ANALYSE 3 NA LECTURE NOTES

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14 December 2020

Contents

1. Ordinary Differential Equations

1.1. Basic concepts and definitions.

All material in this and subsequent sections on ordinary differential equations was inspired by [3, 5, 6, 23, 29]. We presume familiarity with vector spaces (Appendix C.1) and linear operators (Appendix C.2).

1.1.1 Definition.

If $f\colon I \subseteq \mathbb{R} \to \mathbb{C}$ is differentiable, we denote its derivative by f' or $\partial_x[f]$. If f is k-times differentiable, we denote its kth derivative by $f^{(k)}$ or $\partial_x^k[f]$. For simplicity, we will always use the letter x as the independent variable of functions when studying differential equations. We denote by $\mathcal{C}^n(I)$ the vector space of n-times differentiable functions $f\colon I \subseteq \mathbb{R} \to \mathbb{C}$ such that $\partial_x^n[f]$ is continuous on I. We write $\mathcal{C}(I) := \mathcal{C}^0(I)$ to denote the functions that are merely continuous on I. We denote by $\mathcal{C}^\infty(I)$ the space of infinitely differentiable functions on I, i.e., $\mathcal{C}^\infty(I) = \cap_{n=0}^\infty \mathcal{C}^n(I)$.

The following is the most natural definition of a differential equation, and it is a wrong definition.

1.1.2 Undefinition.

An ORDINARY DIFFERENTIAL EQUATION (ODE) *is an equation involving one or more derivatives of a function of a single variable.*

1.1.3 Example.

Each of the following equations is an ODE according to Undefinition 1.1.2, as each is an equation involving one or more derivatives of the function $f = f(x)$.

(i) $f' + xf = 0$;

(ii) $2f'' + 4f' + 2f = 0$;

(iii) $e^x f'' + f = e^{2x}$;

(iv) $f'' + \sin(f) = 0$;

(v) $f''(x) + f(x+1) - 2f(x) + f(x-1) = 0$;

(vi) $f'(x) + \displaystyle\int_0^\infty f(\xi - x)e^{-\xi}\, d\xi = \frac{1}{1+x^2}$.

Undefinition 1.1.2, however, is too broad, as not all the equations in Example 1.1.3 are "pure" differential equations. Equation (v) requires knowledge of f and its derivative not only at one point x but also at $x \pm 1$. Physically, we might think of this equation as demanding knowledge of f not just at the present moment/location x but also ahead/in the future at $x+1$ and behind/in the past at $x-1$. This is an ADVANCE-DELAY DIFFERENTIAL EQUATION. Equation (vi) includes not only a derivative but also an integral and therefore is an INTEGRO-DIFFERENTIAL EQUATION. Both (v) and (vi) are NONLOCAL differential equations, in that they demand knowledge of the solution at multiple points simultaneously (the integro-differential equation here is particularly egregious, requiring us to know how f behaves on all of $[0, \infty)$!). Both integro-differential equations and advance-delay equations raise mathematically rich, stimulating questions in their own right, and, much later, we will see how techniques from Fourier theory can help us solve some of them.

Equations (i) through (iv), however, are "differential equations" as we will ultimately like to think of them. Namely, we can always solve for the highest derivative in the equation as a function of the other, lower derivatives and x. We rewrite them as follows.

(i)$_{\textbf{new}}$ $f' + xf = 0 \iff f' = xf \iff y' = \mathcal{N}_1(x, f)$, where $\mathcal{N}_1(x, y) := xy$;

(ii)$_{\textbf{new}}$ $2f'' + 4f' + 2f = 0 \iff f'' = -2f' - f = \mathcal{N}_2(x, f, f')$, where $\mathcal{N}_2(x, y_0, y_1) := -2y_1 - y_0$;

(iii)$_{\textbf{new}}$ $e^x f'' + f = e^{2x} \iff f'' = e^x - e^x f = \mathcal{N}_3(x, f, f')$, where $\mathcal{N}_3(x, y_0, y_1) := e^x - e^x y_0$;

(iv)$_{\textbf{new}}$ $f'' + \sin(f) = 0 \iff f'' = -\sin(f) = \mathcal{N}_4(x, f, f')$, where $\mathcal{N}_4(x, y_0, y_1) := -\sin(y_0)$.

Here is what these four equations have in common: *we can write them in the form "highest derivative = function of x and the other derivatives, where this function is a map defined on (a subset of) $\mathbb{R}^n$, or even $\mathbb{C}^n$, for some positive integer n."* This commonality ultimately drives much of the existence theory of ODEs[1] and separates these four equations from the distinct challenges of equations like (v) and (vi). While it will be important (for reasons that will not be obvious until we study complex analysis (see Examples 3.5.29 and 3.7.12) for us to take the independent variable x to be real, there is no harm in assuming that coefficients in our differential equations are complex, which may require the solution to be a *complex-valued* function of a *real variable*. And so we come to a better[2] definition of a differential equation.

1.1.4 Definition.

An nTH ORDER ORDINARY DIFFERENTIAL EQUATION **(ODE)** *is an equation for an unknown function f of the form*

$$\partial_x^n[f] = \mathcal{N}\big(x, \partial_x[f], \ldots, \partial_x^{n-1}[f]\big). \tag{1.1.1}$$

Here $\mathcal{N}$ is a function $\mathcal{N}\colon I \times \mathcal{D} \to \mathbb{C}$, where $I \subseteq \mathbb{R}$ is an interval and $\mathcal{D} \subseteq \mathbb{C}^{n-1}$.

A SOLUTION *to (1.1.1) is a function $f \in \mathcal{C}^n(J)$, where $J \subseteq I$ is an interval*[3]*, such that $\partial_x^n[f](x) = \mathcal{N}\big(x, \partial_x[f](x), \ldots, \partial_x^{n-1}[f](x)\big)$ for all $x \in I$.*

[1] Some people abbreviate the plural "ordinary differential equations" as ODE, too. We will not be like those people.

[2] Whether or not this is really a "better" definition is admittedly subjective. One reason to view this as "better" is that the majority of "equations involving derivatives" that we can solve explicitly and/or easily have the form (1.1.1). Certainly all such equations that we will meet in this course do. And the form (1.1.1) ultimately can be shown to possess powerful theoretical advantages for proving the existence of solutions.

[3] There are considerable variations in the literature on what the domain of the solution to an ODE should be; see [26] for a fascinating overview. In more theoretical treatments, obtaining a solution

We often do not meet differential equations by themselves; instead, we will look for a solution taking a particular value at a given point.

1.1.5 Definition.

An n**TH ORDER INITIAL VALUE PROBLEM (IVP)** *consists of an nth order ODE* (1.1.1) *together with an* **INITIAL CONDITION** *on the solution* f*: for some points* $x_0 \in I$ *and* $y_0, \ldots, y_{n-1} \in \mathbb{C}$*, the function* f *must satisfy*

$$\partial_x^k[f](x_0) = y_k, \ \ k = 0, \ldots, n-1.$$

Sometimes we will study ODEs paired with a demand for behavior of f and/or some of its derivatives at more than one point in an interval. Such a problem is called a **BOUNDARY VALUE PROBLEM (BVP)**, as the points involved are typically endpoints of a closed bounded interval. We will study a toy BVP in Example 1.2.6 and, more thoroughly, second-order BVPs in Section 2.7, so we do not give a formal definition here.

The differential equations that we will study will be almost exclusively *linear* differential equations.

1.1.6 Definition.

(i) *An* n**TH ORDER LINEAR ODE** *is an equation of the form*

$$\underbrace{a_n(x)\partial_x^n[f] + a_{n-1}(x)\partial_x^{n-1}[f] + \cdots + a_0(x)f}_{(\mathcal{A}f)(x)} = \sum_{k=0}^{n} a_k(x)\partial_x^k[f](x) = g(x), \ \ x \in I, \tag{1.1.2}$$

where $I \subseteq \mathbb{R}$ *is an interval, the mappings* g, $a_k \colon I \to \mathbb{C}$ *are continuous, and* $a_n(x) \neq 0$ *for all* $x \in \mathbb{R}$*. Since* $a_n(x) \neq 0$*, we may solve for* $\partial_x^n[f]$ *as*

$$\partial_x^n[f] = \frac{g(x)}{a_n(x)} - \sum_{k=0}^{n-1} \frac{a_k(x)}{a_n(x)}\partial_x^k[f] =: \mathcal{N}\big(x, f, \partial_x[f], \ldots, \partial_x^{n-1}[f]\big),$$

where $\mathcal{N} \colon I \times \mathbb{C}^{n-1} \to \mathbb{C}$ *is given by*

$$\mathcal{N}(x, y_1, \ldots, y_{n-1}) := \frac{g(x)}{a_n(x)} - \sum_{k=0}^{n-1} \frac{a_k(x)}{a_n(x)} y_k.$$

(ii) *The linear ODE* (1.1.2) *is* **HOMOGENEOUS** *if* $g(x) = 0$ *for all* x *and* **NONHOMOGENEOUS** *otherwise. The function* g *is sometimes called a* **FORCING** *or* **DRIVING**[4] *term.*

whose domain is an interval is simply the natural way of proceeding. That is, when one invokes an abstract existence theorem, the solution produced is inherently defined on an interval. Physically, if an ODE represents a time- or space-dependent process, requiring the solution to be defined on an interval represents the natural "unbrokenness" or continuity (not to be confused with calculus continuity) of time or space. At a purely symbolic level, this will not concern us too much, though; the function $f(x) = \ln(|x|)$ clearly satisfies $f'(x) = 1/x$ on the non-interval $(-\infty, 0) \cup (0, \infty)$.

[4]In physical problems, e.g., a mass-spring oscillator, the function g can represent an external force on the system.

The operator $\mathcal{A}$ defined in (1.1.2) is a linear map from $\mathcal{C}^n(I)$ to $\mathcal{C}(I)$, which we call a **LINEAR DIFFERENTIAL OPERATOR**. Equivalently, in the parlance of Appendix C.2, $\mathcal{A}$ is a linear operator in $\mathcal{C}(I)$ with domain $\mathfrak{D}(\mathcal{A}) = \mathcal{C}^n(I)$. We will use the linearity of $\mathcal{A}$ in almost all of our calculations and proofs. In particular, if $\mathcal{A}f_1 = g_1$ and $\mathcal{A}f_2 = g_2$ for some f_1, $f_2 \in \mathcal{C}^n(I)$ and g_1, $g_2 \in \mathcal{C}(I)$, then

$$\mathcal{A}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{A} f_1 + c_2 \mathcal{A} f_2 = c_1 g_1 + c_2 g_2 \tag{1.1.3}$$

for any c_1, $c_2 \in \mathbb{C}$. The relation (1.1.3) is often called the **SUPERPOSITION PRINCIPLE** (even though it is just the defining property of a linear operator).

1.2. First-order linear ODEs.

From Definition 1.1.6, a first-order linear differential equation has the form

$$f' + p(x)f = g(x), \tag{1.2.1}$$

where p and g are given functions that are continuous on some interval[5] $I \subseteq \mathbb{R}$.

This is a fairly remarkable equation for a number of reasons.

- We can solve it explicitly, getting an actual formula for its solution(s) in terms of p and q, using only techniques from elementary calculus.

- Our method of solution inherently shows that, up to one free parameter (whose role is also explicit), the solution is unique.

- The precise form of this solution will motivate what might at the outset appear to be a bizarre strategy for solving higher-order linear differential equations.

- The simple structure of this equation will serve as a toy model for series solutions and Fourier transform techniques for more complicated equations.

- And, depending on some additional behavior that we will demand of f, the simple linear mapping $f \mapsto f' + p(x)f$ exhibits a rich linear algebraic structure that will give us insight into application-heavy boundary value and eigenvalue problems later in the course.

1.2.1. Direct integration.

The easiest version of (1.2.1) arguably occurs when $p(x) = 0$ for all $x \in I$. In that case, (1.2.1) becomes

$$f'(x) = g(x), \ x \in I,$$

and so f must be an antiderivative of q. That is,

$$f(x) = \int g(x) \ dx + C. \tag{1.2.2}$$

[5]For the most part, this interval I can be *any* subinterval of $\mathbb{R}$ — open, closed, bounded, unbounded, none of the above.

1.2.1 Example.

Solve

$$\frac{dy}{dx} = \cos(x).$$

Solution. We have, of course,

$$y = \int \frac{dy}{dx}\,dx + C = \int \cos(x)\,dx + C = \sin(x) + C. \qquad \blacktriangle$$

For quick and dirty calculations, the *indefinite* integral is fine. But if we want to be mathematically precise, we will come to dislike the "indefinite integral" notation of (1.2.2) for several reasons. First, in standard calculus, the symbol $\int g(x)\,dx$ denotes a set of functions, all of which differentiate to q, rather than one single function. Second, we are taught in calculus that the "dummy variable" of integration is arbitrary, and so

$$\int g(x)\,dx = \int g(\xi)\,d\xi = \int q(s)\,ds,$$

for all of these symbols just represent the set of antiderivatives of q. Then (1.2.2) introduces the quandary

$$f(x) = \int g(x)\,dx + C = \int g(\xi)\,d\xi + C \overset{?!}{=} f(\xi) + C.$$

And, third, if $\int g(x)\,dx$ represents the set of *all* antiderivatives of q, is writing the $+C$ redundant? So, whenever we will attempt something rigorous, we will use a *definite* integral. But, in examples, when we want a quick and dirty formula, we will write indefinite integrals.

1.2.2. The integrating factor method.

Now consider (1.2.1) with $p(x) \neq 0$, so we cannot use direct integration. The next best thing is to *modify* our ODE so that direct integration will be possible. The classical[6] method of doing this is to observe that its left side somewhat resembles the product rule: if μ is another function[7] of x, then

$$\partial_x[f\mu] = f'\mu + f\mu'.$$

If we could rewrite the left side of (1.2.1) as such a "perfect derivative," then we could simply antidifferentiate both sides and possibly solve for f. One thing that is missing in (1.2.1) is a factor μ on f', so let us multiply both sides of this equation by some unknown g:

$$f' + p(x)f = g(x) \Longrightarrow f'(x)\mu(x) + p(x)\mu(x)f(x) = \mu(x)g(x). \tag{1.2.3}$$

Now, if $p\mu$ is the derivative of μ, then the left side will be a perfect derivative (and so μ is called an **INTEGRATING FACTOR**, because we can easily integrate perfect derivatives). That is, we want μ to satisfy

$$\mu'(x) = p(x)\mu(x). \tag{1.2.4}$$

[6]"A mathematical technique is CLASSICAL if it was proved before I went to graduate school." —Anon.
[7]For nebulous historical and cultural reasons, the letter μ is often used in this approach.

On the surface, this is no better than (1.2.1), since (1.2.4) is equivalent to

$$\mu'(x) - p(x)\mu(x) = 0,$$

which looks almost like (1.2.1), which is what we want to solve in the first place. On the other hand, with enough experience in calculus, one can almost guess[8] what a solution of (1.2.4) should be: $\mu(x) = e^{P(x)}$, where P is any antiderivative[9] of p on I, i.e., $P'(x) = p(x)$, $x \in I$.

This is a good guess, not only because it does solve (1.2.4), but also because (1.2.1) is *equivalent* to the second ODE in (1.2.3), since $e^{P(x)} > 0$ for all x. Thus, with $P' = p$, we have

$$f' + p(x)f = g(x) \iff f'(x)e^{P(x)} + p(x)e^{P(x)}f(x) = e^{P(x)}g(x) \iff \partial_x[fe^P](x) = e^{P(x)}g(x). \tag{1.2.5}$$

To solve for f, let us fix a point $x_0 \in I$ and integrate both sides:

$$\begin{aligned} \partial_x[fe^P](x) = e^{P(x)}g(x) &\iff \int_{x_0}^x \partial_\xi[fe^P](\xi)\ d\xi = \int_{x_0}^x e^{P(\xi)}g(\xi)\ d\xi \\ &\iff f(x)e^{P(x)} - f(x_0)e^{P(x_0)} = \int_{x_0}^x e^{P(\xi)}g(\xi)\ d\xi \qquad (1.2.6) \\ &\iff f(x) = f(x_0)e^{P(x_0)-P(x)} + e^{-P(x)}\int_{x_0}^x e^{P(\xi)}g(\xi)\ d\xi. \end{aligned}$$

Every step above was an “if and only if” step, so we know that f as defined here does solve[10] (1.2.1). But there is one piece of information missing: the value of $f(x_0)$. No data in our problem will help us specify it, so let us write it as an arbitrary constant $c \in \mathbb{C}$. Also, since we can always adjust an antiderivative of p by a constant, let us assume[11] that $P(x_0) = 0$. Then, for any $c \in \mathbb{C}$, the function

$$f(x) = ce^{-P(x)} + e^{-P(x)}\int_{x_0}^x e^{P(\xi)}g(\xi)\ d\xi \tag{1.2.7}$$

solves (1.2.1).

While this is a perfectly good abstract solution — and, indeed, this is the formula we will exploit repeatedly later — in practice we rarely deal with a definite integral over $[x_0, x]$.

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[8] If one cannot almost guess this, fear not. This whole method hinges on the wholly nonobvious observation that (1.2.1) is only a multiple away from being a product rule situation. As much as it is good in mathematical education to see where ideas naturally originate, sometimes one must simply take a leap of faith across the intellectual Rubicon.

[9] Since p is continuous on I, the fundamental theorem of calculus ensures that p has an antiderivative valid on all of I. For example, we might fix a point $x_0 \in I$ and take $P(x) = \int_{x_0}^x p(s)\ ds$.

[10] Alternatively, differentiate f directly using the product rule, the fundamental theorem of calculus, the chain rule, and our assumption $P'(x) = p(x)$.

[11] In fact, by the fundamental theorem of calculus, this is the only possible choice of P. If $P'(x) = p(x)$ and $P(x_0) = 0$, then the FTC implies $P(x) = P(x_0) + \int_{x_0}^x p(s)\ ds = \int_{x_0}^s p(s)\ ds$.

1.2.2 Example.

Let $\lambda \in \mathbb{C}$. Solve

$$f' - \lambda f = e^{\lambda x}.$$

Solution. We have $p(x) = -\lambda$ and $g(x) = e^{\lambda x}$, so an integrating factor is

$$\mu(x) = e^{\int -\lambda \, dx} = e^{-\lambda x}.$$

We elected not to keep a constant of integration here; remember that P can be *any* antiderivative of p. Then we have

$$f' - \lambda f = e^{\lambda x} \iff (f' - \lambda f)e^{-\lambda x} = e^{\lambda x} e^{-\lambda x} \iff \underbrace{f' e^{-\lambda x} - \lambda f e^{-\lambda x}}_{\partial_x[fe^{-\lambda \cdot}]} = 1.$$

Thus

$$\partial_x[fe^{-\lambda \cdot}] = 1 \Longrightarrow f(x)e^{-\lambda x} = \int 1 \, dx + C = x + C \Longrightarrow f(x) = xe^{\lambda x} + Ce^{\lambda x}. \qquad \blacktriangle$$

1.2.3 Method: solve $f'(x) + p(x)f = g(x)$

1. Let P be an antidervative of p. Take the constant of integration to be 0.
2. Multiply both sides by $e^{P(x)}$.
3. Recognize the resulting left side as a product-rule style derivative.
4. Antidifferentiate. Put the constant of integration on the right side.
5. Solve for f by dividing both sides by $e^{P(x)}$.

1.2.3. Four questions and an existence/uniqueness theorem.

Now let us return to the abstract solution (1.2.7) of our ODE (1.2.1).

1. *Does every solution to* (1.2.1) *have the form* (1.2.7)*?* Yes. This is the logic of (1.2.5) and (1.2.7). If we start with a function f satisfying $f' + pf = q$, then these two chains of "if and only if" statements force f to have this form.

2. *Is the solution to* (1.2.1) *unique?* No. Just choose two different values of c in (1.2.7).

3. *Does our choice of P matter? After all, the function p has infinitely many antiderivatives.* Let $\widetilde{P}$ be another antiderivative of p, so $\widetilde{P}'(x) = p(x)$ for all $x \in I$. Then calculus implies the existence of some constant C such that $\widetilde{P}(x) = P(x) + C$ for all $x \in I$. If we replace P with $\widetilde{P}$ throughout the formula for f given in (1.2.7), our solution will now be

$$\widetilde{f}(x) := ce^{\widetilde{P}(x_0) - \widetilde{P}(x)} + e^{-\widetilde{P}(x)} \int_{x_0}^{x} e^{\widetilde{P}(\xi)} g(\xi) \, d\xi$$

$$= ce^{P(x_0) + C - (P(x) + C)} + e^{-(P(x)+C)} \int_{x_0}^{x} e^{P(\xi) + C} g(\xi) \, d\xi$$

$$= ce^{P(x_0)} + e^{-P(x)} \int_{x_0}^{x} e^{P(\xi)} g(\xi)\ d\xi,$$

and so the "new" solution $\widetilde{f}$ agrees with f from (1.2.6). Thus changing the antiderivative of p will not change the solution to (1.2.1).

4. *Does the point x_0 from which we integrated in* (1.2.6) *matter? If we integrate from a different point, say, x_1, will we get a solution different from that in* (1.2.7)*?* Yes, but we will just have to change the value of c. By the calculations in (1.2.6), for any $\kappa \in \mathbb{C}$ the function

$$f_1(x) := \kappa e^{P(x_1)-P(x)} + e^{-P(x)} \int_{x_1}^{x} e^{P(\xi)} g(\xi)\ d\xi$$

also solves (1.2.1). After some slight rearranging, we find

$$f_1(x) = \left(\kappa e^{P(x_1)} + \int_{x_1}^{x_0} e^{P(\xi)} g(\xi)\ d\xi\right) e^{-P(x)} + \int_{x_0}^{x} e^{P(\xi)} g(\xi)\ d\xi.$$

This is precisely the form of the solution given by (1.2.7) with $c = \kappa e^{P(x_1)} + \int_{x_1}^{x_0} e^{P(\xi)} g(\xi)\ d\xi$.

The calculations that led to (1.2.7) and our election $P(x_0) = 0$ now bring us to our first existence and uniqueness theorem.

1.2.4 Theorem.

Let $I \subseteq \mathbb{R}$ be an interval and let $p, g \in \mathcal{C}(I)$. Let $x_0 \in I$ and set $P(x) := \int_{x_0}^{x} p(s)\ ds$.

(i) [Duhamel's formula] *For any $c \in \mathbb{C}$, the function*

$$f(x) := \underbrace{ce^{-P(x)} + e^{-P(x)} \int_{x_0}^{x} e^{P(\xi)} g(\xi)\ d\xi}_{\mathcal{T}[g;x_0,c](x)} \tag{1.2.8}$$

solves

$$f'(x) + p(x)f(x) = g(x), \ \ x \in I.$$

(ii) *Conversely, if f solves this ODE, there exists $c \in \mathbb{C}$ such that $f(x) = \mathcal{T}[g;x_0,c](x)$ for all $x \in I$. Thus the set of all solutions to this ODE is precisely $\{\mathcal{T}[g;x_0,c]\}_{c\in\mathbb{C}}$.*

(iii) *In particular, the IVP*

$$\begin{cases} f'(x) + p(x)f(x) = g(x), \ \ x \in I \\ f(x_0) = c \end{cases}$$

has the unique solution $f = \mathcal{T}[g;x_0,c]$.

