases}$

(ii) $\begin{cases} y' = \dfrac{e^{x^2}}{\cos(y)} \\ y(0) = \pi \end{cases}$

Solution. (i) This differential equation has the form $y' = \mathcal{N}(x,y)$, where $\mathcal{N}(x,y) = x^2y + xy^2$. This function $\mathcal{N}$ is a polynomial in y, and its coefficients are continuous functions of x, where x can be any real number. That is, $\mathcal{N}(x,y) = p_1(x)y + p_2(x)y^2$, where $p_1(x) = x^2$ and $p_2(x) = x$. Take $I = J = (-\infty, \infty)$. Part (i) of Theorem 1.6.7 then guarantees that the initial value problem has a unique solution on some subinterval I_0 of $(-\infty, \infty)$ where 0 is a point of I_0.

(ii) This differential equation has the form $y' = \mathcal{N}(x,y)$, where $\mathcal{N}(x,y) = e^{x^2}/\cos(y^2)$. Take $p(x) = e^{x^2}$ and $q(y) = 1/\cos(y)$. Then p is continuous on $I = (-\infty, \infty)$, and q is continuous and differentiable, with q' itself continuous, on any interval J, such that $\cos(y^2) \neq 0$. Determining all such intervals J could be messy, but we are just concerned with the well-posedness of the initial value problem when $x_0 = 0$ and $y_0 = \pi$. Since $\cos(y) \neq 0$ for $\pi/2 < y < 3\pi/2$, we try $J = (\pi/2, 3\pi/2)$. Then q is continuous and differentiable on J, and $q'(y) = \sin(y)[\cos(y)]^{-2}$ is also continuous on J. Thus part (ii) of Theorem 1.6.7 guarantees that the initial value problem has a unique solution on some subinterval I_0 of $(-\infty, \infty)$ where 0 is a point of I_0. ▲

The size of the interval I_0 on which the unique solution to an initial value problem is defined is quite delicate. Such an interval may be very small or very large, and this depends on both the map $\mathcal{N}$ governing the differential equation and the value x_0 that determines when/where the initial condition occurs.

This is where we finished on Friday, February 18, 2022.

2. Higher-Order Linear Differential Equations

2.1. Linear constant-coefficient homogeneous second-order equations.

2.1.1. An exponential ansatz.

The following example, while seemingly trite, will offer us valuable insight into our new problem.

2.1.1 Example.

Let a and b be real numbers with $a \neq 0$.

(i) *Find all real numbers r such that $ar + b = 0$.*

(ii) *Find all functions f satisfying $af' + bf = 0$.*

Solution. (i) We subtract b from both sides to obtain $ar = -b$ and then, since $a \neq 0$, divide by a to find $r = -b/a$.

(ii) First we divide both sides by a (which is permissible since $a \neq 0$) to obtain

$$f' + \frac{b}{a}f = 0.$$

This is a first-order linear homogeneous equation. We could solve it with the integrating factor method, or we could use something like Corollary 1.3.4 to find that the solution is

$$f(x) = Ce^{\int -(b/a)\ dx} = Ce^{-(b/a)x},$$

where C is an arbitrary constant. ▲

Although the algebraic problem $ar + b = 0$ has only one solution, the real number $r = -b/a$, while the differential equation $af' + bf = 0$ has infinitely many solutions (each of which is a function, i.e., a collection of ordered pairs of real numbers), the quotient $-b/a$ appears in both solutions. This, combined with the fact that the solution to the differential equation is an exponential, is one source of the ideas that will help us solve the following problem.

2.1.2 Definition.

Let a, b, and c be real numbers, and suppose $a \neq 0$. A **second-order constant-coefficient linear homogeneous ordinary differential equation** *is an equation of the form*

$$af''(x) + bf'(x) + cf(x) = 0. \tag{2.1.1}$$

We comment on some of the adjectives present in this definition.

- The equation (2.1.1) is "constant-coefficient" because the prefactors on f'', f', and f are real numbers, i.e., constants. We did not single out constant-coefficient *first*-order equations for particularly special attention; the integrating factor method applies just as well to a problem of the form $f'(x) + p(x)f(x) = 0$, where p is an arbitrary function, as it does to $f'(x) + p_0 f(x) = 0$, where p_0 is an arbitrary *number*. It turns out that "variable-coefficient" equations that involve derivatives of order 2 or higher are quite challenging to solve; in particular, there is no general "integrating factor" by which we can multiply both sides of (2.1.1) to obtain a perfect derivative.

• Of course, (2.1.1) is "second-order" because of the af'' term. The requirement of $a \neq 0$ prevents this term from disappearing, which would reduce our problem to the happy first-order linear case.

• As before, (2.1.1) is "homogeneous" because the right side is 0. Later we will see how to solve nonhomogeneous problems.

• Informally, (2.1.1) is "linear" because no powers (other than 1 or 0) or compositions involving f'', f', or f appear. We will give a somewhat more precise definition of "linear" later.

Linear second-order differential equations are tremendously valuable objects of study. Here are some of their virtues.

• Many "physical" phenomena are governed by Newton's second law, "force equals mass times acceleration." Since acceleration is the second derivative of position, Newton's law immediately induces a second-order differential equation, i.e., an equation involving a second derivative. Many times, the right model is not just some arbitrary second-order differential equation but a linear constant-coefficient equation. Of course, variable-coefficient problems (replace, say, the constant b by a nonconstant function $b(x)$) do come up, but constant-coefficient problems have a lot of mileage.

• Mathematically, (2.1.1) is extremely versatile. It will be possible to construct solutions to a nonhomogeneous version of (2.1.1) using just the solutions to the homogeneous problem. The solution techniques for (2.1.1) generalize in a very direct way to higher-order problems in a way that our first-order techniques simply do not. (There is a sort of "discontinuity of difficulty" when one transitions from first-order problems to second-order and higher.) In short, if one can solve (2.1.1), one has a lot of power.

So, how does one go about solving $af'' + bf' + cf = 0$? Since[15] the corresponding first-order problem $af' + bf = 0$ has an exponential-type solution, we make the very lucky guess $f(x) = e^{rx}$ for the second-order problem. (A guess that a solution to a differential equation has a particular "form" is often called an **ANSATZ**.) Our task is now to figure out what r should be.

We calculate

$$f'(x) = re^{rx} \quad \text{and} \quad f''(x) = r^2 e^{rx}$$

[15]Here is another reason to make this guess. What function f interacts so nicely with its successive derivatives that adding af'', bf', and cf should cancel everything out? Certainly the exponential's derivatives "look a lot alike," so one might think that a suitable exponential could cause the whole sum to collapse to 0.

so that

$$af''(x) + bf'(x) + cf(x) = ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}. \tag{2.1.2}$$

So, if $f(x) = e^{rx}$ solves $af'' + bf' + cf = 0$, then $(ar^2 + br + c)e^{rx} = 0$. Since $e^{rx} > 0$ for all x and r, it must then be the case that $ar^2 + br + c = 0$. This is a quadratic equation, which we may solve with the quadratic formula, and we will do so momentarily. (Pause for a moment to appreciate the power of our exponential ansatz: we have turned a calculus problem into an algebraic problem!)

Conversely, suppose that r satisfies $ar^2 + br + c = 0$. Then, by (2.1.2), the function $f(x) = e^{rx}$ satisfies

$$af''(x) + bf'(x) + cf(x) = (ar^2 + br + c)e^{rx} = 0 \cdot e^{rx} = 0.$$

That is, $f(x) = e^{rx}$ solves our differential equation. Summing up, we have proved a nice little lemma.

2.1.3 Lemma.

Let a, b, c, and r be real numbers. The function $f(x) = e^{rx}$ solves the differential equation

$$af''(x) + bf'(x) + cf(x) = 0 \tag{2.1.3}$$

if and only if the real number r solves the quadratic equation

$$ar^2 + br + c = 0. \tag{2.1.4}$$

The quadratic equation (2.1.4) *is the* **CHARACTERISTIC (AUXILIARY) EQUATION** *of the differential equation* (2.1.3).

To solve this quadratic equation, we call upon the dreaded **QUADRATIC FORMULA**, which tells us that $ar^2 + br + c = 0$ if and only if

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The behavior of the solution(s) to this quadratic equation, and thus of the solution(s) to the differential equation, hinges on the so-called **DISCRIMINANT** of the quadratic equation $ar^2 + br + c = 0$, which is the number $b^2 - 4ac$. Since a, b, and c are real numbers, so is $b^2 - 4ac$, and so there are three possibilities for the discriminant:

$$b^2 - 4ac > 0 \quad \text{or} \quad b^2 - 4ac = 0 \quad \text{or} \quad b^2 - 4ac < 0.$$

Each of these possibilities will induce different behavior, both from the mathematical and physical viewpoints, in the solutions to $af'' + bf' + cf = 0$. We will start with the case of the positive discriminant, which is, in some sense, the simplest; moreover, many of the techniques that we will use in this case carry over very nicely to the other two cases.

2.1.2. The discriminant of the characteristic equation is positive.

Suppose that a, b, and c are real numbers with $a \neq 0$ and $b^2 - 4ac > 0$. Then there are two distinct real solutions to the quadratic equation $ar^2 + br + c = 0$; these are

$$r_+ := \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_- := \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2.1.5}$$

These are "distinct" in that $r_+ \neq r_-$; in fact, $r_- < r_+$. Lemma 2.1.3 then tells us that two solutions to $af'' + bf' + cf = 0$ are

$$f_1(x) := e^{r_1 x} \quad \text{and} \quad f_2(x) := e^{r_2 x}.$$

2.1.4 Example.

Find exponential-type solutions to $f'' - f = 0$.

Solution. This equation has the form $af'' + bf' + cf = 0$ with $a = 1$, $b = 0$, and $c = -1$. Hence its characteristic equation is $r^2 - 1 = 0$. This equation factors as the difference of perfect squares: $r^2 - 1 = (r+1)(r-1)$, and so its solutions are $r = 1$ and $r = -1$. Thus two solutions to the differential equation are $f_1(x) = e^x$ and $f_2(x) = e^{-x}$. ▲

Are these two solutions the only ones? (The eagle-eyed reader might spot a third, the "trivial" solution $f(x) = 0$ for all x.) Our experience with first-order equations, linear or not, taught us to expect a "free constant" in the solution; usually this constant arose as some kind of constant of integration. In these exponential solutions, there are no apparent constants.

Here we need a new (really, old) idea: the **LINEARITY** of the derivative. Recall that the derivative[16] is linear in the following sense. Suppose that g and h are differentiable functions and c is a real number. Then

$$(g + h)' = g' + h' \quad \text{and} \quad (cg)' = cg'.$$

Informally, derivatives "split up" over sums and derivatives "pull out" constants; these are rules for differentiation that we have probably used every day of our calculus-based lives without contemplating them too much. The linearity of the derivative suggests that we try adding constant multiples of solutions to a differential equation together.

2.1.5 Example.

Let c_1 and c_2 be real numbers. Show that $f(x) := c_1 e^x + c_2 e^{-x}$ solves $f'' - f = 0$.

Solution. Put $f_1(x) = e^x$ and $f_2(x) = e^{-x}$. We just saw that $f_1'' - f_1 = 0$ and $f_2'' - f_2 = 0$. Since

$$f' = (c_1 f_1 + c_2 f_2)' = (c_1 f_1)' + (c_2 f_2)' = c_1 f_1' + c_2 f_2'$$

by the linearity of the derivative, and similarly

$$f'' = c_1 f_1'' + c_2 f_2'',$$

[16] And the integral and limits in general.

a little algebra gives

$$f'' - f = c_1(f_1'' - f_1) + c_2(f_2'' - f_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0. \qquad \blacktriangle$$

2.1.6 Example.

Find as many solutions as possible to $f'' - 3f' + 2f = 0$.

Solution. The characteristic equation is $r^2 - 3r + 2 = 0$. Perhaps the factorization $r^2 - 3r + 2 = (r-1)(r-2)$ is obvious; if not, use the quadratic formula. Either way, the solutions are $r = 1$ and $r = 2$, so $f_1(x) = e^x$ and $f_2(x) = e^{2x}$ solve the differential equation. Then

$$f(x) = c_1 e^x + c_2 e^{2x}$$

is a solution for any choice of c_1 and c_2. ▲

This is where we finished on Monday, February 21, 2022.

2.1.7 Example.

Find as many solutions as possible to $f'' - 2f' - 5f = 0$.

Solution. The characteristic equation is $r^2 - 2r - 5 = 0$. This may not seem to factor easily, so we use the quadratic formula to solve for

$$r_{\pm} := \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)} = \frac{2 \pm \sqrt{24}}{2} = \frac{2 \pm 2\sqrt{6}}{2} = 1 \pm \sqrt{6}.$$

Thus $f_1(x) = e^{(1+\sqrt{6})x}$ and $f_2(x) = e^{(1-\sqrt{6})x}$ are solutions, and so

$$f(x) = c_1 e^{(1+\sqrt{6})x} + c_2 e^{(1-\sqrt{6})x}$$

is a solution for any choice of real numbers c_1 and c_2. ▲

We summarize our recent work.

2.1.8 Lemma.

Let a, b, and c be real numbers with $a \neq 0$ and assume $b^2 - 4ac > 0$, so the quadratic equation $ar^2 + br + c = 0$ has the two real roots $r_{\pm}$, given by (2.1.5). *Then*

$$f(x) := c_1 e^{r_+ x} + c_2 e^{r_- x} \tag{2.1.6}$$

solves $af'' + bf' + cf = 0$ for any real numbers c_1 and c_2. (In particular, taking $c_1 = c_2 = 0$ gives the **TRIVIAL** *solution $f(x) = 0$.)*

A natural question is now if, assuming $b^2 - 4ac > 0$, *all* solutions to $af'' + bf' + cf = 0$ have the form (2.1.6) for some constants c_1 and c_2. Recall that in our comprehensive theoretical treatment of the integrating factor method (Theorem 1.3.14), we showed that the solutions to $f' + p(x)f = 0$ that arose from the integrating factor method were the *only* solutions. Is the same true for these "linear combinations" of exponentials for our second-order problem? The answer is yes, but before we see how to arrive at it, it will pay to make a detour to (yet) another worthwhile concern: solving initial value problems.

Without further data imposed on our differential equation, there is no real way to see how to choose the constants c_1 and c_2 in (2.1.6). If, however, we demand that the solution and its derivative satisfy certain values at a certain input, then we can learn more.

2.1.9 Example.

Find a solution to

$$\begin{cases} f'' - f = 0 \\ f(0) = 1 \\ f'(0) = -1. \end{cases}$$

Solution. We know that taking $f(x) = c_1e^x + c_2e^{-x}$ solves $f'' - f = 0$ for any choice of constants c_1 and c_2. Let us try to pick the constants so that $f(0) = 1$ and $f'(0) = -1$. First, we calculate

$$f'(x) = \frac{d}{dx}[c_1e^x + c_2e^{-x}] = c_1e^x - c_2e^{-x}.$$

Thus we need

$$1 = f(0) = c_1e^0 + c_2e^{-0} = c_1 + c_2$$

and

$$-1 = f'(0) = c_1e^0 - c_2e^{-0} = c_1 - c_2.$$

This gives a linear system of equations for c_1 and c_2:

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = -1. \end{cases}$$

There are many ways to solve such a system; here is one. The second equation requires $c_1 = c_2 - 1$, which we substitute into the first equation to find $(c_2 - 1) + c_2 = 1$. That is, $2c_2 = 2$, and so $c_2 = 1$; hence $c_1 = 0$. The function $f(x) = e^{-x}$ thus solves the initial value problem. ▲

Later we will see the physical value of choosing initial conditions of the form

$$f(x_0) = y_0 \quad \text{and} \quad f'(x_0) = y_1$$

in modeling problems; for now, we just practice the mathematics.

2.1.10 Example.

Solve

$$\begin{cases} f'' - 3f' + 2f = 0 \\ f(0) = 0 \\ f'(0) = 1. \end{cases}$$

Solution. We know from Example 2.1.6 that taking $f(x) = c_1e^x + c_2e^{2x}$ solves $f'' - 3f' + 2f = 0$. We want

$$0 = f(0) = c_1 + c_2,$$

and we compute

$$f'(x) = c_1e^x + 2c_2e^{2x}$$

to find that we also need

$$1 = f'(0) = c_1 + 2c_2.$$

Thus c_1 and c_2 need to satisfy the linear system

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 + 2c_2 = 1. \end{cases}$$

The first equation requires $c_1 = -c_2$, so the second equation becomes $c_2 = 1$, which yields $c_1 = -1$. Thus a solution is $f(x) = -e^x + e^{2x}$. ▲

In the preceding two examples, we have actually done a little more than just construct a solution to an initial value problem: we showed that there was only one choice for the constants c_1 and c_2 such that a function of the form (2.1.6) would be a solution. This also generalizes nicely.

2.1.11 Lemma.

Let a, b, and c be real numbers with $a \neq 0$ and assume $b^2 - 4ac > 0$, so the quadratic equation $ar^2 + br + c = 0$ has the two real roots $r_\pm$, given by (2.1.5). Let x_0, y_0, and y_1 be real numbers. Then there exist unique c_1 and c_2 such that the function

$$f(x) := c_1e^{r_+x} + c_2e^{r_-x} \tag{2.1.7}$$

solves the initial value problem

$$\begin{cases} af'' + bf' + cf = 0 \\ f(x_0) = y_0 \\ f'(x_0) = y_1. \end{cases}$$

Proof. Lemma 2.1.8 tells us that f as defined in (2.1.7) solves $af'' + bf' + cf = 0$. We need to determine the right values for c_1 and c_2 and show that only one value works for each constant.

The only way for a function f of the form (2.1.7) to meet both $f(x_0) = y_0$ and $f'(x_0) = y_1$ is for c_1 and c_2 to satisfy the system

$$\begin{cases} e^{r_+ x_0}c_1 + e^{r_- x_0}c_2 = y_0 \\ r_+e^{r_+ x_0}c_1 + r_-e^{r_- x_0}c_2 = y_1. \end{cases} \tag{2.1.8}$$

There are a lot of letters and not many numbers in (2.1.8). But recall that x_0, y_0, and y_1 are all fixed numbers that are given to us in the statement of the initial value problem and that we have constructed r_1 and r_2 from the quadratic formula. Thus the only unknowns in (2.1.8) are c_1 and c_2.

Since $r_1 \neq r_2$, it is possible to solve (2.1.8) uniquely for c_1 and c_2. That is, we can show that (1) there are values for c_1 and c_2 that make (2.1.8) true and (2) these are the only values that work. We leave this as an exercise, but it is essentially the sort of work that we just did in Examples 2.1.9 and 2.1.10. ■

We now know that an initial value problem has only one solution that is a linear combination of exponential-type solutions. Incidentally, initial value problems are not the only worthwhile way to impose data on a differential equation; for example, one could demand some sort of "boundary" condition like $f(x_0) = f(x_1)$ at two different points $x_0 \neq x_1$. We will not treat such "boundary value problems" in this course, as both mathematically and physically they are more suited to a study of *partial* differential equations.

But, in general, are there solutions to $af'' + bf' + cf = 0$ that are not exponential-type? We claim that there are not, and we will show why by conjuring up an initial value problem. First, however, we need a specialized result.

2.1.12 Lemma.

Let a, b, c, and x_0 be real numbers with $a \neq 0$. Suppose that f solves the **HOMOGENEOUS INITIAL VALUE PROBLEM**

$$\begin{cases} af'' + bf' + cf = 0 \\ f(x_0) = 0 \\ f'(x_0) = 0. \end{cases}$$

Then $f(x) = 0$ for all x.

The proof is fairly technical, and so we leave it as optional reading in Appendix B.1. For contrast, recall that establishing that the only solution to the corresponding homogeneous first-order linear initial value problem (1.3.16) was so brief that we left it as an exercise. This is another instance of the, perhaps, unexpected difficulties that arise when studying second-order and higher-order equations. We also remark that Lemma 2.1.12 is valid for all constant-coefficient second-order linear homogeneous problems, not just those whose characteristic equation has a positive discriminant.

This is where we finished on Wednesday, February 23, 2022.

2.1.13 Theorem.

Let a, b, and c be real numbers with $a \neq 0$ and assume $b^2 - 4ac > 0$, so the quadratic equation $ar^2 + br + c = 0$ has the two real roots $r_\pm$, given by (2.1.5). Then every solution f to the differential equation $af'' + bf' + cf = 0$ has the form $f(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}$ for some constants c_1 and c_2.

Proof. Suppose that f is any solution to $af'' + bf' + cf = 0$. Pick some number x_0 in the domain of f and set $y_0 = f(x_0)$ and $y_1 = f(x_1)$. By Lemma 2.1.11, there exist (unique) numbers c_1 and c_2 such that taking $g(x) := c_1 e^{r_+ x} + c_2 e^{r_- x}$ solves the initial value problem

$$\begin{cases} ag'' + bg' + cg = 0 \\ g(x_0) = y_0 \\ g'(x_0) = y_1. \end{cases}$$

Now define $h(x) := f(x) - g(x)$. Our goal is to show that $h(x) = 0$ for all x, as this will give $f(x) = g(x)$ for all x, and thus f will have the desired form $f(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}$. We leave it as an exercise to check that h solves

$$\begin{cases} ah'' + bh' + ch = 0 \\ h(x_0) = 0 \\ h'(x_0) = 0, \end{cases} \tag{2.1.9}$$

and so Lemma 2.1.12 implies that $h(x) = 0$ for all x. ■

Theorem 2.1.13 guarantees that we can find all solutions to a linear homogeneous constant-coefficient second-order differential equation by studying its characteristic equation — at least if the characteristic equation has positive discriminant, or, equivalently, two real solutions. What happens if the discriminant behaves differently? It turns out that the "abstract" results above (Lemma 2.1.11 and Theorem 2.1.13) that seemed to rely very much on the exponential-type functions $e^{r_\pm x}$ are still valid for zero and negative values of the discriminant, if we replace $e^{r_\pm x}$ by "something else."

2.1.3. Interlude: the Wronskian and fundamental solution sets.

It may appear that all our work in Section 2.1.2 on the equation $af'' + bf' + cf = 0$ hinged on having solutions of the form $f_1(x) = e^{r_1 x}$ and $f_2(x) = e^{r_2 x}$ with $r_1 \neq r_2$. This is only superficially true. Far more important than the precise exponential form of these solutions was being able to solve a linear system of the form

$$\begin{cases} f_1(x_0)c_1 + f_2(x_0)c_2 = y_0 \\ f_1'(x_0)c_1 + f_2'(x_0)c_2 = y_1 \end{cases} \tag{2.1.10}$$

where x_0, y_0, and y_1 were given numbers. Such a system arose in treating the concrete initial value problems in Examples 2.1.9 and 2.1.10 and the general initial value problem in Lemma 2.1.11 and in proving that *all* solutions to $af'' + bf' + cf = 0$ had the form

$f(x) = c_1e^{r_1x} + c_2e^{r_2x}$, at least for the case of a positive discriminant in the quadratic equation $ar^2 + br + c = 0$ (this was Theorem 2.1.13).

Systems like (2.1.10) abound in contexts wholly unrelated to differential equations, and being able to solve them is quite worthwhile. The following lemma, whose proof we punt to Appendix B.2, tells us exactly when we can solve a system like (2.1.10).

2.1.14 Lemma.

Let d_{11}, d_{12}, d_{21}, d_{22}, y_0, and y_1 be real numbers. There exist unique numbers c_1 and c_2 that solve

$$\begin{cases} d_{11}c_1 + d_{12}c_2 = y_0 \\ d_{21}c_1 + d_{22}c_2 = y_1 \end{cases}$$

if and only if

$$d_{11}d_{22} - d_{12}d_{21} \neq 0.$$

Consequently, to be able to solve (2.1.10), it suffices to have

$$f_1(x_0)f_2'(x_0) - f_2(x_0)f_1'(x_0) \neq 0.$$

The quantity on the left above is really a function of three variables: the two functions f_1 and f_2 and the real number x_0. It has a special name, and some special properties.

2.1.15 Definition.

Let f_1 and f_2 be differentiable functions. The **Wronskian** *of f_1 and f_2 is the function*

$$\mathcal{W}[f_1, f_2](x) := f_1(x)f_2'(x) - f_1'(x)f_2(x)$$

2.1.16 Example.

Calculate the Wronskian $\mathcal{W}[f_1, f_2]$ of each pair of functions f_1 and f_2.

(i) $f_1(x) = e^{r_1x}$, $f_2(x) = e^{r_2x}$, *where r_1 and r_2 are numbers.*

(ii) $f_1(x) = e^{rx}$ *and* $f_2(x) = xe^{rx}$, *where r is a number.*

Solution. **(i)** We have

$$f_1'(x) = r_1e^{r_1x} \quad \text{and} \quad f_2'(x) = r_2e^{r_2x},$$

so

$$\begin{aligned} \mathcal{W}[f_1, f_2](x) &= e^{r_1x}(r_2e^{r_2x}) - e^{r_2x}(r_1e^{r_1x}) \\ &= r_2e^{r_1x+r_2x} - r_1e^{r_1x+r_2x} \\ &= (r_2 - r_1)e^{(r_1+r_2)x}. \end{aligned}$$

(ii) We have

$$f_1'(x) = re^{rx} \quad \text{and} \quad f_2'(x) = 1 \cdot e^{rx} + x(re^{rx}) = e^{rx} + rxe^{rx},$$

so

$$\begin{aligned}\mathcal{W}[f_1, f_2](x) &= e^{rx}[e^{rx} + rxe^{rx}] - xe^{rx}(re^{rx}) \\ &= e^{2rx} + rxe^{2rx} - rxe^{2rx} \\ &= e^{2rx}.\end{aligned}$$

▲

We can now recast what Lemma 2.1.14 says about the problem (2.1.10) in the language of the Wronskian.

2.1.17 Lemma.

Let f_1 and f_2 be differentiable functions and x_0, y_0, and y_1 be numbers. There exist unique numbers c_1 and c_2 that satisfy (2.1.10) *if and only if $\mathcal{W}[f_1, f_2](x_0) \neq 0$.*

We now make a bold claim: the proofs of Lemma 2.1.11 and Theorem 2.1.13 relied only on the fact that if $f_1(x) := e^{r_1 x}$ and $f_2(x) := e^{r_2 x}$ with $r_1 \neq r_2$, then $\mathcal{W}[f_1, f_2](x) \neq 0$ for all x. That is, these proofs did not really need f_1 and f_2 to be exponentials!

2.1.18 Theorem.

Suppose that f_1 and f_2 are functions that solve $af'' + bf' + cf = 0$ and, moreover, that $\mathcal{W}[f_1, f_2](x) \neq 0$ for all x.

(i) *Let x_0, y_0, and y_1 be numbers. There exist unique numbers c_1 and c_2 such that $f(x) = c_1 f_1(x) + c_2 f_2(x)$ solves*

$$\begin{cases} af'' + bf' + cf = 0 \\ f(x_0) = y_0 \\ f'(x_0) = y_1. \end{cases}$$

(ii) *Every solution to $af'' + bf' + cf = 0$ has the form $f(x) = c_1 f_1(x) + c_2 f_2(x)$ for some (unique) numbers c_1 and c_2.*

A pair of functions as in the theorem above deserves a special name.

2.1.19 Definition.

A FUNDAMENTAL SOLUTION SET FOR $af'' + bf' + cf = 0$ *is a pair of functions f_1 and f_2 such that*

$$af_1'' + bf_1' + cf_1 = 0 \quad \textit{and} \quad af_2'' + bf_2' + cf_2 = 0$$

with $\mathcal{W}[f_1, f_2](x) \neq 0$ for all x.

It appears, then, that solving $af'' + bf' + cf = 0$ just amounts to finding a fundamental solution set. We have already found a fundamental solution set in the event that the characteristic equation has positive discriminant; the fundamental solution set is $f_1(x) = e^{r_1 x}$ and $f_2(x) = e^{r_2 x}$, where $r_1 \neq r_2$ are the distinct roots of the characteristic equation. (That $\mathcal{W}[f_1, f_2](x) \neq 0$ in this case follows from part (i) of Example 2.1.16.) We now turn to finding fundamental solution sets in the case of a nonpositive discriminant.

2.1.20 Remark.

The reader with a background in linear algebra will note that if f_1 and f_2 are differentiable functions, then the condition $\mathcal{W}[f_1, f_2](x) \neq 0$ is equivalent to the invertibility of the matrix

$$\begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix}.$$

This, in turn, is equivalent to the linear independence of the set of vectors

$$\left\{ \begin{pmatrix} f_1(x) \\ f_1'(x) \end{pmatrix}, \begin{pmatrix} f_2(x) \\ f_2'(x) \end{pmatrix} \right\}$$

in $\mathbb{R}^2$. It turns out that — if f_1 and f_2 both solve the same differential equation $af'' + bf' + cf = 0$ — then the linear independence of this set is equivalent to a third condition: the linear independence of the set $\{f_1, f_2\}$, where we consider f_1 and f_2 as vectors in some function space; that is, f_1 is not a constant multiple of f_2. We will not use any of these concepts, however, in our course.

The reader without a background in linear algebra is strongly encouraged to obtain one; it is life-changing.

This is where we finished on Friday, February 25, 2022.

2.1.4. The discriminant of the characteristic equation is negative.

We will spend some time in mathematical explorations before actually getting to the point. The reader uninterested in these reflections can skip to Lemma 2.1.23 to see how to handle our problem $af'' + bf' + cf = 0$ in the event that the characteristic equation $ar^2 + br + c = 0$ has a negative discriminant.

Suppose that a, b, and c are real numbers with $a \neq 0$ and $b^2 - 4ac < 0$. In solving the quadratic equation $ar^2 + br + c = 0$, we want to use the quadratic formula,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

but this begs the question of how to interpret the square root of a negative number. The answer is to introduce a new *species* of number.