1.2.5 Example.

Solve the IVP

$$\begin{cases} f' + \dfrac{1}{x}f = x, \ x > 0 \\ f(1) = 1. \end{cases}$$

Solution. We construct the integrating factor

$$\mu(x) = e^{\int (1/x) \ dx} = e^{\ln(|x|)} = |x| = x,$$

since we are assuming $x > 0$. Then

$$f' + \frac{1}{x}f = x \iff xf'(x) + f(x) = x^2 \iff \partial_x[xf(x)] = x^2 \iff xf(x) = \frac{x^3}{3} + C.$$

Hence

$$f(x) = \frac{x^2}{3} + \frac{C}{x}.$$

We use the initial condition to solve for C:

$$f(1) = 1 \iff \frac{1}{3} + \frac{C}{1} = 1 \iff C = \frac{2}{3}.$$

So, the (unique) solution to the IVP is

$$f(x) = \frac{x^2}{3} + \frac{2}{3x}.$$

▲

If, however, we demand that our solution to (1.2.1) satisfy conditions at two or more distinct points in I, we may not be able to find a solution at all, or we may be able to do so only for select $g \in \mathcal{C}(I)$. Such a problem is a **BOUNDARY VALUE PROBLEM (BVP)**, and we will study them, with motivation from PDE, later in the context of second-order ODEs.

1.2.6 Example.

Fix a number $\alpha \in \mathbb{C}$*, real numbers* $0 < 1$*, and* $g \in \mathcal{C}([0,1])$*. Discuss solutions to the BVP*

$$\begin{cases} f'(x) = g(x), \ 0 \le x \le 1 \\ f(0) = \alpha f(1). \end{cases}$$

Solution. 1. First, direct integration always guarantees that putting

$$f(x) := f(0) + \int_0^x g(\xi) \ d\xi \tag{1.2.9}$$

solves the ODE. If $\alpha = 0$, then the boundary condition reduces to $f(0) = 0$, and so we are dealing with an initial value problem. Theorem 1.2.4 guarantees the existence and uniqueness of a solution to this IVP, but of course we know that just from the integral form of f above.

2. More interesting is the case $\alpha \neq 0$. In that case, to meet the boundary condition, we need

$$f(0) = \alpha f(1) \Longrightarrow \int_0^0 g(\xi)\, d\xi = \alpha \int_0^1 g(\xi)\, d\xi \Longrightarrow \alpha \int_0^1 g(\xi)\, d\xi = 0. \tag{1.2.10}$$

Since $\alpha \neq 0$, we may divide to find $\int_0^1 g(\xi)\, d\xi = 0$. So, we have found a *necessary* condition for the existence of a solution to this BVP: *if* we can solve it, then g must satisfy $\int_0^1 g(\xi)\, d\xi = 0$. Certainly there are some $g \in \mathcal{C}([0,1])$ for which this integral condition fails, e.g., $g(x) = 1$. Immediately, then, we see that existence of a solution is not always guaranteed for a BVP.

3. Next, is this integral condition *sufficient* for the existence of a solution? That is, if $g \in \mathcal{C}([0,1])$ and $\int_0^1 g(\xi)\, d\xi = 0$, does there exist a solution to the BVP? First, by the fundamental theorem of calculus, the solution f must satisfy (1.2.9). That is, the only possible formula for f is (1.2.9). But this is not really a "closed" formula for f, because we still have not determined $f(0)$. Let us try to use the boundary condition to do so. We must have

$$f(0) = \alpha f(1) = \alpha \left(f(0) + \int_0^1 g(\xi)\, d\xi \right) = \alpha f(0). \tag{1.2.11}$$

Then

$$(1 - \alpha) f(0) = 0. \tag{1.2.12}$$

If $\alpha \neq 1$, then $f(0) = 0$, and so from (1.2.9)

$$f(x) = \int_0^x g(\xi)\, d\xi. \tag{1.2.13}$$

Now let us check our work. With f defined by (1.2.13), the fundamental theorem of calculus gives $f' = g$, and we calculate $f(0) = \int_0^0 g(\xi)\, d\xi = 0$, while $f(1) = \int_0^1 g(\xi)\, d\xi = 0$ by our hypothesis on g. So, $f(0) = 0 = \alpha \cdot 0 = \alpha f(1)$, and therefore f satisfies the boundary conditions.

4. In the case $\alpha \neq 0$, $\alpha \neq 1$, and $\int_0^1 g(\xi)\, d\xi = 0$, is the solution to the BVP unique? Yes: by the fundamental theorem of calculus, the only possible solution is (1.2.9). Assuming these conditions on α and g, the calculations in (1.2.11) and (1.2.12) force $f(0) = 0$, and thus f must have the form (1.2.13).

5. What about the case $\alpha = 1$ and $\int_0^1 g(\xi)\, d\xi = 0$? In this case, we still know that f must have the form (1.2.9), and we use $\int_0^1 g(\xi)\, d\xi$ to calculate $f(0) = f(1) = 1 \cdot f(1)$. That is, regardless of what $f(0)$ is, (1.2.9) will always solve the BVP. And so in this case the BVP has *infinitely many* solutions of the form

$$f(x) = c + \int_0^x g(\xi)\, d\xi.$$

▲

1.2.7 Remark.

We see from Example 1.2.6 that a BVP for a linear first-order ODE may have no solutions, a unique solution, or infinitely many solutions. This is wildly unlike the well-behaved theory for linear first-order IVPs!

1.2.8 Linear algebraic viewpoint: first order IVPs and BVPs

Given an interval $I \subseteq \mathbb{R}$ and a continuous function $p\colon I \to \mathbb{C}$, define

$$(\mathcal{A}f)(x) := f'(x) + p(x)f(x), \ x \in I, \ f \in \mathcal{C}^1(I).$$

Then $\mathcal{A}$ is a linear operator from $\mathcal{C}^1(I)$ to $\mathcal{C}(I)$. Equivalently, $\mathcal{A}$ is a linear operator in (see Appendix C.2) $\mathcal{X} := \mathcal{C}(I)$ with domain $\mathfrak{D}(\mathcal{A}) = \mathcal{C}^1(I)$. We will consider several standard properties of $\mathcal{A}$ as a linear operator and see how both abstract linear algebra theory and the particular ODE theory that we developed in Theorem 1.2.4 interact.

1. *The kernel of* $\mathcal{A}$. First let us determine the kernel of $\mathcal{A}$ in $\mathcal{X}$. Let P be an antiderivative of p. If $\mathcal{A}f = 0$, then f solves the ODE

$$f' + p(x)f = 0,$$

and Theorem 1.2.4 then tells us that $f(x) = ce^{-P(x)}$. Hence $\ker(\mathcal{A}) = \operatorname{span}(\{e^{-P(\cdot)}\})$, and so $\ker(\mathcal{A})$ is one-dimensional.

2. *The range of* $\mathcal{A}$. Next, Theorem 1.2.4 tells us that for any $g \in \mathcal{X}$, there exists $f \in \mathfrak{D}(\mathcal{A})$ with $\mathcal{A}f = g$. That is, the range of $\mathcal{A}$ as an operator in $\mathcal{X}$ with domain $\mathfrak{D}(\mathcal{A})$ is all of $\mathcal{X}$.

To see this, just pick any $x_0 \in I$ and $c \in \mathbb{C}$ and set $f = \mathcal{T}[g; x_0, c]$ from (1.2.8). This solution is not unique: since $\mathcal{A}$ has a nontrivial kernel, spanned by $e^{-P(\cdot)}$, we may take any $\alpha \in \mathbb{C}$ and calculate

$$\mathcal{A}\big(\alpha e^{-P(\cdot)} + \mathcal{T}[g; x_0, c]\big) = \alpha\mathcal{A}e^{-P(\cdot)} + \mathcal{A}\mathcal{T}[g; x_0, c] = 0 + g = g.$$

Conversely, any two solutions of $\mathcal{A}f = g$ differ by a scalar multiple of $e^{-P(\cdot)}$: if $\mathcal{A}f_j = q$ for $j = 1$, 2, then

$$\mathcal{A}(f_1 - f_2) = \mathcal{A}f_1 - \mathcal{A}f_2 = q - q = 0, \tag{1.2.14}$$

hence $f_1 - f_2 \in \ker(\mathcal{A})$, and so $f_1 - f_2 = \beta e^{-P(\cdot)}$ for some $\beta \in \mathbb{C}$.

One way to force the solution of $\mathcal{A}f = g$ to be unique is to restrict $\mathcal{A}$ to a subspace $\mathfrak{D}_0(\mathcal{A})$ of $\mathfrak{D}(\mathcal{A})$ such that $\mathfrak{D}_0(\mathcal{A}) \cap \ker(\mathcal{A}) = \{0\}$. For then if there exist f_1, $f_2 \in \mathfrak{D}_0(\mathcal{A})$ with $\mathcal{A}f_j = 0$, the calculation in (1.2.14) shows $f_1 - f_2 \in \ker(\mathcal{A})$, hence $f_1 - f_2 = 0$. On the other hand, we already know from Theorem 1.2.4 how to force a solution of $\mathcal{A}f = g$ to be unique: fix points $x_0 \in I$ and $c \in \mathbb{C}$ and add the requirement that $f(x_0) = c$. However, the set

$$\big\{f \in \mathcal{C}^1(I) \ \big| \ f(x_0) = c\big\}$$

will only be a subspace of $\mathfrak{D}(\mathcal{A})$ if $c = 0$. So, $\mathcal{A}$ is one-to-one when restricted to $\mathfrak{D}_{x_0}(\mathcal{A}) := \big\{f \in \mathcal{C}^1(I) \ \big| \ f(x_0) = 0\big\}$. Moreover, $\mathcal{A}$ is still onto $\mathcal{C}(I)$ when restricted to $\mathfrak{D}_{x_0}(\mathcal{A})$: given $g \in \mathcal{C}(I)$, we have $\mathcal{A}\mathcal{T}[g; x_0, 0] = g$.

3. *The inverse of* $\mathcal{A}$ *on* $\mathfrak{D}_{x_0}(\mathcal{A})$. Since $\mathcal{A}$ is both one-to-one on $\mathfrak{D}_{x_0}(\mathcal{A})$ and onto $\mathcal{C}(I)$, it is invertible. Specifically, there is a (necessarily unique) linear operator $\mathcal{A}^{-1}\colon \mathcal{C}(I) \to \mathfrak{D}_{x_0}(\mathcal{A})$ such that $\mathcal{A}\mathcal{A}^{-1}g = g$ for any $g \in \mathcal{C}(I)$ and $\mathcal{A}^{-1}\mathcal{A}f = f$ for any $f \in \mathfrak{D}_{x_0}(I)$. The work above tells us that $\mathcal{A}^{-1}g = \mathcal{T}[g; x_0, 0]$.

Let us examine the formula for $\mathcal{A}^{-1}$ more closely: we have

$$(\mathcal{A}^{-1}g)(x) = \mathcal{T}[g; x_0, 0](x) = \int_{x_0}^{x} \mathcal{V}(x, \xi)g(\xi)\, d\xi, \qquad \mathcal{V}(x, \xi) := e^{P(\xi) - P(x)}.$$

Thus $\mathcal{A}^{-1}$ is an **integral operator**, and the function $\mathcal{V}$ is called the **kernel** of this operator (not to be confused with the kernel in the sense of the subspace (C.2.1)!). That the inverse of $\mathcal{A}$ involves an integral is unsurprising; after all, $\mathcal{A}$ involves differentiation, and the integral is the inverse of the derivative. We will meet integral operators again when we study the Fourier transform.

4. *Eigenvalues of $\mathcal{A}$ (as an operator in $\mathcal{C}(I)$ with domain $\mathcal{C}^1(I)$).* Can we find scalars $\lambda \in \mathbb{C}$ and functions $f \in \mathfrak{D}(\mathcal{A}) \setminus \{0\}$ such that $\mathcal{A}f = \lambda f$? This is equivalent to solving the ODE

$$f' + p(x)f = \lambda f \iff f' + (p(x) - \lambda)f = 0. \tag{1.2.15}$$

Replacing $p(x)$ by $p(x) - \lambda$, we can use Theorem 1.2.4 to solve this ODE. Namely, any solution to (1.2.15) has the form

$$f(x) = ce^{\lambda x - P(x)},$$

for some $c \in \mathbb{C}$. And so every point of $\mathbb{C}$ is an eigenvalue of $\mathcal{A}$. That is, $\sigma_{\text{pt}}(\mathcal{A}) = \mathbb{C}$.

Moreover, every eigenvalue of $\mathcal{A}$ is simple: let $\mathcal{E}(\mathcal{A}, \lambda)$ be the eigenspace

$$\mathcal{E}(\mathcal{A}, \lambda) = \{f \in \mathfrak{D}(\mathcal{A}) \mid Af = \lambda f\}.$$

If we define $\nu_\lambda(x) := e^{\lambda x - P(x)}$, then the analysis above shows $\mathcal{E}(\mathcal{A}, \lambda) = \text{span}(\{\nu_\lambda\})$.

5. *Eigenvalues of $\mathcal{A}$ (as an operator in $\mathcal{C}(I)$ with domain equal to a certain subspace of $\mathcal{C}^1(I)$* Later, in Example 3.2.7, we will take $\mathcal{A}$ to have the smaller domain

$$\{f \in \mathcal{C}^1([0,1]) \mid f(0) = \alpha f(1)\}.$$

There, we will show that if $\alpha = 0$, then $\sigma_{\text{pt}}(\mathcal{A}) = \varnothing$, while if $\alpha \neq 0$, then $\sigma_{\text{pt}}(\mathcal{A})$ can be enumerated as $\{\lambda_k(\alpha)\}_{k \in \mathbb{Z}}$, where $\lambda_{k_1}(\alpha) \neq \lambda_{k_2}(\alpha)$ for $k_1 \neq k_2$. (Doing so requires some properties of the complex logarithm that we will not assume here.) This shows that changing the domain of a linear operator can radically change its eigenvalues!

The material in this and subsequent linear algebraic viewpoints on ODEs was inspired by [19].

1.3. Some nonlinear first-order equations.

An ODE that is not linear (in the sense of Definition 1.1.6) is called, of course, **NONLINEAR**. In the most general sense, a nonlinear first-order ODE has the form

$$f'(x) = \mathcal{N}(x, f(x)), \tag{1.3.1}$$

where $\mathcal{N}$ *cannot* be written in the form $\mathcal{N}(x, y) = p(x)y + g(x)$, for then the ODE would be linear, as in (1.2.1). There is a robust existence and uniqueness theory for first- (and higher-) order nonlinear ODEs and IVPs that we will not pursue, but we will examine two special situations.

First, it may be the case that we can find some y_0 such that $\mathcal{N}(x, y_0) = 0$ for all x in an interval I. Then defining $f(x) = y_0$ for $x \in I$, we see that

$$f'(x) = 0 = \mathcal{N}(x, y_0) = \mathcal{N}(x, f(x)).$$

This solution $f(x) = y_0$ is an **EQUILIBRIUM SOLUTION** of the ODE (1.3.1), since it does not change value as x changes.

1.3.1 Example.

Fix $y_0 \in \mathbb{C}$. Find all equilibrium solutions to the ODE

$$y' = y(1-y)(y-y_0).$$

Solution. This ODE has the form $y' = \mathcal{N}(x,y)$, where $\mathcal{N}(x,y) = y(1-y)(y-y_0)$, That is, $\mathcal{N}(x,\cdot)$ is a polynomial in y, so it has finitely many (y-)roots. Specifically, they are $y = 0, 1, y_0$. And so the equilibrium solutions are $y(x) = 0$, $y(x) = 1$, and $y(x) = y_0$. ▲

Second, it may be the case that we can write $\mathcal{N}(x,y)$ as a product of a function depending solely on x and a function depending solely on y. Such a situation leads to, unsurprisingly, a "separable" ODE.

1.3.2 Definition.

A first-order ODE

$$f'(x) = \mathcal{N}(x, f(x))$$

is **SEPARABLE** *if F has the form $\mathcal{N}(x,y) = \phi(x)\psi(y)$, where $\phi(\cdot)$ and $\psi(\cdot)$ are functions of a single real variable.*

1.3.1. Formal solutions to separable equations.

We provide examples of a *formal* method for solving separable ODEs in the next examples. In these cases, we prefer the more suggestive Leibniz notation dy/dx in lieu of $f'(x)$.

1.3.3 Example.

Solve $\dfrac{dy}{dx} = 2x(1+y^2)$.

Solution. We "separate variables" to find

$$\frac{dy}{1+y^2} = 2x\ dx.$$

Integrating both sides, we have

$$\int \frac{dy}{1+y^2} = \int 2x\ dx \Longrightarrow \arctan(y) = x^2 + C. \tag{1.3.2}$$

Thus

$$y = \tan(x^2 + C).$$

Because $\tan(\cdot)$ is not defined at odd integer multiples of π, this solution will only be defined on a finite interval depending on what C we choose. ▲

Of course, there is no rigor to the highly useful symbol-pushing of this example: the symbol dy/dx is not a fraction to be separated into a numerator dy and a denominator dx. Rather, one should check at the end (by differentiating directly) that putting $f(x) = \tan(x^2 + C)$ for some $C \in \mathbb{R}$ gives $f'(x) = 2x(1 + f(x)^2)$.

1.3.4 Example.

Solve the **EXPONENTIAL GROWTH** *equation* $\dfrac{dy}{dx} = y$.

Solution. Again we separate variables and integrate both sides:

$$\frac{dy}{dx} = y \overset{(1)}{\Longrightarrow} \frac{dy}{y} = dx \overset{(2)}{\Longrightarrow} \int \frac{dy}{y} = \int dx \overset{(3)}{\Longrightarrow} \ln(|y|) = x + C \overset{(4)}{\Longrightarrow} |y| = e^C e^x. \quad (1.3.3)$$

This suggests that taking $y = e^C e^x$ or $y = -e^C e^x$ will solve the exponential growth equation. A direct calculation verifies that these are indeed solutions. Since $\pm e^C$ can take any nonzero real value, if C is chosen appropriately, we are inclined to think that $y = ce^x$ solves the ODE for any $c \in \mathbb{R}$, not just c nonzero. Indeed, putting $y = 0$ gives $y' = 0$, so $y = 0$ solves the ODE, too. (This is not surprising: the ODE has the form $y' = \mathcal{N}(x, y)$, where $\mathcal{N}(x, y) = y$, so $y = 0$ is obviously an equilibrium solution.) ▲

Thinking too hard about Example 1.3.4 may lead to some weird questions.

1. If $y = 0$, then we cannot divide in $\overset{(1)}{\Longrightarrow}$ to separate variables in the first place (…assuming that "multiplying" both sides by dx makes sense…). This did not happen in the division in Example 1.3.3, since there $1 + y^2 > 0$ for all $y \in \mathbb{R}$. This leads us to "miss" the zero solution $y = 0$ when we reach $\overset{(4)}{\Longrightarrow}$. Here is the warning: naive separation of variables may not reveal all the solutions.

2. At $\overset{(4)}{\Longrightarrow}$, we arrived at $|y(x)| = e^C e^x$. For a general function y, this does not imply that y is strictly positive ($y(x) = e^C e^x$ for all x) or strictly negative ($y(x) = -e^c e^x$ for all x). For example, the function

$$y_?(x) = \begin{cases} -e^2 e^x, & x < 0 \\ e^2 e^x, & x \geq 0 \end{cases}$$

satisfies $|y(x)| = e^2 e^x$ for all $x \in \mathbb{R}$. But y is clearly discontinuous at $x = 0$, and so y does not satisfy the exponential growth ODE on $\mathbb{R}$. (Recall, from Definition 1.1.4, that we require a solution to an nth order differential equation to be $\mathcal{C}^n$ on an interval. The function $y_?$ here satisfies the ODE on $(-\infty, 0)$ and $(0, \infty)$ but not all of $\mathbb{R}$.)

Here is the moral lesson: *separation of variables is an extremely useful technique for producing formulas for solutions to separable equations, but the method is not mathematically rigorous, and we should not expect too much deep analysis from it.*

1.3.5 Method: solve a separable equation $f'(x) = \phi(x)f(\chi)$

1. Rewrite the problem in Leibniz notation as

$$\frac{1}{\psi(y)} \frac{dy}{dx} = \phi(x).$$

2. Formally separate variables:

$$\frac{dy}{\psi(y)} = \phi(x) \ dx.$$

3. Integrate both sides. Put the constant of integration on the "x"-side.

$$\int \frac{dy}{\psi(y)} = \int \phi(x)\ dx + C.$$

4. Try to solve for y explicitly as a function of x. This may not be possible, or there may be several ways to do this.

5. Check for missing "equilibrium" solutions that do not appear because of division by ψ. These are solutions of $\psi(y) = 0$.

1.3.2. *Some theory for separable equations.*

Now we cast the formal separation of variables calculations above into rigorous language. Consider a separable ODE

$$f'(x) = \phi(x)\psi(f(x)), x \in I \tag{1.3.4}$$

where $\phi\colon I \subseteq \mathbb{R} \to \mathbb{C}$ and $\psi\colon J \subseteq \mathbb{R} \to \mathbb{C}$. Any root of ψ yields an equilibrium solution: if $\psi(y_0) = 0$ for some $y_0 \in J$, set $f(x) = y_0$, $x \in I$, to find

$$f'(x) = 0 = \phi(x)\psi(y_0) = \phi(x)\psi(f(x)),\ \ x \in I.$$

Conversely, assume $\psi(y) \neq 0$ for $y \in J$. We will work backwards: *assume that we already have a function* $f\colon I \to J$ *satisfying* (1.3.4). *What can we divine about* f*?*

Since $f(x) \in J$ for all $x \in I$, $\psi(f(x)) \neq 0$, and so we may divide by $\psi(f(x))$ to find that f must satisfy

$$\frac{f'(x)}{\psi(f(x))} = \phi(x). \tag{1.3.5}$$

The left side looks vaguely like a chain rule-type product, especially if we split up the factors as

$$\frac{f'(x)}{\psi(f(x))} = \left(\frac{1}{\psi(f(x))}\right) f'(x),\ \ x \in I.$$

Indeed, let Ψ be an antiderivative of $1/\psi$ on J, so

$$\Psi'(y) = \frac{1}{\psi(y)}.$$

(Of course, we could be explicit, fix $y_0 \in J$, and define $\Psi(y) := \int_{y_0}^{y} (1/\psi(s))\ ds$.) Then the chain rule gives

$$\partial_x[\Psi(f(x))] = \frac{1}{\psi(f(x))} f'(x).$$

And so we rewrite (1.3.5) to see that f must satisfy

$$\partial_x[\Psi(f(x))] = \phi(x). \tag{1.3.6}$$

Then for some antiderivative[12] Φ of ϕ, we have

$$\Psi(f(x)) = \Phi(x). \tag{1.3.7}$$

[12]More precisely, fix $x_0 \in I$ and integrate both sides of (1.3.6) from x_0 to x to find

$$\Psi(f(x)) = \underbrace{\Psi(f(x_0)) + \int_{x_0}^{x} \phi(s)\ ds}_{\Phi(x)}.$$

This is an **implicit relation** for f; depending on Ψ, we may be able to solve for $f(x)$ explicitly — though there is no guarantee that we can do so uniquely. Such an implicit relation appeared in (1.3.2) in Example 1.3.3 and step $\overset{(3)}{\Longrightarrow}$ of (1.3.3) in Example 1.3.4.

Now, we could reverse the flow of the logic above: assume that $f \colon I \to \mathbb{C}$ is a differentiable function satisfying (1.3.7), with Φ and Ψ having the aforementioned properties. Then, differentiating and rearranging, we would find that f solves (1.3.4). In other words, solving the implicit relation (1.3.7) is a *sufficient* condition for a function to solve the separable ODE (1.3.4). We summarize the results of this section in the next theorem.

1.3.6 Theorem.

Suppose that I, $J \subseteq \mathbb{R}$ and $\phi \in \mathcal{C}(I)$, $\psi \in \mathcal{C}(J)$.