2.1.21 Undefinition.

We will use the symbol i to denote the expression satisfying $i^2 = -1$. A **COMPLEX NUMBER** *is an expression of the form $x + iy$, where x and y are real numbers. The* **REAL PART** *of $x + iy$ is $\mathrm{Re}(x + iy) := x$ and the* **IMAGINARY PART** *of $x + iy$ is $\mathrm{Im}(x + iy) := y$.*

All arithmetical operations on complex numbers are performed "as usual," e.g.,

$$(1 + 2i) + (3 + 4i) = (1 + 3) + (2i + 4i) = 4 + 6i$$

and

$$i(2-i) = 2i - i^2 = 2i - (-1) = 1 + 2i.$$

The vast, beautiful field of **COMPLEX ANALYSIS** concerns itself with the calculus of functions that are both defined on sets of complex numbers and that output complex numbers; in particular, complex analysis provides detailed answers to questions like "What does juxtaposing i and y in the expression $x + iy$ actually *mean*?"

The more mundane value of complex numbers for us is that they give meaning to square roots of negative numbers. Specifically, if $w \geq 0$, then we *define*

$$\sqrt{-w} := i\sqrt{w}. \tag{2.1.11}$$

Equivalently, if $v \leq 0$, then, again, we *define*

$$\sqrt{v} = \sqrt{-|v|} = i\sqrt{|v|}. \tag{2.1.12}$$

We wish to emphasize that (2.1.11) and (2.1.12) are *definitions*; previously we had (at least in this course) no firm grasp of how to take the square root of a negative number.

Our more mundane task is to determine a fundamental solution set for $af'' + bf' + cf = 0$ when the discriminant $b^2 - 4ac$ of the characteristic equation $ar^2 + br + c = 0$ is negative. The roots of the characteristic equation are

$$r_1 := -\frac{b}{2a} + i\frac{\sqrt{|b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 := -\frac{b}{2a} - i\frac{\sqrt{|b^2 - 4ac|}}{2a}.$$

Abbreviate

$$\alpha := -\frac{b}{2a} \quad \text{and} \quad \beta := \frac{\sqrt{|b^2 - 4ac|}}{2a} \tag{2.1.13}$$

to write the roots as

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta.$$

The exponential ansatz of Section 2.1.1 suggests that

$$f_1(x) := e^{(\alpha + i\beta)x} \quad \text{and} \quad f_2(x) := e^{(\alpha - i\beta)x}$$

solve $af'' + bf' + cf = 0$. However, what does $e^{(\alpha \pm i\beta)x}$ mean?

Here it is worthwhile to ask ourselves a more fundamental question: what does the symbol e^x mean when x is a real number? After all, we have been using exponentials since the beginning of our study of differential equations. We prefer to think of exponentials as *power series.*

2.1.22 Theorem.

Let x be a real number. Then the sum

$$e^x = \exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!}$$

always converges. Defining e^x in this way yields all the usual properties of exponentials, *including*

$$e^{x+y} = e^x e^y, \qquad (e^x)^y = e^{xy}, \qquad \textit{and} \qquad \frac{d}{dx}[e^x] = e^x.$$

Thus if x and y are real numbers, a reasonable idea is to *define*

$$e^{x+iy} := \sum_{k=0}^{\infty} \frac{(x+iy)^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(x+iy)^k}{k!}.$$

This, of course, begs the question if this limit actually converges in the manner of Theorem 2.1.22, and what this limit means, since, presumably, we have only ever defined limits involving *real* numbers. We will not get into these questions except to claim that, yes, the limit exists.

More remarkable is that if we split the sum over even and odd integers and use the identities

$$i^{2j} = (i^2)^j = (-1)^j \quad \text{and} \quad i^{2j+1} = (i^{2j})i = i(-1)^j,$$

valid for any integer j, then we have

$$\sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = \sum_{j=0}^{\infty} \frac{(iy)^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{(iy)^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} (-1)^j \frac{y^{2j}}{(2j)!} + i \sum_{j=0}^{\infty} (-1)^j \frac{y^{2j+1}}{(2j+1)!}.$$

But these are just the Taylor series[17] for cosine and sine:

$$\sin(y) = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos(y) = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!}$$

Then we have **Euler's formula**:

$$e^{iy} = \cos(y) + i\sin(y).$$

Now, we expect (although this, like everything else, also needs proof), that we have the familiar property

$$e^{x+iy} = e^x e^{iy}.$$

And so we can (more or less confidently) conclude

$$e^{x+iy} = e^x[\cos(y) + i\sin(y)] = e^x \cos(y) + ie^x \sin(y).$$

Thus, with α and β defined in (2.1.13), it appears that we have solutions to $af'' + bf' + cf = 0$ of the form

$$f_1(x) = e^{(\alpha+i\beta)x} = e^{\alpha x + i\beta x} = e^{\alpha x}[\cos(\beta x) + i\sin(\beta x)] = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x) \quad (2.1.14)$$

and (using the oddness of the sine to rewrite $\sin(-\beta x) = -\sin(\beta x)$)

$$f_2(x) = e^{\alpha x}\cos(\beta x) - ie^{\alpha x}\sin(\beta x). \quad (2.1.15)$$

[17] More precisely, these are the true *definitions* of cosine and sine.

This is strange. First, we are claiming that f_1 and f_2 are differentiable (actually, twice differentiable). What does it mean for a function that maps real numbers to complex numbers to be differentiable? Second, everything in the *statement* of the equation $af'' + bf' + cf = 0$ is real-valued; specifically, a, b, and c are real numbers. Is it fair for an equation with real coefficients to have a complex-valued solution? (Possibly: $r^2 + 1 = 0$ is a quadratic equation with real coefficients and the nonreal solutions $r = \pm i$.) If we are going to model something "real" with a second-order linear constant-coefficient differential equation, then having complex solutions is, at best, fishy.

There are several ways to pass from (2.1.14) and (2.1.15) to real-valued solutions of $af'' + bf' + cf = 0$.

- Observe that

$$\frac{f_1(x) + f_2(x)}{2} = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad \frac{f_1(x) - f_2(x)}{2i} = e^{\alpha x}\sin(\beta x)$$

If we believe that $(f_1 + f_2)/2$ and $(f_1 - f_2)/2i$ solve $af'' + bf' + cf = 0$, then we have two real-valued solutions above.

- If we believe that $\mathrm{Re}[f]$ and $\mathrm{Im}[f]$ solve $af'' + bf' + cf = 0$ whenever f does, then we have, again, the two real-valued solutions

$$\mathrm{Re}[f_1(x)] = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad \mathrm{Im}[f_1(x)] = e^{\alpha x}\sin(\beta x).$$

Both of the claims above are true, but they need justification. Toward the first point, we have never tried to add constant multiples of *complex-valued* solutions to $af'' + bf' + cf = 0$ to obtain new solutions to this problem. Toward the second, why should taking real/imaginary parts "commute with derivatives"?

Fortunately, we do not need to justify either point. (It is possible to do so, since performing calculus on functions that take complex, not just real, values is essentially the same as familiar, beloved real-valued calculus.) Rather, we now have a decent idea of what a fundamental solution set to $af'' + bf' + cf = 0$ in the case of negative discriminant *could* be, and once we have a candidate for a fundamental solution set, it is only a matter of time and derivatives until we know for sure that it works.

2.1.23 Lemma.

Suppose that the quadratic equation $ar^2 + br + c = 0$ has the distinct complex roots $r = \alpha \pm i\beta$, where α and β are real numbers and $\beta \neq 0$. Specifically, α and β are defined in (2.1.13)*. Then the functions*

$$f_1(x) := e^{\alpha x}\cos(\beta x) \quad \textit{and} \quad f_2(x) := e^{\alpha x}\sin(\beta x)$$

form a fundamental solution set for the equation $af'' + bf' + cf = 0$.

Proof. We need to show two things:

$$af_1'' + bf_1' + cf_1 = af_2'' + bf_2' + cf_2 = 0$$

and $\mathcal{W}[f_1, f_2](x) \neq 0$ for all x. Both are fairly thankless calculations.

First we claim that

$$af_1''(x) + bf_1'(x) + cf_1(x) = -e^{\alpha x}\left(\left(a(\beta^2 - \alpha^2) - \alpha b - c\right)\cos(\beta x) + 2\beta\left(a\alpha + \frac{b}{2}\right)\sin(\beta x)\right)$$

Earlier, in (2.1.13), we defined α and β in terms of a and b. Those definitions show[18] that $af_1'' + bf_1' + cf_1 = 0$, and the calculation for f_2 is similar.

Next, we claim that $\mathcal{W}[f_1, f_2](x) = \beta e^{2\alpha x}$. Since $\beta \neq 0$, we have $\mathcal{W}[f_1, f_2](x) \neq 0$ for all x. The full verifications of both calculations are worthwhile exercises in differentiation and algebra. ■

2.1.24 Example.

Find all solutions to each differential equation.

(i) $f'' + f = 0$

(ii) $f'' - 2f' + 5f = 0$

Solution. (i) The characteristic equation is $r^2 + 1 = 0$. We could solve this with the quadratic formula, but this equation is essentially why complex numbers were invented, so we just solve it in the straightforward way: $r^2 = -1$ and so $r = \pm\sqrt{-1} = \pm i$. In particular, $r = i = 0 + (1 \cdot i)$ is a root of the characteristic equation, so we take $\alpha = 0$ and $\beta = 1$ to find that

$$f_1(x) = e^{0 \cdot x}\cos(1 \cdot x) = \cos(x) \quad \text{and} \quad f_2(x) = e^{0 \cdot x}\sin(1 \cdot x) = \sin(x)$$

form a fundamental solution set for $f'' + f = 0$. Thus every solution to this problem has the form

$$f(x) = c_1\cos(x) + c_2\sin(x)$$

for some constants c_1 and c_2.

This is where we finished on Monday, February 28, 2022.

(ii) The characteristic equation is $r^2 - 2r + 5 = 0$, and we use the quadratic formula to find its roots:

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Here $\alpha = 1$ and $\beta = 2$, so

$$f_1(x) = e^{1 \cdot x}\cos(2x) = e^x\cos(2x) \quad \text{and} \quad f_2(x) = e^{1 \cdot x}\sin(2x) = e^x\sin(2x)$$

form a fundamental solution set. Thus every solution has the form

$$f(x) = c_1e^x\cos(2x) + c_2e^x\sin(2x)$$

[18] That $\beta^2 = (4ac - b^2)/4a^2$ is important, and maybe subtle.

for some constants c_1 and c_2. The interested reader might compare this solution to that of Example 2.1.7, where we studied $f'' - 2f' - 5f = 0$, and note the difference that $\pm 5f$ makes. ▲

2.1.25 Example.

Let ω be a real number with $\omega \neq 0$. Solve

$$\begin{cases} f'' + \omega f = 0 \\ f(0) = 0 \\ f'(0) = 1. \end{cases}$$

Solution. This is a differential equation with a parameter, ω, and so our solutions will depend on both x, the independent variable, and ω.

The characteristic equation is $r^2 + \omega^2 = 0$, so $r^2 = -\omega^2$, and therefore

$$r = \pm\sqrt{-\omega^2} = \pm i\sqrt{\omega^2} = \pm i|\omega|.$$

Since we do not know that $\omega > 0$, all we can say is $\sqrt{\omega^2} = |\omega|$. (For example, $\sqrt{(-4)^2} = 4 = |-4|$.)

Here $\alpha = 0$ and $\beta = |\omega|$, so all solutions have the form

$$f(x) = c_1 \cos(|\omega|x) + c_2 \sin(|\omega|x)$$

for some constants c_1 and c_2. We want to choose c_1 and c_2 so that $f(0) = 0$ and $f'(0) = 1$. Since $\sin(0) = 0$ and $\cos(0) = 1$, we calculate

$$0 = f(0) = c_1.$$

Thus, whatever c_2 is, we know $f(x) = c_2 \sin(|\omega|x)$. Then $f'(x) = c_2|\omega|\cos(|\omega|x)$, and so

$$1 = f'(0) = |\omega|c_2.$$

Hence $c_2 = 1/|\omega|$, and therefore the (only) solution is

$$f(x) = \frac{1}{|\omega|}\sin(|\omega|x).$$

Here it was important that $\omega \neq 0$. In fact, we have

$$\frac{1}{|\omega|}\sin(|\omega|x) = \frac{1}{\omega}\sin(\omega x);$$

we leave the details as an exercise. ▲

This is where we finished on Wednesday, March 2, 2022.

2.1.5. The discriminant of the characteristic equation is negative.

Since we are considering second-order equations with real coefficients a, b, and c, the discriminant $b^2 - 4ac$ is a real number as well and can be positive, or negative, or zero. We treat this last case here. If $b^2 - 4ac = 0$, then the only solution to the characteristic equation $ar^2 + br + c = 0$ is, per the quadratic formula,

$$r = r_* := -\frac{b}{2a}.$$

Such a root is sometimes called a **REPEATED** or **DOUBLE** root of the quadratic.

This immediately tells us that $f_1(x) := e^{r_* x}$ is a solution to $af'' + bf' + cf = 0$, but we know that we need another solution to construct a full fundamental solution set. By linearity, any scalar multiple of f_1 will also be a solution, but we can check that $\mathcal{W}[f_1, cf_1] = 0$ for any constant c. So, we need a more "interesting" second solution.

There are any number of ways to motivate the following result, and arguably none of them are convincing unless one has already been convinced of what is the right answer. We will sketch some of these motivations as exercises and elsewhere.

2.1.26 Lemma.

Suppose that a, b, and c are real numbers with $a \neq 0$ and that $r_* := -b/2a$ is the only root of the quadratic equation $ar^2 + br + c = 0$. Then the functions

$$f_1(x) := e^{r_* x} \quad \text{and} \quad f_2(x) = xe^{r_* x}$$

form a fundamental solution set for the equation $af'' + bf' + cf = 0$.

Proof. As before, we need to check that both

$$af_1'' + bf_1' + cf_1 = af_2'' + bf_2' + cf_2 = 0$$

and $\mathcal{W}[f_1, f_2](x) \neq 0$ for all x. We already know that f_1 is a solution to our homogeneous problem thanks to our preparatory work in Section 2.1.1. For f_2, we calculate

$$af_2''(x) + bf_2'(x) + cf_2(x) = \big((ar_*^2 + br_* + c)x + (2ar_* + b)\big)e^{r_* x}. \tag{2.1.16}$$

Since $r = r_*$ is a solution to (indeed, the only solution of) the quadratic equation $ar^2+br+c = 0$, we have

$$ar_*^2 + br_* + c = 0.$$

Moreover, since $r_* = -b/2a$, we have $2ar_*+b = 0$. Thus by (2.1.16) we have $af_2''+bf_2'+cf_2 = 0$.

As for the Wronskian, happily we have already calculated that. Part (ii) of Example 2.1.16 tells us that

$$\mathcal{W}[f_1, f_2](x) = e^{2r_* x} \neq 0$$

for all x. ■

2.1.27 Example.

Find all solutions to each differential equation.

(i) $f'' - 2f' + f = 0$

(ii) $f'' - 4f' + 4f = 0$

Solution. **(i)** The characteristic equation is $r^2 - 2r + 1 = 0$, and this factors to $(r-1)^2 = 0$. Hence the only root is $r = 1$, and so the functions $f_1(x) := e^x$ and $f_2(x) := xe^x$ form a fundamental solution set. Thus all solutions are

$$f(x) = c_1 e^x + c_2 x e^x.$$

(ii) The characteristic equation is $r^2 - 4r + 4 = 0$, and this factors to $(r-2)^2 = 0$. Hence the only root is $r = 2$, and so all solutions have the form

$$f(x) = c_1 e^{2x} + c_2 x e^{2x}.$$ ▲

2.1.6. Summary.

The form of solutions to the second-order linear constant-coefficient homogeneous problem $af'' + bf' + cf = 0$ hinges precisely on the behavior of the roots of the associated characteristic equation $ar^2 + br + c = 0$. First we wrap up all our recent hard work into one pat statement, which first relies on Definition B.1.4.

2.1.28 Theorem.

Let a, b, and c be real numbers with $a \neq 0$. Then there exists a fundamental solution set $\{f_1, f_2\}$ for $af'' + bf' + cf = 0$, so that every solution to this equation has the form $f = c_1 f_1 + c_2 f_2$ for some constants c_1 and c_2.

Proof. Consider the following cases on the discriminant $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$. Apply Theorem 2.1.13.

Case 2: $b^2 - 4ac < 0$. Apply Lemma 2.1.23.

Case 3: $b^2 - 4ac = 0$. Apply Lemma 2.1.26. ■

Here is the algorithmic summary of our work.

2.1.29 Method: Solve $af'' + bf' + cf = 0$

1. Find the roots of the characteristic equation $ar^2 + br + c = 0$. Factor the equation if that is easy; the quadratic formula always works.

2. The full solution depends on the sign of the discriminant, $b^2 - 4ac$. Below, c_1 and c_2 are always arbitrary constants.

(i) Distinct real roots ($b^2 - 4ac > 0$): if the characteristic equation has the real roots $r = r_1$ and $r = r_2$, with $r_1 \neq r_2$, then all solutions have the form

$$f(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(ii) Distinct complex roots ($b^2 - 4ac < 0$): if the characteristic equation has the complex roots $r = \alpha + i\beta$ and $r = \alpha - i\beta$, where $\beta \neq 0$ (and $i^2 = -1$), then all solutions have the form

$$f(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x).$$

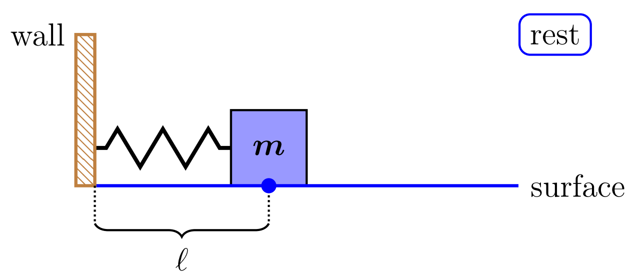
(iii) Repeated (double) real root ($b^2 - 4ac = 0$): if the characteristic equation has only one root, $r = r_*$, then all solutions have the form

$$f(x) = c_1 e^{r_* x} + c_2 x e^{r_* x}.$$

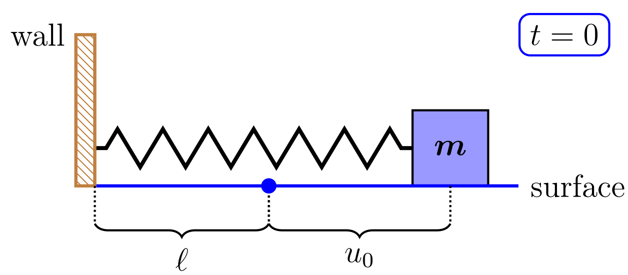
2.2. The undriven harmonic oscillator.

2.2.1. Construction of the model.

Suppose that an object of uniform mass m is placed on a level surface and attached by a spring to a nearby wall. When the spring is at rest the object is ℓ units away[19] from the wall. We call this "mass+spring" construct a **HARMONIC OSCILLATOR**.



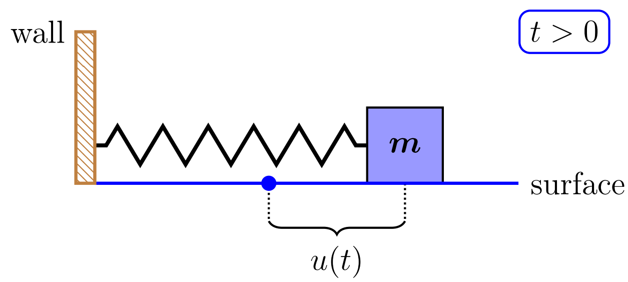
We pull (or, if you prefer, push) the object u_0 units away from this equilibrium position and let it go. What happens?



[19] We will not provide units in this treatment; feel free to bring your own from home. Also, we will act as though our "object" has zero spatial dimensions, so that the spring is ℓ units long at rest. We might also try saying that the object's "center of mass" is ℓ units from the wall, but that would require us to define a center of mass. Best not to think too hard about this.

No doubt the oscillator moves back and forth along the surface. (Let us suppose that we have rigged the universe so that the oscillator can only move horizontally, i.e., left and right, not up, down, in, or out.) Physical experience suggests that the oscillator probably will not move forever and also that the material properties of the oscillator and the environment — the size of the mass, the stiffness of the spring, the friction that the surface may or may not induce — will somehow affect the oscillator's motion.

In order to say anything really meaningful, we should introduce some kind of mathematical language and start asking mathematical questions. Specifically, we will measure how far the oscillator is from its equilibrium position. For time $t \geq 0$, let $u(t)$ denote the distance of the oscillator from equilibrium. In particular, $u(0) = u_0$.



Our point of departure is now **Newton's law**:

$$\text{force} = \text{mass} \times \text{acceleration}.$$

"Mass" is definitely m, and since acceleration is the second derivative of position, we can expect

$$mu''(t) = \text{force}.$$

What forces is the oscillator experiencing? The answer to this question completely determines the oscillator's behavior. Since we are studying in a mathematical world, which is not the real world, we can start very simply: assume that only the spring exerts a force on the oscillator. For now, there will be no friction, no air resistance, no random cat attacks — just the tugging of the spring.

2.2.2. The undamped, undriven harmonic oscillator.

Here we assume that the oscillator only experiences force due to the spring. The oscillator is **undamped** because there is no force of friction to "damp" or suppress its motion; it is **undriven** or **free** because there is no external[20] force "driving" its motion. We will consider damped oscillators shortly and driven oscillators not so shortly.

Experience suggests that when a spring is stretched, it pulls back in the opposite direction; that is, it exerts a force opposite to the direction of pulling. Experience also suggests that it becomes more difficult to stretch a spring once it has already been stretched a long distance. Thus if we pull the spring a distance r from its equilibrium length, we expect it to pull back with a force $\mathsf{F}_{\text{spr}}(r)$ in the direction opposite to the pull, and $\mathsf{F}_{\text{spr}}(r)$ should get larger as r

[20] External to the universe drawn above, consisting of surface, spring, and oscillator.

gets larger. There are many possible choices for the spring force function F_{spr}; perhaps the simplest comes from **Hooke's law**, which takes

$$\mathsf{F}_{\text{spr}}(r) := -\kappa r$$

for some $\kappa > 0$. Here the spring force is directly proportional to the distance pulled, which is r, and the negative sign reflects the notion that the spring pulls in the opposite direction.

Newton's law then tells us that

$$mu''(t) = -\kappa u(t), \tag{2.2.1}$$

since $u(t)$ is the distance that the spring has been pulled from its equilibrium length (ℓ). This is the second-order linear constant-coefficient homogeneous equation

$$mu'' + \kappa u = 0.$$

Its characteristic equation is

$$m\lambda^2 + \kappa = 0,$$

where now we are using λ instead of r for the variable in the quadratic equation. Thus

$$\lambda = \pm\sqrt{-\frac{\kappa}{m}} = \pm i\sqrt{\frac{\kappa}{m}} = \pm i\omega, \qquad \omega := \sqrt{\frac{\kappa}{m}}.$$

Here it is important that $\kappa/m > 0$.

All solutions to (2.2.1) therefore have the form

$$u(t) = c_1\cos(\omega t) + c_2\sin(\omega t). \tag{2.2.2}$$

2.2.1 Example.

Suppose that the oscillator is pulled a distance u_0 from its equilibrium position and then released "very gently," so that no force at all is exerted in letting it go. What is its displacement from equilibrium at time t?

Solution. One way to interpret the phrase "very gently" is to assume that the oscillator has no initial velocity (no extra kick/flick/oomph at the start). Then its displacement u must satisfy

$$\begin{cases} mu'' + \kappa u = 0 \\ u(0) = u_0 \\ u'(0) = 0. \end{cases}$$

By (2.2.2), we have $u(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$ for some constants c_1 and c_2, where $\omega = (\kappa/m)^{1/2} > 0$. We need

$$u_0 = u(0) = c_1\cos(0) + c_2\sin(0) = c_1$$

and $u'(0) = 0$, where

$$u'(t) = -c_1\omega\sin(\omega t) + c_2\omega\cos(\omega t).$$

That is, we need

$$0 = u'(0) = -c_1\omega \sin(0) + c_2\omega \cos(0) = c_2\omega,$$

and since $\omega > 0$ we can divide to solve for $c_2 = 0$. Thus the displacement satisfies

$$u(t) = u_0 \cos(\omega t) = u_0 \cos\left(\sqrt{\frac{\kappa}{m}}t\right). \tag{2.2.3}$$

▲

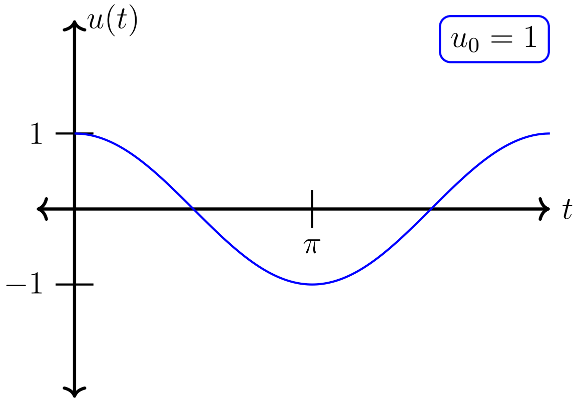
This is where we finished on Monday, March 14, 2022 ($\pi/\Pi/\varpi$-Day).

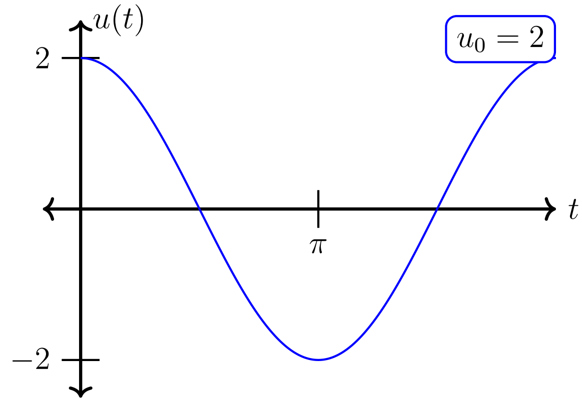
What are the effects of the three different parameters u_0, m, and κ in the solution (2.2.3)? How should they physically affect the oscillator's motion, and how is that reflected mathematically? Will the conclusions that we draw be physically realistic in (i) the context of our lived experience or (ii) under the very limited assumptions that we have made on the oscillator (in particular, the absence of all forces except the spring's force)? One way to address these questions is just to think about the function (2.2.3) with one parameter changing and the other two fixed.

2.2.2 Example.

How does changing the value of u_0 affect the behavior of the oscillator?

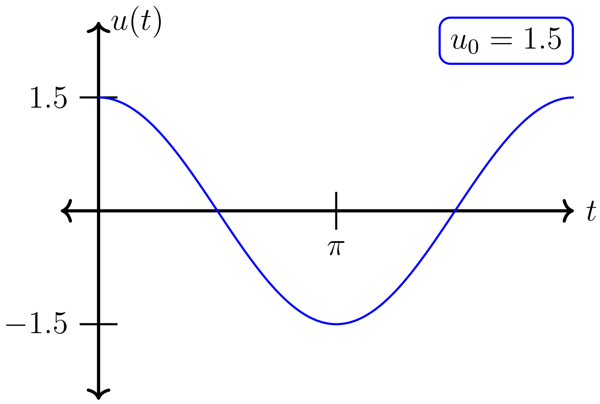
Solution. We graph $u(t) = u_0 \cos(t)$, as defined in (2.2.3) with $\kappa = m = 1$, for several values of u_0. All graphs are over the same time interval $[0, 2\pi]$.

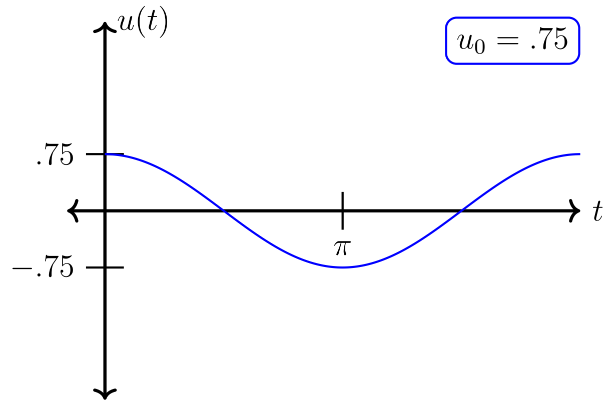




The values of $u(t) = u_0 \cos(t)$ can be any number in the interval $[-u_0, u_0]$, and u takes these values with 2π-periodicity. That is, the oscillator "oscillates" forever between u_0 units to the right and to the left of its equilibrium position. Note in particular that the oscillator has reached the equilibrium position whenever the graph of u crosses the t axis; since $u(t)$ is the distance of the oscillator from equilibrium, the oscillator is *at* equilibrium if and only if $u(t) = 0$.

We should immediately object to the notion of an oscillator that oscillates forever; surely this is impossible, as all things slow down eventually. However, recall that here we are studying an oscillator without friction or any other force to sap, or "damp," its motion.



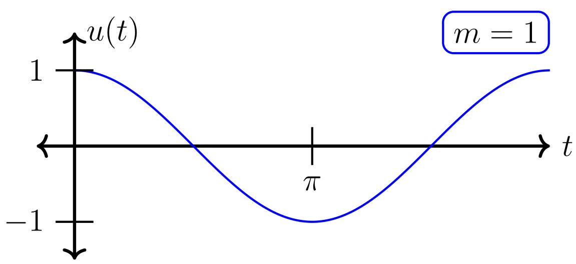


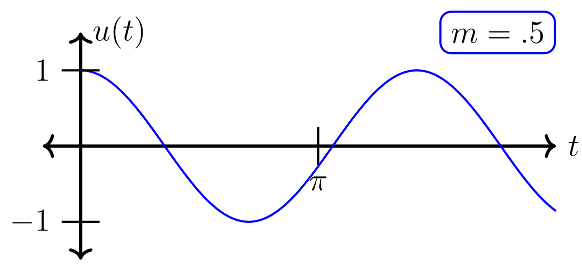
Also, recall that the oscillator's equilibrium position is ℓ units from the wall. This model says that the oscillator can move to u_0 units to the left of equilibrium, which is to say, $\ell - u_0$ units from the wall. If $\ell - u_0 < 0$, then the oscillator will be inside, or maybe even through, the wall! This too is physically unrealistic (unless the oscillator is the dense wall-smashing kind) and even more unrealistic when we consider that passing through the wall does not stop the oscillator's motion at all! ▲

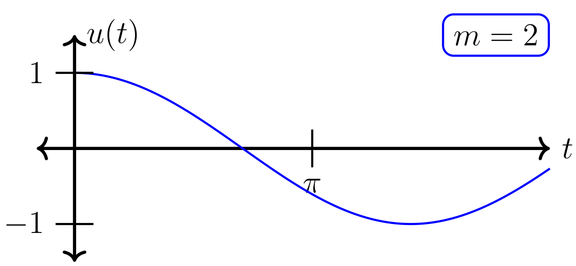
2.2.3 Example.