(i) *Suppose there exist $\Psi \in \mathcal{C}^1(J)$ with $\Phi' = \phi$ and $\Psi' = 1/\psi$. If $f \in \mathcal{C}^1(I, J)$ satisfies*

$$\Psi(f(x)) = \Phi(x) + C, \quad x \in I,$$

for some $C \in \mathbb{C}$, then

$$f'(x) = \phi(x)\psi(f(x)), \quad x \in I. \tag{1.3.8}$$

(ii) *If there exists $y_0 \in J$ such that $\psi(y_0) = 0$, then the function $f(x) := y_0$, $x \in I$, satisfies* (1.3.8) *for all $x \in I$.*

1.4. Second-order linear ODEs.

The general form of a second-order linear ODE is

$$\underbrace{a_2(x)f'' + a_1(x)f + a_0(x)}_{\mathcal{A}f} = g(x),$$

where a_2, a_1, a_0, $g \in \mathcal{C}(I)$ for some interval $I \subseteq \mathbb{R}$ and $a_2(x) \neq 0$. There are several good reasons to study second-order ODEs in detail.

1. They are ubiquitous in applications; after all, force = mass × acceleration, and acceleration = ∂_x^2[position].

2. They are just complicated enough to be interesting, but not so complicated that we will drown in notation. Most of the techniques that we will use on second-order problems easily generalize to higher-order linear ODEs, except the notation there becomes more burdensome.

3. We are able to solve certain types of second-order ODEs completely and explicitly. Conversely, unlike our robust results on first-order ODEs, we will often fail to solve other kinds of second-order problems in any explicit way, even though we will know a great deal about the existence, uniqueness, and behavior of their solutions. This heady oscillation between success and failure, between explicit formula and implicit qualitative reckonings, is mathematically thrilling.

1.4.1. Constant-coefficient second-order linear homogeneous ODEs.

A **constant-coefficient** second-order linear ODE has the form

$$a_2 f'' + a_1 f' + a_0 f = g(x), \tag{1.4.1}$$

where a_2, a_1, $a_0 \in \mathbb{C}$ and $a_2 \neq 0$. If $g = 0$, the ODE is **HOMOGENEOUS**; this is simpler, so let us begin with studying

$$a_2 f'' + a_1 f' + a_0 f = 0. \tag{1.4.2}$$

We might look back to first-order equations for inspiration: the solution of

$$a_1 f' + a_0 f = 0 \iff f' + \frac{a_0}{a_1} f = 0$$

is $f(x) = ce^{-(a_0/a_1)x}$ for any $c \in \mathbb{C}$. (Of course, in the first-order case we need to assume $a_1 \neq 0$, which is not necessary in (1.4.2).) There is no apparent algebraic combination A of the coefficients a_2, a_1, a_0 such that $f(x) := e^{-Ax}$ solves (1.4.2), but historically this led to the good idea that we *guess* that (1.4.2) has a solution of the form $f_\lambda(x) = e^{\lambda x}$, where λ is a free parameter whose value we will determine. Since

$$\partial_x^k [e^{\lambda \cdot}](x) = \lambda^k e^{\lambda x},$$

we find that f_λ solves (1.4.2) if and only if

$$(a_2\lambda^2 + a_1\lambda + a_0)e^{\lambda x} = 0 \iff \underbrace{a_2\lambda^2 + a_1\lambda + a_0}_{\chi_{\mathcal{A}}(\lambda)} = 0. \tag{1.4.3}$$

This polynomial $\chi_{\mathcal{A}}(\lambda)$ is the **CHARACTERISTIC POLYNOMIAL** for the second-order differential operator $\mathcal{A}$, and the equation $\chi_{\mathcal{A}}(\lambda) = 0$ is the **CHARACTERISTIC EQUATION** for $\mathcal{A}$. The fundamental theorem of algebra tells us that $\chi_{\mathcal{A}}$ has two (possibly equal) roots λ_1, $\lambda_2 \in \mathbb{C}$, and so $\mathfrak{J}_1(x) := e^{\lambda_1 x}$ and $\mathfrak{J}_2(x) := e^{\lambda_2 x}$ both solve (1.4.2). Since $\mathcal{A}$ is linear, any linear combination $\alpha_1\mathfrak{J}_1 + \alpha_2\mathfrak{J}_2$ of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ also solves (1.4.2).

1.4.1 Example.

Find solutions for each of the following ODEs.

(i) $f'' - f = 0$.

(ii) $f'' + f = 0$.

Solution. (i) The characteristic polynomial for $\mathcal{A}f = f'' - f$ is

$$\chi_{\mathcal{A}}(\lambda) = \lambda^2 - 1,$$

so its roots are ± 1, and therefore solutions are $\mathfrak{J}_1(x) = e^x$, $\mathfrak{J}_2(x) = e^{-x}$.

(ii) The characteristic polynomial for $\mathcal{A}f = f'' + f$ is

$$\chi(\lambda) = \lambda^2 + 1,$$

so its roots are $\pm i$, and therefore one solution set is $\mathfrak{J}_1(x) = e^{ix}$, $\mathfrak{J}_2(x) = e^{-ix}$. Since the coefficients in this ODE are all real, one can check that given a function $f \colon I \subseteq \mathbb{R} \to \mathbb{C}$, we have $\mathcal{A}f = 0$ if and only if $\mathcal{A}[\mathrm{Re}(f)] = \mathcal{A}[\mathrm{Im}(f)] = 0$. By Euler's formula (Theorem A.3.1), we have $\mathrm{Re}[\mathfrak{J}_1(x)] = \cos(x)$, $\mathrm{Im}[\mathfrak{J}_1(x)] = \sin(x)$, $\mathrm{Re}[\mathfrak{J}_2(x)] = \cos(x)$, and $\mathrm{Im}[\mathfrak{J}_2(x)] = -\sin(x)$. And so $\widetilde{\mathfrak{J}}_1(x) = \cos(x)$, $\widetilde{\mathfrak{J}}_2(x) = \sin(x)$ are also solutions. ▲

1.4.2 Example.

Solve the IVP

$$\begin{cases} f'' + 5f' + 4f = 0 \\ f(0) = 0 \\ f'(0) = -3. \end{cases}$$

Solution. The characteristic polynomial is

$$\chi_{\mathcal{A}}(\lambda) = \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4),$$

so solutions are $\mathfrak{J}_1(x) = e^{-x}$ and $\mathfrak{J}_2(x) = e^{-4x}$. Then $c_1\mathfrak{J}_1 + c_2\mathfrak{J}_2$ will be a solution for any c_1, $c_2 \in \mathbb{C}$. Let us see if we can determine c_1 and c_2 to match the initial conditions. First, we need

$$0 = c_1\mathfrak{J}_1(0) + c_2\mathfrak{J}_2(0) = c_1 + c_2 \Longrightarrow c_1 = -c_2.$$

Next, taking derivatives, we need

$$-3 = \partial_x[c_1\mathfrak{J}_1 + c_2\mathfrak{J}_2](0) = -c_1e^0 - 4c_2e^0 = -(c_1 + 4c_2) = -3c_2,$$

hence $c_2 = 1$ and $c_1 = -1$. That is, $y(x) = -e^{-x} + e^{-4x}$ solves the IVP. ▲

The computations in this last example can be cast more abstractly to prove the following existence and uniqueness theorem.

1.4.3 Theorem.

Let $\mathcal{A}f = a_2f'' + a_1f' + a_0f$ with $a_2 \neq 0$. Suppose that the characteristic polynomial $\chi_{\mathcal{A}}(\lambda) = a_2\lambda^2 + a_1\lambda + a_0$ has the distinct roots λ_1, $\lambda_2 \in \mathbb{C}$. Then $\mathfrak{J}_1(x) := e^{\lambda_1 x}$ and $\mathfrak{J}_2(x) := e^{\lambda_2 x}$ satisfy $\mathcal{A}\mathfrak{J}_k = 0$, $k = 1, 2$. Moreover, given $x_0 \in \mathbb{R}$ and y_0, $y_1 \in \mathbb{C}$, there exist unique c_1, $c_2 \in \mathbb{C}$ such that putting $f(x) = c_1\mathfrak{J}_1(x) + c_2\mathfrak{J}_2(x)$ solves the IVP

$$\begin{cases} \mathcal{A}f = 0 \\ f(x_0) = y_0 \\ f'(x_0) = y_1. \end{cases} \tag{1.4.4}$$

Note what this theorem does and does not say.

- It *does* allow us to construct a solution to $\mathcal{A}f = 0$ satisfying an[13] initial condition.

- It *does* give uniqueness in the sense that no other linear combination of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ will satisfy both the ODE and the initial conditions.

- But it *does not* give uniqueness in the sense that if $g \in \mathcal{C}^2(\mathbb{R})$ also satisfies (1.4.4), then $f(x) = g(x)$ for all x.

Now, here are two natural questions for us to consider.

[13]Technically, there are two conditions: $f(x_0) = y_0$ and $f'(x_0) = y_1$. But the technical parlance is to refer to these two requirements as *one* initial condition.

1. The characteristic polynomials in Examples 1.4.1 and 1.4.2 have two distinct roots. What happens if the characteristic polynomial has one repeated root? For example, this is the case for $\mathcal{A}f = f''-2f'+f$, as then the characteristic polynomial is $\chi_{\mathcal{A}}(\lambda) = (\lambda-1)^2$. Here, the procedure of Example 1.4.2 does not produce enough "free parameters" for a linear combination that would satisfy the initial conditions, since we only have the one exponential solution $\mathfrak{J}_1(x) = e^x$.

2. How do we guarantee uniqueness of solutions? Since our second-order differential operators are linear, any scalar multiple of an existing solution is also a solution, so we expect that we must at least impose initial conditions, as in Theorem 1.4.3.

The first question is relatively straightforward to address, and we do it here. The second is more involved, and the theory becomes no more complicated if we assume that the ODE has nonconstant coefficients, so we defer it momentarily.

So, consider the operator

$$\mathcal{A}f = f'' + a_1 f' + a_0 f$$

such that the characteristic polynomial

$$\chi_{\mathcal{A}}(\lambda) = \lambda^2 + a_1\lambda + a_0$$

has the repeated root $\lambda = \lambda_1$. Then $\chi_{\mathcal{A}}(\lambda) = (\lambda - \lambda_1)^2$ as well, by the fundamental theorem of algebra (see Theorem 3.8.6). Putting $f_1(x) = e^{\lambda_1 x}$ implies $\mathcal{A}f_1 = 0$ by all the work above. We want a second solution, preferably something more interesting than a scalar multiple of f_1.

Let us write the characteristic polynomial in two ways

$$\lambda^2 + a_1\lambda + a_0 = (\lambda - \lambda_1)^2 = \lambda^2 - 2\lambda_1\lambda + \lambda_1^2.$$

Equating coefficients in the polynomials, we have

$$a_1 = -2\lambda_1 \quad \text{and} \quad a_0 = \lambda_1^2.$$

This will help us "factor" $\mathcal{A}$:

$$\begin{aligned}
\mathcal{A}f &= f'' + a_1 f' + a_0 f \\
&= f'' - 2\lambda_1 f' + \lambda_1^2 f \\
&= (f'' - \lambda_1 f') - (\lambda_1 f' - \lambda^2 f) \\
&= \partial_x[f' - \lambda_1 f] - \lambda_1[f' - \lambda_1 f] \\
&= (\partial_x - \lambda_1)[f' - \lambda_1 f] \\
&= (\partial_x - \lambda_1)(\partial_x - \lambda_1)f \\
&= (\partial_x - \lambda_1)^2[f].
\end{aligned}$$

Formally, then, $\mathcal{A} = \chi_{\mathcal{A}}(\partial_x)$!

If $(\partial_x - \lambda_1)[f] = 0$, then certainly $\mathcal{A}f = (\partial_x - \lambda_1)^2[f] = 0$. This is what happens with $f_1(x) = e^{\lambda_1 x}$, which satisfies $(\partial_x - \lambda_1)[f_1] = 0$. (We knew this back in Example 1.2.2.) Now we cross an intellectual Rubicon: if we have a function f_2 that satisfies $(\partial_x - \lambda_1)[f_2] = f_1$, then[14]

$$\mathcal{A}f_2 = (\partial_x - \lambda_1)^2[f_2] = (\partial_x - \lambda_1)(\partial_x - \lambda_1)[f_2] = (\partial_x - \lambda_1)[f_1] = 0.$$

[14] In linear-algebraic terms, we are looking for a generalized eigenvector f_2 for 0 as an eigenvalue of $\mathcal{A}$, which is to say that we are trying to construct a Jordan chain (f_1, f_2) for 0; see Appendix C.6.

So how can we find such an f_2? We want to solve

$$(\partial_x - \lambda_1)[f] = f_1(x) = e^{\lambda_1 x}.$$

But this is simply a linear first-order ODE, and from Theorem 1.2.4 we know that any solution has the form

$$f(x) = Ce^{\lambda_1 x} + e^{\lambda_1 x}\int_0^x e^{-\lambda_1 \xi}e^{\lambda_1 \xi}\, d\xi = Ce^{\lambda_1 x} + xe^{\lambda_1 x} = Cf_1(x) + xe^{\lambda_1 x}.$$

The first term is redundant, since $(\partial_x - \lambda_1)[f_1] = 0$. So, we set $f_2(x) = xe^{\lambda_1 x}$ to conclude that $(\partial_x - \lambda_1)[f_2] = f_1$ and $\mathcal{A}f_2 = 0$.

1.4.4 Example.

Solve the IVP

$$\begin{cases} f'' - 4f' + 4f = 0 \\ f(1) = 1 \\ f'(1) = 1. \end{cases}$$

Solution. The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0 \iff (\lambda - 2)^2 = 0 \iff \lambda = 2,$$

so two solutions are $\mathfrak{J}_1(x) = e^{2x}$ and $\mathfrak{J}_2(x) = xe^{2x}$. As before, we will find constants c_1 and c_2 such that $f(x) = c_1e^{2x} + c_2xe^{2x}$ solves the IVP. We differentiate:

$$f'(x) = 2c_1e^{2x} + c_2e^{2x} + 2c_2xe^{2x} = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x}.$$

So c_1 and c_2 must solve

$$\begin{cases} 1 = f(1) = c_1e^2 + c_2e^2 = (c_1 + c_2)e^2 \\ 1 = f'(1) = (2c_1 + c_2)e^2 + 2c_2e^2 = (2c_1 + 3c_2)e^2. \end{cases}$$

That is,

$$\begin{cases} c_1 + c_2 = e^{-2} \\ 2c_1 + 3c_2 = e^{-2}. \end{cases}$$

The solution to this system is $c_1 = 2e^{-2}$ and $c_2 = -e^{-2}$, so a solution to the IVP is

$$f(x) = 2e^{-2}e^{2x} - e^{-2}xe^{2x} = 2e^{2x-2} - xe^{2x-2}. \qquad \blacktriangle$$

Here is the analogue of Theorem 1.4.3 for the case of repeated real roots.

1.4.5 Theorem.

Let $\mathcal{A}f = a_2f'' + a_1f' + a_0f$ *with* $a_2 \neq 0$. *Suppose that the characteristic polynomial* $\chi_{\mathcal{A}}(\lambda) = a_2\lambda^2 + a_1\lambda + a_0$ *has the repeated root* $\lambda_1 \in \mathbb{C}$. *Then the functions* $\mathfrak{J}_1(x) = e^{\lambda_1 x}$ *and* $\mathfrak{J}_2(x) = xe^{\lambda_1 x}$ *solve* $\mathcal{A}\mathfrak{J}_k = 0$. *Moreover, given* $x_0 \in \mathbb{R}$ *and* y_0, $y_1 \in \mathbb{C}$, *there exist unique* c_1, $c_2 \in \mathbb{C}$ *such that* $f = c_1\mathfrak{J}_1 + c_2\mathfrak{J}_2$ *solves the IVP* (1.4.4).

1.4.6 Method: solve $af'' + bf' + cf = 0$ $(a \neq 0)$

1. Solve the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

2. If the characteristic equation has distinct complex roots $\lambda_1 \neq \lambda_2$, two solutions are

$$\mathfrak{z}_1(x) = e^{\lambda_1 x} \quad \text{and} \quad \mathfrak{z}_2(x) = e^{\lambda_2 x}.$$

3. If the characteristic equation has a repeated real root λ_1, two solutions are

$$\mathfrak{z}_1(x) = e^{\lambda_1 x} \quad \text{and} \quad \mathfrak{z}_2(x) = xe^{\lambda_1 x}.$$

4. If the coefficients a, b, c are all real, and the characteristic equation has distinct nonreal roots, then necessarily the roots are $\alpha \pm \beta i$ with α, $\beta \in \mathbb{R}$ and $\beta \neq 0$. Two solutions are also

$$\widetilde{\mathfrak{z}}_1(x) = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad \widetilde{\mathfrak{z}}_2(x) = e^{\alpha x}\sin(\beta x).$$

1.4.2. Interlude: uniqueness for the homogeneous IVP.

Fix $I \subseteq \mathbb{R}$, $x_0 \in I$, y_0, $y_1 \in \mathbb{C}$, and a_1, $a_0 \in \mathcal{C}(I)$. Write $\mathcal{A}f := f'' + a_1(x)f' + a_0(x)f$. Our goal is to study the existence and uniqueness of solutions to

$$\begin{cases} \mathcal{A}f = g \\ f(x_0) = y_0 \\ f'(x_0) = y_1 \end{cases} \tag{1.4.5}$$

for a given $g \in \mathcal{C}(I)$. We do this in stages. First, in this section, we prove existence and uniqueness in the special homogeneous case $g(x) = 0$, $y_0 = y_1 = 0$.

Why would we study this case first? Assume that we have two solutions to the nonhomogeneous problem (1.4.5). Call the solutions f_1 and f_2. Let us try to show $f_1 - f_2 = 0$. What do we know about f_1 and f_2? Only that each satisfies (1.4.5). What does this tell us about their difference $h := f_1 - f_2$? That h satisfies the *homogeneous* IVP

$$\begin{cases} \mathcal{A}h = 0 \\ h(x_0) = 0 \\ h'(x_0) = 0. \end{cases} \tag{1.4.6}$$

Now, clearly a solution to (1.4.6) is $h = 0$. On the other hand, if we can prove that the *only* solution to (1.4.6) is $h = 0$, then we will have our desired equality $f_1 = f_2$. This is indeed the case, although we will not prove it (a proof can be found in [8]).

1.4.7 Lemma (Uniqueness of solutions for zero initial conditions).

If $f \in \mathcal{C}^2(I)$ solves

$$\begin{cases} \mathcal{A}f = 0 \\ f(x_0) = 0 \\ f'(x_0) = 0 \end{cases}$$

for some $x_0 \in I$, then $f(x) = 0$ for all $x \in I$.

1.4.3. Linear independence and the Wronskian.

We need two auxiliary concepts to enact our proofs of existence and uniqueness for second-order IVPs. The first is familiar from linear algebra (Definition C.1.3: a set of functions $\{f_1, \dots, f_n\} \subseteq \mathcal{C}^r(I)$ is **LINEARLY INDEPENDENT ON** I if whenever $\sum_{k=1}^n \alpha_k f_k(x) = 0$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and all $x \in I$, then $\alpha_1 = \cdots = \alpha_n = 0$.

1.4.8 Example.

Show that each set of functions is linearly independent on any interval $I \subseteq \mathbb{R}$.

(i) $f_1(x) = e^{\lambda_1 x}$, $f_2(x) = e^{\lambda_2 x}$, *where* $\lambda_1 \neq \lambda_2$

(ii) $f_1(x) = e^{\lambda x}$, $f_2(x) = xe^{\lambda x}$ *for any* $\lambda \in \mathbb{C}$

Solution. (i) Suppose $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$\alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} = 0 \tag{1.4.7}$$

for all $x \in I$. We want to show that $\alpha_1 = \alpha_2 = 0$.

There are a number of ways to do this. For example, we could rewrite (1.4.7) as

$$\alpha_1 e^{(\lambda_1 - \lambda_2)x} = -\alpha_2.$$

Since $\lambda_1 \neq \lambda_2$, the left side will not be constant, unless $\alpha_1 = 0$. But then $\alpha_2 = 0$, too.

This method relied on having only two functions involved. A method that generalizes better to show the linear independence of arbitrary many exponentials with different powers (cf. Example C.1.4) is the following. Differentiate both sides of (1.4.7) to find

$$\alpha_1 \lambda_1 e^{\lambda_1 x} + \alpha_2 \lambda_2 e^{\lambda_2 x} = 0, \ x \in I.$$

Fix some $x_0 \in I$, so that we have a system of linear equations for α_1 and α_2:

$$\begin{cases} \alpha_1 e^{\lambda_1 x_0} + \alpha_2 e^{\lambda_2 x_0} & = 0 \\ \alpha_1 \lambda_1 e^{\lambda_1 x_0} + \alpha_2 \lambda_2 e^{\lambda_2 x_0} & = 0. \end{cases}$$

Rewrite this system as a matrix-vector equation:

$$\begin{bmatrix} e^{\lambda_1 x_0} & e^{\lambda_2 x_0} \\ \lambda_1 e^{\lambda_1 x_0} & \lambda_2 e^{\lambda_2 x_0} \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of this matrix is

$$e^{(\lambda_1 + \lambda_2)x_0}(\lambda_2 - \lambda_1) \neq 0,$$

so the matrix is invertible, and therefore $\alpha_1 = \alpha_2 = 0$.

(ii) The proof is similar to the system of equations constructed above and left as an exercise. The matrix methods for these proofs easily generalize to show that more complicated families of functions are linearly independent, e.g., differentiating m times, one could construct an $m \times m$ system to show that

$$e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \dots, x^{m-1} e^{\lambda x}, x^m e^{\lambda x}$$

are linearly independent on any $I \subseteq \mathbb{R}$. ▲

The second concept is a special determinant.

1.4.9 Definition.

The **Wronskian determinant** *of the functions* $f_1, f_2 \in \mathcal{C}^1(I)$ *is*

$$\mathcal{W}[f_1, f_2](x) := \det\left(\begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix}\right) = f_1(x)f_2'(x) - f_1'(x)f_2(x). \tag{1.4.8}$$

For the remainder of this section, fix an interval $I \subseteq \mathbb{R}$, functions $a_1, a_0 \in \mathcal{C}(I)$, and set

$$\mathcal{A}f := f'' + a_1(x)f' + a_0(x)f. \tag{1.4.9}$$

1.4.10 Lemma (Abel).

Suppose that $f_1, f_2 \in \mathcal{C}^2(I)$ *satisfy* $\mathcal{A}f_k = 0$, $k = 1, 2$. *Then their Wronskian satisfies the first-order ODE*

$$\partial_x \mathcal{W}[f_1, f_2] + a_1(x)\mathcal{W}[f_1, f_2] = 0. \tag{1.4.10}$$

In particular, one of the following two (mutually exclusive) alternatives holds.

(i) $\mathcal{W}[f_1, f_2](x) \neq 0$ *for all* $x \in I$.

(ii) $\mathcal{W}[f_1, f_2](x) = 0$ *for all* $x \in I$.

Proof. The proof that $\mathcal{W}[f_1, f_2]$ satisfies (1.4.10) is a direct computation using the hypothesis (1.4.9) and left to the exercises. The theory of linear first-order ODEs then implies

$$\mathcal{W}[f_1, f_2](x) = \mathcal{W}[f_1, f_2](x_0)e^{-\int_{x_0}^{x} a_1(\xi)\ d\xi}$$

for any $x_0 \in I$, and so either $\mathcal{W}[f_1, f_2]$ is identically zero on I or nonzero[15] on all of I. ∎

1.4.11 Theorem.

The following are equivalent when $f_1, f_2 \in \mathcal{C}^2(I)$ *satisfy* $\mathcal{A}f_k = 0$, $k = 1, 2$*:*

(i) f_1 *and* f_2 *are linearly independent on* I.

(ii) $\mathcal{W}[f_1, f_2](x) \neq 0$ *for all* $x \in I$.

(iii) *There exists* $x \in I$ *such that* $\mathcal{W}[f_1, f_2](x) \neq 0$.

Proof. (i) $\Longrightarrow$ (ii) Suppose, by way of contradiction, that $\mathcal{W}[f_1, f_2](x_0) = 0$ for some $x_0 \in I$. Then the columns of the matrix

$$\begin{bmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{bmatrix}$$

are linearly dependent, so there exists $\alpha \in \mathbb{R}$ such that

$$\begin{pmatrix} f_1(x_0) \\ f_1'(x_0) \end{pmatrix} = \alpha \begin{pmatrix} f_2(x_0) \\ f_2'(x_0) \end{pmatrix}.$$

[15] We can say more if $a_1(x)$ is real for all x: then either $\mathcal{W}[f_1, f_2](x)$ is positive for all x, or $\mathcal{W}[f_1, f_2](x) = 0$ for all x, or $\mathcal{W}[f_1, f_2](x) < 0$ for all x.

Then the function $f_1 - \alpha f_2$ solves the IVP

$$\begin{cases} \mathcal{A}[f_1 - \alpha f_2] = 0 \\ (f_1 - \alpha f_2)(x_0) = 0 \\ \partial_x[f_1 - \alpha f_2](x_0) = 0, \end{cases}$$

and so, by Lemma 1.4.7, $f_1 - \alpha f_2 = 0$. This contradicts the linear independence of f_1 and f_2.

(ii) $\Longrightarrow$ (iii) This is obvious.

(iii) $\Longrightarrow$ (i) Suppose that α, $\beta \in \mathbb{R}$ with

$$\alpha f_1(x) + \beta f_2(x) = 0 \tag{1.4.11}$$

for all $x \in I$. We want to show that $\alpha = \beta = 0$. Since (1.4.11) holds over the interval I, we can differentiate both sides to obtain

$$\alpha f_1'(x) + \beta f_2'(x) = 0 \tag{1.4.12}$$

for all $x \in I$. These two equalities rearrange to

$$\alpha \begin{pmatrix} f_1(x) \\ f_1'(x) \end{pmatrix} = -\beta \begin{pmatrix} f_2(x) \\ f_2'(x) \end{pmatrix}. \tag{1.4.13}$$

Suppose that one of α, β is nonzero. Then the columns of the matrix

$$\mathcal{M}(x) := \begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix}$$

are linearly dependent. Taking $x = x_0$, we have $\mathcal{W}[f_1, f_2](x_0) = \det(\mathcal{M}(x_0)) = 0$, a contradiction. Consequently, we must have $\alpha = \beta = 0$. ■

The condition in the preceding proposition that f_1 and f_2 solve $\mathcal{A}f = 0$ is essential; there are linearly dependent functions f_1 and f_2 whose Wronskian vanishes on an interval, but these functions do not (indeed, cannot) satisfy any ODE $\mathcal{A}f = 0$. An example is constructed in the exercises.

1.4.4. Existence and uniqueness theory for second-order homogeneous ODEs.

Now we return to the existence and uniqueness problem. In Lemma 1.4.7 we studied the second-order linear homogeneous IVP with *zero* initial conditions. We will still consider the homogeneous ODE but now allow for *nonzero* initial conditions. Once again, we fix an interval $I \subseteq \mathbb{R}$ and functions a_1, $a_0 \in \mathcal{C}(I)$ and set $\mathcal{A}f := f'' + a_1(x)f' + a_0(x)f$.

We begin with an existence lemma.

1.4.12 Lemma (Existence of fundamental solution set).

There exist linearly independent functions $\mathfrak{J}_1$, $\mathfrak{J}_2 \in \mathcal{C}(I)$ such that $\mathcal{A}\mathfrak{J}_k = 0$, $k = 1, 2$. In particular, such functions satisfy $\mathcal{W}[\mathfrak{J}_1, \mathfrak{J}_2](x) \neq 0$ for all $x \in I$.

We will not prove this lemma in full, but we already know it to be true when a_1 and a_0 are constant. (See [8] again for the general proof.) In that case, we can summon up the functions $\mathfrak{z}_1$ and $\mathfrak{z}_2$ from either Theorem 1.4.3 or Theorem 1.4.5, depending on the roots of the characteristic equation for $\mathcal{A}$.

Such a pair of linearly independent functions $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{C}(I)$ satisfying $\mathcal{A}\mathfrak{z}_1 = \mathcal{A}\mathfrak{z}_2 = 0$ is called a **FUNDAMENTAL SOLUTION SET FOR $\mathcal{A}$ ON I**. Why such a set is "fundamental" will be clear shortly.

1.4.13 Theorem.

Given $x_0 \in I$ and y_0, $y_1 \in \mathbb{C}$, there exists a unique function $f \in \mathcal{C}^2(I)$ such that

$$\begin{cases} \mathcal{A}f = 0 \\ f(x_0) = y_0 \\ f'(x_0) = y_1. \end{cases} \tag{1.4.14}$$

Specifically, this solution has the form $f(x) = c_1\mathfrak{z}_1(x) + c_2\mathfrak{z}_2(x)$ for (unique) c_1, $c_2 \in \mathbb{C}$, where $\{\mathfrak{z}_1, \mathfrak{z}_2\}$ is a fundamental solution set for $\mathcal{A}$.

Proof. First we prove existence. Let us look for a solution of the IVP in the form $f(x) = c_1\mathfrak{z}_1(x) + c_2\mathfrak{z}_2(x)$ for some c_1, $c_2 \in \mathbb{C}$. Clearly any such f satisfies $\mathcal{A}f = 0$ by linearity of $\mathcal{A}$. If we can choose c_1 and c_2 to meet the initial conditions, then we have a solution to the IVP. To do so, we will construct a linear system of equations[16] that c_1 and c_2 must satisfy. Fix any $x_0 \in I$. Then

$$y_0 = f(x_0) = c_1\mathfrak{z}_1(x_0) + c_2\mathfrak{z}_2(x_0) \quad \text{and} \quad y_1 = f'(x_0) = c_1\mathfrak{z}_1'(x_0) + c_2\mathfrak{z}_2'(x_0).$$

That is,

$$\begin{bmatrix} \mathfrak{z}_1(x_0) & \mathfrak{z}_2(x_0) \\ \mathfrak{z}_1'(x_0) & \mathfrak{z}_2'(x_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

The determinant of this matrix is $\mathcal{W}[\mathfrak{z}_1, \mathfrak{z}_2](x_0) \neq 0$. Since $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are linearly independent, the Wronskian is nonzero, the matrix is invertible, and therefore we have (uniquely[17]) determined c_1 and c_2.

Now we prove uniqueness[18] of this solution. Suppose $g \in \mathcal{C}^2(I)$ also solves

$$\begin{cases} \mathcal{A}g = 0 \\ g(x_0) = y_0 \\ g'(x_0) = y_1. \end{cases}$$

Now let $h = f - g$. Then h solves

$$\begin{cases} \mathcal{A}h = 0 \\ h(x_0) = 0 \\ h'(x_0) = 0, \end{cases}$$

[16] This is the same system that appeared in Examples 1.4.2 and 1.4.4. The abstract argument here is the same that would appear in the (omitted) proofs of Theorems 1.4.3 and 1.4.5. But now we can prove uniqueness, not merely of c_1 and c_2, but of the solution to (1.4.14), regardless of the solution's form.

[17] We already know that c_1 and c_2 are unique because $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are linearly independent. That is, if we have a function f that satisfies both $f(x) = c_1\mathfrak{z}_1(x) + c_2\mathfrak{z}_2(x)$ and $f(x) = d_1\mathfrak{z}_1(x) + d_2\mathfrak{z}_2(x)$ for all $x \in I$, then $c_1 = d_1$ and $c_2 = d_2$.

[18] Above we proved that a solution of the form $f(x) = c_1\mathfrak{z}_1(x) + c_2\mathfrak{z}_2(x)$ exists. Linear independence assures us that we cannot write $f = d_1\mathfrak{z}_1 + d_2\mathfrak{z}_2$ for $d_1 \neq c_1$ and $d_2 \neq c_2$. But linear independence *does not* preclude the possibility that we could have another solution to this IVP *not* given by a linear combination of $\mathfrak{z}_1$ and $\mathfrak{z}_2$.

and so by Lemma 1.4.7, $h = 0$. That is, $f(x) = g(x) = c_1\mathfrak{J}_1(x) + c_2\mathfrak{J}_2(x)$ for all $x \in I$. ■

Observe that Theorem 1.4.13 generalizes Lemma 1.4.7 by allowing the initial conditions to be nonzero. To be clear about the flow of our logic, however, we *needed* Lemma 1.4.7 first in order to prove this theorem.

1.4.14 Linear algebraic viewpoint: second-order kernels

1. Lemma 1.4.12 implies that $\ker(\mathcal{A})$ is *at least* two-dimensional when we consider $\mathcal{A}$ as an operator in $\mathcal{X} = \mathcal{C}(I)$ with domain $\mathfrak{D}(\mathcal{A}) = \mathcal{C}^2(I)$.

2. The constructive proof of Theorem 1.4.13 proves that $\ker(\mathcal{A})$ is *exactly* two-dimensional when $\mathcal{A}$ has domain $\mathcal{C}^2(I)$. For, if $\mathcal{A}f = 0$, then we may fix $x_0 \in I$ and set $y_0 = f(x_0)$ and $y_1 = f'(x_0)$ to find that f satisfies an IVP of the form (1.4.14). The proof of Theorem 1.4.13 then produces c_1, $c_2 \in \mathbb{C}$ such that $f = c_1\mathfrak{J}_1 + c_2\mathfrak{J}_2$. And so

$$\ker(\mathcal{A}) = \operatorname{span}(\{\mathfrak{J}_1, \mathfrak{J}_2\}). \tag{1.4.15}$$

This is what is "fundamental" about the fundamental solution set $\{\mathfrak{J}_1, \mathfrak{J}_2\}$.

3. Lemma 1.4.7, however, says that $\ker(\mathcal{A})$ is trivial *if* we restrict $\mathcal{A}$ to a subspace of the form

$$\mathfrak{D}_{x_0}(\mathcal{A}) := \{f \in \mathcal{C}^2(I) \mid f(x_0) = f'(x_0) = 0\}$$

for some $x_0 \in I$. Note, however, that the set

$$\{f \in \mathcal{C}^2(I) \mid f(x_0) = y_0,\ f'(x_0) = y_1\}$$

is *not* a subspace of $\mathcal{C}^2(I)$ unless both $y_0 = 0$ and $y_1 = 0$, and so it does not make sense to talk about the kernel of $\mathcal{A}$ on this *set*.

1.4.5. Second-order nonhomogeneous ODEs: variation of parameters.

The culmination of our second-order theory will be the solution of the **NONHOMOGENEOUS** problem

$$\underbrace{f'' + a_1(x)f' + a_0(x)f}_{\mathcal{A}f} = g \tag{1.4.16}$$

for a given $g \in \mathcal{C}(I)$. As in the preceding section, we will develop our theory with a_1 and a_0 as arbitrary continuous functions on I, but in practice we will mostly compute examples with a_1 and a_0 constant.

Now, unless we impose initial conditions, we should not expect this solution to be unique. Indeed (supposing, of course, a_1, $a_0 \in \mathcal{C}(I)$), recall that Lemma 1.4.12 and (1.4.15) give $\mathfrak{J}_1$, $\mathfrak{J}_2 \in \mathcal{C}^2(I)$ such that if $\mathcal{A}h = 0$, then $h = c_1\mathfrak{J}_1 + c_2\mathfrak{J}_2$ for some c_1, $c_2 \in \mathbb{C}$. Then if we have a solution $f \in \mathcal{C}^2$ to (1.4.16), we can set $\widetilde{f} := f + d_1\mathfrak{J}_1 + d_2\mathfrak{J}_2$ for some d_1, $d_2 \in \mathbb{C}$ and calculate

$$\mathcal{A}\widetilde{f} = \mathcal{A}f + d_1\mathcal{A}\mathfrak{J}_1 + d_2\mathcal{A}\mathfrak{J}_2 = g.$$

Conversely, if f_1, $f_2 \in \mathcal{C}^2(I)$ with $\mathcal{A}f_1 = \mathcal{A}f_2 = g$, then

$$\mathcal{A}(f_1 - f_2) = \mathcal{A}f_1 - \mathcal{A}f_2 = g - g = 0,$$

hence $f_1 - f_2 = c_1\mathfrak{z}_1 + c_2\mathfrak{z}_2$ for some $c_1, c_2 \in \mathbb{C}$. In other words, if we know one "particular solution" to (1.4.16), then we know them all.

We formalize this in a lemma and definition.

1.4.15 Lemma.

Suppose that $f \in \mathcal{C}^2(I)$ with $\mathcal{A}f = g$. Then if $\widetilde{f} \in \mathcal{C}^2(I)$ also satisfies $\mathcal{A}\widetilde{f} = g$, there exist $c_1, c_2 \in \mathbb{C}$ such that $\widetilde{f} = f + c_1\mathfrak{z}_1 + c_2\mathfrak{z}_2$.

1.4.16 Definition.

Given a particular solution f_0 to $\mathcal{A}f = g$ and a fundamental solution set $\{\mathfrak{z}_1, \mathfrak{z}_2\}$ to $\mathcal{A}f = 0$, the expression

$$f = c_1\mathfrak{z}_1 + c_2\mathfrak{z}_2 + f_0, \ c_1, c_2 \in \mathbb{C},$$

is the **GENERAL SOLUTION** *to $\mathcal{A}f = g$.*

Remarkably, if we know a fundamental solution set for the homogeneous problem, there is a general formula for the solution to the nonhomogeneous problem that depends only on the nonhomogeneity. To motivate the development of this formula, let us review the more transparent situation for a linear *first*-order nonhomogeneous ODE.

Let $p, g \in \mathcal{C}(I)$, let P be an antiderivative of p on I, and fix $x_0 \in I$. Recall from Theorem 1.2.4 that all solutions to $f' + p(x)f = g(x)$ have the form

$$f(x) = ce^{-P(x)} + e^{-P(x)} \int_{x_0}^{x} e^{P(\xi)} g(\xi) \, d\xi,$$

for an appropriate $c \in \mathbb{C}$. Taking $g = 0$, we recall that any solution to the homogeneous problem $f' + p(x)f = 0$ has the form $f(x) = c\nu(x)$, where $\nu(x) := e^{-P(x)}$. If $g \neq 0$, we abbreviate $\eta(x) := \int_{x_0}^{x} e^{P(\xi)} g(\xi) \, d\xi$, so that we can write any solution to the nonhomogeneous problem $f' + p(x)f = g(x)$ as

$$f(x) = c\nu(x) + \nu(x)\eta(x). \tag{1.4.17}$$

In words, *any solution to the nonhomogeneous problem is the sum of a scalar multiple of a solution to the homogeneous problem (ν) and a variable-coefficient multiple of a solution to the homogeneous problem* ($\nu\eta$). In particular, taking $c = 0$ in (1.4.17), we see that $f(x) := \nu(x)\eta(x)$ is a solution of the nonhomogeneous problem. And this particular solution is a variable-coefficient multiple of a solution ν to the homogeneous problem!

The natural outgrowth of these remarks on nonhomogeneous first-order problems is that one guesses that (1.4.16) has a particular solution of the form

$$f(x) = c_1(x)\mathfrak{z}_1(x) + c_2(x)\mathfrak{z}_2(x),$$

where c_1 and c_2 are nonconstant functions. One then determines formulas for c_1 and c_2, with some latitude — after all, we just need *one* solution of (1.4.16). The precise derivation of these formulas is not very enlightening (but see [6, 23]), so we just state the result.

1.4.17 Theorem (Variation of parameters).

Let $\{\mathfrak{J}_1, \mathfrak{J}_2\}$ be a fundamental solution set for $\mathcal{A}$ on $I \subseteq \mathbb{R}$, and let $g \in \mathcal{C}(I)$. Let $\mathcal{V}_1$, $\mathcal{V}_2 \in \mathcal{C}^1(I)$ satisfy[19]

$$\mathcal{V}_1'(x) = -\frac{\mathfrak{J}_2(x)g(x)}{\mathcal{W}[\mathfrak{J}_1, \mathfrak{J}_2](x)} \quad \text{and} \quad \mathcal{V}_2'(x) = \frac{\mathfrak{J}_1(x)g(x)}{\mathcal{W}[\mathfrak{J}_1, \mathfrak{J}_2](x)}. \tag{1.4.18}$$

Then putting

$$f(x) = \mathcal{V}_1(x)\mathfrak{J}_1(x) + \mathcal{V}_2(x)\mathfrak{J}_2(x) \tag{1.4.19}$$

solves $\mathcal{A}f = g$.

One can prove this theorem directly by calculating f' and f'' using the relations (1.4.18) and the definition of the Wronskian. The formula in (1.4.19) is clearly a variable-coefficient linear combination of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. It is important to be consistent with what we label as $\mathfrak{J}_1$ and $\mathfrak{J}_2$. The exercises will explore what happens if one selects a different fundamental solution set $\{\widetilde{\mathfrak{J}}_1, \widetilde{\mathfrak{J}}_2\}$.

1.4.18 Example.

(i) *Determine a fundamental solution set for $\mathcal{A} = \partial_x^2 + 3\partial_x + 2$ on $\mathbb{R}$.*

(ii) *Find all functions $f \in \mathcal{C}^2(\mathbb{R})$ such that $\mathcal{A}f = 4x$.*

(iii) *Solve the initial value problem*

$$\begin{cases} f'' + 3f' + 2f = 4x \\ f(0) = 4 \\ f'(0) = 2. \end{cases}$$

Solution. The characteristic polynomial for $\mathcal{A}$ is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2),$$

so a fundamental solution set on $\mathbb{R}$ is

$$\mathfrak{J}_1(x) = e^{-x}, \qquad \mathfrak{J}_2(x) = e^{-2x}.$$

The Wronskian is

$$\mathcal{W}[\mathfrak{J}_1, \mathfrak{J}_2](x) = \det\left(\begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix}\right) = -2e^{-3x} + e^{-3x} = -e^{-3x}.$$

Now we calculate antiderivatives:

$$\int \frac{\mathfrak{J}_1(x)(4x)}{\mathcal{W}[\mathfrak{J}_1, \mathfrak{J}_2](x)}\, dx = \int \frac{4xe^{-x}}{-e^{-3x}}\, dx = -4\int xe^{2x}\, dx = -(2x-1)e^{2x}. \tag{1.4.20}$$

[19] The existence of these functions $\mathcal{V}_1$ and $\mathcal{V}_2$ is predicated on the continuity of the right sides of the equations in (1.4.18), which in turn hinges on the continuity of g. This is because the fundamental theorem of calculus guarantees that all continuous functions have antiderivatives, whereas a discontinuous function may not have an antiderivative. Fourier and Laplace transform methods may be useful in treating nonhomogeneous ODEs with discontinuous forcing functions. For a familiar problem, see Example 2.5.20.

and

$$\int \frac{\mathfrak{Z}_2(x)(4x)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](x)}\, dx = \int \frac{4xe^{-2x}}{-e^{-3x}}\, dx = -4\int xe^x\, dx = -4(x-1)e^x \tag{1.4.21}$$

(When taking antiderivatives for variation of parameters, we will commit the cardinal sin of omitting the constant of integration. Including them would just add a scalar multiple of $\mathfrak{Z}_1$ or $\mathfrak{Z}_2$, which is irrelevant for a particular solution.)

Thus variation of parameters implies that

$$f(x) = -\mathfrak{Z}_1(x)[-4(x-1)e^x] + \mathfrak{Z}_2(x)[-(2x-1)e^{2x}] = 4e^{-x}(x-1)e^x - e^{-2x}(2x-1)e^{2x}$$

$$= 4(x-1) - (2x-1) = 2x - 3 \tag{1.4.22}$$

solves $\mathcal{A}f = 4x$.

Hence the general solution to the nonhomogeneous problem $\mathcal{A}f = 4x$ is

$$f(x) = \alpha_1\mathfrak{Z}_1(x) + \alpha_2\mathfrak{Z}_2(x) + 2x - 3 = \alpha_1 e^{-x} + \alpha_2 e^{-2x} + 2x - 3. \tag{1.4.23}$$

To choose α_1 and α_2 to satisfy the initial conditions, we differentiate to find

$$f'(x) = -\alpha_1 e^{-x} - 2\alpha_2 e^{-2x} + 2,$$

and thus we require

$$\begin{cases} 4 = f(0) = \alpha_1 + \alpha_2 - 3 \\ 2 = f'(0) = -\alpha_1 - 2\alpha_2 + 2. \end{cases}$$

That is,

$$\alpha_1 + 2\alpha_2 = 0 \implies \alpha_1 = -2\alpha_2 \implies -2\alpha_2 + \alpha_2 = 7 \implies \alpha_2 = -7 \implies \alpha_1 = 14,$$

and so our particular solution is

$$f(x) = 14e^{-x} - 7e^{-2x} + 2x - 3.$$

▲

1.4.19 Method: solve $f'' + a_1(x)f' + a_0(x)f = g(x)$

1. Assume that a fundamental solution set $\{\mathfrak{Z}_1, \mathfrak{Z}_2\}$ is known. (If a_1 and a_0 are constant, use Method 1.4.6.)

2. Calculate the Wronskian determinant

$$\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](x) = \det\left(\begin{bmatrix} \mathfrak{Z}_1(x) & \mathfrak{Z}_2(x) \\ \mathfrak{Z}_1'(x) & \mathfrak{Z}_2'(x) \end{bmatrix}\right).$$

3. Antidifferentiate

$$\mathcal{V}_1 := -\int \frac{\mathfrak{Z}_2(x)g(x)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](x)}\, dx \quad \text{and} \quad \mathcal{V}_2 := \int \frac{\mathfrak{Z}_1(x)g(x)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](x)}\, dx.$$

Do not include constants of integration.

4. A particular solution is

$$f(x) = \mathcal{V}_1(x)\mathfrak{Z}_1(x) + \mathcal{V}_2(x)\mathfrak{Z}_2(x).$$

The general solution is

$$f(x) = c_1\mathfrak{J}_1(x) + c_2\mathfrak{J}_2(x) + \mathcal{V}_1(x)\mathfrak{J}_1(x) + \mathcal{V}_2(x)\mathfrak{J}_2(x).$$

The chief advantage to variation of parameters is that it gives us a *formula* for the solution to any nonhomogeneous second-order ODE provided that we have a fundamental solution set for the homogeneous problem. And so we can effectively reduce our workload to solving homogeneous problems. But, correspondingly, there are two obvious disadvantages to variation of parameters.

1. We must know a fundamental solution set for the ODE. This is easy if the ODE is constant-coefficient, for then we can use the methods of Section 1.4.1. If the ODE has variable coefficients, then there is no standard, uniform method for producing a fundamental solution set. When the coefficients are real analytic, we can use the power series method (Section 1.6), but this is often cumbersome and rarely yields a "formula" in terms of "elementary" functions for the fundamental solution set.

2. We need to calculate antiderivatives in (1.4.18). Of course, we could use a definite integral, as we do below in (1.4.25). But these antiderivatives may be difficult, or impossible, to calculate explicitly. We will explore an alternative method for solving $\mathcal{A}f = g$ in Section 1.5.2, where we assume that g has one of several very particular forms. This "method of undetermined coefficients" is often more expedient than brute-force antidifferentiation.