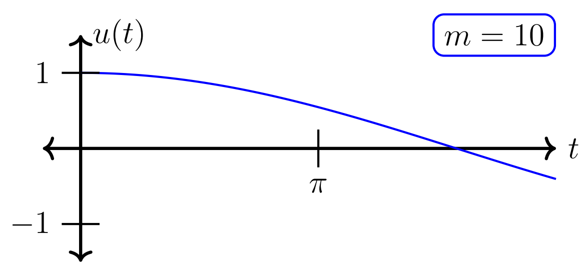
How does changing the value of m affect the behavior of the oscillator?

Solution. We fix $u_0 = \kappa = 1$ and graph $u_m(t) := \cos(t/\sqrt{m})$ for several values of m. All graphs are over the same time interval $[0, 2\pi]$.









Since $u_0 = 1$, the oscillator always moves between one unit to the right and the left of the equilibrium position. Perhaps the most obvious difference among the graphs is that u_m has more roots in the interval $[0, 2\pi]$ when m is small. That is, the oscillator passes through its equilibrium point more often in a fixed time interval for small masses than for large masses.

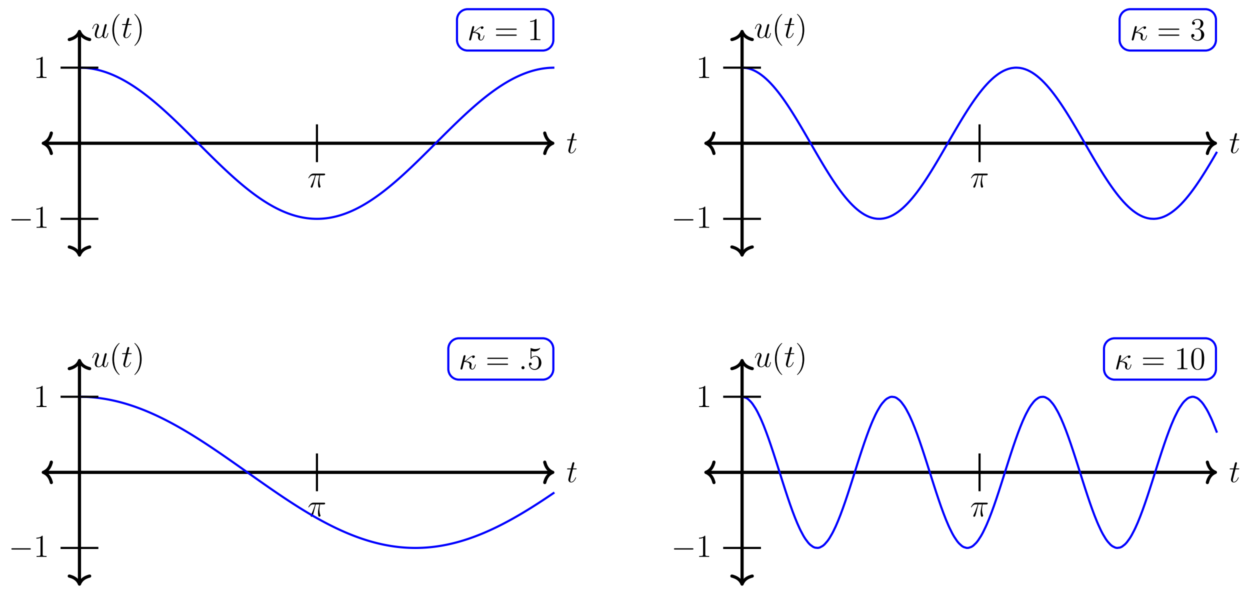
Hopefully this makes sense: a small mass moves more quickly than a large mass; a large mass moves more slowly than a small mass.

At the level of functions, the value of m alters the periodicity (equivalently, the frequency) of u_m. Specifically, u_m is $2\pi\sqrt{m}$-periodic for any $m > 0$. If m is "large," this means that u_m has a "large" period and therefore will take a long time to complete one "repetition" of its motion (between the extremes of one unit to the right and to the left of equilibrium); if m is small, then the period is small, and so u_m cycles through its motion very quickly. ▲

2.2.4 Example.

How does changing the value of κ affect the behavior of the oscillator?

Solution. We fix $u_0 = m = \kappa$ and graph $u_\kappa(t) := \cos(\sqrt{\kappa}t)$ for several values of κ. All graphs are over the same time interval $[0, 2\pi]$.



Again, since $u_0 = 1$, the oscillator moves between one unit to the right and the left of the equilibrium position. We see that u_κ has more roots in the interval $[0, 2\pi]$ when κ is large. Hence the oscillator passes through equilibrium more frequently in a fixed time interval for springs with large κ than with small. Perhaps "large κ" is not as evocative as "large mass m" in the previous example, so we should remember that the spring force is $\mathsf{F}_{\text{spr}}(r) = -\kappa r$; when κ is large, the spring pulls back with more force when stretched a length r than it does when stretched the same length for κ. In other words, a larger κ corresponds to a "stiffer" spring. Now, a stiff spring is harder for us to pull, but, once pulled, it should shoot back more quickly than a loose spring.

Again, the value of κ changes the period of u_κ, which is always $2\pi/\sqrt{\kappa}$. If κ is "large," then $2\pi/\sqrt{\kappa}$ is "small," and so the oscillator completes one "cycle" of its motion (between ± 1 units away from equilibrium) more times over the same time interval $[0, 2\pi]$ than it does if κ is "small," in which case $2\pi/\sqrt{\kappa}$ is "large." ▲

2.2.3. The damped, undriven harmonic oscillator.

Now we will incorporate friction into our model — this is the **DAMPED** oscillator, since friction should "damp" or suppress motion — but we will assume, as before, that there are no "external" forces (thus the oscillator is still "undriven"). Experience suggests that the friction that a moving body experiences is proportional to its velocity. Thus we assume there is $b > 0$ such that at time t, the force of friction is

$$\mathsf{F}_{\mathrm{fr}}(t) := -bu'(t).$$

Assuming that the total force that an object experiences is the sum of the individual forces that it experiences, Newton's law tells us now that

$$mu''(t) = \mathsf{F}_{\mathrm{spr}}(u(t)) + \mathsf{F}_{\mathrm{fr}}(t) = -\kappa u(t) - bu'(t).$$

That is,

$$mu'' + bu' + \kappa u = 0. \tag{2.2.4}$$

Before we attempt any solution of (2.2.4), let us predict what should happen. If friction truly suppresses motion, then over long times the oscillator should stop moving, and its position along the surface should become roughly constant. However, we are not working in true "position along the surface" coordinates[21] but rather "displacement from equilibrium" coordinates. Experience, perhaps, suggests that the oscillator should settle down around its equilibrium position, in which case $u(t)$ should be very small for large t. That is, we predict that if u solves (2.2.4), then

$$\lim_{t\to\infty} u(t) = 0. \tag{2.2.5}$$

Let us try to make this rigorous. Since $b \neq 0$, (2.2.4) is a rather more complicated differential equation than (2.2.1), and really its solution will depend on the sign (and value) of the discriminant, $b^2 - 4m\kappa$. As usual, we study the characteristic equation for (2.2.4), which is

$$m\lambda^2 + b\lambda + \kappa = 0$$

and find

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4m\kappa}}{2m}.$$

We will consider just one possibility on the roots and leave the others as exercises.

2.2.5 Example.

What happens to the oscillator over long times in the **OVERDAMPED** *case of* $b^2 - 4m\kappa > 0$*?*

Solution. Here the roots of the characteristic equation are

$$\lambda_+ := \frac{-b + \sqrt{b^2 - 4m\kappa}}{2m} \quad \text{and} \quad \lambda_- := \frac{-b - \sqrt{b^2 - 4m\kappa}}{2m}.$$

[21] If we take the zero coordinate to be the wall, then the oscillator's true position along the surface at time t is $p(t) := \ell + u(t)$ units from the wall. As we observed in Example 2.2.2, this can lead to the squicky situation $p(t) < 0$, in which case the oscillator is passing through the wall.

Since $b^2 - 4m\kappa > 0$, these are both real numbers, and so displacement satisfies

$$u(t) = c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t}$$

for some constants c_1 and c_2. The signs of λ_+ and λ_- determine the long-term behavior of u, and if we are to have (2.2.5), then we need $\lambda_+ < 0$ and $\lambda_- < 0$.

The second inequality is a bit easier to see: since $\sqrt{b^2 - 4m\kappa} > 0$ and $b > 0$, we have $-b < 0$ and $-\sqrt{b^2 - 4m\kappa} < 0$. Adding and dividing by $2m > 0$, we find $\lambda_- > 0$, as desired.

For the first inequality, we cleverly manipulate some auxiliary inequalities. Since $m > 0$ and $\kappa > 0$, we have $4m\kappa > 0$, thus $-4m\kappa < 0$, and so

$$0 < b^2 - 4m\kappa < b^2.$$

Since the square root is strictly increasing, we have

$$\sqrt{b^2 - 4m\kappa} < \sqrt{b^2} = |b| = b,$$

where the last equality holds because $b > 0$. That is, $\sqrt{b^2 - 4m\kappa} < b$, and so $-b + \sqrt{b^2 - 4m\kappa} < 0$. Dividing by $2m > 0$, we conclude $\lambda_+ < 0$.

Since $\lambda_+ < 0$ and $\lambda_- < 0$, we have

$$\lim_{t \to \infty} e^{\lambda_+ t} = \lim_{t \to \infty} e^{\lambda_- t} = 0,$$

and thus the expected behavior (2.2.5) of u is true. ▲

In addition to studying the cases $b^2 - 4m\kappa < 0$ and $b^2 - 4m\kappa = 0$ and assuring ourselves that $u(t) \to 0$ as $t \to \infty$, there are many other interesting[22] questions that we could take up. For example, the undamped oscillator passes through its equilibrium position infinitely many times as $t \to \infty$. How often does the damped oscillator do so?

2.2.4. Toward more complicated differential equations.

The harmonic oscillator also motivates the study of differential equations with a number of more challenging features than our beloved constant-coefficient linear homogeneous problem possesses.

1. What happens if the oscillator experiences forces in addition to the spring force and friction? Assuming that the spring force is still $\mathsf{F}_{\text{spr}}(r) = -\kappa r$ and the friction force is still $\mathsf{F}_{\text{fr}}(t) = -bu'(t)$, let us say that all the other forces on the oscillator at time t have the value $g(t)$. Then the displacement must satisfy

$$mu''(t) + bu'(t) + \kappa u(t) = g(t).$$

Since g is, presumably, not 0 for all t, this is a **NONHOMOGENEOUS** equation. While we quickly learned how to solve nonhomogeneous linear *first*-order problems with the integrating factor, all of our work with the characteristic equation above has been predicated on having a homogeneous second-order problem. Treating nonhomogeneous second-order problems will occupy much of the rest of our attention.

[22] This is, as always, something of a matter of opinion.

2. What happens if some of the "material" properties of the oscillator and its environment change over time? For example, what if the spring rusts (and therefore gets stiffer) or is lubricated (and therefore gets looser)? Then instead of a constant κ in the spring force, we could have a time-dependent function $\kappa(t)$. Or what if the mass of the oscillator somehow "leaks" and decreases, or maybe "aggregates" and increases? Then instead of the constant m we have a function $m(t)$. Thus we are led to study the **VARIABLE-COEFFICIENT** equation

$$m(t)u''(t) + b(t)u'(t) + \kappa(t)u(t) = g(t).$$

Variable-coefficient equations tend to be difficult, or at least quite laborious, to solve; often one can assume that m, b, κ, g, and u have power series expansions in t and then, given coefficients for m, b, κ, and g, one can eke out formulas for the coefficients in u. Typically these power series solutions do not converge to "familiar" functions but instead define new creations.

3. What if we quibble with Hooke's law and reject the notion that the spring force is directly, linearly proportional to the length that the spring is pulled from equilibrium? Say that instead of $\mathsf{F}_{\text{spr}}(r) = -\kappa r$, we take $\mathsf{F}_{\text{spr}}(r) = -\kappa r - \beta r^2$, where now $\beta > 0$ is another constant. Then the oscillator's displacement satisfies the **NONLINEAR** equation

$$mu'' + bu' + \kappa u + \beta u^2 = g(t). \tag{2.2.6}$$

As difficult as variable-coefficient equations are to solve, there are, typically, even fewer "explicit" formulas for nonlinear equations — our success with first-order separation of variables being something of an exception. Nonetheless, an explicit formula is not necessarily equivalent to solving, let alone understanding, a differential equation. Often "abstract" existence theorems can assure us that solutions to nonlinear problems like (2.2.6) *exist*; then softer "qualitative" methods can tell us something about their properties. (By the way, m, b, κ, and β could all be t-dependent in (2.2.6), too!)

This is where we finished on Wednesday, March 16, 2022.

2.3. Fundamental properties of linear nonhomogeneous equations.

Let g be a function and a, b, and c be real numbers with $a \neq 0$. Much of our efforts will now be devoted to solving and analyzing the **NONHOMOGENEOUS** problem

$$af'' + bf' + cf = g(x). \tag{2.3.1}$$

In principle, we could have $g(x) = 0$ for all x, and then we would be back to the homogeneous problem; we will emphasize the dependence of g on x in (2.3.1) to remind ourselves that we are no longer dealing with strictly homogeneous equations. Sometimes we will call the function g the **NONHOMOGENEITY** of (2.3.1); with an eye toward harmonic oscillators, we might also call g the **FORCING** or **DRIVING** function of (2.3.1).

Before we learn some specific algorithmic methods for solving (2.3.1), it will be worthwhile to examine two properties of nonhomogeneous problems in general.

2.3.1 Theorem.

Let a, b, and c be real numbers. Suppose that g_1 and g_2 are functions and f_1 and f_2 solve

$$af_1'' + bf_1' + cf_1 = g_1(x) \quad \text{and} \quad af_2'' + bf_2' + cf_2 = g_2(x).$$

Let c_1 and c_2 be real numbers. Then the function $h(x) := c_1 f_1(x) + c_2 f_2(x)$ solves

$$ah'' + bh' + ch = c_1 g_1(x) + c_2 g_2(x).$$

The proof, which we leave as an exercise, uses the linearity of the derivative, e.g.,

$$h' = (c_1 f_1 + c_2 f_2)' = (c_1 f_1)' + (c_2 f_2)' = c_1 f_1' + c_2 f_2'.$$

One consequence of superposition is that we can "break up" a problem with a "complicated" nonhomogeneity if the nonhomogeneity is the sum of two or more functions. That is, we solve the same differential equation set equal to each of the terms of the nonhomogeneity and then add all the results together.

A more rewarding consequence of superposition, and our hard work with homogeneous problems, is that we can find *all* solutions to the nonhomogeneous problem (2.3.1) if we know but two things: one "particular" solution to the nonhomogeneous problem and a fundamental solution set for the homogeneous problem (Definition 2.1.19). We can always construct the latter using the characteristic equation techniques of Section 2.1; we will learn how to produce particular solutions shortly.

2.3.2 Theorem.

Let a, b, and c be real numbers, and let g be a function. Suppose that f_* solves

$$af_*'' + bf_*' + cf_* = g(x)$$

and that f_1 and f_2 form a fundamental solution set for the corresponding homogeneous problem

$$af'' + bf' + cf = 0.$$

Suppose that f also solves

$$af'' + bf' + cf = g(x).$$

Then there are constants c_1 and c_2 such that

$$f = f_* + c_1 f_1 + c_2 f_2. \tag{2.3.2}$$

Proof. Put $h(x) := f(x) - f_*(x)$. Superposition tells us that h solves

$$ah'' + bh' + ch = g(x) - g(x) = 0. \tag{2.3.3}$$

Part (ii) of Theorem 2.1.18 then tells us that $h = c_1 f_1 + c_2 f_2$ for some constants c_1 and c_2. That is, $f - f_* = c_1 f_1 + c_2 f_2$, which implies (2.3.2). ■

In the following examples, we will be given a particular nonhomogeneous solution f_*, which we will learn how to construct later. We will then use (2.3.2) to find all solutions to the nonhomogeneous problem.

2.3.3 Example.

Given that $f_*(x) = x$ *solves*

$$f'' + f = x,$$

find all solutions to this equation.

Solution. It is worthwhile to take a moment to check that the claim about f_* is true. Since $f_*'(x) = 1$ and $f_*''(x) = 0$, we have

$$f_*''(x) + f_*(x) = 0 + x = x,$$

so f_* does indeed solve the nonhomogeneous problem.

Now we find a fundamental solution set for the homogeneous problem $f'' + f = 0$. Actually, we did this already in part (i) of Example 2.1.24, where we saw that a fundamental solution set is $f_1(x) := \cos(x)$ and $f_2(x) := \sin(x)$. Thus any solution to the nonhomogeneous problem has the form

$$f(x) = x + c_1 \cos(x) + c_2 \sin(x)$$

for some constants c_1 and c_2. ▲

2.3.4 Example.

Given that $f_*(x) = e^{2x}/3$ *solves*

$$f'' - f = e^{2x},$$

solve the initial value problem

$$\begin{cases} f'' - f = e^{2x} \\ f(0) = 0 \\ f'(0) = 1. \end{cases}$$

Solution. As before, we first check that f_* is a solution. Since

$$f_*'(x) = \frac{2e^{2x}}{3} \quad \text{and} \quad f_*''(x) = \frac{4e^{2x}}{3},$$

we have

$$f_*''(x) - f_*(x) = \frac{4e^{2x}}{3} - \frac{e^{2x}}{3} = \frac{4e^{2x} - e^{2x}}{3} = \frac{3e^{2x}}{3} = e^{2x}.$$

Next, we find a fundamental solution set for $f'' - f = 0$. Actually, we did this in Example 2.1.4 and therefore just reread that solution to see that $f_1(x) := e^x$ and $f_2(x) := e^{-x}$ form a fundamental solution set. Thus every solution to $f'' - f = e^{2x}$ has the form

$$f(x) = \frac{e^{2x}}{3} + c_1 e^x + c_2 e^{-x}$$

for some constants c_1 and c_2.

Now we choose c_1 and c_2 to meet the initial conditions. We need

$$0 = f(0) = \frac{e^{2\cdot 0}}{3} + c_1e^0 + c_2e^{-0} = \frac{1}{3} + c_1 + c_2.$$

We pause to differentiate f and find

$$f'(x) = \frac{2e^{2x}}{3} + c_1e^x - c_2e^{-x},$$

and so we want

$$1 = f'(0) = \frac{2e^{2\cdot 0}}{3} + c_1e^0 - c_2e^{-0} = \frac{2}{3} + c_1 - c_2.$$

Thus c_1 and c_2 must satisfy the linear system

$$\begin{cases} c_1 + c_2 = -\frac{1}{3} \\ c_1 - c_2 = \frac{1}{3}. \end{cases} \tag{2.3.4}$$

As usual, there are any number of ways to solve this system; one method is simply to add the two equations together to find that $2c_1 = 0$ and thus $c_1 = 0$. Then the first equation gives $c_2 = -1/3$, and so the solution to the initial value problem is

$$f(x) = \frac{e^{2x}}{3} - \frac{e^{-x}}{3}.$$

▲

2.3.5 Method: Solve $af'' + bf' + cf = g(x)$

1. Find a fundamental solution set for the homogeneous problem $af'' + bf' + cf = 0$. This will involving finding the roots of the characteristic equation $ar^2 + br + c = 0$ and interpreting their behavior. Call the functions in this fundamental solution set f_1 and f_2.

2. Find one "particular" solution f_* to the nonhomogeneous problem $af'' + bf' + cf = 0$.

3. All solutions to the nonhomogeneous problem have the form

$$f = f_* + c_1f_1 + c_2f_2$$

for some constants c_1 and c_2.

This is where we finished on Friday, March 18, 2022.

2.4. Variation of parameters.

We should now be convinced of the superb value of having just one "particular" solution to the nonhomogeneous problem

$$af'' + bf' + cf = g(x). \tag{2.4.1}$$

So how do we find one?

It turns out that we do not need much more than a fundamental solution set for the corresponding homogeneous problem to build a particular solution to a given nonhomogeneous problem. To see how the construction might proceed, it pays to revisit the familiar first-order linear problem.

2.4.1 Example.

Let g be a continuous function on the interval I and let a and b be real numbers with $a \neq 0$. Study the structure of solutions to the problem $af' + bf = g(x)$.

Solution. We could repeat the integrating factor method on this problem, but we will instead just read off the solution from Theorem 1.3.14. To do so, we first rewrite our problem as

$$f' + \frac{b}{a}f = \frac{g(x)}{a}.$$

The coefficient function on f is $p(x) := b/a$, and an antiderivative for p is $P(x) := (b/a)x$. Replacing $g(x)$ by $g(x)/a$ throughout Theorem 1.3.14, we find that every solution to $f' + (b/a)f = g(x)/a$ has the form

$$f(x) = u(x)e^{-(b/a)x} + Ce^{-(b/a)x},$$

where C is a constant and the function u satisfies

$$u'(x) = \frac{e^{(b/a)x}g(x)}{a}.$$

Motivated by Remark 1.3.15, we abbreviate $f_0(x) := e^{-(b/a)x}$ and take $C = 0$ to find that

$$f_*(x) := u(x)e^{-(b/a)x} = u(x)f_0(x)$$

is a particular solution to $af' + bf = g$. The function f_0 solves the homogeneous problem $af_0' + bf_0 = 0$. *Thus a particular solution to the nonhomogeneous problem is a "variable-coefficient" multiple of a (nontrivial) solution to the homogeneous problem.* More precisely, this function u satisfies

$$u'(x) = \frac{g(x)}{af_0(x)}.$$

In other words, u is completely determined by the nonhomogeneity g and the solution f_0 to the homogeneous problem. ▲

The structure of solutions to the first-order nonhomgeneous problem (loosely) motivates the following idea for solving the second-order nonhomogeneous problem (2.4.1). Suppose that f_1 and f_2 form a fundamental solution set for the corresponding homogeneous problem $af''+bf'+cf=0$; such functions always exist, thanks to our knowledge of the characteristic equation. For (2.4.1), make the ansatz

$$f(x) = u_1(x)f_1(x) + u_2(x)f_2(x). \tag{2.4.2}$$

We will figure out what u_1 and u_2 should be by working backwards. Specifically, we will evaluate (2.4.1) with f in this form and see what u_1 and u_2 should be[23] (or, more precisely, what they should *do*). Once we have good candidates for u_1 and u_2, we can check that such candidates actually produce a solution to (2.4.1) via (2.4.2). As it turns out, the calculations are messy and tedious and not all that enlightening, and we defer them to Appendix C.

Here is what we find. If u_1 and u_2 satisfy the differential equations

$$u_1'(x) = -\frac{g(x)f_2(x)}{a\mathcal{W}[f_1,f_2](x)} \quad \text{and} \quad u_2'(x) = \frac{g(x)f_1(x)}{a\mathcal{W}[f_1,f_2](x)}, \tag{2.4.3}$$

then defining f by (2.4.2) does indeed yield a solution of (2.4.1).

How do we know that these differential equations have solutions? First, they are first-order direct integration problems. So, when does a first-order direct integration problem have solutions? When all the functions involved are continuous (Theorem 1.1.3).

And are the functions here in (2.4.3) continuous? Since f_1 and f_2 form a fundamental solution set for $af''+bf'+cf=0$, they are twice differentiable, therefore differentiable, and therefore continuous. Since $\mathcal{W}[f_1,f_2] = f_1f_2' - f_1'f_2$, and since f_1' and f_2' are differentiable, all the components of $\mathcal{W}[f_1,f_2]$ are continuous. Since $a \neq 0$ (as otherwise we would not be considering a second-order problem) and since $\mathcal{W}[f_1,f_2](x) \neq 0$ by definition of a fundamental solution set, we have no concerns about dividing by zero in (2.4.3). It therefore falls to g to be continuous (note that we did not assume continuity of the nonhomogeneity g in Section 2.3).

2.4.2 Theorem (Variation of parameters[24]).

Let a, b, and c be real numbers with $a \neq 0$. Let g be continuous on the interval I and let f_1 and f_2 form a fundamental solution set for the homogeneous problem $af''+bf'+cf=0$. Suppose that u_1 and u_2 satisfy (2.4.3) on I. Then the function $f_* := u_1f_1 + u_2f_2$ solves $af_*''+bf_*'+cf_* = g(x)$ on I.

---

[23] This was exactly our philosophy in solving the homogeneous problem via the exponential ansatz. We guessed that $f(x) = e^{rx}$ would solve $af''+bf'+cf=0$ for some real number r; we substituted this formula for f into the differential equation; and then we tried to learn about r. Specifically, we learned that r had to solve the characteristic equation $ar^2+br+c=0$.

[24] So named because we are "varying" the "parameters" c_1 and c_2 in the homogeneous solution $f_0(x) = c_1f_1(x)+c_2f_2(x)$ to get the nonhomogeneous solution $f_*(x) = u_1(x)f_1(x)+u_2(x)f_2(x)$. Since we usually do not think of c_1 and c_2 as "parameters" in the differential equation $af''+bf'+cf=g(x)$ — if anything, a, b, and c are the parameters — perhaps a more evocative name for the method is "variation of constants," which is certainly in vogue in some circles.

Proof. The work preceding the statement of this theorem is not, in fact, a proof of it. Rather, that work suggested the definitions of u_1 and u_2 via (2.4.3). The real proof is somewhat more mundane, but still important: with $f_* = u_1 f_1 + u_2 f_2$, we need to use the product rule and the properties (2.4.3) to show that $af_*''(x) + bf_*'(x) + cf_*(x) = g(x)$ for all $x \in I$. This, like so many other things, is just a matter of calculus, algebra, and time. Our ability to construct u_1 and u_2 in the first place hinges on continuity as discussed in the paragraph preceding the statement of this theorem. For example, we could define

$$u_1(x) := -\int_{x_0}^{x} \frac{g(s)f_2(s)}{a\mathcal{W}[f_1, f_2](s)}\, ds \quad \text{and} \quad u_2(x) := \int_{x_0}^{x} \frac{g(s)f_1(s)}{a\mathcal{W}[f_1, f_2](s)}\, ds$$

for a given point x_0 of I. The continuity of g ensures that the integrals exist and, moreover, that they differentiate as in (2.4.3), per the fundamental theorem of calculus. ■

2.4.3 Example.

Revisit Example 2.3.4 and find all solutions to $f'' - f = e^{2x}$.

Solution. A fundamental set for the homogeneous problem $f'' - f = 0$ is $f_1(x) := e^x$ and $f_2(x) := e^{-x}$. (In particular, note that the coefficient a here is $a = 1$.) The Wronskian is

$$\mathcal{W}[f_1, f_2](x) = f_1(x)f_2'(x) - f_1'(x)f_2(x) = e^x(-e^{-x}) - e^x e^{-x} = -1 - 1 = -2.$$

This agrees, by the way, with what we calculated in part (i) of Example 2.1.16.

Then a particular solution to $f'' - f = e^{2x}$ is

$$f_*(x) = u_1(x)f_1(x) + u_2(x)f_2(x) = u_1(x)e^x + u_2(x)e^{-x},$$

where u_1 and u_2 satisfy

$$u_1'(x) = -\frac{g(x)f_2(x)}{1 \cdot \mathcal{W}[f_1, f_2](x)} = -\frac{e^{2x}e^{-x}}{-2} = \frac{e^x}{2}$$

and

$$u_2'(x) = \frac{g(x)f_1(x)}{1 \cdot \mathcal{W}[f_1, f_2](x)} = \frac{e^{2x}e^x}{-2} = -\frac{e^{3x}}{2}.$$

We solve these auxiliary differential equations by direct integration:

$$u_1(x) = \int \frac{e^x}{2}\, dx + C_1 = \frac{e^x}{2} + C_1$$

and

$$u_2(x) = \int -\frac{e^{3x}}{2}\, dx + C_2 = -\frac{e^{3x}}{6} + C_2.$$

For the moment, we are keeping the constants of integration present, and we are giving ourselves some freedom by denoting them by the different variables C_1 and C_2.

We see that f_* has the form

$$f_*(x) = \left(\frac{e^x}{2} + C_1\right) e^x + \left(-\frac{e^{3x}}{6} + C_2\right) e^{-x} = \frac{e^{2x}}{2} - \frac{e^{2x}}{6} + (C_1 e^x + C_2 e^{-x}) = \frac{e^{2x}}{3} + (C_1 e^x + C_2 e^{-x}).$$

In fact, since C_1 and C_2 are arbitrary constants of integration, this form of f_* really contains every possible solution to our problem.

When $C_1 = C_2 = 0$, we obtain the solution $f_*(x) = e^{2x}/3$ that we were previously given in Example 2.3.4. We could have obtained this solution by omitting the constants of integration in solving for u_1 and u_2 above; we kept them in to hedge our bets, but note that the discussion preceding the statement of Theorem 2.4.2 just calls for u_1 and u_2 to be *some* antiderivatives, and we are free to set the constants of integration to be what we wish. If we do take $C_1 = C_2 = 0$ when computing u_1 and u_2 above, then we would need to remember to add an arbitrary linear combination of the functions in the fundamental solution set to get all solutions to the problem. ▲

This is where we finished on Monday, March 21, 2022.

2.4.4 Method: Solve $af'' + bf' + cf = g(x)$

1. Find a fundamental solution set for the corresponding homogeneous problem $af'' + bf' + cf = 0$. (Use Method 2.1.29.) Call the functions in this fundamental solution set f_1 and f_2.