Notwithstanding these disadvantages, we should take comfort (and pride) in that variation of parameters *always* provides a particular solution. In fact, we can combine variation of parameters for nonhomogeneous equations and the existence and uniqueness theory for homogeneous ODEs from Section 1.4.4 to produce the following existence and uniqueness result for nonhomogeneous second-order ODEs. Observe how much extra work we had to do to produce this theorem, compared with the straightforward, explicit analysis for first-order ODEs that gave us Theorem 1.2.4.

1.4.20 Theorem.

Suppose that $I \subseteq \mathbb{R}$, a_1, a_0, $g \in \mathcal{C}(I)$, and y_0, $y_1 \in \mathbb{C}$. Let $\mathcal{A}f = f'' + a_1(x)f' + a_0(x)f$. Then for any $x_0 \in I$, the INITIAL VALUE PROBLEM

$$\begin{cases} \mathcal{A}f = g(x) \\ f(x_0) = y_0 \\ f'(x_0) = y_1 \end{cases}$$

has a unique solution in $\mathcal{C}^2(I)$.

Proof. Let $\{\mathfrak{J}_1, \mathfrak{J}_2\}$ be a fundamental solution set for $\mathcal{A}$ from Lemma 1.4.12. By variation of parameters, we may construct $f_0 \in \mathcal{C}^2(I)$ such that $\mathcal{A}f_0 = g$. Note, though, that variation of parameters guarantees nothing about the initial conditions. However, for any c_1, $c_2 \in \mathbb{C}$, we have

$$\mathcal{A}(c_1\mathfrak{J}_1 + c_2\mathfrak{J}_2 + f_0) = g,$$

so we may choose c_1 and c_2 to satisfy the initial conditions.

Specifically, c_1 and c_2 must satisfy

$$\begin{cases} f_0(x_0) + c_1\mathfrak{Z}_1(x_0) + c_2\mathfrak{Z}_2(x_0) = y_0 \\ f_0'(x_0) + c_1\mathfrak{Z}_1'(x_0) + c_2\mathfrak{Z}_2'(x_0) = y_1. \end{cases}$$

This is equivalent to the matrix-vector system

$$\begin{bmatrix} \mathfrak{Z}_1(x_0) & \mathfrak{Z}_2(x_0) \\ \mathfrak{Z}_1'(x_0) & \mathfrak{Z}_2'(x_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 - f_0(x_0) \\ y_1 - f_0'(x_0) \end{pmatrix}.$$

Observe that the determinant of the matrix on the left is $\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](x_0) \neq 0$, since $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are linearly independent kernel elements of $\mathcal{A}$, so the matrix is invertible, and we can solve (uniquely) for c_1 and c_2. So, we have constructed a solution

$$f := c_1\mathfrak{Z}_1 + c_2\mathfrak{Z}_2 + f_0. \tag{1.4.24}$$

Now we prove uniqueness[20]. Suppose that h is another solution to the IVP. Put $\phi := f - h$. Then ϕ satisfies the homogeneous IVP

$$\begin{cases} \mathcal{A}\phi = 0 \\ \phi(x_0) = 0 \\ \phi'(x_0) = 0, \end{cases}$$

and so Lemma 1.4.7 implies that $\phi = 0$, hence $f = h$. ■

1.4.21 Linear algebraic viewpoint: variation of parameters

1. So that we may speak explicitly about the functions $\mathcal{V}_1$ and $\mathcal{V}_2$ in Theorem 1.4.17, fix $x_0 \in I$ and set

$$\mathcal{V}_1(x) = -\int_{x_0}^{x} \frac{\mathfrak{Z}_2(\xi)g(\xi)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](\xi)}\, d\xi \quad \text{and} \quad \mathcal{V}_2(x) = \int_{x_0}^{x} \frac{\mathfrak{Z}_1(\xi)g(\xi)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](\xi)}\, d\xi. \tag{1.4.25}$$

Then for any $x_0 \in I$, the function

$$f(x) = \left(-\int_{x_0}^{x} \frac{\mathfrak{Z}_2(\xi)g(\xi)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](\xi)}\, d\xi\right)\mathfrak{Z}_1(x) + \left(\int_{x_0}^{x} \frac{\mathfrak{Z}_1(\xi)g(\xi)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](\xi)}\, d\xi\right)\mathfrak{Z}_2(x) \tag{1.4.26}$$

solves $\mathcal{A}f = g$.

We can combine the integrals in (1.4.19) and write

$$f(x) = \int_{x_0}^{x} \left(\frac{\mathfrak{Z}_1(\xi)\mathfrak{Z}_2(x) - \mathfrak{Z}_1(x)\mathfrak{Z}_2(\xi)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](\xi)}\right) g(\xi)\, d\xi.$$

Putting

$$\mathcal{V}[\mathfrak{Z}_1, \mathfrak{Z}_2](x, \xi) := \frac{\mathfrak{Z}_1(\xi)\mathfrak{Z}_2(x) - \mathfrak{Z}_1(x)\mathfrak{Z}_2(\xi)}{\mathcal{W}[\mathfrak{Z}_1, \mathfrak{Z}_2](\xi)}, \tag{1.4.27}$$

[20] As before, we have guaranteed that $f := c_1\mathfrak{Z}_1 + c_2\mathfrak{Z}_2 + f_0$ solves the IVP, and if $h := d_1\mathfrak{Z}_1 + d_2\mathfrak{Z}_2 + f_0$ also solves the IVP, then $c_1 = d_1$ and $c_2 = d_2$. We have yet to rule out the possibility that there could be a function h *not* of the form $d_1\mathfrak{Z}_1 + d_2\mathfrak{Z}_2 + f_0$ that also solves the IVP.

we see that the function

$$\mathcal{R}[g](x) := \int_{x_0}^{x} \mathcal{V}(x,\xi)g(\xi)\, d\xi \tag{1.4.28}$$

solves[21] $\mathcal{A}\mathcal{R}[g] = g$. Observe that $\mathcal{R}$ is a linear operator on $\mathcal{C}(I)$. Specifically, $\mathcal{R}$ is an **integral operator** with kernel $\mathcal{V}$; recall Item 3 in Linear Algebraic Viewpoint 1.2.8.

2. We know $\mathcal{A}\mathcal{R}[g] = g$ for any $g \in \mathcal{C}(I)$. Thus $\mathcal{A}$ is onto $\mathcal{C}(I)$. And Item 3 in Linear Algebraic Viewpoint 1.4.14 tells us $\ker(\mathcal{A})$ is trivial if we restrict $\mathcal{A}$ to the domain

$$\mathfrak{D}_{x_0}(\mathcal{A}) := \left\{ f \in \mathcal{C}^2(I) \;\middle|\; f(x_0) = f'(x_0) = 0 \right\}.$$

Thus $\mathcal{A}$ is invertible from $\mathfrak{D}_{x_0}(\mathcal{A})$ to $\mathcal{C}(I)$ with inverse $\mathcal{A}^{-1} = \mathcal{R}$.

1.5. Glimpses of higher-order ODEs and beyond.

Virtually all of the theory, formulas, and computations that we have developed for second-order linear ODEs extends in a natural way to higher-order problems (order $n \geq 3$); this is discussed at length and in various contexts in [2, 3, 6, 8, 16, 23]. There are generalizations of the Wronskian and variation of parameters for nonhomogeneous problems; they are conceptually simple, especially if one has mastered the material in Section 1.4, but often notationally and computationally cumbersome. We will not cover these generalizations in this course, with one exception.

1.5.1. Constant-coefficient homogeneous equations.

A nth order linear constant-coefficient homogeneous ODE has the form

$$\mathcal{A}f := a_n\partial_x^n[f] + a_{n-1}\partial_x^{n-1}[f] + \cdots + a_1\partial_x[f] + a_0 f = \sum_{k=0}^{n} a_k \partial_x^k[f] = 0,$$

where $a_k \in \mathbb{C}$ and $a_n \neq 0$. We can introduce the characteristic polynomial

$$\chi_{\mathcal{A}}(\lambda) := \sum_{k=0}^{n} a_k \lambda^k,$$

and, in a computation analogous to that at the start of Section 1.4.1, conclude that if $\chi_{\mathcal{A}}(\lambda_\star) = 0$ for some $\lambda_\star \in \mathbb{C}$, then $f(x) := e^{\lambda_\star x}$ solves $\mathcal{A}f = 0$. The only difference is that if $n > 2$, then $\chi_{\mathcal{A}}$ can have more than two (distinct) roots, and so we may find more than two exponential solutions to $\mathcal{A}f = 0$.

In fact, we can always construct n linearly independent solutions to $\mathcal{A}f = 0$. In the following theorem, we say that a root $\lambda_\star$ of $\chi_{\mathcal{A}}$ has multiplicity $m \geq 1$ if we can write

$$\chi_{\mathcal{A}}(\lambda) = (\lambda - \lambda_\star)^m \widetilde{\chi}_{\mathcal{A}}(\lambda),$$

where $\widetilde{\chi}_{\mathcal{A}}(\lambda_\star) \neq 0$. (See also Section 3.8.3.)

[21] We know from our work above with the variation of parameters formula that $\mathcal{R}[g]$ is twice differentiable, but that may not be obvious from the form of $\mathcal{R}$ in (1.4.28). With suitable hypotheses on an abstract kernel $\mathcal{V}$ and the function g, a mapping of the form $x \mapsto \int_{x_0}^{x} \mathcal{V}(x,\xi)g(\xi)\, d\xi$ can be shown to be differentiable by combining Leibniz's rule for differentiating under the integral (Theorem 2.4.35) with the chain rule for functions of two variables.

1.5.1 Theorem.

Suppose that the characteristic polynomial $\chi_{\mathcal{A}}$ of the constant-coefficient linear differential operator $\mathcal{A} = \sum_{k=0}^{n} a_k \partial_x^k$, $a_n \neq 0$, has the r distinct roots $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$, with $1 \leq r \leq n$. Suppose that each root has multiplicity m_k, $1 \leq k \leq r$, with $\sum_{k=1}^{r} m_k = n$.

(i) *The characteristic polynomial, $\chi_{\mathcal{A}}(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$, factors as*

$$\chi_{\mathcal{A}}(\lambda) = a_n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_r)^{m_r} = a_n \prod_{k=1}^{r} (\lambda - \lambda_k)^{m_k},$$

and the differential operator $\mathcal{A}$ factors as

$$\mathcal{A} = a_n(\partial_x - \lambda_1)^{m_1}(\partial_x - \lambda_2)^{m_2} \cdots (\partial_x - \lambda_r)^{m_r} = a_n \prod_{k=1}^{r} (\partial_x - \lambda_k)^{m_k}.$$

(ii) *Denote by $\mathfrak{z}_1, \ldots, \mathfrak{z}_n$ the n functions below:*

$$\underbrace{\underbrace{e^{\lambda_1 x}, xe^{\lambda_1 x}, \ldots, x^{m_1-1}e^{\lambda_1 x}}_{m_1 \text{ functions}}, \quad \underbrace{e^{\lambda_2 x}, xe^{\lambda_2 x}, \ldots, x^{m_2-1}e^{\lambda_2 x}}_{m_2 \text{ functions}}, \quad \ldots, \quad \underbrace{e^{\lambda_r x}, xe^{\lambda_r x}, \ldots, x^{m_r-1}e^{\lambda_r x}}_{m_r \text{ functions}}}_{m_1 + m_2 + \cdots + m_r = n \text{ functions}}.$$

Then $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_n\}$ forms a fundamental solution set for $\mathcal{A}$ in the sense that $\mathcal{A}\mathfrak{z}_k = 0$ for each k; the set $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_n\}$ is linearly independent; and if $\mathcal{A}f = 0$ for some $f \in \mathcal{C}^n(I)$, then there exist (unique) $c_1, \ldots, c_n \in \mathbb{C}$ such that $f = \sum_{k=1}^{n} c_k \mathfrak{z}_k$.

(iii) *Suppose that $a_k \in \mathbb{R}$ for all k. Then $\chi_{\mathcal{A}}(\lambda_k) = 0$ if and only if $\chi_{\mathcal{A}}(\overline{\lambda}_k) = 0$. In this case, one can replace the $2m_k$ functions*

$$e^{\lambda_k x}, xe^{\lambda_k x}, \ldots, x^{m_k-1}e^{\lambda_k x}, e^{\overline{\lambda}_k x}, xe^{\overline{\lambda}_k x}, \ldots, x^{m_k-1}e^{\overline{\lambda}_k x}$$

in the fundamental solution set from part (ii) by the $2m_k$ functions

$$\cos(\mu_k x), x\cos(\mu_k x), \ldots, x^{m_k-1}\cos(\mu_k x), \sin(\nu_k x), x\sin(\nu_k x), \ldots, x^{m_k-1}\sin(\nu_k x),$$

where $\lambda_k = \mu_k + i\nu_k$ for μ_k, $\nu_k \in \mathbb{R}$.

1.5.2 Example.

Find a fundamental solution set for $\mathcal{A}f := \partial_x^4[f] - 16f$ on $\mathbb{R}$.

Solution. The characteristic polynomial for $\mathcal{A}f = \partial_x^4[f] - 16f$ is

$$\chi_{\mathcal{A}}(\lambda) = \lambda^4 - 16 = (\lambda^2 + 4)(\lambda^2 - 4),$$

and so the roots of $\chi_{\mathcal{A}}$ are $\pm 2i$ and ± 2. Hence a fundamental solution set is

$$\mathfrak{z}_1(x) = e^{2ix}, \quad \mathfrak{z}_2(x) = e^{-2ix}, \quad \mathfrak{z}_3(x) = e^{2x}, \quad \mathfrak{z}_4(x) = e^{-2x}.$$

Since $\mathcal{A}$ has real coefficients, we can also form a fundamental solution set using strictly real-valued functions:

$$\widetilde{\mathfrak{z}}_1(x) = \cos(2x), \quad \widetilde{\mathfrak{z}}_2(x) = \sin(2x), \quad \mathfrak{z}_3(x) = e^{2x}, \quad \mathfrak{z}_4(x) = e^{-2x}.$$

▲

1.5.3 Method: solve $a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_2 f'' + a_1 f' + a_0 f = 0$

0. Assume $a_n \neq 0$.

1. Form the characteristic polynomial

$$\chi(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

2. Find the roots of χ.

3. Compare the roots with part (ii) of Theorem 1.5.1 to construct a fundamental solution set. In general, if a root $\lambda_\star$ has multiplicity m, the "part" of the fundamental solution set that "arises" from $\lambda_\star$ will be the m functions

$$\underbrace{e^{\lambda_\star x}, xe^{\lambda_\star x}, \ldots, x^{m-1}e^{\lambda_\star x}}_{m \text{ functions}}.$$

If the coefficients a_k are all real, then $\lambda_\star = \mu_\star + i\nu_\star$ is a root of χ if and only if $\overline{\lambda}_\star = \mu_\star - i\nu_\star$ is, in which case one can use the $2m$ functions

$$\cos(\mu_\star x), x\cos(\mu_\star x), \ldots, x^{m-1}\cos(\mu_\star x), \sin(\mu_\star x), x\sin(\mu_\star x), \ldots, x^{m-1}\sin(\mu_\star x)$$

in lieu of the $2m$ exponentials with *both* $\lambda_\star$ and $\overline{\lambda}_\star$ from above.

1.5.2. The method of undetermined coefficients.

Consider the following abstract situation. Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces and $\mathcal{A}\colon \mathcal{X} \to \mathcal{Y}$ be a linear operator. Suppose that we want to solve the equation $\mathcal{A}f = g$ for $f \in \mathcal{X}$, given some $g \in \mathcal{Y}$. And suppose we are in the happy situation of having another vector space $\mathcal{Z}$ and another linear operator $\mathcal{B}\colon \mathcal{Y} \to \mathcal{Z}$ such that $\mathcal{B}g = 0$. So, *if* $\mathcal{A}f = g$, *then* $\mathcal{B}\mathcal{A}f = 0$. That is, any solution to $\mathcal{A}f = g$ must be in the kernel of $\mathcal{B}\mathcal{A}$. So, perhaps we could determine $\ker(\mathcal{B}\mathcal{A})$ and then see which elements of the kernel solve $\mathcal{A}f = g$.

Now, ostensibly, we have not made our work any easier. We have gone from trying to solve the particular (nonhomogeneous) equation $\mathcal{A}f = g$ to the (homogeneous) problem $\mathcal{B}\mathcal{A}f = 0$. The operator $\mathcal{B}\mathcal{A}$ is probably more complicated than just $\mathcal{A}$. But, maybe the equation $\mathcal{B}\mathcal{A}f = 0$ is easy to solve!

This is the situation with certain kinds of constant-coefficient linear ODEs $\mathcal{A}f = g$. We know precisely what the kernel (that is, the fundamental solution set) of such a problem is from Theorem 1.5.1. If the nonhomogeneous function g has a special form, then it is "easy" to find a constant-coefficient linear differential operator $\mathcal{B}$ such that $\mathcal{B}g = 0$. Then any solution to $\mathcal{A}f = g$ must satisfy $\mathcal{B}\mathcal{A}f = 0$, and if $\mathcal{A}$ and $\mathcal{B}$ are both constant-coefficient operators, $\mathcal{B}\mathcal{A}$ will be, too.

Specifically, we will solve ODEs when g has the special form

$$g(x) = e^{\lambda x}\,\mathrm{trig}(\alpha x)\sum_{k=0}^{m} a_k x^k. \tag{1.5.1}$$

Here λ, $a_k \in \mathbb{C}$, $\alpha \in \mathbb{R}$, and $\mathrm{trig}(X) = \sin(X)$ or $\cos(X)$. Such "forcing" functions appear naturally in models of vibration, e.g, mass-spring systems and harmonic oscillators. Although the differential operators $\mathcal{A}$ that we study will be second-order, the product $\mathcal{B}\mathcal{A}$ will be higher-order, hence our deferment of this topic until only now. Although variation of parameters always applies for nonhomogeneities like (1.5.1), calculating the requisite antiderivatives can be enervating.

The method that we will develop is best illustrated through a number of examples. Throughout, we will obtain ample further practice in solving constant-coefficient homogeneous problems and using Lemma 1.4.15 to express the general solution to a nonhomogeneous ODE as the sum of a particular solution to the nonhomogeneous problem and a linear combination of the underlying fundamental solution set.

1.5.4 Example.

Find the general solution to $f'' + f = e^{2x}$.

Solution. Put $\mathcal{A}f = f''+f$. We know that a fundamental solution set for $\mathcal{A}$ is $\{e^{ix}, e^{-ix}\}$, so from Lemma 1.4.15 and Definition 1.4.16, the general solution has the form

$$f(x) = c_1 e^{ix} + c_2 e^{-ix} + f_0(x),$$

where f_0 is some function such that $\mathcal{A}f_0 = e^{2x}$. Another fundamental solution set is $\{\cos(x), \sin(x)\}$, so, if we know a particular nonhomogeneous solution f_0, we could also write the general solution as

$$f(x) = d_1 \cos(x) + d_2 \sin(x) + f_0(x).$$

Whether one works with complex exponentials or sines and cosines is largely a matter of taste and personal preference.

Let us focus on finding this particular solution f_0 to the nonhomogeneous problem. The method outlined above tells us that we should find a constant-coefficient differential operator $\mathcal{B}$ such that $\mathcal{B}[e^{2\cdot}] = 0$. A moment's thought (or Example 1.2.2) suggests $\mathcal{B}g := g' - 2g$, and, indeed, we check

$$\mathcal{B}[e^{2\cdot}](x) = 2e^{2x} - 2e^{2x} = 0.$$

So, any function satisfying $\mathcal{A}f = f'' + f = e^{2x}$ must satisfy

$$0 = \mathcal{B}\mathcal{A}f = (\partial_x - 2)(\partial_x^2 + 1)f.$$

Note that $\mathcal{B}\mathcal{A}$ is a third-order differential operator.

We do not bother expanding $\mathcal{B}\mathcal{A}$ but instead read off its characteristic polynomial:

$$\chi_{\mathcal{B}\mathcal{A}}(\lambda) := (\lambda - 2)(\lambda^2 + 1).$$

The roots are $\lambda = 1, \pm i$. Each has multiplicity 1, so $\mathcal{B}\mathcal{A}f = 0$ if and only if

$$f(x) = c_1 e^{ix} + c_2 e^{-ix} + c_3 e^{2x} \tag{1.5.2}$$

for some c_1, c_2, $c_3 \in \mathbb{C}$. These numbers are the "undetermined coefficients" of f; let us determine them.

We need f in this form (1.5.2) to satisfy $\mathcal{A}f = e^{2x}$. That is, we need

$$e^{2x} = \mathcal{A}f = \mathcal{A}\big[c_1 e^{ix} + c_2 e^{-ix} + c_3 e^{2x}\big] = c_3 \mathcal{A}[e^{2x}] = 4c_3 e^{2x} + c_3 e^{2x} = 5c_3 e^{2x}.$$

Here we used the fact that $\mathcal{A}[e^{\pm ix}] = 0$. Hence c_3 must satisfy

$$(5c_3 - 1)e^{2x} = 0,$$

and so we may divide by e^{2x} to find $c_3 = 1/5$. Thus

$$f(x) = c_1 e^{ix} + c_2 e^{-ix} + \frac{e^{2x}}{5}.$$

solves $\mathcal{A}f = e^{2x}$. Note that we did not determine c_1 and c_2. This is unsurprising, since $e^{\pm ix}$ solve the homogeneous problem $\mathcal{A}f = 0$.

We are also free to take $c_1 = c_2 = 0$ and conclude that $f_0(x) := e^{2x}/5$ satisfies $\mathcal{A}f_0 = e^{2x}$. Thus another way to write the general solution to $\mathcal{A}f = e^{2x}$ is the "real" form

$$f(x) = d_1 \cos(x) + d_2 \sin(x) + \frac{e^{2x}}{5}.$$ ▲

1.5.5 Example.

Find the general solution to $f'' + 9f = \sin(2x)$.

Solution. Let $\mathcal{A}f = f'' + 9f$. We first observe that a fundamental solution set for $\mathcal{A}$ is $\{e^{3ix}, e^{-3ix}\}$. Another is $\{\cos(3x), \sin(3x)\}$.

Now, we need a constant-coefficient linear differential operator $\mathcal{B}$ such that $\mathcal{B}[\sin(2\cdot)] = 0$. Perhaps recalling that $\sin(2\cdot) = \operatorname{Im}[e^{2i\cdot}]$, we take $\mathcal{B}g := g'' + 4g$. And so we will see which solutions of $\mathcal{B}\mathcal{A}f = 0$ might solve $\mathcal{A}f = \sin(2x)$. First,

$$\mathcal{B}\mathcal{A} = (\partial_x^2 + 4)(\partial_x^2 + 9),$$

and so the characteristic polynomial is

$$\chi_{\mathcal{B}\mathcal{A}}(\lambda) = (\lambda^2 + 4)(\lambda^2 + 9).$$

Its roots are $\lambda = \pm 2i, \pm 3i$, each of which has multiplicity 1. Then f must have the form

$$f(x) = c_1 e^{3ix} + c_2 e^{-3ix} + c_3 e^{2ix} + c_4 e^{-2ix}.$$

To determine these coefficients, we will need to compute $\mathcal{A}f$ and set that equal to $\sin(2x)$. We can save ourselves some time, however, by recalling that $\mathcal{A}[e^{\pm 3i\cdot}] = 0$. We find that $\mathcal{A}f = \sin(2x)$ if and only if

$$\sin(2x) = \mathcal{A}\big[c_1 e^{3ix} + c_2 e^{-3ix} + c_3 e^{2ix} + c_4 e^{-2ix}\big] = \mathcal{A}\big[c_3 e^{2ix} + c_4 e^{-2ix}\big] = 5c_3 e^{2ix} + 5c_4 e^{-2ix}. \tag{1.5.3}$$

Unsurprisingly, the coefficients c_1 and c_1, which correspond to kernel elements of $\mathcal{A}$, no longer appear in our equation, and so we will not be able to specify them.