2. Find functions u_1 and u_2 satisfying

$$u_1' = -\frac{gf_2}{a\mathcal{W}[f_1, f_2]} \quad \text{and} \quad u_2' = \frac{gf_1}{a\mathcal{W}[f_1, f_2]}.$$

Use direct integration; if necessary, express u_1 and u_2 as definite integrals. It is not necessary to include constants of integration.

3. A particular solution to the nonhomogeneous problem is $f_* := u_1 f_1 + u_2 f_2$.

4. All solutions to the nonhomogeneous problem are $f = f_* + c_1 f_1 + c_2 f_2$ for constants c_1 and c_2.

2.4.5 Example.

Revisit Example 2.3.3 and find all solutions to $f'' + f = x$.

Solution. We work through the four steps of Method 2.4.4.

1. The corresponding homogeneous problem is $f'' + f = 0$, and the characteristic equation is $r^2 + 1 = 0$; its roots are $r = \pm\sqrt{-1} = \pm i$. As before, is now, and always shall be, a fundamental solution set consists of $f_1(x) := \cos(x)$ and $f_2(x) := \sin(x)$.

2. We have $f_1'(x) = -\sin(x)$ and $f_2'(x) = \cos(x)$, so the Wronskian is

$$\mathcal{W}[f_1, f_2](x) = f_1(x)f_2'(x) - f_1'(x)f_2(x) = \cos(x)\cos(x) - [-\sin(x)]\sin(x) = \cos^2(x) + \sin^2(x) = 1.$$

It is worthwhile to make sure that the Wronskian is never 0; this must be the case if f_1 and f_2 form a fundamental solution set (after all, that is part of the *definition* of a fundamental solution set), so checking can help us spot errors.

Now we need to solve

$$u_1'(x) = -\frac{x\sin(x)}{1\cdot 1} = -x\sin(x)$$

and

$$u_2' = \frac{x\cos(x)}{1\cdot 1} = x\cos(x).$$

Integrating by parts[25] gives

$$u_1(x) = \int -x\cos(x)\ dx + C_1 = -\sin(x) + x\cos(x) + C_1$$

and

$$u_2(x) = \int x\sin(x)\ dx + C_2 = \cos(x) + x\sin(x) + C_2.$$

We can feel free to leave out the constants of integration ($C_1 = C_2 = 0$).

3. A particular solution is then

$$\begin{aligned} f_*(x) &= [-\sin(x) + x\cos(x)]\cos(x) + [\cos(x) + x\sin(x)]\sin(x) \\ &= -\sin(x)\cos(x) + x\cos^2(x) + \cos(x)\sin(x) + x\sin^2(x) = x[\cos^2(x) + \sin^2(x)] = x. \end{aligned}$$

4. All solutions have the form

$$f(x) = x + c_1\cos(x) + c_2\sin(x)$$

for some constants c_1 and c_2. (Had we kept the constants of integration C_1 and C_2 earlier, we would have seen this pattern of solution emerge in the preceding step.) ▲

2.4.6 Example.

Find all solutions to $f'' + 2f' + f = e^x\ln(x)$. Where are the solutions defined?

Solution. We first work step-by-step.

1. The corresponding homogeneous problem is $f''+2f'+f = 0$; its characteristic equation is $r^2+2r+1 = 0$; we have $r^2+2r+1 = (r+1)^2$; and so the only root of the characteristic equation is $r = -1$. This is a double root, so a fundamental solution set consists of $f_1(x) := e^{-x}$ and $f_2(x) := xe^{-x}$.

[25] At the risk of overworking the vowel u, take $u = x$ and $dv = \sin(x)$ for the first integral. Then $du = dx$ and $v = -\cos(x)$, so

$$\int x\cos(x)\ dx = x[-\cos(x)] - \int -\cos(x)\ dx = -x\cos(x) + \int \cos(x)\ dx = \sin(x) - x\cos(x) + C.$$

2. We have $f_1'(x) = -e^{-x}$ and $f_2'(x) = e^{-x} - xe^{-x}$, so the Wronskian is

$$\mathcal{W}[f_1, f_2](x) = e^{-x}[e^{-x} - xe^{-x}] + (-e^{-x})[xe^{-x}] = e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x}.$$

This agrees with part (ii) of Example 2.1.16.

Now we must solve

$$u_1'(x) = -\frac{e^x \ln(x)[xe^{-x}]}{1 \cdot e^{-2x}} = -xe^{2x} \ln(x)$$

and

$$u_2'(x) = \frac{e^x \ln(x)[e^{-x}]}{1 \cdot e^{-2x}} = e^{2x} \ln(x).$$

We could try integrating by parts, but for variety's sake, let us represent the antiderivatives as definite integrals. For example, we could take u_1 to be

$$u_1(x) = -\int_{x_0}^{x} se^{2s} \ln(s)\ ds,$$

provided that the integrand is continuous on an interval containing x_0 and x. The factor of $\ln(\cdot)$ means that we must exclude zero negative values from this interval. In particular, our solutions will be defined on, at best, $(0, \infty)$.

So, let us be specific and take

$$u_1(x) := -\int_1^x se^{2s} \ln(s)\ ds \quad \text{and} \quad u_2(x) := \int_1^x e^{2s} \ln(s)\ ds.$$

3. A particular solution is

$$f_*(x) = \left(-\int_1^x se^{2s} \ln(s)\ ds\right) e^{-x} + \left(\int_1^x e^{2s} \ln(s)\ ds\right) xe^{-x}.$$

4. Every solution has the form

$$f(x) = \left(-\int_1^x se^{2s} \ln(s)\ ds\right) e^{-x} + \left(\int_1^x e^{2s} \ln(s)\ ds\right) xe^{-x} + c_1 e^{-x} + c_2 xe^{-x}$$

for some constants c_1 and c_2. ▲

This is where we finished on Wednesday, March 23, 2022.

All of our hard work on homogeneous equations and variation of parameters can be summarized in the following result.

2.4.7 Theorem.

Let g be continuous on the interval I. Let x_0 be a point in I, and let y_0, y_1, a, b, and c be *real numbers with $a \neq 0$. There exists a unique solution f defined on I to*

$$\begin{cases} af'' + bf' + cf = g(x) \\ f(x_0) = y_0 \\ f'(x_0) = y_1. \end{cases}$$

Proof. We sketch the proof and leave the details as an exercise. Here is how we show that a solution *exists*. Use variation of parameters to find a particular solution f_* satisfying $af_*'' + bf_*' + cf_* = g(x)$ on I. Then posit that a solution to the initial value problem has the form $f = f_* + c_1 f_1 + c_2 f_2$, where f_1 and f_2 form a fundamental solution set for the corresponding homogeneous problem. Solve a linear system for c_1 and c_2 along the lines of (2.3.4); here it will be important that $\mathcal{W}[f_1, f_2](x_0) \neq 0$.

To prove *uniqueness*, suppose that h is another solution of the initial value problem; consider the initial value problem that $f - h$ solves and use Lemma 2.1.12. ∎

Nonetheless, more questions still linger. The integrals that resolve (2.4.3) can be cumbersome and exhausting, especially if many rounds of integration by parts are needed — and there is no guarantee that we can find an explicit formula for the antiderivative in terms of elementary functions, anyway. Is there a faster method than variation of parameters for solving nonhomogeneous problems? The answer is yes, but we will have to restrict the permitted nonhomogeneities g to have a fairly special structure (a structure that, nonetheless, arises in plenty of applications). This leads to the dreaded method of undetermined coefficients, toward which we will move in the next section.

Second, variation of parameters (like, not incidentally, the integrating factor method) presumes that the nonhomogeneity g is continuous, precisely to ensure that the coefficients u_1 and u_2 can be defined via integration. What happens if g is discontinuous? We have not considered many (if any) discontinuous terms or coefficients in our equations yet, but there are good physical reasons to do so for applications. This line of inquiry will lead to the method of Laplace transforms.

2.5. Higher-order linear constant-coefficient homogeneous problems.

Let $k \geq 0$ be an integer. We denote by $f^{(k)}$ the kth derivative of a function f. In particular,

$$f^{(0)} = f, \qquad f^{(1)} = f', \qquad f^{(2)} = f'', \qquad \text{and} \qquad f^{(3)} = f'''.$$

For derivatives of order $k \geq 4$, this notation is much more efficient than primes.

We will study differential equations of the form

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0. \tag{2.5.1}$$

The coefficients $a_0, a_1, \ldots, a_n$ are constants, and in particular we assume $a_n \neq 0$. The equation (2.5.1) is then a nTH ORDER CONSTANT-COEFFICIENT LINEAR HOMOGENEOUS differential equation. We will often write (2.5.1) using sigma notation[26]:

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f = \sum_{k=0}^{n} a_k f^{(k)}.$$

While such equations naturally arise in various modeling scenarios, we will view them mostly as tools for an alternative method of solving certain *nonhomogeneous* equations without the ordeal of variation of parameters. We begin by attempting to find just some solutions to (2.5.1) using an idea that served us well before. Guess that a solution has the form $f(x) = e^{rx}$ for some number r to see that the kth derivative must be

$$f^{(k)}(x) = \frac{d^k}{dx^k}[e^{rx}] = r^k e^{rx}. \tag{2.5.2}$$

Thus we must have

$$0 = \sum_{k=0}^{n} a_k r^k e^{rx} = \left(\sum_{k=0}^{n} a_k r^k\right) e^{rx}, \tag{2.5.3}$$

and so r must satisfy

$$\sum_{k=0}^{n} a_k r^k = 0. \tag{2.5.4}$$

Conversely, if r solves (2.5.4), then by (2.5.2) and (2.5.3), it must be the case that $f(x) = e^{rx}$ solves (2.5.1).

The equation (2.5.4) is the **CHARACTERISTIC EQUATION** for the differential equation (2.5.1), while the polynomial

$$p(r) := \sum_{k=0}^{n} a_k r^k = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$$

is the **CHARACTERISTIC POLYNOMIAL** for (2.5.1). (Note that we never really talked about "characteristic quadratics" before in the second-order regime, but now we will need to think about polynomials as independent entities for a while.) Such a polynomial has **DEGREE** n since the highest power of r that appears is r^n; this is because $a_n \neq 0$.

[26]If, for each integer $k \geq 0$, we have a number A_k, then we *recursively* define

$$\sum_{k=0}^{n} A_k := \begin{cases} A_0, \ n = 0 \\ \left(\sum_{k=0}^{n-1} A_k\right) + A_n, \ n \geq 1. \end{cases}$$

The point of sigma notation is to make precise the intuitive idea of "adding all the numbers A_0 through A_n," and so we euphemistically write

$$\sum_{k=0}^{n} A_k = A_0 + \cdots + A_n.$$

2.5.1 Example.

Find solutions to $f''' - f' = 0$.

Solution. This equation has the form

$$\sum_{k=0}^{3} a_k f^{(k)} = 0, \qquad a_3 = 1,\ a_2 = 0,\ a_1 = -1,\ a_0 = 0,$$

although it may be silly to pass to sigma notation for an equation this short. The characteristic polynomial is

$$p(r) = r^3 - r = r(r^2 - 1) = r(r+1)(r-1),$$

and so its roots are $r = 0$, $r = -1$, and $r = 1$. Thus three solutions to the problem are

$$f_1(x) = e^{0\cdot x} = 1, \qquad f_2(x) = e^{1\cdot x} = e^x, \quad \text{and} \quad f_3(x) = e^{-1\cdot x} = e^{-x}. \tag{2.5.5}$$

If an equation is labeled "linear," it better be the case that linear combinations of solutions are also solutions, and indeed one can check that putting

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = c_1 + c_2 e^x + c_3 e^{-x}$$

solves $f''' - f' = 0$ for any choice of constants c_1, c_2, and c_3. ▲

Of course we would like to say that all solutions to the preceding problem are linear combinations of the three functions in (2.5.5). A rigorous proof that this is true would proceed largely along the lines of all our work in Section 2.1.2, but, frankly, working through the details would not teach us all that much new. Instead, we will use this opportunity to explore some properties of polynomials and just state the corresponding theory. First we have the following classical statement on the roots of polynomials.

2.5.2 Theorem (Fundamental theorem of algebra I).

Let $p(r) = \sum_{k=0}^{r} a_k r^k$ *be a polynomial of degree* n *(in particular,* $a_n \neq 0$*). There exist numbers* $r_1, \ldots, r_n$ *such that the only zeros (roots) of* p *are* $r_1, \ldots, r_n$*. That is,*

$$p(r) = 0 \iff r = r_1, \ldots, r = r_n.$$

The careful wording of this result does not specify if all of the r_k are *all* real numbers or if there are n *distinct* roots. Neither needs to be the case.

2.5.3 Example.

(i) *The roots of* $p(r) = r^2 - 1 = (r+1)(r-1)$ *are* $r = 1$ *and* $r = -1$.

(ii) *The only root of* $p(r) = r^2 - 2r + 1 = (r-1)^2$ *is* $r = 1$.

(iii) *The roots of* $p(r) = r^3 + 4r = r(r^2 + 4)$ *are* $r = 0$, $r = 2i$, *and* $r = -2i$.

For the purposes of solving differential equations, perhaps the simplest situation is the one that occurred in Example 2.5.1.

2.5.4 Theorem.

Suppose that the characteristic polynomial $p(r) = \sum_{k=0}^{n} a_k r^k$ *of the differential equation* (2.5.1) *has* n *distinct real roots* $r_1, \ldots, r_n$. *(That is,* $r_j \neq r_k$ *if* $j \neq k$.*) Then every solution* f *to* (2.5.1) *has the form*

$$f(x) = \sum_{k=1}^{n} c_k e^{r_k x}$$

for some constants $c_1, \ldots, c_n$.

2.5.5 Example.

Suppose that the characteristic polynomial for some linear constant-coefficient homogeneous differential equation is

$$p(r) = r(r-5)(r+6)(r-8).$$

Find all solutions to this homogeneous equation.

Solution. The roots of p, thanks to its nicely factored form, are $r = 0$, $r = 5$, $r = -6$, and $r = 8$. That is, p has four distinct roots. Since p is the product of four linear factors, the degree of p is 4. Thus the corresponding differential equation is fourth-order. Since we have four distinct roots to the characteristic polynomial, all solutions to this differential equation must have the form

$$f(x) = c_1 e^{0 \cdot x} + c_2 e^{5x} + c_3 e^{-6x} + c_4 e^{8x} = c_1 + c_2 e^{5x} + c_3 e^{-6x} + c_4 e^{8x}. \qquad \blacktriangle$$

This is where we finished on Friday, March 25, 2022.

To handle the case where the characteristic polynomial does not have n distinct roots, we need a more detailed version of the fundamental theorem of algebra.

2.5.6 Theorem (Fundamental theorem of algebra II).

Let $p(r) = \sum_{k=0}^{n} a_k r^k$ *be an nth degree polynomial. There exist an integer* d *satisfying* $1 \leq d \leq n$; *integers* $m_1, \ldots, m_d$ *satisfying*

$$1 \leq m_k \leq n \quad \text{and} \quad \sum_{k=1}^{d} m_k = n;$$

and numbers $r_1, \ldots, r_d$ *such that*

$$p(r) = a_n (r - r_1)^{m_1} \cdots (r - r_d)^{m_d}.$$

The number m_k *is the* MULTIPLICITY *of the root* r_k *of* p.

Informally, but evocatively, the multiplicity of a root r_* of a polynomial $p = p(r)$ is the number of times that $r - r_*$ appears as a factor in the factorization of p. Such a factor appears at least once (otherwise r_* would not be a root of p) but no more times than the degree of p (otherwise p would have higher degree than it really does). It will be worthwhile to keep the two identities

$$r + a = r - (-a) \quad \text{and} \quad r^2 + a^2 = (r + ia)(r - ia) = (r - (-ia))(r - ia)$$

in mind in the future.

2.5.7 Example.

Find the roots of each polynomial and their multiplicities.

(i) $p(r) = (r-1)(r-2)^3$

(ii) $p(r) = r^6 + r^4$

(iii) $p(r) = (r^2-1)^2(r^2+1)^3$

Solution. (i) The roots are $r_1 = 1$ and $r_2 = 2$. The multiplicity of r_1 is $m_1 = 1$ and the multiplicity of r_2 is $m_2 = 3$.

(ii) First we factor

$$r^6 + r^4 = r^4(r^2+1) = (r-0)^4(r-(-i))(r-i).$$

Then the roots are $r_1 = 0$ with multiplicity $m_1 = 4$, $r_2 = -i$ with multiplicity $m_2 = 1$, and $r_3 = i$ with multiplicity $m_3 = 1$. Note that $m_1 + m_2 + m_3 = 4 + 1 + 1 = 6$, which is the degree of p.

(iii) We factor

$$p(r) = [(r+1)(r-1)]^2[(r-(-i))(r-i)]^3 = (r-(-1))^2(r-1)^2(r-(-i))^3(r-i)^3.$$

The roots are $r_1 = -1$, $r_2 = 1$, $r_3 = -i$, and $r_4 = i$. The respective multiplicities are $m_1 = m_2 = 2$ and $m_3 = m_4 = 3$. ▲

Now that we are more fully equipped with the vocabulary of roots and multiplicities, we may apply them to our study of homogeneous problems.

2.5.8 Theorem.

Any nth order homogeneous problem $\sum_{k=0}^n a_k f^{(k)} = 0$ *has a* **FUNDAMENTAL SOLUTION SET** *in the following sense. There exist functions* $f_1, \ldots, f_n$ *such that if* $\sum_{k=0}^n a_k f^{(k)} = 0$, *then* $f = \sum_{k=1}^n c_k f_k$ *for some constants* c_k.

Here is how the functions $f_1, \ldots, f_n$ *are determined. Suppose that* $p(r) = \sum_{k=0}^n a_k r^k$ *is the characteristic polynomial for the nth order equation* $\sum_{k=0}^n a_k f^{(k)} = 0$. *Let* $r_1, \ldots, r_d$ *be the roots of* p, *each with multiplicity* m_k. *Each root generates functions in the fundamental solution set as follows.*

(i) *If* r_k *is real and* $m_k = 1$*, then* r_k *generates the single function*

$$x \mapsto e^{r_k x}.$$

(ii) *If* r_k *is real and* $m_k > 1$*, then* r_k *generates the* m_k *functions*

$$x \mapsto e^{r_k x}, \; x \mapsto xe^{r_k x}, \; \ldots, \; x \mapsto x^{m_k-1}e^{r_k x}.$$

(iii) *If* $r_k = \alpha_k \pm i\beta_k$ *is a complex conjugate pair with* $\beta_k \neq 0$ *and* $m_k = 1$*, then* r_k *generates the two functions*

$$x \mapsto e^{\alpha_k x}\cos(\beta_k x) \quad \text{and} \quad x \mapsto e^{\alpha_k x}\sin(\beta_k x).$$

(iv) *If* $r_k = \alpha_k \pm i\beta_k$ *is a complex conjugate pair with* $\beta_k \neq 0$ *and* $m_k > 1$*, then* r_k *generates the* $2m_k$ *functions*

$$x \mapsto e^{\alpha_k x}\cos(\beta_k x), \; x \mapsto xe^{\alpha_k x}\cos(\beta_k x), \; \ldots, \; x \mapsto x^{m_k-1}e^{\alpha_k x}\cos(\beta_k x)$$

and

$$x \mapsto e^{\alpha_k x}\sin(\beta_k x), \; x \mapsto xe^{\alpha_k x}\sin(\beta_k x), \; \ldots, \; x \mapsto x^{m_k-1}e^{\alpha_k x}\sin(\beta_k x).$$

This theorem is the ultimate nth order generalization of our prior result in Method 2.1.29 for the second-order homogeneous problem. We will first practice using it, as it is as much an exercise in reading mathematical language and notation as it is a series of truths about calculus. Later we will explore why it works.

2.5.9 Example.

Find all solutions to each of the following problems.

(i) $f^{(4)} + f''' = 0$

(ii) $f''' - f'' + 4f' - 4f = 0$

(iii) $f^{(5)} + 2f''' + f' = 0$

(iv) $f^{(4)} + f = 0$

Solution. **(i)** The characteristic equation is $r^4 + r^3 = 0$. We factor $r^4 + r^3 = r^3(r+1)$, so the roots of the characteristic polynomial are $r_1 = 0$ with multiplicity $m_1 = 3$ and $r = -1$ with multiplicity $m_2 = 1$. Both of these roots are real. Thus a fundamental solution set consists of

$$f_1(x) = e^{0 \cdot x} = 1, \qquad f_2(x) = xe^{0 \cdot x} = x, \qquad f_3(x) = x^2 e^{0 \cdot x} = x^2, \qquad \text{and} \qquad f_4(x) = e^{-x}.$$

Every solution therefore has the form

$$f(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

for some constants c_1, c_2, c_3, and c_4.

This is where we finished on Monday, March 28, 2022.

(ii) The characteristic polynomial is $p(r) = r^3 - r^2 + 4r - r$, which factors as

$$p(r) = r^2(r-1) + 4(r-1) = (r^2+4)(r-1).$$

Then the roots are $r_1 = 1$ with multiplicity $m_1 = 1$ and the complex conjugate pair $r_2 = 2i$ and $r_3 = -2i$ with multiplicities $m_2 = m_3 = 1$. The functions in the fundamental solutions set are then $f_1(x) = e^x$, $f_2(x) = \cos(2x)$, and $f_3(x) = \sin(2x)$, so every solution has the form

$$f(x) = c_1 e^x + c_2 \cos(2x) + c_3 \sin(2x).$$

(iii) The characteristic polynomial is

$$r^5 + 2r^3 + r = r(r^4 + 2r^2 + 1) = r(r^2+1)^2.$$

The roots are $r_1 = 0$ with multiplicity $m_1 = 1$ and the complex conjugate pair $r_2 = i$ and $r_3 = -i$, with multiplicities $m_2 = m_3 = 2$. The functions in the fundamental solution set are $f_1(x) = 1$, $f_2(x) = \cos(x)$, $f_3(x) = x\cos(x)$, $f_4(x) = \sin(x)$, and $f_5(x) = x\sin(x)$, so every solution has the form

$$f(x) = c_1 + c_2\cos(x) + c_3 x\cos(x) + c_4\sin(x) + c_5 x\sin(x).$$

(iv) The characteristic polynomial is $p(r) = r^4 + 1$, which does not have any apparent factorization. Methods of complex analysis, however, reveal that the roots are the two complex conjugate pairs

$$r_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \qquad r_2 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \qquad r_3 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \qquad \text{and} \qquad r_4 = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.$$

Each root turns out to have multiplicity 1. Assuming this to be true (and not worrying further about exactly *how* the complex analysis works), a fundamental solution set is

$$f_1(x) = e^{x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right), \qquad f_2(x) = e^{x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right), \qquad f_3(x) = e^{-x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)$$
$$\text{and} \qquad f_4(x) = e^{-x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right).$$

Thus every solution has the form

$$f(x) = c_1 e^{x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right) + c_2 e^{x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right), + c_3 e^{-x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right) + c_4 e^{-x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right).$$

This is a bulky expression, and it may be worthwhile to factor some of the trig and/or exponentials out. ▲

2.6. Linear differential operators.

The study of nth order homogeneous equations, worthwhile in its own right, is one of the two tools that we will need for our alternative approach to solving nonhomogeneous problems. The other tool is the notion of the linear differential operator, which we explore here. Broadly, we can view the derivative f' of a function f as a new function constructed from f, but we can also view the *act* of differentiating as a *function on functions*. After all, any function is just a rule that pairs elements of one set (numbers, people, cats) with elements of another set (other numbers, months, numbers) in a unique way (e.g., $x \mapsto 2x$, human $\mapsto$ month in which human was born, cat $\mapsto$ length of tail in inches, if one is permitted to measure). Every differentiable function has a unique derivative, and so the derivative is a function on a set of differentiable functions!

2.6.1. *Operator theory.*

Let f be differentiable. Write

$$Df := f'.$$

For example, if $f(x) = 2x$, then $(Df)(x) = 2x$. That is, f is a function, and Df is another function[27], specifically, the derivative of f. But the act of associating f with Df is itself a function, which we will call D.

Since functions have domains, we should specify what the domain of D is. Depending on the precise problem at hand, there are lots of possibilities. One choice is to specify an interval I (after all, the good parts of calculus always play out on an interval of some form) and consider functions on I. We write $f\colon I \to \mathbb{R}$ to indicate that f is a real-valued function defined on I.

Now let $\mathcal{C}(I)$ denote the set of all continuous functions on I, and let $\mathcal{C}^1(I)$ denote the set of all differentiable functions on I whose derivatives are also continuous. (Recall from Definition 1.6.4 and Example 1.6.5 that we like our solutions f to differential equations to be differentiable with f' continuous.) If f is a function in $\mathcal{C}^1(I)$, then f' is continuous, so Df is a function in $\mathcal{C}(I)$. Thus $D\colon \mathcal{C}^1(I) \to \mathcal{C}(I)$ is a function.

Before calculus, we primarily met functions one at a time. After calculus, we can consider whole classes of functions at once — and we relate those classes by *functions whose inputs and outputs are themselves functions*. For example, solving the exponential growth equation

$$f' = f$$

is really the same as solving

$$Df - f = 0.$$

If we agree to factor $Df - f = (D-1)f$, then solving exponential growth is the same as asking for the roots, or zeros, of $D - 1$. Of course, the way to find these zeros is our known techniques of separation of variables or integrating factors, so there is no apparent *computational* advantage to writing $(D-1)f = 0$ instead of $f' = f$. Yet.

[27] This is why we write the parentheses around Df in $(Df)(x)$, to indicate that the name of this new function is Df.

We need some further notation to reach this advantage. For an integer $k \geq 0$, define

$$D^k f = f^{(k)}. \tag{2.6.1}$$

The function D^k is then defined on the set k-times differentiable functions. Using the word "function" twice in the previous sentence is probably awkward, and uncomfortable, and so we will typically refer to D^k as a **MAP** or **OPERATOR**.

2.6.1 Example.

Let $f(x) = \cos(x)$. Explain why $D^2 f = -f$.

Solution. We have $D^2 f = f''$, and here $f'(x) = -\sin(x)$ and $f''(x) = -\cos(x) = -f(x)$. Saying $f''(x) = -f(x)$ for all x is the same as abbreviating $f'' = -f$, and in our new notation this is $D^2 f = -f$. ▲

This is where we finished on Wednesday, March 30, 2022.

Of course, we do not usually meet one derivative by itself. Instead, just as we can add one function of a real variable to another, or multiply a function by a constant, producing in each case a new function, so too can we add the operators D^k and multiply them by real numbers.

2.6.2 Definition.

An n**TH ORDER LINEAR DIFFERENTIAL OPERATOR**[28] *is a map of the form*

$$\mathcal{A} = \sum_{k=0}^{n} a_k D^k, \tag{2.6.2}$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are real numbers and $a_n \neq 0$. If f is n-times differentiable, then we define

$$\mathcal{A}f := \sum_{k=0}^{n} a_k D^k f = \sum_{k=0}^{n} a_k f^{(k)}.$$

We emphasize that if $\mathcal{A}$ is defined by (2.6.2) and f is an n-times differentiable function, then the symbol $\mathcal{A}f$ denotes a *new function* defined pointwise by

$$(\mathcal{A}f)(x) = \sum_{k=0}^{n} a_k f^{(k)}(x).$$

These operators are *linear* differential operators precisely because they satisfy the following "linearity" condition.

[28] Strictly speaking, we should emphasize that these are **CONSTANT-COEFFICIENT** linear differential operators, since the coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ are constant real numbers. We worked extensively with variable-coefficient first-order linear differential operators, e.g., $(\mathcal{A}f)(x) := f'(x) + xf(x)$, but our higher-order work has been strictly constant-coefficient, for the reasons that we discussed in Section 2.2.4.

2.6.3 Lemma.

Let $\mathcal{A}$ be an nth order linear differential operator, let f and g be n-times differentiable, and let c be a real number. Then

$$\mathcal{A}(f+g) = \mathcal{A}f + \mathcal{A}g \quad \text{and} \quad \mathcal{A}(cf) = c\mathcal{A}f.$$

In particular,

$$\mathcal{A}[0] = 0.$$

Any nth order linear homogeneous equation, then, is really asking us to solve $\mathcal{A}f = 0$ for some nth order linear differential operator. That is, homogeneous problems are quests for the roots or zeros of an nth order linear differential operator. For example, solving

$$f'' - f = 0 \tag{2.6.3}$$

is the same as solving

$$(D^2 - 1)f = 0.$$

This particular equation is worth exploring quite a bit further. (Of course, we know that every solution has the form $f(x) = c_1e^x + c_2e^{-x}$, but that is not quite what we are after now.) The characteristic polynomial of (2.6.3) is $p(r) = r^2 - 1$, which looks an awful lot like $D^2 - 1$. In fact, we might be tempted to write $D^2 - 1 = p(D)$.

More generally, we have a one-to-one correspondence of polynomials and differential operators via the identification of a linear differential operator with its characteristic polynomial.

2.6.4 Definition.

Let $\mathcal{A} = \sum_{k=0}^{n} a_kD^k$ be an nth order linear differential operator. The **CHARACTERISTIC POLYNOMIAL OF** $\mathcal{A}$ *is the polynomial*

$$p_{\mathcal{A}}(r) := \sum_{k=0}^{n} a_kr^k.$$

Returning to the concrete problem (2.6.3), its characteristic polynomial factors as

$$p(r) = r^2 - 1 = (r+1)(r-1).$$

We might ask about first the meaning of the expression $(D+1)(D-1)$ and then if it equals $D^2 - 1$. Here is the meaning of this expression.

2.6.5 Definition.

Let $\mathcal{A}$ be an nth order linear differential operator and let $\mathcal{B}$ be an mth order linear differential operator. The **COMPOSITION** $\mathcal{AB}$ *of $\mathcal{A}$ and $\mathcal{B}$ is defined by*

$$\mathcal{AB}f := \mathcal{A}(\mathcal{B}f) \tag{2.6.4}$$

for any $(n+m)$-times differentiable function f.