Now, we are mixing complex exponentials and trig functions, so let us rewrite

$$\sin(2x) = \frac{e^{2ix} - e^{-2ix}}{2i}.$$

If we rewrite (1.5.3) using this complex form of $\sin(2x)$ and collect like terms, we find that c_3 and c_4 must satisfy

$$\left(5c_3 - \frac{1}{2i}\right) e^{2ix} + \left(5c_4 + \frac{1}{2i}\right) e^{-2ix} = 0.$$

Since the functions $e^{\pm 2ix}$ are linearly independent, we must have

$$5c_3 - \frac{1}{2i} = 5c_4 + \frac{1}{2i} = 0.$$

Thus

$$c_3 = \frac{1}{10i} = -\frac{i}{10} \quad \text{and} \quad c_4 = -\frac{1}{10i} = \frac{i}{10}.$$

We have bowed to the convention that one avoids i in the denominator of simplified expressions.

A general solution to $\mathcal{A}f = \sin(2x)$, therefore, is

$$f(x) = c_1 e^{3ix} + c_2 e^{-3ix} - \frac{ie^{2ix}}{10} + \frac{ie^{-2ix}}{10}. \tag{1.5.4}$$

A particular solution is

$$f_0(x) := -\frac{ie^{2ix}}{10} + \frac{ie^{-2ix}}{10} = \frac{i}{5}\left(\frac{e^{-2ix} - e^{2ix}}{2}\right) = -\frac{i^2}{5}\left(\frac{e^{2ix} - e^{-2ix}}{2i}\right) = \frac{1}{5}\sin(2x).$$

And so we can also write the general solution in the form

$$f(x) = d_1 \cos(3x) + d_2 \sin(3x) + \frac{1}{5}\sin(2x).$$ ▲

1.5.6 Example.

Find the general solution to $f'' + f = 5x^2 e^{2x}$.

Solution. Put $\mathcal{A}f = f'' + f$. Once again, a fundamental solution set is $\{e^{ix}, e^{-ix}\}$, and another is $\{\cos(x), \sin(x)\}$.

The nonhomogeneity here should remind us of, per Theorem 1.5.1, the fundamental solution set e^{2x}, xe^{2x}, x^2e^{2x} for the differential operator $\mathcal{B} := (\partial_x - 2)^3$. Then

$$\mathcal{B}\mathcal{A} = (\partial_x - 2)^3(\partial_x^2 + 1),$$

so its characteristic polynomial is

$$\chi_{\mathcal{B}\mathcal{A}}(\lambda) = (\lambda - 2)^3(\lambda^2 + 1),$$

and so the roots are 2, with multiplicity 3, and $\pm i$, each with multiplicity 1. Hence $\mathcal{B}\mathcal{A}f = 0$ if and only if

$$f(x) = c_1 e^{ix} + c_2 e^{-ix} + c_3 e^{2x} + c_4 x e^{2x} + c_5 x^2 e^{2x}.$$

We compute $\mathcal{A}f = 5x^2e^{2x}$ if and only if

$$5x^2e^{2x} = \mathcal{A}\big[c_3e^{2x} + c_4xe^{2x} + c_5x^2e^{2x}\big] = (5c_3 + 4c_4 + 2c_5)e^{2x} + (5c_4 + 8c_5)xe^{2x} + 5c_5x^2e^{2x}.$$

That is,

$$(5c_3 + 4c_4 + 2c_5)e^{2x} + (5c_3 + 8c_4)xe^{2x} + 5(c_5 - 1)x^2e^{2x} = 0.$$

By linear independence, c_3, c_4, and c_5 must satisfy the linear system

$$\begin{cases} 5c_3 + 4c_4 + 2c_5 & = 0 \\ 5c_3 + 8c_4 & = 0 \\ 5(c_5 - 1) & = 0. \end{cases}$$

This is fairly easy to solve for

$$c_3 = \frac{22}{25}, \qquad c_4 = -\frac{8}{5}, \qquad \text{and} \qquad c_5 = 1.$$

Thus the general solution is

$$f(x) = +c_1 e^{ix} + c_2 e^{-ix} + \frac{22}{25}e^{2x} - \frac{8}{5}xe^{2x} + x^2 e^{2x}.$$

A particular solution is

$$f_0(x) = \frac{22}{25}e^{2x} - \frac{8}{5}xe^{2x} + x^2 e^{2x},$$

and so another way to write the general solution is

$$f(x) = d_1 \cos(x) + d_2 \sin(x) + \frac{22}{25}e^{2x} - \frac{8}{5}xe^{2x} + x^2 e^{2x}.$$ ▲

1.5.7 Example.

Find the general solution to $f'' - f = e^{-x}$.

Solution. Here $\mathcal{A}f = f'' - f$, and a fundamental solution set is $\{e^x, e^{-x}\}$. Then since $\mathcal{A}[e^{-x}] = 0$, we look first for solutions of $\mathcal{A}^2 f = 0$, where

$$\mathcal{A}^2 = (\partial_x^2 - 1)^2 \qquad \text{and} \qquad \chi_{\mathcal{A}^2}(x) = (\lambda^2 - 1)^2 = \chi_{\mathcal{A}}(\lambda)^2.$$

The roots of $\chi_{\mathcal{A}^2}$ are ± 1, each with multiplicity 2. So our solution f must have the form

$$f(x) = c_1 e^x + c_2 e^{-x} + c_3 x e^x + c_4 x e^{-x}.$$

Then f must satisfy

$$e^{-x} = \mathcal{A}f = \mathcal{A}\big[c_3 x e^x + c_4 x e^{-x}\big] = 2c_3 e^x - 2c_4 e^{-x}.$$

That is,

$$2c_3 e^x - (2c_4 + 1)e^{-x} = 0,$$

and so $c_3 = 0$ and $c_4 = -1/2$. Then the general solution is

$$f(x) = c_1 e^x + c_2 e^{-x} - \frac{xe^{-x}}{2}.$$ ▲

Our success in all of these examples of solving $\mathcal{A}f = g$ was predicated on knowing (or guessing, finding, sussing out...) a constant-coefficient linear differential operator $\mathcal{B}$ such that $\mathcal{B}g = 0$. Such an operator $\mathcal{B}$ is sometimes said to be an **ANNIHILATOR** for g. For the classes of nonhomogeneities g that often arise, there are two particular annihilator operators worth knowing.

1.5.8 Lemma.

Let $k \geq 1$ be an integer, $\alpha \in \mathbb{R}$, and $\lambda \in \mathbb{C}$. Then

(i) $(\partial_x - \lambda)^{k+1}[x^k e^{\lambda x}] = 0$.

(ii) $(\partial_x^2 + \alpha^2)^{k+1}[x^k \sin(\alpha x)] = (\partial_x^2 + \alpha^2)^{k+1}[x^k \cos(\alpha x)] = 0$.

1.5.9 Method: solve $\mathcal{A}f = e^{\lambda x} p(x) \operatorname{trig}(\alpha x)$

0. Assume that $\mathcal{A}$ is a constant-coefficient linear differential operator, $\lambda \in \mathbb{C}$; $p(x) = \sum_{k=0}^{m} b_k x^k$; $\operatorname{trig}(X) = \sin(X)$, $\operatorname{trig}(X) = \cos(X)$, or $\operatorname{trig}(X) = 1$; and $\alpha \in \mathbb{R}$. This method solves this ODE, term-by-term, i.e., it constructs functions f_k such that $\mathcal{A}f_k = b_k e^{\lambda x} x^k \operatorname{trig}(\alpha x)$ for $k = 0, \ldots, m$. By superposition, a particular solution is $f = \sum_{k=0}^{m} f_k$.

1. To solve $\mathcal{A}f_k = b_k e^{\lambda x} x^k \operatorname{trig}(\alpha x)$, consider the following cases.

(1-i) If $\lambda = 0$, the operator $(\partial_x^2 + \alpha^2)^{k+1}$ satisfies $(\partial_x^2 + \alpha^2)^{k+1}[x^k \operatorname{trig}(\alpha x)]$. Use Method 1.5.3 to find the general solution to $(\partial_x^2 + \alpha^2)^{k+1}\mathcal{A}f = 0$; denote this solution by $f_{\alpha,k}$. It will have undetermined coefficients c_k. Compute $\mathcal{A}f_{\alpha,k}$, neglecting terms in $f_{\alpha,k}$ that correspond to terms in the fundamental solution set for $\mathcal{A}$. Express the right side of $\mathcal{A}f_{\alpha,k} - b_k x^k \operatorname{trig}(\alpha x) = 0$ as a sum of linearly independent functions. Set the coefficients of those functions equal to zero to determine a linear system of equations for c_k. Solve this system.

(1-ii) If $\operatorname{trig}(X) = 1$, the operator $(\partial_x - \lambda)^{k+1}$ satisfies $(\partial_x - \lambda)^{k+1}[x^k e^{\lambda x}] = 0$. Use Method 1.5.3 to find the general solution to $(\partial_x - \lambda)^{k+1}\mathcal{A}f = 0$; denote this solution by $f_{\lambda,k}$. It will have undetermined coefficients c_k. Compute $\mathcal{A}f_{\lambda,k}$, neglecting terms in $f_{\lambda,k}$ that correspond to terms in the fundamental solution set for $\mathcal{A}$. Express the right side of $\mathcal{A}f_{\lambda,k} - b_k x^k e^{\lambda x} = 0$ as a sum of linearly independent functions. Set the coefficients of those functions equal to zero to determine a linear system of equations for c_k. Solve this system.

(1-iii) If $\lambda \neq 0$ and $\operatorname{trig}(X) = \sin(X)$ or $\operatorname{trig}(X) = \cos(X)$, rewrite $\operatorname{trig}(\alpha x)$ using complex identities:

$$\operatorname{trig}(\alpha x) = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2} \quad \text{if} \quad \operatorname{trig}(X) = \cos(X)$$

or

$$\operatorname{trig}(\alpha x) = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i} \quad \text{if} \quad \operatorname{trig}(X) = \sin(X).$$

Multiply and combine the $e^{\lambda x}$ and $e^{\pm i\alpha x}$ exponentials in two separate terms. Multiply and combine the $\pm 1/2$ or $\pm 1/2i$ coefficients and b_k into one coefficient $\widetilde{b}_k$. Solve $\mathcal{A}f = \widetilde{b}_k x^k e^{(\lambda \pm i\alpha)x}$ using Step (ii).

2. Add all solutions from Step 1 together.

3. To find a general solution, add the general solution of $\mathcal{A}f = 0$ to the particular solution constructed in Step 2.

1.5.3. And beyond. . . .

Where might one go after mastering first- and second-order linear ODEs and constant-coefficient higher-order problems?

• In Section 1.6 we will return to the problem of finding a fundamental solution set for variable-coefficient problems with the power series method. That is, we will learn how to solve ODEs like $f'(x) + e^x f(x) = 0$.

• In Section 2.7 we will explore second-order linear ODEs with boundary conditions instead of initial conditions, e.g., we might demand that a solution f be defined on $[0, 1]$ with a relation between $f(0)$ and $f(1)$, not separate conditions on just $f(0)$ and $f'(0)$.

• Variation of parameters provides solutions to nonhomogeneous problems $\mathcal{A}f = g$ when the nonhomomgeneity g is *continuous*. This ultimately has to do with the fundamental theorem of calculus, which guarantees that continuous functions have antiderivatives. A discontinuous function g may prevent us from finding the antiderivatives in (1.4.18) for variation of parameters. Depending on the nature of the discontinuity, we could use Fourier series techniques (Section 2.4.8) or the Laplace transform (which we will not meet in this course).

• Perhaps we are not interested in the "local" behavior of a solution at an initial value but rather something more "global." We could demand that a function both solve an ODE and be improperly integrable (Appendix B) over $\mathbb{R}$. Such an integrability condition allows us to measure the "energy" of the function in a meaningful way. In that situation, the Fourier transform (Section 2.5.6) is the right tool.

There are plenty of other interesting problems for differential equations that we will not consider in this course.

• One could study *systems* of linear or nonlinear equations. The rates of change of two or more quantities could be related to each other via some sort of natural coupling, e.g., the motion of two or more coupled harmonic oscillators (perhaps with different masses and/or spring potentials).

• Any single nth order ODE naturally reduces to a system of n first-order ODEs via a clever substitution, like the one we now describe. (Conversely, *not* every system of n ODEs needs to correspond to an nth order ODE.) Consider the familiar ODE

$$f'' + 2f' + f = 0$$

and put $v = f'$. Then

$$v' = f'' = -(2f' + f),$$

and so the functions f and v satisfy

$$\begin{cases} f' = v \\ v' = -(2f' + f) = -(2v + f). \end{cases}$$

This is a system of differential equations. In fact, if we put $\mathbf{f}(x) = (f(x), v(x))$ and set

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},$$

then the system is equivalent to

$$\mathbf{f}'(x) = A\mathbf{f}(x).$$

This looks a great deal like the first-order problem $f' = af$, and, indeed, we can solve it if we know what the **MATRIX EXPONENTIAL** e^{Ax} means for $A \in \mathbb{C}^{2\times 2}$. Much of the theory of Section 1.2.2 carries over to vector-valued first-order systems. In fact, most proofs of the existence and/or uniqueness of solutions to general ODEs (like that of Lemma 1.4.7) hinge on first reducing the ODE to a system of first-order ODEs and then proving something about first-order systems.

• One can ask not merely for formulas for solutions, as we have largely done here, but for more "qualitative" information about the behavior of solutions. For example, just given the ODE, can one determine the behavior as $x \to \infty$ of all solutions? Are all solutions bounded? Given an initial condition at a point x_0, what is the largest interval containing x_0 on which a solution exists? If that interval is bounded, what happens to the solution as x approaches the boundary of the interval? Often the behavior of a related, easier "linearized" problem can help.

1.5.10 Example.

One of the many incarnations of the **ONE-DIMENSIONAL SCHRÖDINGER EQUATION** *with potential* σ *is*

$$f''(x) + (1 + \sigma(x))f(x) = 0,$$

where $\sigma \in \mathcal{C}(\mathbb{R})$ *and* $\lim_{x\to\pm\infty} \sigma(x) = 0$. *Such an equation has two linearly independent solutions by Lemma 1.4.12. Use knowledge of a related linear equation to conjecture how these solutions behave at* $\pm\infty$.

Solution. When $|x|$ is large, $\sigma(x)$ is close to 0, so let us discount its effects on Schrödinger's equation, which then becomes

$$f''(x) + f(x) = 0. \tag{1.5.5}$$

This equation has solutions $\widetilde{\mathfrak{J}}_1(x) = e^{ix}$ and $\widetilde{\mathfrak{J}}_2(x) = e^{-ix}$, and, moreover, any solution to (1.5.5) is a linear combination of $\widetilde{\mathfrak{J}}_1$ and $\widetilde{\mathfrak{J}}_2$. We therefore conjecture that if f solves Schrödinger's equation, then, for $|x|$ large, f becomes arbitrarily close to a linear combination of $\widetilde{\mathfrak{J}}_1$ and $\widetilde{\mathfrak{J}}_2$. That is, given a solution f of Schrödinger's equation, there are constants c_1, $c_2 \in \mathbb{C}$ such that

$$\lim_{x\to\pm\infty} \left|f(x) - \left(c_1 e^{ix} + c_2 e^{-ix}\right)\right| = 0.$$

Such a function f is sometimes said to be **ASYMPTOTICALLY SINUSOIDAL**. The solutions $\mathfrak{J}_1$, $\mathfrak{J}_2$ to Schrödinger's equation are its **JOST SOLUTIONS** [27, 28]. ▲

1.6. Series solutions.

Variation of parameters ensures that if we know a fundamental solution set for the operator $\mathcal{A}f = f'' + a_1(x)f' + a_0(x)f$, with a_0, a_1 continuous, then we can always solve the nonhomogeneous problem $\mathcal{A}f = g$. We know how to construct a fundamental solution set when $\mathcal{A}$ has constant coefficients. In general, constructing a fundamental solution set when $\mathcal{A}$ has continuous, but not constant, coefficients may be very difficult. However, when the coefficients are real analytic, it is possible to construct explicitly a fundamental solution set via a **POWER SERIES ANSATZ**. We will explore this technique for second-order variable-coefficient linear ODEs, although it generalizes immediately to higher-order problems.

Recall that a function $f\colon I \subseteq \mathbb{R} \to \mathbb{C}$ is **REAL ANALYTIC**[22] on the interval I if at every point in I, f has a power series expansion valid on a subinterval of I centered at that point. That is, for each $x_0 \in I$, there exist a number $\delta > 0$ and a sequence (a_k) in $\mathbb{C}$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k \quad \text{if} \quad x \in I \text{ and } x_0 - \delta < x < x_0 + \delta.$$

We presume familiarity with the basic calculus of power series from Appendix A.4. We mention here only two of the most important properties of power series: the differentiation rule

$$f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k \Longrightarrow f'(x) = \sum_{k=1}^{\infty} ka_k(x - x_0)^k$$

and the **IDENTITY PRINCIPLE** (part (iv) of Theorem A.4.5),

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k = 0 \text{ for all } x \in (x_0 - \delta, x_0 + \delta) \Longrightarrow a_k = 0 \text{ for all } k.$$

We begin with two simple, illustrative examples for equations that we know how to solve already.

1.6.1 Example.

Solve the first-order linear ODE

$$f' = f$$

by making the ansatz $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and determining a formula for* a_k*. (Of course, we expect to recover* $f(x) = Ce^x$*.)*

Solution. If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then differentiating term-by-term we have

$$f'(x) = \sum_{k=1}^{\infty} ka_k x^{k-1},$$

and so

$$0 = f' - f = \sum_{k=1}^{\infty} ka_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k - \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \big[(k+1)a_{k+1} - a_k\big]x^k.$$

If this is to hold for all x, then the identity principle for power series implies

$$(k + 1)a_{k+1} - a_k = 0, \ k \geq 0.$$

[22] See Section 3.7.1 for more details on real analyticity. The reader familiar with differentiation and analyticity for a function of a complex variable should note that all the results of this section, including Fuchs's theorem (Theorem 1.6.3) below remain true for ODEs for functions defined on open subsets of $\mathbb{C}$. In general, just substitute z for x throughout.

Thus
$$a_{k+1} = \frac{a_k}{k+1}.$$
This is a **RECURSION RELATION** for a_{k+1} in terms of a_k, i.e., we have written $a_{k+1} = \Phi(a_k, k)$ for a function $\Phi \colon \mathbb{C} \times \mathbb{N} \to \mathbb{R}$. Specifically, $\Phi(z, k) = z/(k+1)$.

This recursion relation will not tell us a_0, but we expect one free parameter in our solution since this is a first-order ODE. We can, however, do better and write $a_{k+1} = \Psi(k)$ for a function $\Psi \colon \mathbb{N} \to \mathbb{C}$. That is, we will not need to determine a_k in order to find a_{k+1}.

We calculate a few terms for k small, starting with $k = 1$, since the formula does not tell us what a_0 should be:
$$a_1 = a_{0+1} = \frac{a_0}{0+1} = \frac{a_0}{1}$$
$$a_2 = a_{1+1} = \frac{a_1}{1+1} = \frac{a_0}{2}$$
$$a_3 = a_{2+1} = \frac{a_2}{2+1} = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2}$$
$$a_4 = a_{3+1} = \frac{a_3}{3+1} = \frac{a_3}{4} = \frac{a_0}{4 \cdot 3 \cdot 2}.$$
Continuing in this fashion, we find[23]
$$a_{k+1} = \frac{a_0}{(k+1)!}, \quad k \geq 0. \tag{1.6.1}$$
Thus
$$f(x) = \sum_{k=0}^{\infty} \frac{a_0}{k!} x^k = a_0 \sum_{k=0}^{\infty} \frac{x^k}{k!} = a_0 e^x.$$
▲

1.6.2 Example.

Solve
$$f'' + f = 0$$
by making a power series ansatz $f(x) = \sum_{k=0}^{\infty} a_k x^k$. *(Of course we expect to find* $f = c_1 \cos(x) + c_2 \sin(x)$.*)*

Solution. Differentiating twice, we have
$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{and} \quad f''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$
Then
$$0 = f'' + f = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} a_k x^k$$

[23] A formal proof of the equality (1.6.1) requires induction.

$$= \sum_{k=0}^{\infty} \big[(k+2)(k+1)a_{k+2} + a_k\big]x^k.$$

The identity principle implies

$$(k+2)(k+1)a_{k+2} + a_k = 0, \ \ k \geq 0,$$

and thus

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, \ \ k \geq 0.$$

We calculate a few coefficients, starting with $k = 0$ (i.e., with a_2), noting that the formula does not tell us what a_0 or a_1 should be. But we expect to have two "free" parameters in this problem, since it is a second-order ODE.

We find

$$a_2 = a_{0+2} = -\frac{a_0}{2 \cdot 1}$$

$$a_3 = a_{1+2} = \frac{a_1}{(1+2)(1+1)} = -\frac{a_1}{3 \cdot 2}$$

$$a_4 = a_{2+2} = -\frac{a_2}{(2+2)(2+1)} = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$a_5 = a_{3+2} = -\frac{a_3}{(3+2)(3+1)} = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}.$$

The pattern that seems to emerge is

$$a_{2k} = a_0 \frac{(-1)^k}{(2k)!} \quad \text{and} \quad a_{2k+1} = a_1 \frac{(-1)^k}{(2k+1)!},$$

and so we are inclined to split our original series $\sum_{k=0}^{\infty} a_k x^k$ into the sum of two series, the even-indexed $\sum_{k=0}^{\infty} a_{2k} x^{2k}$ and the odd-indexed $\sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$.

From Lemma A.2.7, we know that the series $\sum_{k=0}^{\infty} a_k x^k$ converges if and only if the series $\sum_{k=0}^{\infty} \big(a_{2k} x^{2k} + a_{2k+1} x^{2k+1}\big)$ converges, and since the two series

$$\sum_{k=0}^{\infty} a_{2k} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = a_0 \cos(x)$$

and

$$\sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = a_1 \sin(x)$$

both converge, the splitting is legitimate. We conclude

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = a_0 \cos(x) + a_1 \sin(x). \qquad \blacktriangle$$

Now we state precisely how a power series ansatz is always a valid approach for ODEs with analytic coefficients.

1.6.3 Theorem (Fuchs).

Suppose that a_0, a_1, and g are real analytic on $I \subseteq \mathbb{R}$. Let $f \in \mathcal{C}^2(I)$ solve[24]

$$f'' + a_1(x)f' + a_0(x)f = g(x), \ x \in I. \tag{1.6.2}$$

Then f is also real analytic on I, and for any $x_0 \in I$ the power series for f centered at x_0 converges with radius of convergence at least as large as the minimum of the radii of convergence for the series of a_0, a_1, and g centered at x_0.

That is, let $x_0 \in I$ and let r_0, r_1, $r_g > 0$ be the radii of convergence of the power series for a_0, a_1, and g centered at x_0, respectively. Then there is a sequence (a_k) in $\mathbb{R}$ such that $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ for $|x - x_0| < \min\{r_0, r_1, r_g\}$.