In (2.6.4), we read left-to-right but work right-to-left: first we apply $\mathcal{B}$ to f (which costs m derivatives), and then we apply $\mathcal{A}$ to $\mathcal{B}f$ (which costs an additional n derivatives, so f should possess $n+m$ derivatives). Usually when we are thinking about the composition of functions defined for real numbers, we use the symbol $\circ$ (e.g., $(f \circ g)(x) = f(g(x))$ for functions f and g). Here we do not write $\mathcal{A} \circ \mathcal{B}$. One motivation for defining composition by 2.6.4 is that if j and k are integers, then

$$D^{n+m}f = f^{(n+m)} = D^n[f^{(m)}] = D^n[D^m f],$$

and so we desire that whatever $D^n D^m$ means, it should equal D^{n+m}. And $D^{n+m}f$ and $D^n[D^m f]$ certainly are the same for all $(n+m)$-times differentiable functions f.

Returning to the concrete problem $(D^2-1)f = f''-f$, we now can assign meaning to $(D+1)(D-1)$:

$$(D+1)(D-1)f = (D+1)[(D-1)f].$$

We will use our notational baggage from above to show that $(D+1)[(D-1)f]$ equals exactly what we expect. Afterward, we will never again go through such a baroque computation and instead follow our noses. Let f be twice-differentiable. Then

$$\begin{aligned}
(D+1)(D-1)f &= (D+1)[(D-1)f] \text{ by Definition 2.6.5} \\
&= (D+1)(Df-f) \text{ by Definition 2.6.2} \\
&= (D+1)(f'-f) \text{ by (2.6.1)} \\
&= (D+1)f' - (D+1)f \text{ by the linearity of } D+1 \text{ in Lemma 2.6.3} \\
&= f''+f'-(f'-f) \text{ by (2.6.1) again} \\
&= f''-f \text{ by algebra} \\
&= D^2f-f \text{ by (2.6.1) once more} \\
&= (D^2-1)f \text{ by definition 2.6.2 yet again.}
\end{aligned}$$

Recall that functions f and g are equal if they have the same domain and if $f(x)=g(x)$ for all x in this shared domain. Motivated by this definition, our expectation that D^2-1 and $(D+1)(D-1)$ are the same, and the just-established reality that $(D^2-1)f = (D+1)(D-1)f$ for all twice-differentiable f, we make the following definition.

2.6.6 Definition.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both nth order linear differential operators. Then $\mathcal{A} = \mathcal{B}$ if $\mathcal{A}f = \mathcal{B}f$ for all n-times differentiable functions f.

Here is why we phrase this definition as we do. First, $\mathcal{A}$ and $\mathcal{B}$ are really functions (on functions!), and so they need to agree when evaluated at any element of their domains. This is requiring the equality $\mathcal{A}f = \mathcal{B}f$. Second, $\mathcal{A}$ and $\mathcal{B}$ should have the same domain, and so they should take the same number of derivatives. This is why both $\mathcal{A}$ and $\mathcal{B}$ are nth order operators.

Finally, the "factoring" of a composition of operators respects the factoring of the various characteristic polynomials. For example, if

$$\mathcal{A} = D+1 \quad \text{and} \quad \mathcal{B} = D-1,$$

then their composition, by the work above, is

$$\mathcal{AB} = D^2 - 1.$$

The respective characteristic polynomials are

$$p_{\mathcal{A}}(r) = r + 1, \qquad p_{\mathcal{B}}(r) = r - 1, \qquad \text{and} \qquad p_{\mathcal{AB}}(r) = r^2 - 1.$$

We have

$$p_{\mathcal{AB}} = (r+1)(r-1) = p_{\mathcal{A}}(r)p_{\mathcal{B}}(r).$$

This is true in general: the characteristic polynomial of a composition of operators is the product of the respective characteristic polynomials of the operators. This is one reason why we use the "juxtaposition" notation for operator composition, i.e., $\mathcal{AB}$, instead of the classical "composition" notation $\mathcal{A} \circ \mathcal{B}$. Composition of operators proceeds in lockstep with multiplication of characteristic polynomials.

2.6.7 Theorem.

Let $\mathcal{A}$ and $\mathcal{B}$ be differential operators with characteristic polynomials $p_{\mathcal{A}}$ and $p_{\mathcal{B}}$. The characteristic polynomial $p_{\mathcal{AB}}$ of the composition $\mathcal{AB}$ satisfies

$$p_{\mathcal{AB}}(r) = p_{\mathcal{A}}(r)p_{\mathcal{B}}(r)$$

for all real numbers r.

Multiplication of polynomials is commutative: $p_{\mathcal{A}}(r)p_{\mathcal{B}}(r) = p_{\mathcal{B}}(r)p_{\mathcal{A}}(r)$. After all, given a real number r, this is just the multiplication of the two real numbers $p_{\mathcal{A}}(r)$ and $p_{\mathcal{B}}(r)$, and order is irrelevant. The same turns out to be true for differential operators.

2.6.8 Theorem.

Let $\mathcal{A}$ and $\mathcal{B}$ be nth and mth order differential operators, respectively. Then $\mathcal{AB} = \mathcal{BA}$, in the sense that if f is $(n+m)$-times differentiable, then $\mathcal{A}(\mathcal{B}f) = \mathcal{B}(\mathcal{A}f)$.

2.6.9 Example.

Let $f(x) = \sin(x)$, $\mathcal{A} = D$, and $\mathcal{B} = D^2$. Check that $\mathcal{AB}f = \mathcal{BA}f$.

Solution. We have

$$\mathcal{AB}f = \mathcal{A}(\mathcal{B}f) = \mathcal{A}(D^2 f) = \mathcal{A}(f''),$$

and here $f''(x) = -\sin(x) = -f(x)$. Thus

$$\mathcal{A}(f'') = \mathcal{A}(-f) = -\mathcal{A}f = -Df = -f'.$$

That is,

$$(\mathcal{AB}f)(x) = -\cos(x).$$

Similarly,

$$\mathcal{BA}f = \mathcal{B}(\mathcal{A}f) = \mathcal{B}(Df) = \mathcal{B}(f') = D^2(f') = f''',$$

and $f'''(x) = -\cos(x)$. Hence $(\mathcal{A}\mathcal{B}f)(x) = (\mathcal{B}\mathcal{A}f)(x)$ for all x, and so $\mathcal{A}\mathcal{B}f = \mathcal{B}\mathcal{A}f$. In the preceding sentence, the first use of "=" was to state an equality of real numbers, while the second "=" states an equality of functions. Of course, we also know that $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$, and here we are using "=" to denote an equality of operators! ▲

This is where we finished on Friday, April 1, 2022.

We conclude with a surprising application of differential operators to a prior unresolved issue with homogeneous equations.

2.6.10 Example.

Consider the differential equation $f'' - 2f' + f = 0$.

(i) *Explain why this is the same as solving* $(D-1)^2 f = 0$.

(ii) *Check that if* $f_1(x) = e^x$, *then* $(D-1)f_1 = 0$ *and thus* $(D-1)^2 f_1 = 0$.

(iii) *Argue that if* f_2 *solves* $(D-1)f_2 = f_1$, *then* $(D-1)^2 f_2 = 0$.

(iv) *Find such an* f_2.

(v) *What does this all mean?*

Solution. **(i)** We have

$$f'' - 2f' + f = (D^2 - 2D + 1)f = (D-1)^2 f.$$

(ii) We calculate

$$(D-1)f_1 = f_1' - f_1 = e^x - e^x = 0.$$

Thus

$$(D-1)^2 f_1 = (D-1)[(D-1)f_1] = (D-1)[0] = 0.$$

(iii) Suppose we have a function f_2 that solves $(D-1)f_2 = f_1$. Then

$$(D-1)^2 f_2 = (D-1)[(D-1)f_2] = (D-1)f_1 = 0.$$

(iv) We want to find f_2 such that $(D-1)f_2 = f_1$. That is, we need to solve the first-order problem

$$f_2' - f_2 = e^x.$$

We can use the integrating factor method; multiply both sides by $\mu(x) = e^{-\int dx} = e^{-x}$ to find

$$1 = e^x e^{-x} = f_2'(x)e^{-x} - e^{-x}f_2(x) = \frac{d}{dx}[f_2(x)e^{-x}],$$

and so

$$f_2(x)e^{-x} = x + C$$

for some constant C. Thus $f_2(x) = xe^x + Ce^x$. Since C can be any number, and since we want only one function f_2, we take $C = 0$ to find $f_2(x) = xe^x$.

(v) If we try to solve $f'' - 2f' + f = 0$ with the exponential ansatz $f(x) = e^{rx}$, we only come up with $f(x) = e^x$, i.e., $r = 1$, since the characteristic equation has the double root $r = 1$. Viewing the problem $f'' - 2f' + f = 0$ as the operator equation $(D-1)^2 f = 0$ and exploiting the factorization, along with first-order techniques, gave us the second solution $f_2(x) = xe^x$ and motivates the "multiply by x" technique that we deployed without explanation in Section 2.1.5. ▲

2.6.2. Annihilators and nonhomogeneous linear equations.

At last, we may apply our abstract work in this section and our prior work on arbitrary homogeneous problems to the nonhomogeneous equation. Suppose that $\mathcal{A}$ is an nth order linear differential operator and that we want to solve

$$\mathcal{A}f = g(x) \tag{2.6.5}$$

for a given function g. Suppose further — and this is a very big supposition — that there is an mth order linear differential operator $\mathcal{B}$ such that

$$\mathcal{B}g = 0. \tag{2.6.6}$$

Such an operator $\mathcal{B}$ is said to **ANNIHILATE** the function g; equivalently, $\mathcal{B}$ is an **ANNIHILATOR** of g.

Then

$$\begin{aligned} \mathcal{BA}f &= \mathcal{B}(\mathcal{A}f) \text{ by Definition 2.6.5} \\ &= \mathcal{B}g \text{ by (2.6.5)} \\ &= 0 \text{ by (2.6.6)}. \end{aligned}$$

Here is the value of this calculation: $\mathcal{BA}$ is an $(n+m)$th order linear differential operator, and as such we know all the functions f that could satisfy $\mathcal{BA}f = 0$. They are the functions delineated in Theorem 2.5.8. Consequently, f must have the form of a linear combination of those functions — and so now we know more about f even though we did not explicitly solve $\mathcal{A}f = g$. Yet.

2.6.11 Example.

Solve, once again, $f'' + f = x$.

Solution. This equation has the "operator form"

$$(D^2 + 1)f = x.$$

We need to find an annihilator for $g(x) = x$. Differentiating once, we have[29]

$$D[x] = 1,$$

[29] If we are given a formula but no name for a function, we will write D[formula] for applying the differential operator D to the function given by that formula.

and differentiating twice we have

$$D^2[x] = 0.$$

So, a good[30] annihilator is D^2. That is,

$$(D^2 + 1)f = x \Longrightarrow D^2(D^2 + 1)f = 0. \tag{2.6.7}$$

The characteristic polynomial for this homogeneous equation is $p(r) = r^2(r^2 + 1)$, thanks to Theorem 2.6.7, so its roots are $r = 0$ (with multiplicity 2) and $r = \pm i$ (with multiplicity 1). Thus

$$f(x) = c_1 \cos(x) + c_2 \sin(x) + c_3 + c_4 x$$

for some constants $c_1, \ldots, c_4$.

However, we can, and should, do better. It certainly is not the case that *any* choice of constants will work here. For example, if we take all four to be 0, then $f(x) = 0$, and that is not a solution to this *nonhomogeneous* problem. That is,

$$D^2(D^2 + 1)f = 0 \not\Longrightarrow (D^2 + 1)f = x. \tag{2.6.8}$$

To determine more precisely the constants, recall that f still needs to solve $(D^2+1)f = x$. That is, we need

$$(D^2 + 1)[c_1 \cos(x) + c_2 \sin(x) + c_3 + c_4 x] = x.$$

We use linearity to calculate

$$(D^2+1)[c_1 \cos(x)+c_2 \sin(x)+c_3+c_4 x] = (D^2+1)[c_1 \cos(x)+c_2 \sin(x)]+(D^2+1)[c_3+c_4 x].$$

Now,

$$(D^2 + 1)[c_1 \cos(x) + c_2 \sin(x)] = 0,$$

since $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$ form a fundamental solution set for $D^2 + 1$. Thus we really need

$$(D^2 + 1)[c_3 + c_4 x] = x.$$

We calculate

$$(D^2 + 1)[c_3 + c_4 x] = D^2(c_3 + c_4 x] + (c_3 + c_4 x) = c_3 + c_4 x,$$

and so we come down to

$$c_3 + c_4 x = x. \tag{2.6.9}$$

We have two unknowns left, c_3 and c_4. We could plug in two values for x in (2.6.9), say, $x = 0$ and $x = 1$, to find $c_3 = 0$ and $c_4 = 1$. Or we could rewrite (2.6.9) as

$$c_3 + (c_4 - 1)x = 0.$$

This is a polynomial equation, and a polynomial adds up to 0 *for all values of* x if and only if all of its coefficients are 0. Thus we need $c_3 = 0$ and $c_4 - 1 = 0$, hence $c_4 = 1$.

[30]If a differential operator $\mathcal{B}_1$ annihilates g, and $\mathcal{B}_2$ is any differential operator, then $\mathcal{B}_2\mathcal{B}_1 g = 0$, too. Thus the composition $\mathcal{B}_2\mathcal{B}_1$ is also an annihilator for g. But why annihilate with more force than necessary?

However we do it, we must have

$$f(x) = c_1 \cos(x) + c_2 \sin(x) + x,$$

where c_1 and c_2 can remain arbitrary coefficients. This is wholly in agreement with Examples 2.3.3 and 2.4.5. ▲

This is where we finished on Monday, April 4, 2022.

2.6.12 Example.

Solve, for the third (and maybe last) time, $f'' - f = e^{2x}$.

Solution. This equation is

$$(D^2 - 1)f = e^{2x}.$$

We need an annihilator for $g(x) = e^{2x}$. Hopefully our experience suggests that if $g(x) = e^{ax}$ for a constant a, then g solves $g' = ag$, thus $(D - a)g = 0$. So, if we can solve this nonhomogeneous problem, then we must have

$$(D - 2)(D^2 - 1)f = 0.$$

The characteristic polynomial is then $p(r) = (r - 2)(r^2 - 1)$, so its roots are $r = 2$ and $r = \pm 1$, each with multiplicity 1. Thus

$$f(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x},$$

and so we need

$$(D^2 - 1)[c_1 e^x + c_2 e^{-x} + c_3 e^{2x}] = 0.$$

Since $f_1(x) = e^x$ and $f_2(x) = e^{-x}$ form a fundamental solution set for $D^2 - 1$, this collapses to

$$(D^2 - 1)[c_3 e^{2x}] = e^{2x}.$$

We calculate

$$(D^2 - 1)[c_3 e^{2x}] = c_3 (D^2 - 1)[e^{2x}] = c_3 (4e^{2x} - e^{2x}) = 3c_3 e^{2x}.$$

Thus we need

$$3c_3 e^{2x} = e^{2x},$$

and so

$$c_3 = \frac{1}{3}.$$

That is, the solution f has the form

$$f(x) = c_1 e^x + c_2 e^{-x} + \frac{e^{2x}}{3},$$

where c_1 and c_2 are arbitrary constants, exactly as in Examples 2.3.4 and 2.4.3. ▲

This is where we finished on Wednesday, April 6, 2022.

2.6.13 Example.

Let ω be a real number. Find all solutions to $f'' + f = \cos(\omega x)$.

Solution. This equation has the form $(D^2+1)f = \cos(\omega x)$. We need to find a differential operator $\mathcal{B}$ such that $\mathcal{B}[\cos(\omega x)] = 0$. We try differentiating a few times:

$$D[\cos(\omega x)] = -\sin(\omega x)\omega \quad \text{and} \quad D^2[\cos(\omega x)] = -\omega^2\cos(\omega x).$$

The second equality rearranges to

$$0 = D^2[\cos(\omega x)] + \omega^2\cos(\omega x) = (D^2+\omega^2)[\cos(\omega x)].$$

So, the operator $\mathcal{B} := D^2 + \omega^2$ annihilates the nonhomogeneity here. Thus any solution f to $f'' + f = \cos(\omega x)$ must also solve

$$(D^2+\omega^2)(D^2+1)f = 0. \tag{2.6.10}$$

The characteristic polynomial for this homogeneous problem is $p(r) = (r^2+\omega^2)(r^2+1)$. It has the roots $r = \pm i\omega$ and $\pm i$. We must be very careful here: if $\omega = \pm 1$, then really $p(r) = (r^2+1)^2$, and the only roots are $r = \pm i$, with multiplicity 2. So, from now on, we consider two cases.

Case 1. $\omega \neq \pm 1$. Here we have two distinct complex conjugate pairs: $r = \pm i\omega$ and $r = \pm i$. Hence all solutions to (2.6.10) are

$$f(x) = c_1\cos(x) + c_2\sin(x) + c_3\cos(\omega x) + c_4\sin(\omega x).$$

We want such an f to solve $(D^2+1)f = \cos(\omega x)$, and so we evaluate

$$\begin{aligned}
&(D^2+1)[c_1\cos(x) + c_2\sin(x) + c_3\cos(\omega x) + c_4\sin(\omega x)] \\
&\qquad = (D^2+1)[c_1\cos(x) + c_2\sin(x)] + c_3(D^2+1)[\cos(\omega x)] + c_4(D^2+1)[\sin(\omega x)] \\
&\qquad\qquad = c_3(D^2+1)[\cos(\omega x)] + c_4(D^2+1)[\sin(\omega x)].
\end{aligned}$$

Here we have used linearity and the fundamental solution set for D^2+1. Next, we calculate

$$(D^2+1)[\cos(\omega x)] = (1-\omega^2)\cos(\omega x) \quad \text{and} \quad (D^2+1)[\sin(\omega x)] = (1-\omega^2)\sin(\omega x).$$

Thus we need c_3 and c_4 to meet

$$c_3(1-\omega^2)\cos(\omega x) + c_4(1-\omega^2)\sin(\omega x) = \cos(\omega x). \tag{2.6.11}$$

Intuitively, we probably expect that the only way that a combination of cosines and sines can add up to a cosine is for there just to be a cosine in the sum. Let us try to see this more rigorously. If we take $x = 0$, then we find

$$c_3(1-\omega^2)^2 = 1,$$

and so, since $\omega \neq \pm 1$, we may divide to solve for

$$c_3 = \frac{1}{1-\omega^2}.$$

We replace c_3 with this value in (2.6.11) to see that

$$\cos(\omega x) + c_4(1-\omega^2)\sin(\omega x) = \cos(\omega x),$$

and thus

$$c_4(1-\omega^2)\sin(\omega x) = 0.$$

This has to be true for all x, and it can only happen if either $c_4 = 0$ or $\omega = 0$. Either way, we conclude

$$f(x) = c_1\cos(x) + c_2\sin(x) + \frac{\cos(\omega x)}{1-\omega^2}$$

Case 2. $\omega = \pm 1$. In either case, the differential equation is $f'' + f = \cos(x)$, since $\cos(-x) = \cos(x)$. The only roots of the characteristic polynomial $p(r) = (r^2+1)^2$ are the repeated conjugate pair $r = \pm i$ with multiplicity 2, and so f must have the form

$$f(x) = c_1\cos(x) + c_2\sin(x) + c_3 x\cos(x) + c_4 x\sin(x).$$

Again, we want

$$(D^2+1)[c_1\cos(x) + c_2\sin(x) + c_3 x\cos(x) + c_4 x\sin(x)] = \cos(x),$$

and this amounts to requiring c_3 and c_4 to satisfy

$$(D^2+1)[c_3 x\cos(x) + c_4 x\sin(x)] = \cos(x).$$

Believe it or not,

$$(D^2+1)[x\cos(x)] = -2\sin(x) \quad \text{and} \quad (D^2+1)[x\sin(x)] = 2\cos(x).$$

Thus c_3 and c_4 must meet

$$-2c_3\sin(x) + 2c_4\cos(x) = \cos(x)$$

for all x.

We choose x cleverly to make some of the terms above equal to 0. If $x = 0$, this means $2c_4 = 1$, so $c_4 = 1/2$. If $x = \pi/2$, then (because $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$), we have $-2c_3 = 0$, thus $c_3 = 0$. In this case

$$f(x) = c_1\cos(x) + c_2\sin(x) + \frac{x\sin(x)}{2}.$$

Thus depending on the choice of the parameter ω, we have two radically different kinds of solutions to $f'' + f = \cos(\omega x)$:

$$f(x) = \begin{cases} c_1\cos(x) + c_2\sin(x) + \dfrac{\cos(\omega x)}{1-\omega^2}, & \omega \neq \pm 1 \\[2ex] c_1\cos(x) + c_2\sin(x) + \dfrac{x\sin(x)}{2}, & \omega = \pm 1. \end{cases}$$ ▲

2.7. The forced harmonic oscillator.

Recall our construction of the harmonic oscillator from Section 2.2. We attached an object of mass m to a spring that exerted the force $\mathsf{F}_{\text{spr}}(r) := -\kappa r$ for some constant $\kappa > 0$ when stretched a distance r and then attached the spring to a wall. We measured the oscillator's displacement at time t from its equilibrium (rest) position by $u(t)$ and supposed that the oscillator experienced a friction force of the form $\mathsf{F}_{\text{fr}}(t) := -bu'(t)$ for some constant $b \geq 0$. Taking $b = 0$ allowed us to consider the magical situation in which there was no friction. Assuming that no other forces acted on the oscillator, Newton's second law (mass $\times$ acceleration $=$ force told us

$$mu'' = \mathsf{F}_{\text{spr}}(u(t)) + \mathsf{F}_{\text{fr}}(t) = -\kappa u(t) - bu'(t),$$

and thus

$$mu'' + bu'(t) + \kappa u(t) = 0.$$

Suppose now that there are some other external forces acting on the oscillator. By "external" we mean any force that does not arise from friction or the pull of the spring. Maybe a cat is whacking the oscillator with her beefy paw; maybe an annoying downstairs neighbor (or, worse, family member) is playing music loudly, so that the wall to which the oscillator is attached is vibrating. Measure all these external forces at time t by $\mathsf{F}_{\text{ext}}(t)$. Then Newton's law really says

$$mu'' = \mathsf{F}_{\text{spr}}(u(t)) + \mathsf{F}_{\text{fr}}(t) + \mathsf{F}_{\text{ext}}(t),$$

and this rearranges to

$$mu'' + bu' + \kappa u = \mathsf{F}_{\text{ext}}(t).$$

This is a nonhomogeneous constant-coefficient second-order linear differential equation. If F_{ext} is continuous, we can use variation of parameters to solve it; if F_{ext} has the fairly special form "polynomial $\times$ exponential $\times$ trig," then we can use annihilators. Neither way is pretty.

2.7.1 Example.

An has mass 1 and spring constant 1 ($m = \kappa = 1$) and there is no friction ($b = 0$). The external force is periodic: $\mathsf{F}_{\text{ext}}(t) = \cos(\omega t)$ for some real number ω. When we set the oscillator in motion, we pulled it 1 unit to the right of equilibrium and let it go very gently: $u(0) = 1$ and $u'(0) = 0$. What happens?

Solution. The displacement satisfies the initial value problem

$$\begin{cases} u'' + u = \cos(\omega t) \\ u(0) = 1 \\ u'(0) = 0. \end{cases} \tag{2.7.1}$$

Example 2.6.13 tells us that if $u'' + u = \cos(\omega t)$, then

$$u(t) = \begin{cases} c_1\cos(t) + c_2\sin(t) + \dfrac{\cos(\omega t)}{1-\omega^2}, & \omega \neq \pm 1 \\ \\ c_1\cos(t) + c_2\sin(t) + \dfrac{t\sin(t)}{2}, & \omega = \pm 1 \end{cases}$$

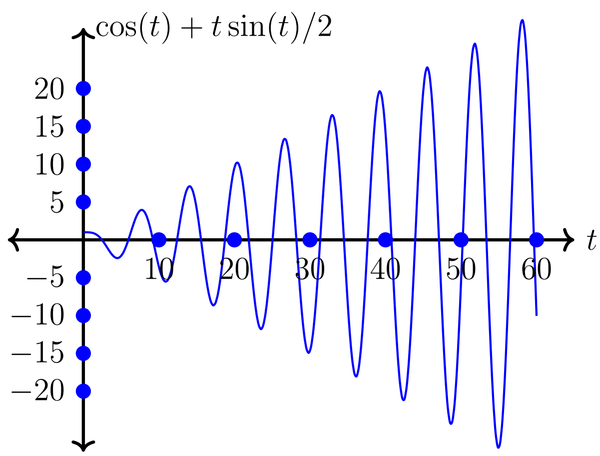
for some constants c_1 and c_2. The key part of that example was recognizing the "bifurcation" in the annihilator method at $\omega = \pm 1$.

To solve the full initial value problem (2.7.1), we need to select the constants c_1 and c_2; some algebra reveals that the solution is

$$u(t) = \begin{cases} \cos(t) + \dfrac{\cos(\omega t) - \cos(t)}{1-\omega^2}, & \omega \neq \pm 1 \\ \\ \cos(t) + \dfrac{t\sin(t)}{2}, & \omega = \pm 1. \end{cases}$$

These solutions have drastically different behavior. A careful reading of their formulas will suggest why, but it is easier (and maybe more fun) to start by looking at the graphs.

Here is the graph of $u(t) = \cos(t) + t\sin(t)/2$ for the $\omega = \pm 1$ case.

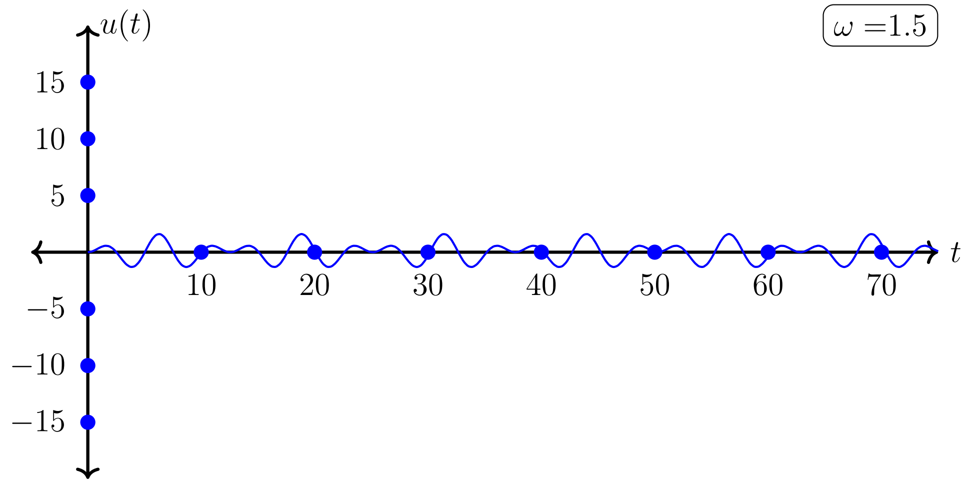


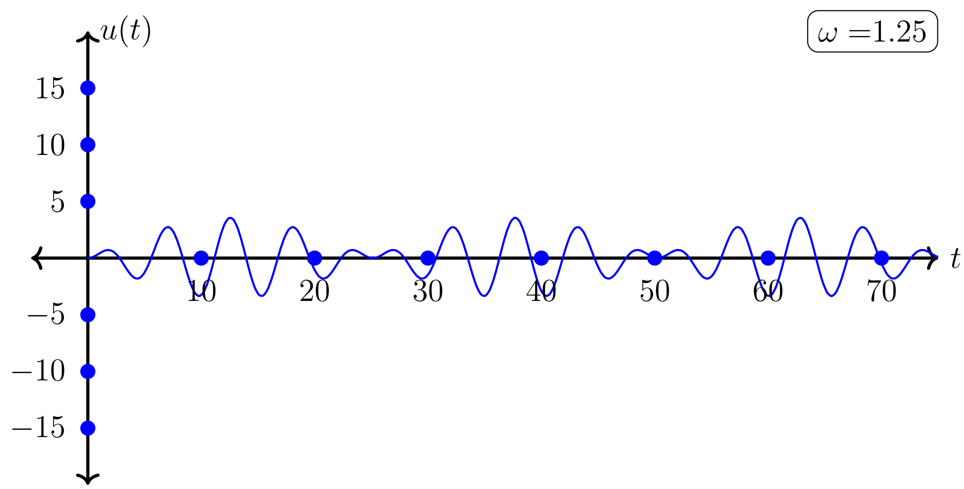
It looks like the oscillations in the graph keep getting larger and larger as $t \to \infty$. The graph still intersects the t-axis, so it is not the case that $\lim_{t\to\infty} u(t) = 0$. That is, the oscillator passes through equilibrium infinitely many times, but after each pass it moves further away from equilibrium than it did before. Physically, this is wholly unrealistic; not only is the oscillator never slowing down (no friction, after all), it is actually getting further away from — and then returning to — equilibrium. The extreme oscillations come from the $t\sin(t)/2$ term; take $t = (2k+1)\pi/2$ to see that

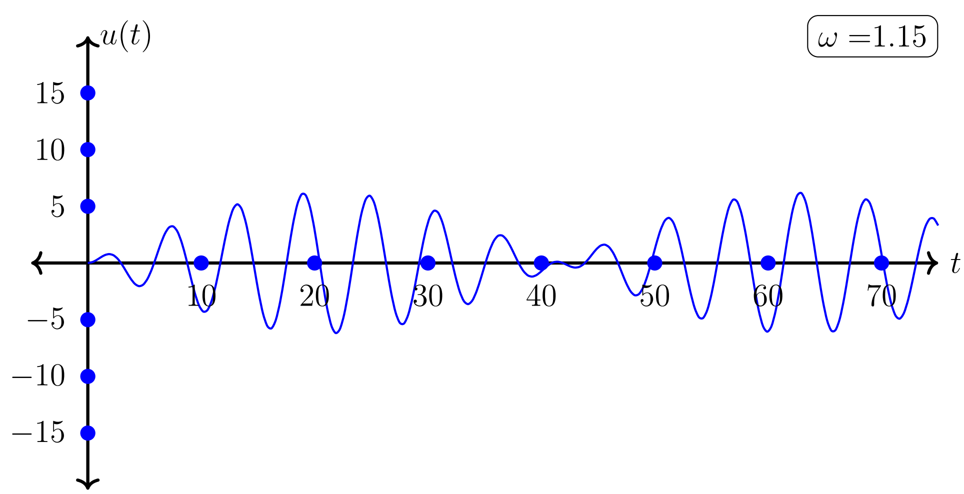
$$\left|u\left(\frac{(2k+1)\pi}{2}\right)\right| = \frac{(2k+1)\pi}{4} \to \infty$$

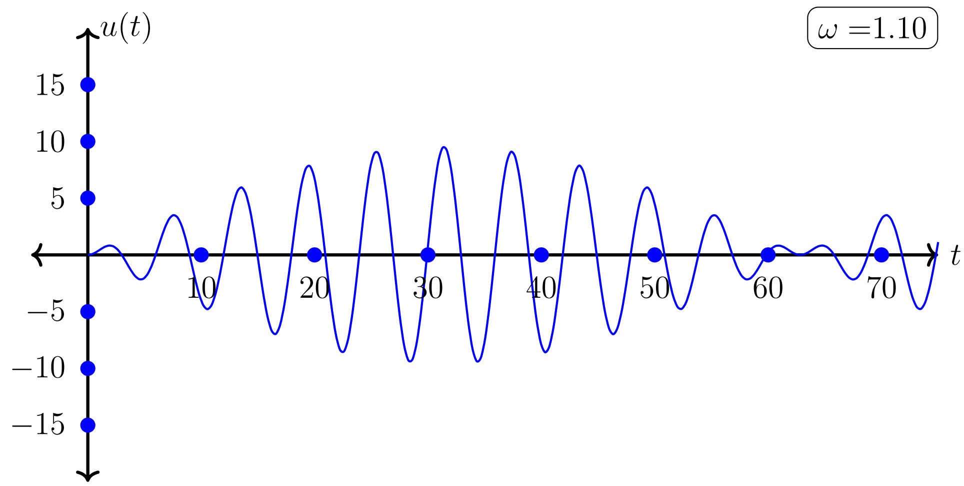
as $k \to \infty$.