In general, the nth derivative of a solution to an ODE will be at least as differentiable as the coefficients of that ODE. For example, if f solves (1.6.2), then $f'' = g(x) - a_1(x)f' - a_0(x)f$. By our convention (Definition 1.1.4), f must be at least $\mathcal{C}^2$. But suppose a_1, a_0, and g are all $\mathcal{C}^1$. Then since f' and f are both $\mathcal{C}^1$, then sum $g - a_1f' - a_0f$ is also $\mathcal{C}^1$. That is, f'' is $\mathcal{C}^1$, and so f is $\mathcal{C}^3$. We can **BOOTSTRAP** in this way to show that if $f \in \mathcal{C}^n(I)$ solves the ODE $\partial_x^n[f] + \sum_{k=0}^{n-1} a_k(x)\partial_x^k[f] = g(x)$ with a_k, $g \in \mathcal{C}^m(I)$, then $f \in \mathcal{C}^{n+m}(I)$. In particular, if $m = \infty$, then such a solution f must be infinitely differentiable. And so it stands to reason (though, of course, it needs further proof!) that if the coefficients of the ODE are real analytic, then the solution should be, too.

1.6.4 Example.

Find a fundamental solution set on $\mathbb{R}$ for **AIRY'S EQUATION**,

$$f''(x) + xf(x) = 0.$$

Solution. Since the coefficients in Airy's equation are real analytic — specifically, they are polynomials — we know that there exist solutions of the form $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where this series converges on $\mathbb{R}$. We make such a power series ansatz and differentiate twice, finding

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{and} \quad f''(x) = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2},$$

and then evaluate

$$0 = f''(x) + xf(x) = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + x\sum_{k=0}^{\infty} a_k x^k = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^{k+1}.$$

We want to combine everything into one series with a factor of x^k. As before, we reindex the first series as

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k.$$

The second series is

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k.$$

[24] Such a solution exists by variation of parameters.

Thus

$$0 = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k = 2a_2 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k$$

$$= 2a_2 + \sum_{k=1}^{\infty}\big[(k+2)(k+1)a_{k+2} + a_{k-1}\big]x^k.$$

Hence $a_2 = 0$ and

$$(k+2)(k+1)a_{k+2} + a_{k-1} = 0, \ k \geq 1,$$

thus

$$a_{k+2} = -\frac{a_{k-1}}{(k+2)(k+1)}, \ k \geq 1.$$

Note that this formula will not give us information on a_2 (which we already know to be 0) nor on a_0 or a_1. But this is a second-order ODE, so we expect two free parameters, which will be a_0 and a_1.

We compute some coefficients for small k:

$$a_3 = a_{1+2} = -\frac{a_{1-1}}{(1+2)(1+1)} = -\frac{a_0}{3\cdot 2}$$

$$a_4 = a_{2+2} = -\frac{a_{2-1}}{(2+2)(2+1)} = -\frac{a_1}{4\cdot 3}$$

$$a_5 = a_{3+2} = -\frac{a_{3-1}}{(3+2)(3+1)} = -\frac{a_2}{5\cdot 4} = 0$$

$$a_6 = a_{4+2} = -\frac{a_{4-1}}{(4+2)(4+1)} = -\frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2}$$

$$a_7 = a_{5+2} = -\frac{a_{5-1}}{(5+2)(5+1)} = -\frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}$$

$$a_8 = a_{6+2} = -\frac{a_{6-1}}{(6+2)(6+1)} = -\frac{a_5}{8\cdot 7} = 0$$

$$a_9 = a_{7+2} = -\frac{a_{7-1}}{(7+2)(7+1)} = -\frac{a_6}{9\cdot 8} = -\frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$$

$$a_{10} = a_{8+2} = -\frac{a_{8-1}}{(8+2)(8+1)} = -\frac{a_7}{10\cdot 9} = -\frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$$

$$a_{11} = a_{9+2} = -\frac{a_{9-1}}{(9+2)\cdot(9+1)} = -\frac{a_8}{11\cdot 10} = 0.$$

If we look closely, we see a pattern repeating in threes, and we can write any integer $k \in \mathbb{Z}$ in the form

$$k = 3j \quad \text{or} \quad k = 3j+1 \quad \text{or} \quad k = 3j+2$$

for some $j \in \mathbb{Z}$. (This is nothing more than division by 3.) Most obviously, then,

$$a_{3j+2} = 0, \ j \geq 0.$$

Next, put $\alpha_0 = \beta_0 = 1$ and, for $j \geq 1$,

$$\alpha_j := \frac{(-1)^j}{\big[(3j)\cdot(3j-1)\big]\cdot\big[(3j-3)\cdot(3j-4)\big]\cdots\big[6\cdot 5\big]\cdot\big[3\cdot 2\big]}$$

and

$$\beta_j := \frac{(-1)^j}{\big[(3j+1)\cdot(3k)\big]\cdot\big[(3j-2)\cdot(3j-3)\big]\cdots\big[7\cdot 6\big]\cdot\big[4\cdot 3\big]}.$$

Then

$$a_{3j} = a_0\alpha_j \quad \text{and} \quad a_{3j+1} =: a_1\beta_j.$$

Rigorous proofs that a_{3j}, a_{3j+1}, and a_{3j+2} satisfy the formulas above would all require induction on j, which we omit.

Now, we know *a priori* that if f solves Airy's equation, then we can write $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where the series $\sum_{k=0}^{\infty} a_k x^k$ converges on $\mathbb{R}$. And from the above we expect that

$$f(x) = \sum_{k=0}^{\infty} a_{3k}x^{3k} + \sum_{k=0}^{\infty} a_{3k+1}x^{3k+1} = a_0 \underbrace{\sum_{k=0}^{\infty} \alpha_k x^{3k}}_{\mathfrak{J}_1(x)} + a_1 \underbrace{\sum_{k=0}^{\infty} \beta_k x^{3k+1}}_{\mathfrak{J}_2(x)}. \tag{1.6.3}$$

Splitting up the series in this manner is valid: the power series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for all $x \in \mathbb{R}$ by Fuchs's theorem and Theorem A.4.3. Then we use Lemma A.2.7 with $N = 3$, recalling that $a_{3j+2} = 0$ for all j.

So, the general solution to Airy's equation is $f(x) = a_0\mathfrak{J}_1(x) + a_1\mathfrak{J}_2(x)$, where a_0 and a_1 are the first coefficients in the power series expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Can a_0 and a_1 be arbitrary? Just because we did not find formulas for a_0 and a_1 with the identity principle analysis above does not preclude that, perhaps, they have restricted values. For that matter, do $\mathfrak{J}_1$ and $\mathfrak{J}_2$ form a fundamental solution set for Airy's equation? The condition $\mathcal{A}(a_0\mathfrak{J}_1 + a_1\mathfrak{J}_2) = \mathcal{A}f = 0$ does not imply[25] $\mathcal{A}\mathfrak{J}_1 = \mathcal{A}\mathfrak{J}_2 = 0$. The answer to both questions is yes; in particular, we will be able to take a_0 and a_1 arbitrary if we just show that $\{\mathfrak{J}_1, \mathfrak{J}_2\}$ is an FSS.

First, the general existence theory implies that the IVP

$$\begin{cases} f'' + xf = 0 \\ f(0) = 1 \\ f'(0) = 0 \end{cases}$$

[25] Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces and $T\colon \mathcal{X} \to \mathcal{Y}$ be a linear operator. Suppose $f \notin \ker(T)$ and set $f_1 = f$ and $f_2 = -f$. Then $T(f_1 + f_2) = 0$ but $Tf_1 \neq 0$ and $Tf_2 \neq 0$.

has a unique solution f_0, and Fuchs's theorem implies that f_0 has a power series expansion of the form $f_0 = \sum_{k=0}^{\infty} a_k x^k$ on $\mathbb{R}$, where

$$a_0 = f_0(0) = 1 \quad \text{and} \quad a_1 = f'(0) = 0.$$

Our analysis above implies that

$$f_0 = a_0 \mathcal{J}_1 + a_1 \mathcal{J}_2 = \mathcal{J}_1.$$

So, $\mathcal{J}_1$ is indeed a solution of Airy's equation. And by similar reasoning with the IVP

$$\begin{cases} f'' + xf = 0 \\ f(0) = 0 \\ f'(0) = 1, \end{cases}$$

so is $\mathcal{J}_2$.

Last, we show that $\mathcal{J}_1$ and $\mathcal{J}_2$ are linearly independent. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ solve a second-order continuous-coefficient ODE, we may do this merely by demonstrating $\mathcal{W}[\mathcal{J}_1, \mathcal{J}_2](0) \neq 0$. Since

$$\mathcal{J}_1(x) = \alpha_0 + \alpha_1 x^3 + \sum_{k=2}^{\infty} \alpha_k x^{3k} \quad \text{and} \quad \mathcal{J}_2(x) = \beta_0 x + \beta_1 x^4 + \sum_{k=2}^{\infty} \beta_k x^{3k+1},$$

we may calculate

$$\mathcal{J}_1(0) = \alpha_0 = 1 \qquad \mathcal{J}_2(0) = 0$$

$$\mathcal{J}_1'(0) = 0 \qquad \mathcal{J}_2'(0) = \beta_0 = 1,$$

and so

$$\mathcal{W}[\mathcal{J}_1, \mathcal{J}_2](0) = \det\left(\begin{bmatrix} \mathcal{J}_1(0) & \mathcal{J}_2(0) \\ \mathcal{J}_1'(0) & \mathcal{J}_2'(0) \end{bmatrix}\right) = (1 \cdot 1) - (0 \cdot 0) = 1.$$

So, $\mathcal{J}_1$ and $\mathcal{J}_2$ are linearly independent solutions to Airy's equation on $\mathbb{R}$ and therefore form a fundamental solution set. ▲

1.6.5 Example.

Make a power series ansatz $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *for*

$$f'' + e^{-x} f = 0$$

and find, but do not solve, a recursion relation for the coefficients a_k.

Solution. With $f(x) = \sum_{k=0}^{\infty} a_k x^k$, we have, as usual,

$$0 = f'' + e^{-x} f = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + e^{-x} \sum_{k=0}^{\infty} a_k x^k.$$

Of course, we also have

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}.$$

Now we want to multiply the power series product

$$\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty}a_kx^k\right).$$

To do so, we need the **Cauchy product formula**: if $\sum_{k=0}^{\infty}A_k$ converges and $\sum_{k=0}^{\infty}|B_k|$ converges, then

$$\left(\sum_{k=0}^{\infty}A_k\right)\left(\sum_{k=0}^{\infty}B_k\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_kB_{n-k}. \tag{1.6.4}$$

Thus

$$\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty}a_kx^k\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{x^k}{k!}a_{n-k}x^{n-k}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)a_{n-k}x^n.$$

Plugging this into the above, we have

$$\begin{aligned}0&=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}a_{n-k}\right)x^n\\&=\sum_{k=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+\sum_{k=0}^{n}\frac{(-1)^k}{k!}a_{n-k}\right]x^n,\end{aligned}$$

and so

$$(n+2)(n+1)a_{n+2}+\sum_{k=0}^{n}\frac{(-1)^k}{k!}a_{n-k}=0.$$

Our recursion relation is therefore

$$a_{n+2}=\frac{1}{(n+2)(n+1)}\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k!}a_{n-k}.$$

It is unlikely that we could ever solve this recursion relation in the explicit form $a_n=\Phi(a_0,a_1,n)$ for some function $\Phi\colon\mathbb{R}^2\times\mathbb{N}\to\mathbb{R}$, so we would need to use a computer to calculate however many terms we need for an accurate solution. ▲

1.6.6 Method: solve $f''+\phi(x)f'+\psi(x)f=0$ with power series

0. Assume ϕ and ψ are real analytic at 0.

1. Substitute $f(x)=\sum_{k=0}^{\infty}a_kx^k$ and differentiate

$$f'(x)=\sum_{k=1}^{\infty}ka_kx^{k-1}\quad\text{and}\quad f''(x)=\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2}.$$

2. With f' and f'' in power series form, multiply $\phi(x)f'$ and $\psi(x)f$, possibly expressing ϕ and ψ themselves as power series and using the Cauchy product formula (1.6.4).

3. Reindex the left side of the ODE to write it in the form $\sum_{k=0}^{\infty}\alpha(a_k,k)x^k$, where $\alpha(a_k,k)$

depends on both a_k and k. (There may be "straggler" terms for small k that do not fit nicely into a formula $\alpha(a_k, k)$ valid only for larger k.)

4. Set $\alpha(a_k, k) = 0$ and try to solve explicitly for a_k as a function of k, a_1, and a_0. An explicit solution may not be possible, or there may be different formulas for k even and k odd, etc.

2. Partial Differential Equations and Fourier Analysis

2.1. Fundamentals of PDE.

Broadly, all material on PDEs was inspired by [4, 15, 24]. As with ordinary differential equations, we will not give a precise definition of a **partial differential equation (PDE)** but merely say that it is an equation involving one or more derivatives of a function of one or more variables. If u is a function of the independent variables x and t, we will write $u = u(x,t)$ and denote partial derivatives (say, with respect to x) using any one of the following symbols:

$$\frac{\partial u}{\partial x}, \qquad \frac{\partial}{\partial x}[u], \qquad \partial_x[u], \qquad u_x.$$

The **order** of a PDE is, of course, the order of the highest derivative in the equation.

2.1.1 Example.

The general form of a **first-order linear PDE** *for a function of two variables is*

$$a_0(x,t)u(x,t) + a_1(x,t)u_x(x,t) + a_2(x,t)u_t(x,t) = g(x,t),$$

and the general form of a **second-order linear PDE** *for a function of two variables is*

$$a_0(x,t)u(x,t) + a_1(x,t)u_x(x,t) + a_2(x,t)u_t(x,t) + b_{11}(x,t)u_{xx}(x,t) + b_{12}(x,t)u_{xt}(x,t) + b_{22}(x,t)u_{tt} = g(x,t).$$

Unlike ODEs, there is no general theory of solutions for "all" nth order linear PDEs. Solutions depend greatly on a number of circumstances, including

- The order of the PDE: a higher-order PDE contains more terms;

- The *geometry* of the domain in $\mathbb{R}^2$ (or $\mathbb{R}^n$) on which the PDE is intended to be solved: there is much more flexibility in choosing a domain in $\mathbb{R}^2$ for a PDE than an interval in $\mathbb{R}$ for an ODE;

- The initial conditions and/or boundary conditions that solutions of the PDE must satisfy: these can greatly vary based on the order of the PDE and the domain in which one works.

Due to these challenges, we will cover only very specific PDEs in this course.

2.2. Three elementary techniques.

We collect together three kinds of equations/techniques that are particularly fundamental for our future work.

2.2.1. Equations with a missing variable.

Suppose that $u = u(x,t)$ solves a PDE that only depends on x. Then this PDE is really an ODE, so we can solve it by prior techniques provided that we allow all arbitrary constants to depend on t.

2.2.1 Example.

Find all functions $u = u(x,t)$ such that

$$u_{xx} + u = 0, \ (x,t) \in \mathbb{R}^2.$$

Solution. Fix $t \in \mathbb{R}$ and let $g(x) = u(x,t)$. Then $g'' + g = 0$, so

$$g(x) = c_1 \sin(x) + c_2 \cos(x)$$

for some c_1, $c_2 \in \mathbb{R}$. But g depends on t, so c_1 and c_2 really depend on t, too. Thus the general solution is

$$u(x,t) = c_1(t) \sin(x) + c_2(t) \cos(x),$$

where c_1, $c_2 \colon \mathbb{R} \to \mathbb{R}$ are arbitrary functions. (Since we only care that u is twice differentiable with respect to x, the functions c_1 and c_2 do not even have to be continuous in t!) ▲

2.2.2. The transport equation.

2.2.2 Example.

Suppose that a, $b \in \mathbb{R}$ are not both zero. Solve the **TRANSPORT EQUATION**

$$au_x + bu_t = 0, \ (x,t) \in \mathbb{R}^2$$

for $u = u(x,t)$.

Solution. First suppose $b \neq 0$ and let us work in reverse: suppose that u solves the transport equation. We will deduce a special form for u. It will then be a direct calculation that any function of this form solves the transport equation.

Recall that the **GRADIENT** of $u \colon \mathbb{R}^2 \to \mathbb{R}$ is the vector $\nabla u(x,t) := (u_x(x,t), u_t(x,t)) \in \mathbb{R}^2$. Then

$$0 = au_x + bu_t = (a,b) \cdot (u_x, u_t) = (a,b) \cdot \nabla u \tag{2.2.1}$$

Recall that $(a,b) \cdot \nabla u$ is the directional derivative of u in the direction of the vector[26] (a,b), and so we see that u is constant along vectors parallel to (a,b). More precisely, fix $(x,t) \in \mathbb{R}^2$ and set

$$z(s) = u(x + sa, t + sb).$$

Then (2.2.1) and the chain rule tell us

$$z'(s) = u_x(x+sa, t+sb)a + u_t(x+sa, t+sb)b = \big(u_x(x+sa, t+sb), u_t(x+sa, t+sb)\big) \cdot (a,b)$$

$$= \nabla u(x + sa, t + sb) \cdot (a,b) = 0.$$

That is, z is constant in s, and so for any fixed $s_0 \in \mathbb{R}$, we have

$$u(x + sa, t + sb) = z(s) = z(s_0) = u(x + s_0 a, t + s_0 b), \ s, x, t \in \mathbb{R}.$$

[26] Technically (a,b) should be a unit vector.

In particular, taking $s = 0$, we find

$$u(x,t) = u(x + s_0 a, t + s_0 b).$$

Let us choose s_0 to be some convenient value; if we take $s_0 = -t/b$, then we find

$$u(x,t) = u\left(x + \left(-\frac{t}{b}\right)a, t + \left(-\frac{t}{b}\right)b\right) = u\left(x - \frac{at}{b}, 0\right) = u\left(\frac{bx - at}{b}, 0\right), \quad x, t \in \mathbb{R}.$$

That is, the behavior of u depends only on its first coordinate. And so we may set

$$f(X) := u\left(\frac{X}{b}, 0\right)$$

to find that $f \in \mathcal{C}^1(\mathbb{R})$ and u has the form

$$u(x,t) = f(bx - at). \tag{2.2.2}$$

Conversely, if $f \in \mathcal{C}^1(\mathbb{R})$ is arbitrary, and we define u by (2.2.2), then a direct calculation shows that this function u satisfies the transport equation.

Now suppose $b = 0$ but $a \neq 0$. Then the transport equation is really an ODE, which we solve as

$$au_x = 0 \Longrightarrow u_x = 0 \Longrightarrow u(x,t) = C(t)$$

for some differentiable function $C = C(t)$.

In fact, this solution can be put in the same form as above when $b \neq 0$. If $u(x,t) = C(t)$ and $a \neq 0$, $b = 0$, then

$$u(x,t) = \widetilde{C}(bx - at),$$

where

$$\widetilde{C}(\tau) = C\left(-\frac{\tau}{a}\right).$$

And so, whether or not $b = 0$, the general solution to the transport equation is $u(x,t) = f(bx - at)$, where $f = f(X)$ is an arbitrary differentiable function. ▲

The procedure above illustrates how we often solve PDEs: we assume there is a solution and try to deduce its form. Then we must verify that the deduced form actually solves the PDE. This is not unlike how we solved constant-coefficient ODEs: we posited that a solution would have the form $f(x) = e^{\lambda x}$ for some $\lambda \in \mathbb{C}$, and then we determined the λ that would give a solution. We never actually went back to check that $f(x) = e^{\lambda x}$ solves the ODE (although that was apparent from the characteristic equation factoring, see, e.g., the if and only if statement in (1.4.3).)

2.2.3 Example.

Solve

$$\begin{cases} 4u_x - 3u_t = 0, & (x,t) \in \mathbb{R}^2 \\ u(0,t) = t^3, & t \in \mathbb{R}. \end{cases}$$

Solution. The general solution to the transport equation $4u_x - 3u_t = 0$ has the form

$$u(x,t) = f(-3x - 4t)$$

for some function $f \in \mathcal{C}^1(\mathbb{R})$. To satisfy the boundary condition, we need

$$t^3 = u(0,t) = f(-4t).$$

In fact, this gives a formula for f. Set $X = -4t$, so $t = -X/4$. Then

$$f(X) = f(-4t) = t^3 = \left(-\frac{X}{4}\right)^3 = -\frac{X^3}{64}.$$

Hence

$$u(x,t) = -\frac{(-3x-4t)^3}{64} = \frac{(3x+4t)^3}{64}. \qquad ▲$$

2.2.3. Separation of variables.

An *ansatz* that often yields good results on a PDE is to assume that a solution $u = u(x,t)$ is a product of two functions of a single variable, i.e.,

$$u(x,t) = X(x)T(t)$$

for some functions $X = X(x)$ and $T = T(t)$ of one variable. Then we can often derive simple ODEs that X and T must satisfy.

2.2.4 Example.

Solve the transport equation

$$u_x - 4u_t = 0$$

with the ansatz $u(x,t) = X(x)T(t)$.

Solution. To be clear about what we shall do, we are assuming we have a solution $u(x,t) = X(x)T(t)$ to this PDE, and we want to discern more properties of X and T. We calculate

$$u_x = \partial_x[X(x)T(t)] = X'(x)T(t) \quad \text{and} \quad u_t = \partial_t[X(x)T(t)] = X(x)T'(t),$$

hence

$$0 = u_x - 4u_t = X'(x)T(t) - 4X(x)T'(t).$$

This rearranges to

$$\frac{X'(x)}{X(x)} = \frac{4T'(t)}{T(t)}, \tag{2.2.3}$$

assuming $X(x) \neq 0$ and $T'(t) \neq 0$; note that $u = 0$ is a solution to the PDE, so $X = 0$ or $T = 0$ is allowed. Since (2.2.3) must hold for any x, $t \in \mathbb{R}$, we have

$$\frac{X'(x)}{X(x)} = \frac{4T'(t_0)}{T(t_0)} =: \lambda \tag{2.2.4}$$

for all $x \in \mathbb{R}$ and any fixed $t_0 \in \mathbb{R}$. That is, X must satisfy the ODE

$$X' = \lambda X \Longrightarrow X(x) = C_1 e^{\lambda x}$$

for some constants $C_1 \in \mathbb{R}$. (Note that since X is a function of only x, we do not allow C_1 or λ to depend on t.) Similarly, fixing $x_0 \in \mathbb{R}$ and letting $t \in \mathbb{R}$ be arbitrary, we have

$$\frac{4T'(t)}{T(t)} = \frac{X'(x_0)}{X(x_0)} = \lambda.$$

The first equality is (2.2.3), and the second is (2.2.4). Thus

$$T'(t) = \frac{\lambda}{4}T \Longrightarrow T(t) = C_2 e^{\lambda t/4},$$

and so

$$u(x,t) = X(x)T(t) = \big(C_1 e^{\lambda x}\big)\big(C_2 e^{\lambda t/4}\big) = Ce^{\lambda x}e^{\lambda t/4}.$$

We have combined C_1 and C_2 into one arbitrary constant C, and so we are left with two "free" parameters in our solution, C and λ. This does not surprise us; an nth order ODE in one variable has n free parameters, so a first order PDE in two variables should have two.

Now let us compare this solution to the general solution that we expect from Example 2.2.2. This example tells us that putting $u(x,t) = f(-4x-t)$, where f is *any* differentiable function, will solve $u_x - 4u_t = 0$. We rewrite

$$Ce^{\lambda x}e^{\lambda t/4} = Ce^{\lambda(x+t/4)} = Ce^{-(\lambda/4)(-4x-t)}.$$

Thus putting $f(\xi) := Ce^{-(\lambda/4)\xi}$ allows us to write any product solution as $u(x,t) = f(-4x-t)$.