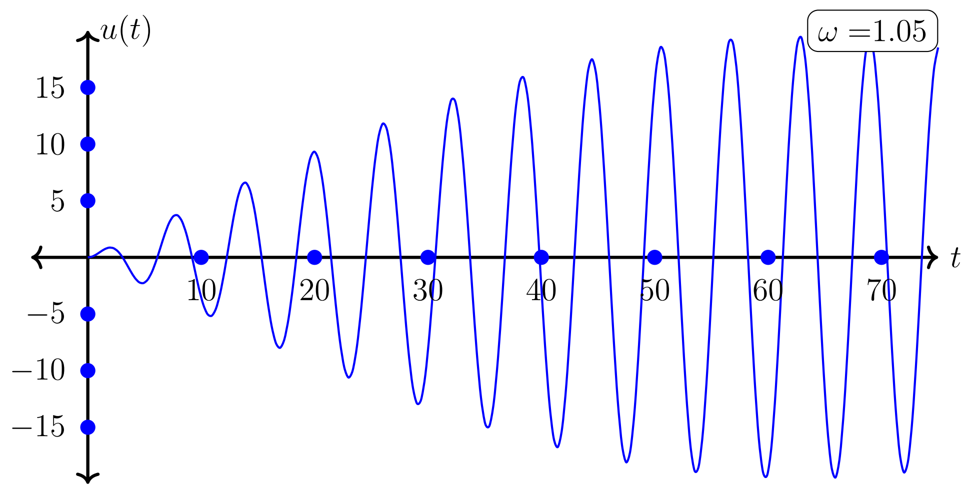
Here are several graphs for the $\omega \neq \pm 1$ case. It is probably more interesting to see what happens when ω is close to but not equal to 1, so we plot some of those sketches.











It seems that as $\omega \to 1^+$, the oscillations remain bounded but become larger as ω gets closer to 1. This is not surprising, since the cosine terms in the sum

$$u_\omega(t) := \cos(t) + \frac{\cos(\omega t) - \cos(t)}{1 - \omega^2}$$

remain bounded between 1 and -1, but the (absolute value of the) denominator $1 - \omega^2$ becomes increasingly large as $\omega \to 1$. That is, if u_ω is the solution for a given $\omega \neq \pm 1$, then

$$\max_{t \geq 0} |u_\omega(t)| < \infty \quad \text{but} \quad \lim_{\omega \to 1^+} \max_{t \geq 0} |u_\omega(t)| = \infty.$$

Why is $\omega = \pm 1$ so special? Our choice of the material data for the oscillator — $m = \kappa = 1$ and $b = 0$ — means that the functions $f_1(t) = \cos(t)$ and $f_2(t) = \sin(t)$ form a fundamental solution set for the undriven (homogeneous) problem $u'' + u = 0$. These functions f_1 and f_2 have frequency $\omega = 1$. Taking the periodic forcing function to have this same frequency excites a "resonance" in the oscillator that leads to the arbitrarily large displacements that we first observed. Taking the periodic forcing function to have frequency very close to but not equal to 1 — a "near resonance" forcing — permitted the resulting oscillations to become increasingly large. ▲

Consider one final forcing situation. Suppose that we initially set up the oscillator (still with $m = \kappa = 1$ and $b = 0$) without the influence of any external forces for some time, say $0 \le t < t_0$ for some $t_0 > 0$. Then from time $t = t_0$ to some time $t = t_1 > t_0$, a fickle cat comes by and whacks the oscillator, so that it is experiencing the force $\cos(\omega t)$ for $t_0 \le t < t_1$. The cat, being fickle, decides at time $t = t_1$ to pursue her important cat business elsewhere, and she stops whacking the oscillator as abruptly as she started. Then the external force that the oscillator experiences is

$$\mathsf{F}_{\text{ext}}(t) := \begin{cases} 0, & 0 \le t < t_0 \\ \cos(\omega t), & t_0 \le t < t_1 \\ 0, & t_1 \le t. \end{cases}$$

We could solve $u'' + u = \mathsf{F}_{\text{ext}}(t)$ "interval by interval" and somehow "piece together by continuity" the different solutions. This, however, is exceedingly dull, as the following toy example of a differential equation with a "discontinuous forcing function" shows.

2.7.2 Example.

Solve the initial value problem

$$\begin{cases} u'(t) = g(t) \\ u(0) = 0, \end{cases} \qquad \textit{where} \qquad g(t) := \begin{cases} 0, & t < 0 \\ 1, & t \ge 0. \end{cases}$$

Solution. For $t < 0$, we need $u'(t) = 0$, thus $u(t) = c$ for some constant c and all $t < 0$. For $t \ge 0$, we need $u'(t) = 1$, thus $u(t) = t + k$ for some constant k and all $t \ge 0$. We also want $u(0) = 0$, so we need $0 = 0 + k = k$. Thus

$$u(t) = \begin{cases} c, & t < 0 \\ t, & t \ge 0, \end{cases}$$

where c is arbitrary. Say $c = 2$. Then the graph of u is as follows.

This means that u is discontinuous at $t = 0$. But surely the solution to a differential equation should be continuous on its domain. If we think for a moment, we see that we can choose $c = 0$ to have

$$u(t) = \begin{cases} 0, & t < 0 \\ t, & t \ge 0 \end{cases}$$

as a continuous function that satisfies $u'(t) = g(t)$ for $t \ne 0$. However, no matter what c we choose, u will not be differentiable at $t = 0$, although the limits

$$\lim_{t \to 0^-} u'(t) \quad \text{and} \quad \lim_{t \to 0^+} u'(t)$$

do exist. The solution u inherits this lack of differentiability at $t = 0$ from the discontinuity at $t = 0$ of the "forcing" function g. ▲

Here is the lesson of this example. Solving differential equations with discontinuous terms/coefficients may be physically worthwhile and meaningful, but symbolically it is annoying. Our final topic, the Laplace transform, will provide us with a different method for handling such "discontinuous" equations (which will be annoying, no doubt, in different ways).

This is where we finished on Monday, April 11, 2022.

3. The Laplace Transform

The Laplace transform offers a very different perspective on differential equations. We start with a special kind of improper integral, see how it talks to derivatives, and convert differential equations into sorts of "algebraic" problems. We do not really need a new way to solve constant-coefficient problems, but the Laplace transform plays particularly well with the discontinuous forcing functions that previously caused us some moaning and gnashing of teeth. It will take a bit of work to see the value of the Laplace transform for differential equations; be patient and trust that it will be worthwhile.

3.1. Definition and elementary properties of the Laplace transform.

In calculus we studied many flavors of the improper integral; here we need just one.

3.1.1 Definition.

(i) *A function f defined on the interval $[0, \infty)$ is* **LOCALLY INTEGRABLE** *if f is integrable on each interval $[0, b]$ for all $b > 0$.*

(ii) *If f is locally integrable on $[0, \infty)$, and if the limit*

$$\lim_{b \to \infty} \int_0^b f(x)\ dx$$

exists, then we say that f is **IMPROPERLY INTEGRABLE** *on $[0, \infty)$, and we define*

$$\int_0^\infty f(x)\ dx := \lim_{b \to \infty} \int_0^b f(x)\ dx.$$

We say that the integral $\int_0^\infty f(x)\ dx$ **CONVERGES** *if the limit above exists and is finite and* **DIVERGES** *if the limit above either does not exist or exists and is $\pm\infty$.*

Incidentally, if f is continuous on $[0, b]$, then f is integrable on $[0, b]$; thus if f is continuous on $[0, \infty)$, then f is locally integrable on $[0, \infty)$. In particular, all differentiable functions on $[0, \infty)$ are locally integrable on $[0, \infty)$.

3.1.2 Definition.

Let f be locally integrable on $[0, \infty)$ and let s be a real number. The **LAPLACE TRANSFORM OF f AT s** *is the number*

$$\mathscr{L}[f](s) := \int_0^\infty f(x)e^{-sx}\ dx,$$

if this improper integral converges. If this improper integral diverges, then we say that $\mathscr{L}[f](s)$ is **UNDEFINED**.

In other words,

$$\mathscr{L}[f](s) = \lim_{b\to\infty} \int_0^b f(x)e^{-sx}\,dx.$$

It is important to remember that x is the dummy variable of integration, s is a parameter in the integrand, and b is the upper limit of integration.

3.1.3 Example.

Let $f(x) = e^{3x}$. Determine all values of s for which the Laplace transform $\mathscr{L}[f](s)$ is defined and calculate a formula for $\mathscr{L}[f](s)$ there.

Solution. By definition,

$$\mathscr{L}[f](s) = \int_0^\infty e^{3x}e^{-sx}\,dx = \lim_{b\to\infty} \int_0^b e^{(3-s)x}\,dx.$$

First we evaluate the definite integral for b fixed:

$$\int_0^b e^{(3-s)x}\,dx = \left.\frac{e^{(3-s)x}}{3-s}\right|_{x=0}^{x=b} = \frac{e^{(3-s)b}}{3-s} - \frac{1}{3-s}.$$

This, by the way, is only valid for $3 - s \neq 0$; we will handle $3 - s = 0$ momentarily. Hence (if $3 - s \neq 0$) we have

$$\mathscr{L}[f](s) = \lim_{b\to\infty} \left(\frac{e^{(3-s)b}}{3-s} - \frac{1}{3-s}\right).$$

Recall that for a given real number $r \neq 0$, we have

$$\lim_{b\to\infty} e^{rb} = \begin{cases} 0, & b < 0 \\ \infty, & b > 0. \end{cases}$$

Thus if $3 - s < 0$, we have

$$\mathscr{L}[f](s) = \lim_{b\to\infty} \left(\frac{e^{(3-s)b}}{3-s} - \frac{1}{3-s}\right) = -\frac{1}{3-s} = \frac{1}{s-3},$$

while if $3 - s > 0$, the limit does not exist; the improper integral does not converge; and the Laplace transform is undefined. That is,

$$\mathscr{L}[f](s) = \begin{cases} \dfrac{1}{s-3}, & s > 3 \\ \\ \text{undefined}, & s < 3. \end{cases}$$

Finally, to handle the case $s - 3 = 0$, or $s = 3$, we appeal to the definition once again:

$$\mathscr{L}[f](3) = \int_0^\infty e^{3x}e^{-3x}\,dx = \int_0^\infty 1\,dx = \lim_{b\to\infty} \int_0^b 1\,dx = \lim_{b\to\infty} b = \infty.$$

Thus $\mathscr{L}[f](3)$ is undefined, and we conclude

$$\mathscr{L}[f](s) = \begin{cases} \dfrac{1}{s-3}, & s > 3 \\ \text{undefined}, & s \leq 3. \end{cases}$$ ▲

We will always use the letter s for the variable of the Laplace transform and x for the variable of the underlying function. It is a bit sloppy, but very evocative, to replace f in $\mathscr{L}[f](s)$ with the formula for f as a function of x; thus we would say

$$\mathscr{L}[e^{3x}](s) = \frac{1}{s-3}, \quad s > 3.$$

More generally (by replacing every instance of 3 in Example 3.1.3 with a) we have the following result.

3.1.4 Lemma.

Let a be a real number. Then

$$\mathscr{L}[e^{ax}](s) = \begin{cases} \dfrac{1}{s-a}, & s > a \\ \textit{undefined}, & s \leq a. \end{cases}$$

At the best of times, the Laplace transform of a function really is another function, possibly on a different domain. As we just saw above, while $f(x) := e^{ax}$ is defined for all real numbers x, its Laplace transform is only defined on the interval (a, ∞). Like the differential operator D, the Laplace transform $\mathscr{L}$ is a map or operator on a set of functions: it turns functions f defined on $[0, \infty)$ into functions $\mathscr{L}[f]$ defined. . .somewhere. We will give a sufficient condition for the Laplace transform of a function to be always defined shortly. For now, we discuss the primary reason that the Laplace transform is important for differential equations.

3.1.5 Example.

Suppose that f is defined and differentiable on $[0, \infty)$. How can we relate the Laplace transforms $\mathscr{L}[f](s)$ and $\mathscr{L}[f'](s)$?

Solution. Without further information on f, the best that we can do is appeal to the definitions:

$$\mathscr{L}[f'](s) = \int_0^\infty f'(x)e^{-sx}\,dx = \lim_{b\to\infty}\int_0^b f'(x)e^{-sx}\,dx,$$

if this limit exists. So, when does this limit exist, and, if it exists, what is its value?

We are dealing here with the integral of a product, and this is exactly why we have integration by parts. Since we want to relate this integral to f, we probably should pick

$dv = f'(x)\ dx$. That is, we set

$$u = e^{-sx} \qquad dv = f'(x)\ dx$$
$$du = -se^{-sx}\ dx \qquad v = f(x)$$

to find

$$\int_0^b f'(x)e^{-sx}\ dx = e^{-sx}f(x)\Big|_{x=0}^{x=b} - \int_0^b f(x)[-se^{-sx}]\ dx = e^{-sb}f(b) - f(0) + s\int_0^b f(x)e^{-sx}\ dx.$$

This is comforting: we see the same integral over $[0, b]$ above as appears in the limit definition of $\mathscr{L}[f](s)$.

Specifically, since

$$\lim_{b\to\infty} s\int_0^b f(x)e^{-sx}\ dx = s\lim_{b\to\infty}\int_0^b f(x)e^{-sx}\ dx = s\mathscr{L}[f](s),$$

we have

$$\lim_{b\to\infty}\left(e^{-sb}f(b) - f(0) + s\int_0^b f(x)e^{-sx}\ dx\right) = \left(\lim_{b\to\infty} e^{-sb}f(b)\right) + \left(s\mathscr{L}[f](s) - f(0)\right),$$

if the limit

$$\lim_{b\to\infty} e^{-sb}f(b)$$

exists. In the particularly nice case that this limit is 0, we can conclude

$$\lim_{b\to\infty}\int_0^b f'(x)e^{-sx}\ dx = s\mathscr{L}[f](s) - f(0).$$

We summarize our work: if the differentiable function f and the number s satisfy

$$\lim_{b\to\infty} e^{-sb}f(b) = 0,$$

then the Laplace transform $\mathscr{L}[f'](s)$ exists and, moreover, we have the identity

$$\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0).$$

This is definitely a relationship between $\mathscr{L}[f']$ and $\mathscr{L}[f]$. ▲

The work of the preceding example proves the following lemma.

3.1.6 Lemma.

Suppose that f is locally integrable and differentiable on $[0, \infty)$ and let s be a real number. If the Laplace transform $\mathscr{L}[f](s)$ converges and if

$$\lim_{b\to\infty} e^{-sb}f(s) = 0,$$

then the Laplace transform $\mathscr{L}[f'](s)$ also converges and

$$\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0). \tag{3.1.1}$$

This is where we finished on Wednesday, April 13, 2022.

The natural question, then, is for what functions f and what real numbers s do we have

$$\lim_{x\to\infty} e^{-sx}f(x) = 0. \tag{3.1.2}$$

One way to approach this is to work backwards: if this limit holds, then for all $C > 0$, there is $M > 0$ such that if $x \geq M$, then

$$|e^{-sx}f(x)| \leq C.$$

Thus for $x \geq M$, we have

$$|f(x)| \leq Ce^{sx}. \tag{3.1.3}$$

However, an inequality of this form does not guarantee that the limit (3.1.2) holds. Indeed, if we take $f(x) := e^{sx}$, then the inequality (3.1.3) is true with $C = 1$ for all x — in fact, it holds with equality — but we have

$$\lim_{x\to\infty} e^{-sx}f(x) = \lim_{x\to\infty} e^{-sx}e^{sx} = 1.$$

The right idea is to avoid overworking s and to introduce a new parameter q.

3.1.7 Definition.

Let q be a real number. A function f defined on $[0,\infty)$ has **EXPONENTIAL ORDER** *q or* **GROWS AT MOST EXPONENTIALLY WITH RATE** *q if there are constants $C > 0$ and $x_0 \geq 0$ such that if $x \geq x_0$, then $|f(x)| \leq Ce^{qx}$.*

In other words, for $x \geq x_0$, the graph of $|f|$ is trapped between the x-axis and the graph of $y = Ce^{qx}$. This property[31] not only guarantees the relationship (3.1.1) between the Laplace transform of a function and the transform of its derivative; it gives us a range of s for which $\mathscr{L}[f](s)$ exists in the first place.

[31] Here is a more leisurely exposition of how Definition 3.1.7 entered the collective mathematical consciousness. It would be nice to have a way of ensuring that the Laplace transform exists without checking the improper integral for each and every real number s. The following version of the **COMPARISON TEST** for improper integrals is useful: if there is a locally integrable function g defined on $[0,\infty)$ and a number $x_0 \geq 0$ such that $e^{-sx}|f(x)| \leq g(x)$ for all $x \geq x_0$, and if $\int_0^\infty g(x)\,dx$ converges, then $\mathscr{L}[f](s)$ also converges. (This is something that we learned in calculus, and we will simply accept that it is true here.)

One such candidate for g is the very nice function $g(x) = Ce^{-rx}$, where $r > 0$ and $C > 0$; we have

$$\int_0^\infty e^{-rx}\,dx = \frac{1}{r}.$$

If we have $e^{-sx}|f(x)| \leq Ce^{-rx}$ for all x greater than some x_0, then $\mathscr{L}[f](s)$ exists. This inequality is the

3.1.8 Lemma.

Suppose that f is locally integrable on $[0, \infty)$ and has exponential order q.

(i) $\mathscr{L}[f](s)$ *converges for all* $s > q$.

(ii) *If f is also differentiable on $[0, \infty)$, then $\mathscr{L}[f'](s)$ converges for all $s > q$ and $\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0)$ for $s > q$.*

Proof. We leave the proof of part (i) as an exercise in the comparison test for improper integrals. For part (ii), we just need to show

$$\lim_{x \to \infty} e^{-sx} f(x) = 0,$$

thanks to Lemma 3.1.6. (Note that we do not need to assume that f' has exponential order q to apply this lemma!) We use the squeeze theorem:

$$0 \leq |e^{-sx} f(x)| \leq Ce^{-sx} e^{qx} = Ce^{(q-s)x},$$

and since $s > q$, we have $q - s < 0$, thus

$$\lim_{x \to \infty} e^{(q-s)x} = 0. \qquad \blacksquare$$

The next natural question is what functions f have exponential order q and what value of q works for those functions. The good news is that "many" of the functions that we meet in calculus do have some exponential order, and so their Laplace transforms are defined.

3.1.9 Example.

What is the exponential order of each function below?

(i) $f(x) = \sin(x)$

(ii) $f(x) = x$

Solution. **(i)** We know that $-1 \leq \sin(x) \leq 1$ for all x, so $|\sin(x)| \leq 1$. And since $1 = e^{0 \cdot x} = 1 \cdot e^{0 \cdot x}$, we have

$$|\sin(x)| \leq 1 \cdot e^{0 \cdot x}$$

for all x. So, taking $x_0 = 0$ (purely for convenience), $C = 1$, and $q = 0$, we see that f has exponential order 0. In particular, $\mathscr{L}[\sin(x)](s)$ is defined for all $s > 0$. We will figure out a formula later.

---

same as saying $|f(x)| \leq Ce^{(s-r)x}$.

Thus if there are C, $r > 0$ and $x_0 \geq 0$, and if $|f(x)| \leq Ce^{(s-r)x}$ for $x \geq x_0$, then $\mathscr{L}[f](s)$ exists. We can clean this up a bit further, actually. Put $q = s - r$. Since $r > 0$, we have $s > s - r = q$.

Now, forget about r, and just suppose that for some $C > 0$, some $x_0 \geq 0$, and some real number q (which need not be positive, or negative), we have $|f(x)| \leq Ce^{qx}$ whenever $x \geq x_0$. Let $s > q$. Then $e^{-sx}|f(x)| \leq Ce^{(q-s)x}$, and since $q - s < 0$, the comparison test forces $\mathscr{L}[f](s)$ to exist.

(ii) Experience from prior calculus might suggest to us that "exponentials grow at ∞ more quickly than polynomials" and so, given $q > 0$, we should be able to find $C > 0$ and $x_0 > 0$ such that if $x > x_0$, then $|x| \le Ce^{qx}$. To make this more precise, we use L'Hospital's rule to calculate

$$\lim_{x\to\infty} \frac{x}{e^{qx}} = \lim_{x\to\infty} \frac{1}{qe^{qx}} = 0,$$

and so there is $x_0 > 0$ such that if $x \ge x_0$, then

$$\left|\frac{x}{e^{qx}}\right| \le 1.$$

(Here we are choosing 1 purely for convenience; we could replace 1 with any positive number and just have to adjust x_0 accordingly.) Thus for $x \ge x_0$, we have $|x| \le e^{qx}$, and so $f(x) = x$ has exponential order q.

What is nice here is that $q > 0$ was arbitrary. So, given $s > 0$, let $q = s/2$. Then $q > 0$ and $s > q$, so $\mathscr{L}[x](s)$ converges. That is, $\mathscr{L}[x](s)$ converges for all $s \ge 0$. ▲

The results in this example can be strengthened: any bounded function has exponential order 0 and any polynomial has exponential order 0, too. Lemma 3.1.8 says that any exponential of the form $f(x) = e^{ax}$ has exponential order a.

3.1.10 Example.

Calculate $\mathscr{L}[x](s)$ without evaluating any integrals. For what values of s does this converge?

Solution. First, $f(x) = x$ is a polynomial, so f has exponential order 0, and therefore $\mathscr{L}[f](s)$ converges for all $s > 0$. We have $f'(x) = 1 = e^{0\cdot x}$, and Lemma 3.1.4 tells us that

$$\mathscr{L}[e^{0\cdot x}](s) = \frac{1}{s}$$

for $s > 0$. On the other hand, we know

$$\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0) = s\mathscr{L}[f](s),$$

since $f(0) = 0$. Thus

$$s\mathscr{L}[f](s) = \mathscr{L}[f'](s) = \mathscr{L}[e^{0\cdot x}](s) = \frac{1}{s},$$

and so

$$\mathscr{L}[x](s) = \frac{1}{s^2}.$$ ▲

There is one other property of the transform that we need, and it is, hopefully, glaringly obvious. The Laplace transform is **LINEAR**.

3.1.11 Lemma.

Suppose that f and g are locally integrable on $[0, \infty)$ and that the Laplace transforms $\mathscr{L}[f](s)$ and $\mathscr{L}[g](s)$ converge for some s.

(i) *The transform $\mathscr{L}[f+g](s)$ converges and $\mathscr{L}[f+g](s) = \mathscr{L}[f](s) + \mathscr{L}[g](s)$.*

(ii) *Let c be any real number. Then the transform $\mathscr{L}[cf](s)$ converges and $\mathscr{L}[cf](s) = c\mathscr{L}[f](s)$.*

The Laplace transform is not **MULTIPLICATIVE**: in general $\mathscr{L}[fg](s) \neq \mathscr{L}[f](s) \cdot \mathscr{L}[g](s)$. That is, "the transform of a product is not the product of the transforms." This should be unsurprising; most things in calculus are linear, but relatively few play nicely with multiplication of nonconstant functions. (The derivative of a product is not the product of derivatives.)

Now that we know a few Laplace transforms and the all-important property $\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0)$, we are ready to solve some (toy) differential equations with the transform. We will introduce more refined properties of the transform and a few more formulas as we need them.

3.2. Solving differential equations with the Laplace transform.

3.2.1 Example.

Pretend that we do not know how to solve

$$\begin{cases} f' + f = e^x \\ f(0) = 0. \end{cases}$$

with the integrating factor method. Instead, make the two large assumptions that (1) this problem does have a solution f and (2) f is of some exponential order, so that f and f' have Laplace transforms for s large enough. What can we learn about the Laplace transform of f, and what does this teach us about f?

Solution. If $f' + f = e^x$, $f(0) = 0$, and both f and f' have Laplace transforms, then we have the following chain of implications:

$$\begin{aligned} f' + f = e^x &\Longrightarrow \mathscr{L}[f' + f](s) = \mathscr{L}[e^x](s) \\ &\Longrightarrow \mathscr{L}[f'](s) + \mathscr{L}[f](s) = \frac{1}{s-1}, \text{ assuming } s > 1 \\ &\Longrightarrow s\mathscr{L}[f](s) - f(0) + \mathscr{L}[f](s) = \frac{1}{s-1} \\ &\Longrightarrow s\mathscr{L}[f](s) + \mathscr{L}[f](s) = \frac{1}{s-1} \text{ since } f(0) = 0 \\ &\Longrightarrow (s+1)\mathscr{L}[f](s) = \frac{1}{s-1} \end{aligned}$$

$$\Longrightarrow \mathscr{L}[f](s) = \frac{1}{(s-1)(s+1)}.$$

So, whatever f is, we know what its Laplace transform is. Now, we only know two transforms so far (those of $g(x) = e^{ax}$ and $h(x) = x$), and the result above does not look like either of those transforms. The right, although perhaps not obvious, idea is to appeal to the dreaded method of partial fractions and rewrite

$$\frac{1}{(s-1)(s+1)} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)}.$$

We recognize that

$$\frac{1}{2(s-1)} = \frac{1}{2}\mathscr{L}[e^x](s) \quad \text{and} \quad \frac{1}{2(s+1)} = \frac{1}{2(s-(-1))} = \mathscr{L}[e^{-x}](s).$$

Thus, by linearity, f satisfies

$$\mathscr{L}[f](s) = \frac{\mathscr{L}[e^x](s)}{2} - \frac{\mathscr{L}[e^{-x}](s)}{2} = \mathscr{L}\left[\frac{e^x - e^{-x}}{2}\right](s).$$

This gives us the strong suspicion that the solution to our problem is

$$f(x) = \frac{e^x - e^{-x}}{2},$$

and a moment of calculus and algebra shows that this is indeed the case. ▲

This is not an exciting result. Of the many things our world needs right now, a new method for solving *constant-coefficient linear first-order equations* (not even variable-coefficient equations!) probably isn't one of them. But it does give us a good idea: if we have no clue about how to solve a differential equation, take the Laplace transform of everything and see what we can learn about the Laplace transform.

This is where we finished on Friday, April 15, 2022.

Here is a slightly more challenging problem.

3.2.2 Example.

Clarice used to raise lambs, but they got to be too noisy, and now she raises rabbits. Clarice started with 10 rabbits, and, as everyone knows, rabbit populations grow exponentially (at least when they are not beset by disease, predators, or internecine strife with nearby totalitarian warrens). Specifically, if the rabbit population at time $t \geq 0$ months is $p(t)$, then p satisfies

$$\begin{cases} p'(t) = 2p(t) \\ p(0) = 10. \end{cases} \tag{3.2.1}$$

Clarice solved this equation pretty quickly using separation of variables (and she double-

checked her work with an integrating factor) and found that after t months, she would have $p(t) = 10e^{2t}$ rabbits. This is a lot of rabbits, and so after six months Clarice plans to sell 50 rabbits per month. Then the rabbit population can be modeled by

$$p'(t) = \begin{cases} 2p(t), & 0 \le t < 6 \\ 2p(t) - 50, & t \ge 6, \end{cases} \qquad p(0) = 10. \tag{3.2.2}$$

Is this a good idea? Will Clarice ever run out of rabbits?

Solution. An effective, but dull, way to approach this problem is to note that Clarice has $10e^{12}$ rabbits at time $t = 6$, using the formula $p(t) = 10e^{2t}$ for the unharvested rabbit population. Then for times $t \ge 6$, the rabbit population (subject to harvesting) must solve the initial value problem

$$\begin{cases} p'(t) = 2p(t) - 50 \\ p(t) = 10e^{12}. \end{cases}$$

We could solve this with separation of variables or an integrating factor.

Suppose, however, that Clarice decided to change her harvesting pattern every six months or so. Then the differential equation (3.2.2) would have more than just two "piecewise pieces," and it could become even more tedious to have to solve that equation "piece by piece." The method that we will now develop generalizes nicely to more varied harvesting patterns.

We will need several perhaps non-intuitive ideas. First, we rewrite (3.2.2) to remove the piecewise notation. We introduce a function h to govern the harvesting:

$$h(t) := \begin{cases} 0, & 0 \le t < 6 \\ -50, & t \ge 6. \end{cases}$$

Then (3.2.2) collapses to

$$\begin{cases} p'(t) = 2p(t) + h(t) \\ p(0) = 10. \end{cases} \tag{3.2.3}$$

This looks much more familiar, but there will be problems if we try to solve it with an integrating factor: h is not continuous, so we cannot just integrate h to get a differentiable antiderivative.

Instead, we take the Laplace transform of (3.2.3) and hope we learn something about p. (Once again, we are working backwards in assuming that (3.2.3) has a solution that has a Laplace transform.) We find

$$\begin{aligned} p'(t) = 2p(t) + h(t) &\Longrightarrow \mathscr{L}[p'](s) = \mathscr{L}[2p + h] \\ &\Longrightarrow s\mathscr{L}[p](s) - p(0) = 2\mathscr{L}[p](s) + \mathscr{L}[h](s) \\ &\Longrightarrow (s-2)\mathscr{L}[p](s) = 10 + \mathscr{L}[h](s) \text{ since } p(0) = 10 \\ &\Longrightarrow \mathscr{L}[p](s) = \frac{10}{s-2} + \frac{\mathscr{L}[h](s)}{s-2}. \end{aligned}$$

This tells us what the Laplace transform of p must do: we have

$$\mathscr{L}[p](s) = \frac{10}{s-2} + \frac{\mathscr{L}[h](s)}{s-2}. \tag{3.2.4}$$

The denominator $s-2$ might look familiar, since

$$\mathscr{L}[e^{2t}](s) = \frac{1}{s-2}.$$

However, the numerator $\mathscr{L}[h](s)$ in the second term in (3.2.4) needs further analysis. ▲

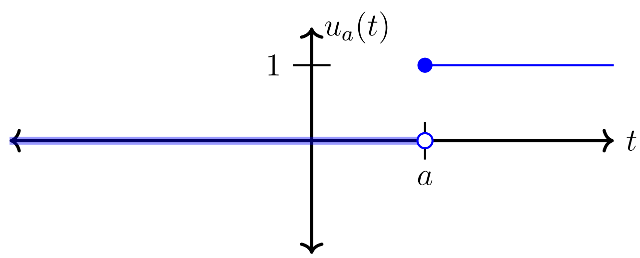
We pause our work with Clarice to study the Laplace transform $\mathscr{L}[h]$ more carefully. It will pay off to introduce a new, but related, function.

3.2.3 Definition.

A STEP FUNCTION *or* HEAVISIDE FUNCTION *is a function of the form*

$$u_a(t) := \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

for a given real number a.



Sometimes the function

$$u_0(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

is called *the* Heaviside function (not "a") or the UNIT STEP FUNCTION. One can check that

$$u_a(t) = u_0(t-a)$$

for all t and a.

Returning to Clarice's problem, the harvesting function

$$h(t) = \begin{cases} 0, & t < 6 \\ -50, & t \geq 6 \end{cases} \tag{3.2.5}$$

satisfies

$$h(t) = -50u_6(t).$$

Thus we really need to find $\mathscr{L}[u_6]$, and we may as well do this for a arbitrary.

3.2.4 Example.

Let a be a real number. Calculate $\mathscr{L}[u_a](s)$ for all s at which this transform is defined.

Solution. Since $|u_a(t)| \le 1$ for all t, and since u_a is piecewise continuous, we see that u_a is locally integrable on $[0, \infty)$ and bounded, and so it should have a Laplace transform defined for $s > 0$. This follows from part (i) of Lemma 3.1.8 with $q > 0$.

It is much more effective, however, just to calculate the Laplace transform from scratch:

$$\mathscr{L}[u_a](s) = \int_0^\infty u_a(t)e^{-st}\,dt = \lim_{b\to\infty}\int_0^b u_a(t)e^{-st}\,dt.$$

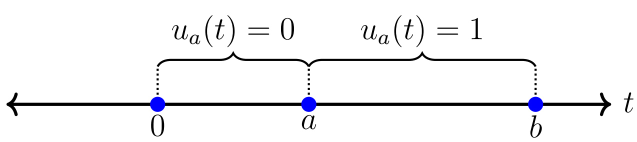
We are integrating a piecewise function here, so we should rig the limits of integration to respect the pieces.

If $b < a$, then $u_a(t) = 0$ for all $0 \le t \le b$. Thus if $b < a$, we have

$$\int_0^b u_a(t)e^{-st}\,dt = \int_0^b 0 \cdot e^{-st}\,dt = 0.$$

However, we are really interested in b large, so assume $b \ge a$. Then $u_a(t) = 0$ for $0 \le t < a$ and $u_a(t) = 1$ for $t \ge a$, and so properties of the integral give

$$\int_0^b u_a(t)e^{-st}\,dt = \int_0^a u_a(t)e^{-st}\,dt + \int_a^b u_a(t)e^{-st}\,dt = \int_0^a 0 \cdot e^{-st}\,dt + \int_a^b 1 \cdot e^{-st}\,dt = \int_a^b e^{-st}\,dt.$$



(assuming $a > 0$)

If $s \ne 0$, we can antidifferentiate as usual:

$$\int_a^b e^{-st}\,dt = \frac{e^{-st}}{-s}\bigg|_{t=a}^{t=b} = \frac{e^{-sb} - e^{-sa}}{-s} = \frac{e^{-sa} - e^{-sb}}{s}.$$

Since

$$\lim_{b\to\infty} e^{-sb} = \begin{cases} 0, & -s < 0 \\ \infty, & s > 0 \end{cases} = \begin{cases} 0, & s > 0 \\ \infty, & s < 0, \end{cases}$$

we conclude that the integral converges if $s > 0$ and diverges if $s < 0$. We leave the divergence at $s = 0$ as an exercise. Thus

$$\mathscr{L}[u_a](s) = \lim_{b\to\infty}\frac{e^{-sa} - e^{-sb}}{s} = \frac{e^{-sa}}{s}, \quad s > 0.$$

▲

Since Clarice's harvesting function h is $h(t) = -50u_6(t)$, we have

$$\mathscr{L}[h](s) = -\frac{50e^{-6s}}{s}, \quad s > 0.$$

Thus her rabbit population p, per (3.2.4), satisfies

$$\mathscr{L}[p](s) = \frac{10}{s-2} - \frac{50e^{-6s}}{s(s-2)}. \tag{3.2.6}$$

Hopefully the factors of s and $s-2$ in the denominator above make us think of transforms of exponentials:

$$\mathscr{L}[e^{0 \cdot t}](s) = \frac{1}{s} \quad \text{and} \quad \mathscr{L}[e^{2t}](s) = \frac{1}{s-2}.$$

This, in turn, should motivate us to perform a partial fractions decomposition:

$$\frac{1}{s(s-2)} = -\frac{1}{2s} + \frac{1}{2(s-2)}.$$

This converts (3.2.6) to

$$\mathscr{L}[p](s) = \frac{10}{s-2} + \frac{25e^{-6s}}{s} - \frac{25e^{-6s}}{s-2} = 10\mathscr{L}[e^{2t}](s) + 25e^{-6s}\mathscr{L}[e^{0 \cdot t}](s) - 25e^{-6s}\mathscr{L}[e^{2t}](s). \tag{3.2.7}$$

It would be nice if we could replace the last two terms by "pure" Laplace transforms. That is, are there functions g_1 and g_2 such that

$$e^{-6s}\mathscr{L}[e^{0 \cdot t}](s) = \mathscr{L}[g_1(t)](s) \quad \text{and} \quad e^{-6s}\mathscr{L}[e^{2 \cdot t}](s) = \mathscr{L}[g_2(t)](s)?$$

If so, then (3.2.7) becomes

$$\mathscr{L}[p](s) = 10\mathscr{L}[e^{2t}](s) + 25\mathscr{L}[g_1(t)](s) - 25\mathscr{L}[g_2(t)](s) = \mathscr{L}[10e^{2t} + 25g_1(t) - 25g_2(t)](s).$$

A good candidate for Clarice's rabbit population would then be

$$p(t) = 10e^{2t} + 25g_1(t) - 25g_2(t).$$

Here, then, is the right (and, for now, final) question to ask.

3.2.5 Example.

Let f be a function whose Laplace transform is defined at the number s, and let a also be a real number. Is there a function g such that

$$e^{-as}\mathscr{L}[f](s) = \mathscr{L}[g](s)?$$

This is where we finished on Monday, April 18, 2022.

Solution. We may as well try to manipulate the expression $e^{-as}\mathscr{L}[f](s)$. Since f has a Laplace transform at s, the improper integral below is defined:

$$e^{-as}\mathscr{L}[f](s) = e^{-as}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(a+t)s}\,dt.$$

We substitute $v = a + t$ to find $dv = dt$ and $t = v - a$ (we are using v, not u, since u is playing a role above with u_a) to find

$$\int_0^\infty f(t)e^{-(a+t)s}\ dt = \int_a^\infty f(v-a)e^{-vs}\ dv.$$

This sort of substitution in improper integrals can be justified by using the limit definition of the improper integral; we leave the details as an exercise.

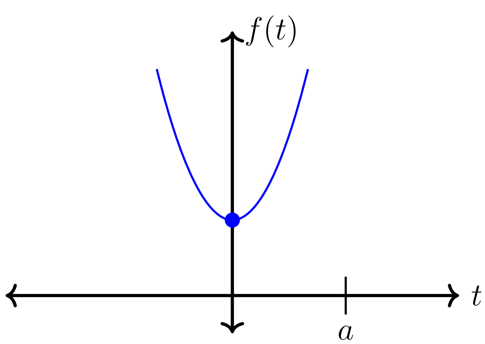
We would like to relate the integral $\int_a^\infty f(v-a)e^{-vs}\ dv$ to a Laplace transform, which is an integral over $[0,\infty)$. But here we are just integrating over $[a,\infty)$. The right, and probably not obvious, idea is to relate everything to a Heaviside function:

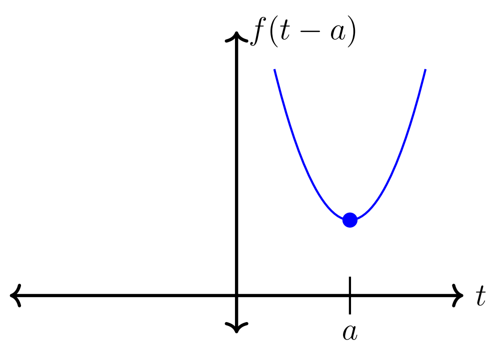
$$\begin{aligned}\int_a^\infty f(v-a)e^{-vs}\ dv &= 0 + \int_a^\infty 1\cdot f(v-a)e^{-vs}\ dv \\ &= \int_0^a 0\cdot f(v-a)e^{-vs}\ dv + \int_a^\infty 1\cdot f(v-a)e^{-vs}\ dv \\ &= \int_0^a u_a(v)f(v-a)e^{-vs}\ dv + \int_a^\infty u_a(v)f(v-a)e^{-vs}\ dv \\ &= \int_0^\infty u_a(v)f(v-a)e^{-vs}\ dv \\ &= \mathscr{L}[u_a(v)f(v-a)](s).\end{aligned}$$

We conclude (going back to our original variable t, not v)

$$e^{-as}\mathscr{L}[f](s) = \mathscr{L}[u_a(t)f(t-a)](s). \qquad \blacktriangle$$

The graphical interpretation of the function $t \mapsto u_a(t)f(t-a)$ may not be immediately obvious from the formula, so it is worth pausing for a moment to draw. For simplicity, take $a > 0$. Then the graph of $t \mapsto f(t-a)$ is the graph of f shifted a units to the right on the t-axis.

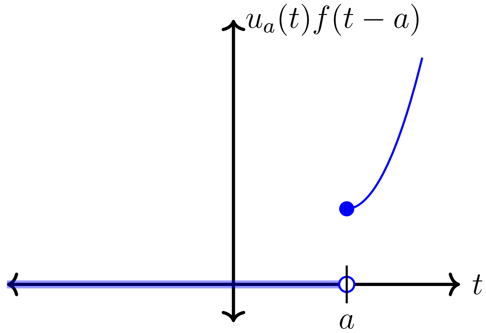




Since

$$u_a(t)f(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a, \end{cases}$$

the graph of $t \mapsto u_a(t)f(t-a)$ is just 0 for $t < a$ and then, starting at $t = a$, the graph of f shifted to the right by t units.



Recall, however dimly, from (3.2.7) that Clarice's rabbit population p satisfies

$$\mathscr{L}[p](t) = 10\mathscr{L}[e^{2t}](s) + 25e^{-6s}\mathscr{L}[e^{0\cdot t}](s) - 25e^{-6s}\mathscr{L}[e^{2t}](s).$$

Now we see that

$$e^{-6s}\mathscr{L}[e^{0\cdot t}](s) = \mathscr{L}[u_6(t)e^{0\cdot(t-6)}](s) = \mathscr{L}[u_6(t)](s)$$

and

$$e^{-6s}\mathscr{L}[e^{2t}](s) = \mathscr{L}[u_6(t)e^{2(t-6)}](s),$$

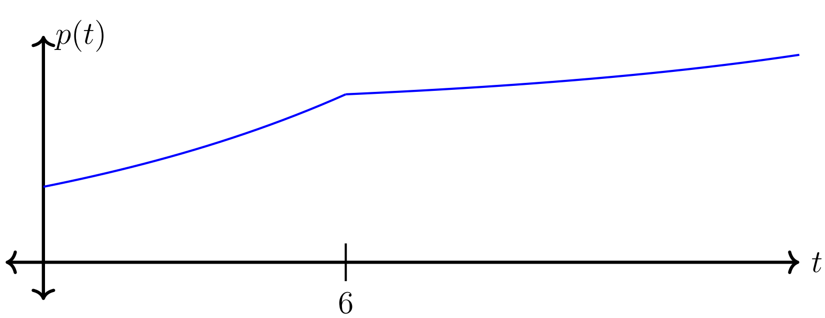
and so

$$\begin{aligned}\mathscr{L}[p](t) = 10\mathscr{L}[e^{2t}](s) + 25\mathscr{L}[u_6(t)](s) - 25\mathscr{L}[u_6(t)e^{2(t-6)}](s) \\ = \mathscr{L}[10e^{2t} + 25u_6(t) - 25u_6(t)e^{2(t-6)}](s).\end{aligned}$$

This suggests that Clarice's rabbit population, with harvesting, is

$$p(t) = 10e^{2t} + 25u_6(t)(1 - e^{2(t-6)}) = \begin{cases} 10e^{2t}, & t < 6 \\ 10e^{2t} + 25(1 - e^{2(t-6)}), & t \geq 6. \end{cases}$$

We can (and should) check directly that $p'(t) = 2p(t) + h(t)$ for $p \neq 6$, and so p solves (3.2.3). Now, this is a bulky formula, so here is a graph (not exactly to scale).



It looks like p is continuous for all t but there is a corner at $t = 6$, so p is probably not differentiable there. This is in line with our toy problem from Example 2.7.2.

Now we give a more rigorous analysis of continuity and differentiability. By inspection of its formula, p is continuous on $(-\infty, 6)$ and $(6, \infty)$, and, since $p(6) = 10e^{12}$ and

$$\lim_{t\to 6^-} p(t) = \lim_{t\to 6^-} 10e^{2t} = 10e^{12} \quad \text{and} \quad \lim_{t\to 6^+} p(t) = \lim_{t\to 6^+} 10e^{2t} + 25(1 - e^{2(t-6)}) = 10e^{12},$$

p is also continuous at $t = 6$. However, we do not expect p to be differentiable at $t = 6$, since

$$\lim_{t\to 6^-} p'(t) \neq \lim_{t\to 6^+} p'(t);$$

we leave the calculation of these one-sided limits as an exercise.

Finally, we still need to figure out what happens to Clarice's rabbit population after harvesting. If she takes 50 hapless rabbits a month, will she ever run out? This seems unlikely from the graph. More precisely, we have

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} 10e^{2t} + 25(1 - e^{2(t-6)}) = \lim_{t\to\infty} e^{2t}[10 + 25e^{-2t} - 25e^{-12}].$$

Since

$$\lim_{t\to\infty} e^{2t} = \infty \quad \text{and} \quad \lim_{t\to\infty} 10 + 25e^{-2t} - 25e^{-12} = 10 - 25e^{-12} > 0,$$

she will always have plenty of rabbits.

This was a long (terrible) story, so we pause to summarize our important auxiliary developments with the Heaviside (step) function.

3.2.6 Lemma.

For a real number a, put

$$u_a(t) := \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$

(i) $\mathscr{L}[u_a(t)](s) = \dfrac{e^{-as}}{s}$, $s > 0$

(ii) *If the function f has a Laplace transform at the number s, then*

$$\mathscr{L}[u_a(t)f(t-a)](s) = e^{-as}\mathscr{L}[f(t)](s).$$

3.2.7 Example.

Use the Laplace transform to find a reasonable candidate for the solution to

$$\begin{cases} f'(x) = f(x) + e^{-x} + u_2(x) \\ f(0) = 1. \end{cases} \tag{3.2.8}$$

This is where we finished on Wednesday, April 20, 2022.

Solution. We assume that the problem has a solution that has a Laplace transform and calculate

$$\begin{aligned}
f'(x) = f(x) + e^{-x} + u_2(x) &\Longrightarrow \mathscr{L}[f'(x)](s) = \mathscr{L}[f(x)](s) + \mathscr{L}[e^{-x}](s) + \mathscr{L}[u_2(x)](s) \\
&\Longrightarrow s\mathscr{L}[f](s) - f(0) = \mathscr{L}[f](s) + \frac{1}{s-(-1)} + \frac{e^{-2s}}{s} \\
&\Longrightarrow s\mathscr{L}[f](s) - \mathscr{L}[f](s) = 1 + \frac{1}{s+1} + \frac{e^{-2s}}{s} \text{ since } f(0) = 1 \\
&\Longrightarrow (s-1)\mathscr{L}[f](s) = 1 + \frac{1}{s+1} + \frac{e^{-2s}}{s} \\
&\Longrightarrow \mathscr{L}[f](s) = \frac{1}{s-1} + \frac{1}{(s-1)(s+1)} + \frac{e^{-2s}}{s(s-1)}.
\end{aligned}$$

We rewrite

$$\frac{1}{(s-1)(s+1)} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)} \quad \text{and} \quad \frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}$$

so that

$$\begin{aligned}
\mathscr{L}[f](s) &= \frac{1}{s-1} + \frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s-1} \\
&= \frac{3}{2(s-1)} - \frac{1}{2(s+1)} - \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s-1} \\
&= \frac{3}{2}\mathscr{L}[e^{x}](s) - \frac{1}{2}\mathscr{L}[e^{-x}](s) - \frac{1}{2}\big(e^{-2s}\mathscr{L}[e^{0\cdot x}](s)\big) + \big(e^{-2s}\mathscr{L}[e^{x}](s)\big) \\
&= \frac{3}{2}\mathscr{L}[e^{x}](s) - \frac{1}{2}\mathscr{L}[e^{-x}](s) - \mathscr{L}[u_2(x)e^{0\cdot(x-2)}](s) + \mathscr{L}[u_2(x)e^{x-2}](s) \\
&= \mathscr{L}\left[\frac{3e^{x}}{2} - \frac{e^{-x}}{2} - u_2(x) + u_2(x)e^{x-2}\right](s).
\end{aligned}$$

This suggests that the solution is

$$f(x) = \frac{3e^{x} - e^{-x}}{2} + u_2(x)(e^{x-2} - 1) \tag{3.2.9}$$

and we can check that this works pointwise in x for $x \neq 2$. ▲

We could have solved the preceding problem (and Clarice's) more or less how we treated Example 2.7.2. Here we would need to use the integrating factor method "piecewise" over the two intervals $[0, 2)$ and $[2, \infty)$; this is a good review exercise. However, what if the problem had many more pieces? Returning to Clarice's problem, suppose she decided to change her harvesting pattern. Before month 6, she harvests no rabbits. Starting in month 6, she harvests 50 rabbits per month; starting in month 12, she harvests 100 rabbits per month. Of course, she could try to harvest even more rabbits on an increasing biannual cycle, but this situation is just complicated enough. If the rabbit population p continues to grow exponentially as in (3.2.1), we have

$$p'(t) = \begin{cases} 2p(t),\ 0 \le t < 6 \\ 2p(t) - 50,\ 6 \le t < 12 \\ 2p(t) - 100, 12 \le t. \end{cases}$$

Put

$$\widetilde{h}(t) := \begin{cases} 0,\ 0 \le t < 6 \\ -50,\ 6 \le t < 12 \\ -100,\ 12 \le t \end{cases}$$

so that the initial value problem now reads

$$\begin{cases} p'(t) = 2p(t) + \widetilde{h}(t) \\ p(0) = 10. \end{cases}$$

Calculations identical to those that gave (3.2.4) now provide

$$\mathscr{L}[p](s) = \frac{10}{s-2} + \frac{\mathscr{L}[\widetilde{h}](s)}{s-2}. \tag{3.2.10}$$

Naturally, we want a formula for the Laplace transform of $\widetilde{h}$. A good idea is to rewrite

$$\widetilde{h}(t) = \begin{cases} 0,\ 0 \le t < 6 \\ -50,\ 6 \le t < 12 \\ -100,\ 12 \le t \end{cases} = \begin{cases} 0,\ 0 \le t < 6 \\ -50,\ 6 \le t < 12 \\ 0,\ 12 \le t \end{cases} + \begin{cases} 0,\ 0 \le t < 6 \\ 0,\ 6 \le t < 12 \\ -100,\ 12 \le t \end{cases}$$

$$= \begin{cases} 0,\ 0 \le t < 6 \\ -50,\ 6 \le t < 12 \\ 0,\ 12 \le t \end{cases} - 100u_{12}(t).$$

Abbreviate

$$w(t) := \begin{cases} 0, t < 6 \\ 1,\ 6 \le t < 12 \\ 0,\ 12 \le t, \end{cases}$$

so that

$$\begin{cases} 0,\ 0 \le t < 6 \\ -50,\ 6 \le t < 12 \\ 0,\ 12 \le t \end{cases} = -50w(t).$$

We therefore have

$$\widetilde{h}(t) = -50w(t) - 100u_{12}(t), \tag{3.2.11}$$

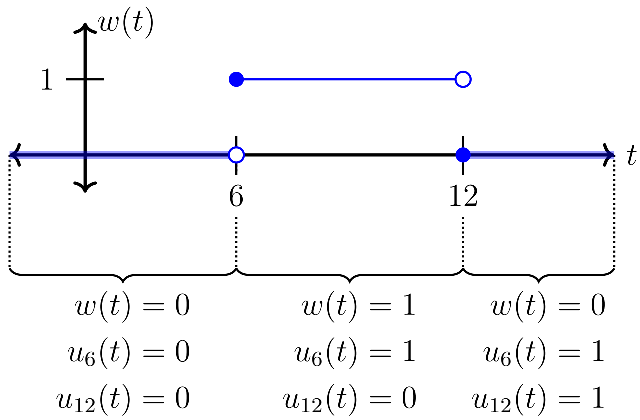
and so

$$\mathscr{L}[h](s) = -50\mathscr{L}[w](s) - 100\mathscr{L}[u_{12}](s) = -50\mathscr{L}[w](s) - \frac{100e^{-12s}}{s}.$$

It then suffices to find a formula for the Laplace transform of w. (Incidentally, w and h are bounded, locally integrable functions, and so their Laplace transforms definitely exist.)

This is where we finished on Friday, April 22, 2022.

Here is the graph of w, along with a few observations.



The picture suggests, and a careful calculation proves, that

$$w(t) = u_6(t) - u_{12}(t).$$

Thus

$$\mathscr{L}[w](s) = \mathscr{L}[u_6 - u_{12}](s) = \mathscr{L}[u_6](s) - \mathscr{L}[u_{12}](s) = \frac{e^{-6s}}{s} - \frac{e^{-12s}}{s} = \frac{e^{-6s} - e^{-12s}}{s}.$$

Then, using the definition of $\widetilde{h}$ in (3.2.11), we have

$$\mathscr{L}[\widetilde{h}](s) = -50\left(\frac{e^{-6s} - e^{-12s}}{s}\right) - \frac{100e^{-12s}}{s} = -\frac{50e^{-6s}}{s} - \frac{50e^{-12s}}{s}.$$

Returning to Clarice's revised problem and the Laplace transform in (3.2.10), we obtain

$$\mathscr{L}[p](s) = \frac{10}{s-2} - \frac{50e^{-6s}}{s(s-2)} - \frac{50e^{-12s}}{s(s-2)}.$$

There is not all that much new to be learned here; after making some partial fractions decompositions analogous to those above, we will recognize the Laplace transforms of products involving exponentials and the *two* Heaviside functions u_6 and u_{12}. We leave the details as an exercise; whatever the final formula, we should not be surprised to see the behavior of the rabbit population "switching" at times $t = 6$ and $t = 12$.

We summarize our recent work more abstractly.

3.2.8 Lemma.

Let a and b be real numbers with $a < b$. Then

$$u_a(t) - u_b(t) = \begin{cases} 0, & t < a \\ 1, & a \le t < b \\ 0, & b \le t, \end{cases} \tag{3.2.12}$$

and

$$\mathscr{L}[u_a - u_b](s) = \frac{e^{-as} - e^{-bs}}{s}, \ s > 0. \tag{3.2.13}$$

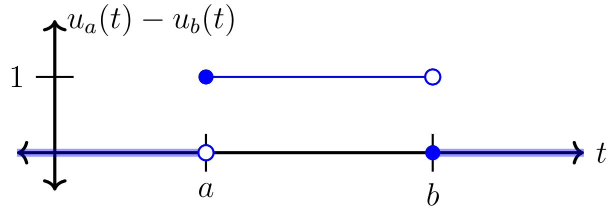
Moreover, if f is any function defined on an interval containing a and b, then

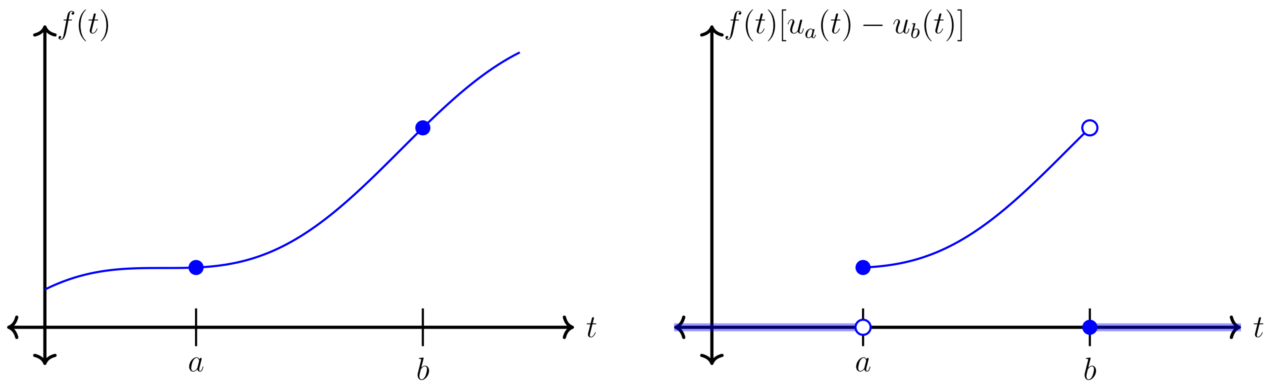
$$f(t) = f(t)[u_a(t) - u_b(t)] \text{ for } a \le t < b \tag{3.2.14}$$

and

$$f(t)[u_a(t) - u_b(t)] = 0 \text{ for } r < a \text{ and } t \ge b. \tag{3.2.15}$$

The identities (3.2.14) and (3.2.15) might think of $u_a - u_b$ as an "indicator" or "windowing" function for the interval $[a, b)$. Multiplying by $u_a - u_b$ "turns f on" in the interval $[a, b)$ and "off" elsewhere.





By the way, there is nothing really special about having the interval "open" at the right endpoint b. This hearkens back to our definition of the Heaviside function u_a as only "turning on" when $t \ge a$. We could have worked with a variant of the Heaviside function, like

$$\widetilde{u}_a(t) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$

and found more or less the same results.

Here is the deeper value of this work. First, any piecewise function can be expressed as a *sum of differences of multiples Heaviside functions*. Consider, for example,

$$f(t) = \begin{cases} 2, & 0 \le t < 4 \\ 4, & 4 \le t < 6 \\ 6, & 6 \le t < 10 \\ 8, & 8 \le t. \end{cases}$$

We just need to capture the separate behaviors of f on the intervals $[0,4)$, $[4,6)$, $[6,10)$, and $[10,\infty)$. We can do this with the following two observations from the formula (3.2.12):

Thus

$$\begin{aligned} f(t) = f(t)[u_2(t) - u_4(t)] + f(t)[u_4(t) - u_6(t)] + f(t)[u_6(t) - u_{10}(t)] + f(t)u_{10}(t) \\ = 2[u_2(t) - u_4(t)] + 4[u_4(t) - u_2(t)] + 6[u_6(t) - u_8(t)] + 8u_8(t). \end{aligned}$$

Imagine now that for some silly reason we had to compute the Laplace transform $\mathscr{L}[f]$. We could just read off the transform from the expansion of f above and the formula (3.2.13)!

3.3. The inverse Laplace transform.

Here has been our strategy for solving differential equations with the Laplace transform.

1. Assume that a solution exists and that it is "nice enough" to have a Laplace transform.

2. Use the differential equation (and initial values) and properties of the Laplace transform to determine a formula for the Laplace transform of the solution.

3. Use partial fractions and Laplace transform properties to express this formula as the sum of Laplace transforms of known functions.

4. Guess that the solution really is the sum of these known functions. Check by differentiating.

For instance, in Example 3.2.7 we saw that the solution f to the problem (3.2.8) had to satisfy

$$\mathscr{L}[f](s) = \mathscr{L}\left[\frac{3e^x}{2} - \frac{e^{-x}}{2} + \frac{u_2(x)}{2} - \frac{u_2(x)e^{x-2}}{2}\right](s), \tag{3.3.1}$$

and so the right idea seemed to be that

$$f(x) = \frac{3e^x}{2} - \frac{e^{-x}}{2} + \frac{u_2(x)}{2} - \frac{u_2(x)e^{x-2}}{2}. \tag{3.3.2}$$

We should not put too much faith in this procedure as a mathematically rigorous method. Remember, we are making the perilous double assumption that (1) a solution exists and (2) it has a Laplace transform, and then we are trying to determine what that solution must be.

As always in differential equations, once we have a candidate for a solution, we can check it promptly.

Here is the underlying question that the passage from (3.3.1) to (3.3.2) should raise: if f_1 and f_2 are functions with

$$\mathscr{L}[f_1](s) = \mathscr{L}[f_2](s)$$

for all s (at least, for all s at which these transforms are defined), do we have $f_1(x) = f_2(x)$ for all x at which f_1 and f_2 are defined? The unsatisfying answer is *no*, and it hinges on a lovely little property of integrals.

This is where we finished on Monday, April 25, 2022.

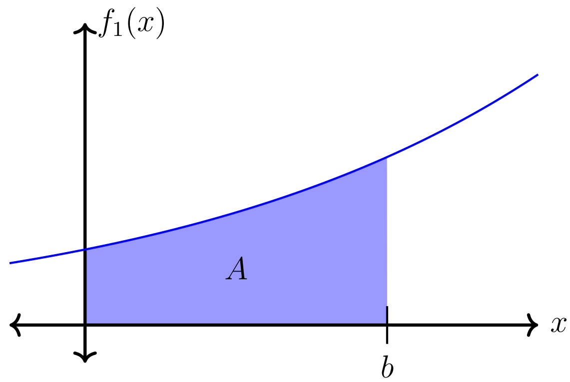
Consider the following situation. Take

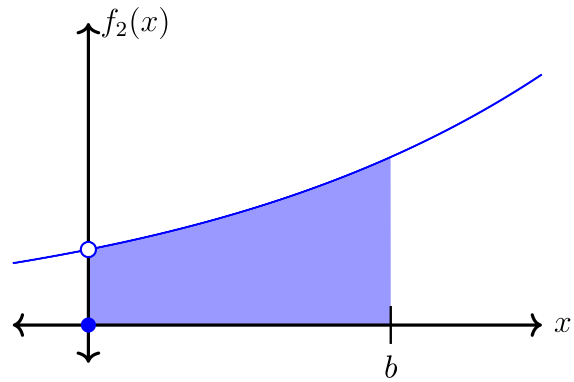
$$f_1(x) = e^x \quad \text{and} \quad f_2(x) = \begin{cases} 0, & x = 0 \\ e^x, & x \neq 0. \end{cases}$$

Note that $f_1 \neq f_2$ since $f_1(0) \neq f_2(0)$. However, it is the case that

$$\int_0^b f_1(x)\,dx = \int_0^b e^x\,dx \quad \text{and} \quad \int_0^b f_2(x)\,dx = \int_0^b e^x\,dx \tag{3.3.3}$$

for all $b \geq 0$. In words, *changing the value of the integrand at one point does not change the value of the integral*, or more geometrically, *the area under a point is zero*. This is suggested by the following pictures (which correspond to the integrand of the Laplace transform for $s = 0$); the area of the shaded region in each picture is the same number A.





From (3.3.3) and the limit definition of the Laplace transform, then, we have

$$\mathscr{L}[f_1](s) = \frac{1}{s-1} = \mathscr{L}[f_2](s) \tag{3.3.4}$$

for all s, even though $f_1 \neq f_2$. If, however, we add the hypothesis of continuity to our functions, then we do get a uniqueness property of the Laplace transform.

3.3.1 Theorem.

Suppose that f_1 and f_2 are continuous functions on $[0, \infty)$ and there is a real number q such that $\mathscr{L}[f_1](s) = \mathscr{L}[f_2](s)$ for $s > q$. Then $f_1(x) = f_2(x)$ for all x.

Embiggened with this uniqueness result, we make a formal definition.

3.3.2 Definition.

Let g be a function defined on an interval (q, ∞) and suppose there is a continuous function f defined on $[0, \infty)$ such that $\mathscr{L}[f](s) = g(s)$ for all $s > q$. (There is at most one such function g by Theorem 3.3.1.) Then we call f the INVERSE LAPLACE TRANSFORM *of g and we write*

$$f(x) = \mathscr{L}^{-1}[g](x).$$

3.3.3 Example.

Put

$$f(x) := e^{2x} \quad \text{and} \quad g(s) := \frac{1}{s-2}, \ s > 2.$$

Then $\mathscr{L}[f](s) = g(s)$ for all $x > 2$, and, moreover, f is continuous on $[0, \infty)$. Thus $f(x) = \mathscr{L}^{-1}[g](s)$; equivalently,

$$\mathscr{L}^{-1}\left[\frac{1}{s-2}\right](x) = e^{2x}.$$

Although

$$h(x) := \begin{cases} e^x, x \neq 5 \\ -1, \ x = 5 \end{cases}$$

also satisfies $\mathscr{L}[h](s) = g(s)$, we do not write $h = \mathscr{L}^{-1}[g]$ since h is not continuous on $[0, \infty)$.

We can therefore amend our strategy for solving differential equations with the Laplace transform to read: *Assume there is a solution f and find a formula for its Laplace transform as $\mathscr{L}[f](s) = g(s)$ for some function g. Then a solution candidate is $f(x) = \mathscr{L}^{-1}[g](x)$.*

Unfortunately, given a function g, there is no "easy" or "transparent" formula for its inverse Laplace transform, if the inverse transform exist. A semester or so of complex analysis will produce a nice theoretical result, but for us it has little computational relevance. Instead, we often calculate inverse Laplace transforms using the same catch-as-catch-can strategy that we do antiderivatives: rewrite the function so that more elementary inverse transforms appear and hope for the best. Often this rewriting involves a number of partial fractions decompositions, and while it is worthwhile to know "conceptually" how to perform such decompositions, any intense calculations are better left for a computer.

3.4. Impulses and delta functions.

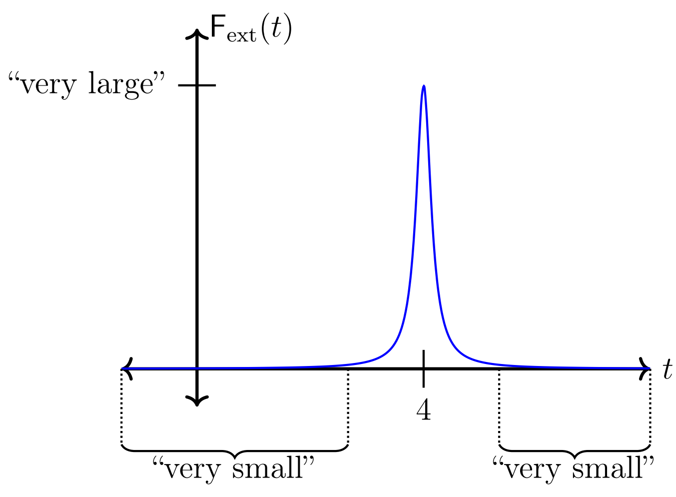
We will consider a new scenario for the harmonic oscillator that will lead not merely to a problem that we *can* solve via the Laplace transform but rather one that we indeed *must* solve with the transform. Our ordinary "time-side" mathematical language will not get us very far with this problem, but our "Laplace-side" perspectives will.

Consider, for simplicity, an undamped harmonic oscillator of mass 1 and spring constant 1. At time $t = 0$ the oscillator is at its equilibrium position, and it experiences virtually no force on being placed into motion. Then if $\mathsf{F}_{\text{ext}}(t)$ is the external force on the oscillator at time t, its displacement $u(t)$ from equilibrium satisfies

$$\begin{cases} u'' + u = \mathsf{F}_{\text{ext}}(t) \\ u(0) = 0 \\ u'(0) = 0. \end{cases}$$

Suppose that at time $t = 4$ the oscillator experiences a very brief but "massive" force. (There is nothing special about $t = 4$, save that 4 does not look like the numbers 1, 2, or 3. The number 1 is too plain; the number 2 will appear elsewhere in our model; and the number 3, when handwritten, can look like the Greek letter ϵ backwards.) For example, rather than the beefy, stubby, paw of a cat paddling the oscillator, someone strikes it with a hammer, or a homicidal coyote tosses a stick of dynamite at it; why any of this would happen is a mystery. Such a force is sometimes called an IMPULSE. What sort of function F_{ext} would model this force, beyond

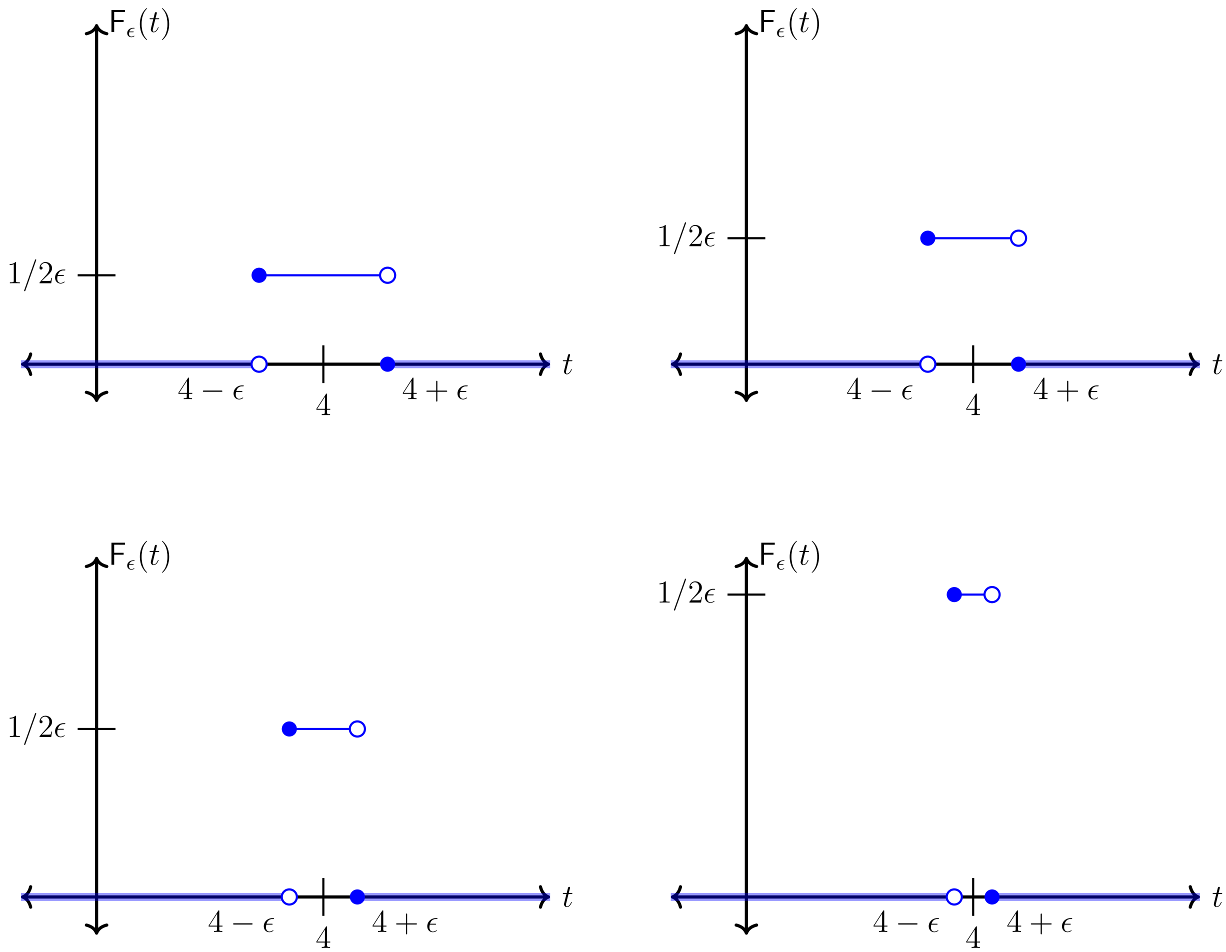
$$\mathsf{F}_{\text{ext}}(t) = \begin{cases} \text{"very large,"} \ t \approx 4 \\ 0, \ t < 4 \text{ or } t > 4? \end{cases}$$



One way of proceeding is to approximate F_{ext} with a function that is "large" over "small" times centered around $t = 1$, solve the problem for one of these approximations, and then take a limit. There are lots of good candidates for such approximating functions; here is a nice and easy one. For $\epsilon > 0$, put

$$\mathsf{F}_\epsilon(t) := \begin{cases} 0, & t < 4 - \epsilon \\ 1/\epsilon, & 4 - \epsilon \le t < 4 + \epsilon \\ 0, & t \ge 4 + \epsilon. \end{cases} \tag{3.4.1}$$

Here are some graphs of F_ϵ for successively smaller ϵ.



It seems that taking $\epsilon \to 0$ does a nice job of modeling a function that is very large when the input is close to 4 and zero everywhere else. (The factor of 2 in the denominator of F_ϵ is there for a convenience that we will exploit later.) In fact, one can show that

$$\lim_{\epsilon \to 0} \mathsf{F}_\epsilon(t) = \begin{cases} \infty, & t = 4 \\ 0, & t \neq 4 \end{cases} =: \mathsf{F}_0(t),$$

and this definition of F_0 certainly fits the bill of a "very brief but 'massive' force." Determining the limit above is a little strange from a calculus perspective, since both the outputs of F_ϵ and the "pieces" are changing with ϵ. Nonetheless, it is feasible, but that symbol ∞ should make us uncomfortable — the function F_0 is not "real-valued," like everything good in calculus, but rather *extended real-valued*. Consequently, none of our prior techniques will allow us to solve the problem

$$\begin{cases} u'' + u = \mathsf{F}_0(t) \\ u(0) = 0 \\ u'(0) = 0 \end{cases} \tag{3.4.2}$$

since we have no results for *extended real-valued* forcing functions.

This is where we finished on Wednesday, April 27, 2022.

Instead, we will try to solve the approximate problem

$$\begin{cases} u'' + u = \mathsf{F}_\epsilon(t) \\ u(0) = 0 \\ u'(0) = 0. \end{cases} \tag{3.4.3}$$

This is a problem with a piecewise forcing function, and we could work "interval by interval," which would be horrible, but feasible. Another approach is to use the Laplace transform; after all, F_ϵ is piecewise. This would require us to interpret what $\mathscr{L}[u''](s)$ is, and that turns out to be an extension of the old idea $\mathscr{L}[u'](s) = s\mathscr{L}[u](s) - u(0)$.

However we do it, suppose that we can solve (3.4.3) and get a solution u_ϵ. Then we would like to study

$$\lim_{\epsilon \to 0} u_\epsilon(t)$$

for each time value t. Does this limit exist as a finite real number? If so, what does the limit say about the solution to the hypothetical problem (3.4.2)?

To begin to answer these questions, we first study (3.4.3) on the Laplace side. Taking the Laplace transform, we find

$$\mathscr{L}[u''](s) + \mathscr{L}[u](s) = \mathscr{L}[\mathsf{F}_\epsilon](s). \tag{3.4.4}$$

We have used the linearity of the transform on the left side. Next, we want to express $\mathscr{L}[u'']$ in terms of $\mathscr{L}[u]$. This is quite easy if we use the definition of the second derivative: $u'' = (u')'$. We find, from our formula for the interaction of a Laplace transform with a first derivative,

$$\begin{aligned} \mathscr{L}[u''](s) = \mathscr{L}[(u')'](s) = s\mathscr{L}[u'](s) - u'(0) = s\big(s\mathscr{L}[u](s) - u(0)\big) - u'(0) \\ = s^2\mathscr{L}[u](s) - su(0) - u'(0). \end{aligned}$$

The initial values $u(0) = u'(0) = 0$ from (3.4.3) then convert (3.4.4) into

$$s^2\mathscr{L}[u](s) + \mathscr{L}[u](s) = \mathscr{L}[\mathsf{F}_\epsilon](s),$$

and so we find

$$\mathscr{L}[u](s) = \frac{\mathscr{L}[\mathsf{F}_\epsilon](s)}{s^2 + 1}.$$

To calculate $\mathscr{L}[\mathsf{F}_\epsilon](s)$, we use the definition of F_ϵ in (3.4.1) to rewrite F_ϵ as a window function, as in Lemma 3.2.8:

$$\mathsf{F}_\epsilon(t) = u_{4-\epsilon}(t) - u_{4+\epsilon}(t).$$

Thus

$$\mathscr{L}[\mathsf{F}_\epsilon](s) = \frac{e^{-(4-\epsilon)s} - e^{-(4+\epsilon)s}}{2\epsilon s} = \left(\frac{e^{-4s}}{s}\right)\left(\frac{e^{\epsilon s} - e^{-\epsilon s}}{2\epsilon}\right). \tag{3.4.5}$$

We have factored $\mathscr{L}[\mathsf{F}_\epsilon]$ in this manner for convenience later. Then

$$\mathscr{L}[u](s) = \frac{1}{s^2+1}\left(\frac{e^{-4s}}{s}\right)\left(\frac{e^{\epsilon s} - e^{-\epsilon s}}{2\epsilon}\right). \tag{3.4.6}$$

Now, we could try to invert this transform and get a solution $u = u_\epsilon$ to (3.4.3). But since we are going to take the limit as $\epsilon \to 0^+$, a worthwhile (if wholly non-obvious) idea is to *take the limit on the Laplace side*. Only the third factor in (3.4.6) is ϵ-dependent, and it turns out that

$$\lim_{\epsilon \to 0^+} \frac{e^{\epsilon s} - e^{-\epsilon s}}{2\epsilon} = s$$

for any real number s. Here we need to use L'Hospital's rule with respect to ϵ. Then, in the limit as $\epsilon \to 0^+$, (3.4.6) becomes

$$\mathscr{L}[u](s) = \frac{e^{-4s}}{s^2+1}. \tag{3.4.7}$$

This *suggests* (but certainly does not *prove*) a formula for the solution to (3.4.2), which is the "real" problem that we want to solve.

Furthermore, our prior experience with exponential factors on the Laplace side suggests to us that if we could find a function g such that

$$\mathscr{L}[g](s) = \frac{1}{s^2+1},$$

then (3.4.7) will become

$$\mathscr{L}[u](s) = e^{-4s}\mathscr{L}[g](s) = \mathscr{L}[u_4(t)g(t-4)](s),$$

and so the solution u to (3.4.2) could be

$$u(t) = u_4(t)g(t-4).$$

So, what is

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right](t)?$$

If we stare at this for some time, the quadratic s^2+1 in the denominator should remind us of the characteristic polynomial of $f'' + f = 0$. And this equation has sines and cosines for solutions. For a lark, we might try computing $\mathscr{L}[\sin(t)](s)$. We could do this via the integral definition of the Laplace transform, but that would involve integration by parts. Instead, we note that if $f(t) = \sin(t)$, then f solves the initial value problem

$$\begin{cases} f'' + f = 0 \\ f(0) = 0 \\ f'(0) = 1, \end{cases}$$

and so (**??**) gives

$$0 = s^2\mathscr{L}[f](s) - sf(0) - f'(0) + \mathscr{L}[f](s) = (s^2+1)\mathscr{L}[f](s) - 1.$$

Hence

$$\mathscr{L}[\sin(t)](s) = \mathscr{L}[f](s) = \frac{1}{s^2+1}.$$

Our lucky guess worked out, and by the reasoning above, the function

$$u(t) = u_4(t)\sin(t-4) = \begin{cases} 0, & t < 4 \\ \sin(t-4), & t \geq 4 \end{cases}$$

is a good candidate for the solution to (3.4.2).

3.4.1 Remark.

We have seen that with F_ϵ *defined in* (3.4.1)*, we have*

$$\lim_{\epsilon \to 0^+} \mathscr{L}[\mathsf{F}_\epsilon](s) = e^{-4s}.$$

Conversely, we viewed the extended real-valued limit

$$\lim_{\epsilon \to 0^+} \mathsf{F}_\epsilon(t)$$

as a good approximation to an impulse force occurring at time $t = 4$*. The convention is then to construct the extended real-valued function* δ_4 *defined by*

$$\delta_4(t) := \begin{cases} \infty, & t = 4 \\ 0, & t \neq 4 \end{cases}$$

and to put $\mathscr{L}[\delta_4](s) := e^{-4s}$*. This Laplace transform has no rigorous definition as an improper integral (since integrals are only defined, as far as we know, for real-valued functions), of course, but it allows us to (1) view the problem*

$$\begin{cases} u'' + u = \delta_4 \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

as a representation of the impulse force on the "time side" and (2) to view the problem

$$\mathscr{L}[u''+u](s) = e^{-4s}, \ u(0) = u'(0) = 0$$

as the incarnation of this problem on the "Laplace side." We can solve the Laplace side problem perfectly well and then invert to the time side.

There is nothing special about doing all this at time $t = 4$. If, for a real number a, we define

$$\delta_a(t) := \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$$

and agree that $\mathscr{L}[\delta_a](s) = e^{-as}$, then the problem

$$u'' + u = \delta_a(t)$$

represents an oscillator suffering an impulse force at time $t = a$. We can interpret this problem in terms of elementary functions on the Laplace side (provided we know initial conditions).

This is where we finished on Friday, April 29, 2022.

A. Calculus Essentials

1. A **function** on a real interval I is a rule that associates or pairs each number in I with a unique (real) number. It is possible to give a much more rigorous definition of "associates" or "pairs." It is also possible to consider functions whose "inputs" and "outputs" are not real numbers but far more abstract objects — including functions themselves! We will eventually consider *complex*-valued functions of a real variable, i.e., rules that pair real numbers in a unique way with complex numbers.

If f is a function on a real interval I, we will write $f(x)$ to refer to the value of f at the number x in I. We will typically *not* write $f(x)$ to refer to the "whole" function, which we denote just by f.

2. Differentiation is **linear** in the following sense. If f and g are differentiable functions and C is a real number, then

$$\frac{d}{dx}[f+g] = \frac{df}{dx} + \frac{dg}{dx} \quad \text{or} \quad (f+g)' = f' + g'$$

and

$$\frac{d}{dx}[Cf] = C\frac{df}{dx} \quad \text{or} \quad (Cf)' = Cf'(x).$$

3. The **product rule** states that if f and g are differentiable functions, then their product fg is a differentiable function, and

$$\frac{d}{dx}[fg] = f'g + fg'.$$

4. Let f be continuous on the interval I. Then f has an antiderivative on I: there is a differentiable function F defined on I such that $F'(x) = f(x)$ for all x in I. Specifically, if a is any point in I, then we can take F to be the definite integral

$$F(x) := \int_a^x f(t)\ dt.$$

That is,

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x).$$

5. Let f be a function. The symbol

$$\int f(x)\ dx$$

denotes the set of all functions F such that $F' = f$, and it is called the **indefinite integral** of f. That is, the symbol $\int f(x)\ dx$ denotes the set of all antiderivatives of f. The letter x is a "dummy" variable in the sense that we can replace it with any other letter:

$$\int f(x)\ dx = \int f(s)\ ds = \int f(\xi)\ d\xi.$$

6. Suppose that f and g are differentiable functions defined on the same interval I. Suppose as well that $f'(x) = g'(x)$ for all x in I. Then there is a real number C such that $f(x) = g(x) + C$ for all x in I.

7. Let f be a differentiable function on the interval I such that $f'(x) = 0$ for all x in I. Then f is constant: there is a real number C such that $f(x) = C$ for all x in I.

B. Assorted Proofs

B.1. The proof of Lemma 2.1.12.

For convenience, we repeat this lemma.

B.1.1 Lemma.

Let a, b, c, and x_0 be real numbers with $a \neq 0$. Suppose that f solves the **HOMOGENEOUS INITIAL VALUE PROBLEM**

$$\begin{cases} af'' + bf' + cf = 0 \\ f(x_0) = 0 \\ f'(x_0) = 0. \end{cases}$$

Then $f(x) = 0$ for all x.

The proof of this lemma is not terribly difficult, but it requires two auxiliary details.

B.1.2 Lemma (Arithmetic-geometric inequality).

Let A and B be real numbers. Then

$$AB \leq \frac{A^2 + B^2}{2}.$$

Proof. First, $0 \leq (A - B)^2$. Second, we expand $(A - B)^2 = A^2 - 2AB + B^2$. Thus $0 \leq A^2 - 2AB + B^2$, and so $2AB \leq A^2 + B^2$. We divide by 2 to conclude the desired inequality. ■

B.1.3 Lemma (Gronwall's inequality).

Suppose that f is a function defined on the interval I with the following properties.

(i) *$f(x) \geq 0$ for all x in I.*

(ii) *There is x_0 in I such that $f(x_0) = 0$.*

(iii) *There is a constant C such that $f'(x) - Cf(x) \leq 0$ for all x in I.*

Then $f(x) = 0$ for all x in I.

Proof. The inequality $f'(x) - Cf(x) \leq 0$ looks very much like a homogeneous linear first-order differential equation, except for the presence of $\leq$ instead of $=$. Motivated by this similarity, we multiply both sides of the inequality by the integrating factor $\mu(x) = e^{-\int C \, dx} = e^{-Cx}$. Since $e^{-Cx} > 0$ for all x, of course this preserves the inequality. Thus we have

$$f'(x)e^{-Cx} + f(x)[-Ce^{-Cx}] \leq 0.$$

The left side is, as usual, a perfect derivative:

$$f'(x)e^{-Cx} + f(x)[-Ce^{-Cx}] = \frac{d}{dx}[f(x)e^{-Cx}].$$

That is, we now know

$$\frac{d}{dx}[f(x)e^{-Cx}] \le 0.$$

We integrate both sides of this inequality from x_0 to x, where x is an arbitrary point of I. Since definite integrals respect inequalities regarding integrands[32] we find[33]

$$\int_{x_0}^{x} \frac{d}{dx}[f(x)e^{-Cx}]\ dx \le \int_{x_0}^{x} 0\ dx. \tag{B.1.1}$$

That is,

$$f(x)e^{-Cx} - f(x_0)e^{-Cx_0} \le 0.$$

Since $f(x_0) = 0$, we have $f(x) \le 0$. Earlier we assumed $f(x) \ge 0$ for all x in I. The only way that we can have both $f(x) \le 0$ and $f(x) \ge 0$ is via $f(x) = 0$. So, it must be the case that $f(x) = 0$ for all x in I. ■

B.1.4 Theorem.

Let a, b, c, and x_0 be real numbers with $a \ne 0$. Suppose that f solves

$$\begin{cases} af'' + bf' + cf = 0 \\ f(x_0) = 0 \\ f'(x_0) = 0. \end{cases}$$

Then $f(x) = 0$ for all x.

Proof. The differential equation $af'' + bf' + cf = 0$ is, since $a \ne 0$, equivalent to the slightly simpler problem $f'' + \beta f' + \gamma f = 0$, where $\beta := b/a$ and $\gamma := c/a$. Assume from now on that $f'' + \beta f' + \gamma f = 0$. Define

$$g(x) := \frac{[f(x)]^2 + [f'(x)]^2}{2}.$$

[32] If g and h are integrable on $[a, b]$ and $g(x) \le h(x)$ for all x in $[a, b]$, then

$$\int_a^b g(x)\ dx \le \int_a^b h(x)\ dx.$$

[33] The notation

$$\int_{x_0}^{x} \frac{d}{dx}[f(x)e^{-Cx}]\ dx$$

in (B.1.1) is terrible grammar, since we are using x both as the upper limit of integration and the dummy variable of integration. Better phrasing might be

$$\int_{x_0}^{x} \frac{d}{ds}[f(s)e^{-Cs}]\ ds.$$

We will show that g satisfies the hypotheses of Gronwall's inequality to conclude $g(x) = 0$ for all x. Since $0 \leq [f(x)]^2 \leq g(x)$ for all x, it follows that $0 \leq [f(x)]^2 \leq 0$ for all x and thus $f(x) = 0$ for all x.

First, since $[f(x)]^2 \geq 0$ and $[f'(x)]^2 \geq 0$, we have $g(x) \geq 0$. Next, since f is twice-differentiable, g is differentiable. It remains for us to show that g satisfies an inequality of the form $g'(x) - Cg(x) \leq 0$ for all x and some C, and this will be the remainder of the proof.

The chain rule gives

$$g'(x) = f(x)f'(x) + f'(x)f''(x).$$

Since $f'' + \beta f' + \gamma f = 0$, this means

$$\begin{aligned} g'(x) & f(x)f'(x) + f'(x)[-\beta f'(x) - \gamma f(x)] \\ &= f(x)f'(x) - \beta[f'(x)]^2 - \gamma f(x)f'(x) \\ &= (1-\gamma)f(x)f'(x) - \beta[f'(x)]^2. \end{aligned} \tag{B.1.2}$$

For any x, the arithmetic-geometric inequality with $A = f(x)$ and $B = f'(x)$gives

$$f(x)f'(x) \leq \frac{[f(x)]^2}{2} + \frac{[f'(x)]^2}{2}. \tag{B.1.3}$$

Combine (B.1.2) and (B.1.3) to obtain

$$\begin{aligned} g'(x) &\leq (1-\gamma)\frac{[f(x)]^2}{2} + (1-\gamma)\frac{[f'(x)]^2}{2} - \beta[f'(x)]^2 \\ &= (1-\gamma)\frac{[f(x)]^2}{2} + (1-\gamma-2\beta)\frac{[f'(x)]^2}{2}. \end{aligned}$$

Put[34]

$$C := \max\{1-\gamma, 1-\gamma-2\beta\}$$

to conclude

$$g'(x) \leq C\frac{[f(x)]^2}{2} + C\frac{[f'(x)]^2}{2} = C\left(\frac{[f(x)]^2}{2} + \frac{[f'(x)]^2}{2}\right) = Cg(x).$$

That is, $g'(x) - Cg(x) \leq 0$. This is the last hypothesis of Gronwall's inequality that we needed to satisfy. ■

B.2. The proof of Lemma 2.1.14.

Coming soon!

[34]To be clear, by $\max\{A, B\}$, where A and B are numbers, we mean the larger of A and B; if $A = B$, then $\max\{A, B\} = A$.

C. The Coefficients for Variation of Parameters

Coming eventually! See the textbook.