However, there are many other solutions not of this form. For example, putting $u(x,t) = \sin(-4x-t)$ also solves $u_x - 4u_t = 0$, but we cannot write $\sin(\xi) = Ce^{(-\lambda/4)\xi}$, no matter how we choose C and λ. The moral, then, is that separation of variables need not give all solutions to a PDE. ▲

2.3. The heat equation.

The **HEAT EQUATION** is the PDE

$$u_t = u_{xx}. \tag{2.3.1}$$

The heat equation models the temperature $u(x,t)$ of an idealized, one-dimensional rod at a horizontal position x along the rod and a point in time t. The rod is assumed to have the same "thermal conductivity" properties throughout its length. Often the heat equation is written as $u_t = \kappa u_{xx}$, where $\kappa > 0$ is a material constant derived from properties of the rod. We will see how to "nondimensionalize" the heat equation in the exercises, so that (2.3.1) is the only version we need to consider. We begin by finding product solutions to the heat equation, which will naturally lead to Fourier series. Later, we will study the heat equation for a rod of "infinite" length via the Fourier transform and also for a rod whose thermal properties vary across the rod, which will lead to the more general theory of boundary value problems for ODEs.

2.3.1 Example.

Determine all real-valued solutions to the heat equation of the form $u(x,t) = X(x)T(t)$.

Solution. We will assume that X and T are real-valued here, since if u is measuring heat, u should be real-valued. We have

$$u_t = X(x)T'(t), \qquad u_x = X'(x)T(t), \qquad u_{xx} = X''(x)T(t),$$

so we find

$$X(x)T'(t) = X''(x)T(t) \Longrightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

Then there is $\lambda \in \mathbb{R}$ (since X and T are real) such that

$$\frac{X''(x)}{X(x)} = \lambda \quad \text{and} \quad \frac{T'(t)}{T(t)} = \lambda,$$

thus

$$X'' = \lambda X \quad \text{and} \quad T'(t) = \lambda T.$$

We conclude $T(t) = Ce^{\lambda t}$, but for X we need to consider cases on λ.

1. $\lambda < 0$. Write $\lambda = -\alpha^2$ for some $\alpha > 0$, so X satisfies

$$X'' + \alpha^2 X = 0$$

and consequently

$$X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

In this case we see that a solution to the heat equation is

$$u(x,t) = X(x)T(t) = Ce^{-\alpha^2 t}\big[c_1 \cos(\alpha x) + c_2 \sin(\alpha x)\big] = e^{-\alpha^2 t}\big[\beta_1 \cos(\alpha x) + \beta_2 \sin(\alpha x)\big].$$

Here we have written $\beta_1 = Cc_1$ and $\beta_2 = Cc_2$.

2. $\lambda = 0$. Then X satisfies

$$X'' = 0,$$

so directly integrating twice we find $X(x) = c_1 x + c_2$. Thus

$$u(x,t) = X(x)T(t) = Ce^{0\cdot t}(c_1 x + c_2) = Cc_1 x + Cc_2 = \beta_1 x + \beta_2.$$

3. $\lambda > 0$. Write $\lambda = \alpha^2$ for some $\alpha > 0$. Then X satisfies

$$X'' - \alpha^2 X = 0,$$

so

$$X = c_1 e^{\alpha x} + c_2 e^{-\alpha x},$$

and therefore

$$u(x,t) = X(x)T(t) = Ce^{\alpha^2 t}\big[c_1 e^{\alpha x} + c_2 e^{-\alpha x}\big] = e^{\alpha^2 t}\big[\beta_1 e^{\alpha x} + \beta_2 e^{-\alpha x}\big]. \qquad \blacktriangle$$

These "product solutions" to the heat equation are far from the only valid solutions; we will derive a very different solution using the Fourier transform.

2.3.2 Example.

Fix $P > 0$. Determine all $f \in \mathcal{C}([0,P])$ for which there is a product solution to

$$\begin{cases} u_t = u_{xx}, \ 0 \le x \le P, \ t \ge 0 \\ u(0,t) = u(P,t) = 0, \ t \ge 0 \\ u(x,0) = f(x), \ 0 \le x \le P. \end{cases} \tag{2.3.2}$$

This initial/boundary value problem (IVP/BVP) roughly models the flow of heat in a one-dimensional rod of length P in which the temperature of both ends of the rod is maintained at 0.

Solution. From Example 2.3.1, a product solution for the heat equation has one of the following three forms:

$$u(x,t) = \begin{cases} e^{-\alpha^2 t}\big[\beta_1 \cos(\alpha x) + \beta_2 \sin(\alpha x)\big] \\ \beta_1 x + \beta_2 \\ e^{\alpha^2 t}\big[\beta_1 e^{\alpha x} + \beta_2 e^{-\alpha x}\big] \end{cases} \tag{2.3.3}$$

for constants $\alpha > 0$ and $\beta_1,\ \beta_2 \in \mathbb{R}$.

We claim that if we fit the latter two cases to the boundary conditions, we obtain only the trivial solution $u(x,t) = 0$. First, suppose $u(x,t) = \beta_1 x + \beta_2$. Then $0 = u(0,t) = \beta_2$ and so $0 = u(P,t) = \beta_1 P$. Since $P > 0$, we have $\beta_1 = 0$, and thus $u = 0$. In particular, we must have $f(x) = u(x,0) = 0$.

Now suppose $u(x,t) = e^{\alpha^2 t}\big[\beta_1 e^{\alpha x} + \beta_2 e^{-\alpha x}\big]$. Then

$$0 = u(0,t) = e^{\alpha^2 t}(\beta_1 + \beta_2) \Longrightarrow \beta_1 + \beta_2 = 0$$

and

$$0 = u(P,t) = e^{\alpha^2 P}\big[\beta_1 e^{\alpha P} + \beta_2 e^{-\alpha P}\big] \Longrightarrow \beta_1 e^{\alpha P} + \beta_2 e^{-\alpha P} = 0.$$

Then β_1 and β_2 satisfy the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\alpha P} & e^{-\alpha P} \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of this matrix is $e^{-\alpha P} - e^{\alpha P}$. Since α and P are nonzero, $\alpha P \neq -\alpha P$, and since the exponential is one-to-one, we have $e^{-\alpha P} \neq e^{\alpha P}$. So, the determinant is nonzero, the matrix is invertible, and $\beta_1 = \beta_2 = 0$. Again, $u = 0$ and $f = 0$.

So, we focus on the first case in (2.3.3). We find

$$0 = u(0,t) = \beta_1 e^{-\alpha^2 t} \Longrightarrow \beta_1 = 0$$

and

$$0 = u(P,t) = \beta_2 e^{-\alpha^2} \sin(\alpha P).$$

To avoid a trivial solution, we require $\sin(\alpha P) = 0$, and thus

$$\alpha P = k\pi \iff \alpha = \frac{k\pi}{P}$$

for some $k \in \mathbb{Z}$. Thus

$$u(x,t) = Ce^{(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right)$$

for some $k \in \mathbb{Z}$, $C > 0$, and $\alpha > 0$. Hence

$$f(x) = u(x,0) = C \sin\left(\frac{k\pi x}{P}\right).$$ ▲

Since the "heat operator"

$$\mathcal{A} := \partial_t - \partial_x^2$$

is linear, we see that if we take the initial condition of the heat equation to be

$$f(x) = \sum_{k=0}^{n} C_k \sin\left(\frac{k\pi x}{P}\right) \tag{2.3.4}$$

for $C_k \in \mathbb{R}$, then

$$u(x,t) := \sum_{k=0}^{n} C_k e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right)$$

satisfies $\mathcal{A}u = 0$ and $u(x,0) = f(x)$. Thus if the initial condition for the heat equation is given as a "trigonometric polynomial" like (2.3.4), we can read off the solution just from the coefficients of f. But trigonometric polynomials constitute a very restrictive class of functions: they are periodic, $\mathcal{C}^\infty$, and bounded.

More broadly, we know from real and complex variable theory that it is often advantageous to represent a function via its Taylor series. This does not require the function to be a trigonometric polynomial (in particular, the function need not be periodic), but it does require the function to be $\mathcal{C}^\infty$ and real analytic (Section 3.7.1). Otherwise the resemblance to a (normal) polynomial disappears; a polynomial necessarily has roots in $\mathbb{C}$ and is unbounded at $\pm\infty$, whereas there are bounded and zero-free (real) analytic functions. And so we might expect that some of the nice, but restrictive, properties of trigonometric polynomials disappear when we consider a trigonometric *series*, thereby allowing us to represent a wider class of functions as a limit of trigonometric polynomials. This brings us to the theory and practice of Fourier series.

2.4. Fourier series.

This presentation draws on [4, 9, 17, 20]. In general, any omitted proofs can be found in [9]. Familiarity with inner product spaces (Appendix C.4) and generalized Fourier series therein (Appendix C.5) will be helpful throughout but necessary only if one wishes to understand some of the proofs and technical language in Sections 2.4.5 and 2.4.6.

2.4.1. Trigonometric polynomials.

We begin by formalizing some of the ideas from the end of Section 2.3. First, observe that for any $P > 0$ and any integer $k \in \mathbb{Z}$, the mapping

$$f(x) := \cos\left(\frac{k\pi x}{P}\right) + i\sin\left(\frac{k\pi x}{P}\right) = e^{ik\pi x/P} \tag{2.4.1}$$

is $2P$-periodic, i.e., $f(x+2P) = f(x)$ for all $x \in \mathbb{R}$. Next, we generalize and name the object from (2.3.4) that proved so useful in solving the heat equation with a superposition of product solutions.

2.4.1 Definition.

Let $P > 0$. A TRIGONOMETRIC POLYNOMIAL ON THE INTERVAL $[-P,P]$ *is a function of the form*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(\frac{k\pi x}{P}\right) + b_k \sin\left(\frac{k\pi x}{P}\right), = \sum_{k=-n}^{n} c_k e^{ik\pi x/P}$$

for some coefficients $\{a_k\}_{k=0}^{n}$, $\{b_k\}_{k=1}^{n} \subseteq \mathbb{C}$. The factor of $1/2$ on a_0 is a useful convention.

We will work on the "symmetric" interval $[-P,P]$ for quite some time. This is not the interval $[0,P]$ that we met in Example 2.3.2, and it is certainly not the only interval we could use — indeed, all of the subsequent theory could be developed on an arbitrary interval $[a, b]$. However, in that case, many calculations and definitions become much messier. We will see in the exercises that one can always adapt, in a natural way, the results on $[-P, P]$ to any interval $[a, b]$. Later, in Section 2.4.7, we will return to the particular case of a "half" interval $[0, P]$, as well as to the situation in Example 2.3.2, in which we only had sines in the trigonometric polynomial. That, too, will incorporate much of our coming hard work on symmetric intervals $[-P, P]$.

It is always possible, and often advantageous, to express a trigonometric polynomial in terms of complex exponentials using Euler's formula (2.4.1).

2.4.2 Lemma.

For any set of coefficients $\{a_k\}_{k=0}^n$, $\{b_k\}_{k=1}^n \subseteq \mathbb{C}$, there are coefficients $\{c_k\}_{k=-n}^n \subseteq \mathbb{C}$ such that

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(\frac{k\pi x}{P}\right) + b_k \sin\left(\frac{k\pi x}{P}\right) = \sum_{k=-n}^{n} c_k e^{ik\pi x/P}.$$

The two sets of coefficients are related via the identities

$$a_0 = 2c_0, \qquad a_k = c_k + c_{-k},\ k \geq 1, \qquad \text{and} \qquad b_k = i(c_k - c_{-k}),\ k \geq 1. \tag{2.4.2}$$

and

$$c_k = \begin{cases} \dfrac{a_k - ib_k}{2}, & k \geq 1 \\[2ex] \dfrac{a_0}{2}, & k = 0 \\[2ex] \dfrac{a_{-k} + ib_{-k}}{2}, & k \leq -1. \end{cases} \tag{2.4.3}$$

From a different point of view, if we know *a priori* that a function is a trigonometric polynomial, then we can always recover its coefficients.

2.4.3 Lemma.

(i) *Let $\{a_k\}_{k=0}^n$, $\{b_k\}_{k=1}^n \subseteq \mathbb{C}$ and define*

$$f(x) := \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(\frac{k\pi x}{P}\right) + b_k \sin\left(\frac{k\pi x}{P}\right).$$

Then[27]

$$a_k = \frac{1}{P}\int_{-P}^{P} f(x)\cos\left(\frac{k\pi x}{P}\right)\,dx \qquad \text{and} \qquad b_k = \frac{1}{P}\int_{-P}^{P} f(x)\sin\left(\frac{k\pi x}{P}\right)\,dx. \tag{2.4.4}$$

(ii) *Let $\{c_k\}_{k=-n}^n \subseteq \mathbb{C}$ and*

$$g(x) := \sum_{k=-n}^{n} c_k e^{ik\pi x/P}.$$

Then

$$c_k = \frac{1}{2P}\int_{-P}^{P} g(x)e^{-ik\pi x/P}\,dx. \tag{2.4.5}$$

We leave the proofs of these two lemmas as (mostly) computational exercises.

2.4.2. Basic definitions.

Now, our goal is to represent a function $f\colon [-P,P] \to \mathbb{C}$ as a **TRIGONOMETRIC SERIES**. That is, we would like to write

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{P}\right) + b_k \sin\left(\frac{k\pi x}{P}\right) \tag{2.4.6}$$

for *two* sequences (a_k) and (b_k) in $\mathbb{C}$. Of course, by "=" in (2.4.6), we mean that the series on the right side of "=" converges for all $x \in [-P,P]$ as a series in $\mathbb{C}$, according to Definition A.2.1, and also equals $f(x)$. That is, we want

$$f(x) = \lim_{n\to\infty} \underbrace{\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(\frac{k\pi x}{P}\right) + b_k \sin\left(\frac{k\pi x}{P}\right)}_{\mathsf{S}_n[f](x)}. \tag{2.4.7}$$

This is a lofty goal, and our ability to achieve it will depend greatly on certain continuity, differentiability, and periodicity properties of f.

Per Lemma 2.4.2, we know that we can write

$$\mathsf{S}_n[f](x) = \sum_{k=-n}^{n} c_k e^{ik\pi x/P} \tag{2.4.8}$$

for some $c_k \in \mathbb{C}$. So, we expect that if we have (2.4.6), then we also have

$$f(x) = \lim_{n\to\infty} \mathsf{S}_n[f](x) = \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ik\pi x/P}. \tag{2.4.9}$$

Ignoring the underlying function f for the moment and supposing that we have a sequence (c_k) in $\mathbb{C}$ indexed by $k \in \mathbb{Z}$, the second limit in (2.4.9) suggests that we should *define*

$$\sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/P} := \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ik\pi x/P}. \tag{2.4.10}$$

Our goal with Fourier series, then, is to write a function $f\colon [-P,P] \to \mathbb{C}$ as a series of either the form (2.4.6), which converges according to our usual definition of series convergence (Definition A.2.1) or as the doubly infinite series (2.4.10), which converges *synchronously* (Definition A.2.11).

Now that we know *how* a trigonometric series will converge, we need candidates for the coefficients a_k and b_k in (2.4.6) and c_k in (2.4.9). Lemma 2.4.3 provides the right intuition. In fact, *if* we have the representation (2.4.6) or (2.4.9), and *if* we can interchange an infinite sum and an integral, then (the proof of) Lemma 2.4.3 shows that the following are the *only* reasonable candidates for the coefficients.

[27] If we did not put the factor of $1/2$ on the a_0 term, then we would have to make a special exception for the case $k=0$ in the integral formula (2.4.4). Namely, if we put $f(x) := a_0 + \sum_{k=1}^{n} a_k \cos(k\pi x/P) + b_k \sin(k\pi x/P)$, then

$$a_0 = \frac{1}{2P}\int_{-P}^{P} f(x)\ dx$$

while we retain (2.4.4) for a_k and b_k with $k \geq 1$.

2.4.4 Definition.

Let $f\colon [-P,P] \to \mathbb{C}$ be integrable (in the sense of Definition A.5.3).

(i) *The* REAL FOURIER COEFFICIENTS OF f ON $[-P,P]$ *are*

$$\mathsf{a}_k[f] := \frac{1}{P}\int_{-P}^{P} f(x)\cos\left(\frac{k\pi x}{P}\right)\,dx, \ \ k \ge 0$$

and

$$\mathsf{b}_k[f] := \frac{1}{P}\int_{-P}^{P} f(x)\sin\left(\frac{k\pi x}{P}\right)\,dx, \ \ k \ge 1.$$

(ii) *The* COMPLEX FOURIER COEFFICIENTS OF f ON $[-P,P]$ *are*

$$\widehat{f}(k) := \frac{1}{2P}\int_{-P}^{P} f(x)e^{-ik\pi x/P}\,dx, \ \ k \in \mathbb{Z}.$$

The notation $\widehat{f}(k)$ is standard; the symbols $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$ are peculiar to these notes. Since we allow f to be complex-valued in Definition 2.4.4, the adjectives "real" and "complex" modifying "Fourier coefficients" *do not* mean that $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$ are necessarily real-valued or that $\widehat{f}(k)$ must be complex and not real. Rather, these adjectives refer to the functions against which we integrate f: the real-valued sine and cosine, or the complex exponential. These symbols have one disadvantage: they do not specify what P is. To add that in would probably overburden our notation, so we will live without it. Euler's formula and (the proofs of) Lemmas 2.4.2 and 2.4.3 give the following relations between the real and complex Fourier coefficients.

2.4.5 Lemma.

Let $f\colon [-P,P] \to \mathbb{C}$ be integrable. Then

$$\mathsf{a}_0[f] = 2\widehat{f}(0), \quad \mathsf{a}_k[f] = \widehat{f}(k) + \widehat{f}(-k), \ \ k \ge 1, \quad \text{and} \quad \mathsf{b}_k[f] = i\big(\widehat{f}(k) - \widehat{f}(-k)\big), \ \ k \ge 1$$

and

$$\widehat{f}(k) = \begin{cases} \dfrac{\mathsf{a}_k[f] - i\mathsf{b}_k[f]}{2}, & k \ge 1 \\[2ex] \dfrac{\mathsf{a}_0[f]}{2}, & k = 0 \\[2ex] \dfrac{\mathsf{a}_{-k}[f] + i\mathsf{b}_{-k}[f]}{2}, & k \le -1. \end{cases}$$

At last, let us give names to the objects in (2.4.6) and (2.4.9).

2.4.6 Definition.

Let $P > 0$ and $f\colon [-P,P] \to \mathbb{C}$ be integrable.

(i) *The* FORMAL REAL FOURIER SERIES *of f on $[-P, P]$ is*

$$\mathsf{FS}[f](x) := \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} \mathsf{a}_k[f]\cos\left(\frac{k\pi x}{P}\right) + \mathsf{b}_k[f]\sin\left(\frac{k\pi x}{P}\right).$$

(ii) *The* FORMAL COMPLEX FOURIER SERIES *of f on $[-P, P]$ is*

$$\mathsf{FS}_{\mathbb{C}}[f](x) := \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ik\pi x/P}.$$

The notation $\mathsf{FS}[f]$ is again peculiar to these notes (it originates in [4]). We use the adjective "formal" to indicate that we are not considering (yet!) whether or not these series converge; as we know from calculus, then, we should simply think of these series as sequences in $\mathbb{C}$ of the partial sums $\big(\mathsf{S}_n[f](x)\big)$ from (2.4.7) or (2.4.8).

2.4.3. Computational examples.

We will defer any questions of the convergence of real and/or complex Fourier series to Section 2.4.4 and practice calculating formal Fourier series here.

2.4.7 Example.

Let $f(x) = x$. Find the real and complex Fourier series for f on $[-P, P]$.

Solution. For the real series, we need to evaluate the integrals

$$\mathsf{a}_k[f] = \frac{1}{P}\int_{-P}^{P} x\cos\left(\frac{k\pi x}{P}\right)\,dx \quad \text{and} \quad \mathsf{b}_k[f] = \frac{1}{P}\int_{-P}^{P} x\sin\left(\frac{k\pi x}{P}\right)\,dx.$$

Integrating by parts, we have the antiderivatives

$$\int x\cos\left(\frac{k\pi x}{P}\right)\,dx = \frac{P^2}{k^2\pi^2}\left[\cos\left(\frac{k\pi x}{P}\right) + \frac{k\pi x}{P}\sin\left(\frac{k\pi x}{P}\right)\right]$$

and

$$\int x\sin\left(\frac{k\pi x}{P}\right)\,dx = \frac{P^2}{k^2\pi^2}\left[\sin\left(\frac{k\pi x}{P}\right) - \frac{k\pi x}{P}\cos\left(\frac{k\pi x}{P}\right)\right].$$

Note that these antiderivatives are only valid for $k \neq 0$. This is a recurring theme with Fourier series: we often need to compute the $k = 0$ coefficient separately. So, we do that first:

$$\mathsf{a}_0[f] = \frac{1}{P}\int_{-P}^{P} x\,dx = 0.$$

For $k \neq 0$, using the facts that

$$\sin(\pm k\pi) = 0 \quad \text{and} \quad \cos(\pm k\pi) = \cos(k\pi) = (-1)^k,$$

we have

$$\mathsf{a}_k[f] = \frac{1}{P}\int_{-P}^{P} x\cos\left(\frac{k\pi x}{P}\right)\,dx$$

$$= \frac{1}{P}\left(\frac{P^2}{k^2\pi^2}\right)[\cos(k\pi) + k\pi\sin(k\pi) - \cos(-k\pi) - k\pi(-1)\sin(-k\pi)]$$

$$= \frac{P}{k^2\pi^2}[(-1)^k + k\pi\cdot 0 - (-1)^k + k\pi\cdot 0]$$

$$= 0$$

and

$$\mathsf{b}_k[f] = \frac{1}{P}\int_{-P}^{P} x\sin\left(\frac{k\pi x}{P}\right)\,dx$$

$$= \frac{1}{P}\left(\frac{P^2}{k^2\pi^2}\right)[\sin(k\pi) - k\pi\cos(k\pi) - \sin(-k\pi) + k\pi(-1)\cos(-k\pi)]$$

$$= \frac{P}{k^2\pi^2}[0 - k\pi(-1)^k - 0 + k\pi(-1)(-1)^k]$$

$$= \frac{P}{k^2\pi^2}[(-1)^{k+1}k\pi + k\pi(-1)^{k+1}]$$

$$= (-1)^{k+1}\frac{2P}{k\pi}.$$

We conclude

$$\mathsf{FS}[f](x) = \frac{2P}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\left(\frac{k\pi x}{P}\right).$$

For the complex series, we need to evaluate

$$\widehat{f}(k) = \frac{1}{2P}\int_{-P}^{P} xe^{-ik\pi x/P}\,dx.$$

Integrating by parts with the complex-valued integrand (i.e., $u = x$, $dv = e^{-ik\pi x/P}\,dx$) we get, for $k \neq 0$,

$$\int xe^{-ik\pi x/P}\,dx = \frac{P(ik\pi x + P)e^{-ik\pi x/P}}{k^2\pi^2}.$$

Then, for $k \neq 0$, we have

$$\widehat{f}(k) = \frac{P}{2P(k^2\pi^2)}\left[(ik\pi(P) + P)e^{-ik\pi} - (ik\pi(-P) + P)e^{ik\pi}\right]$$

$$= \frac{1}{2k^2\pi^2}[(ik\pi P + P)(-1)^k - (-ik\pi P + P)(-1)^k]$$

$$= \frac{1}{2k^2\pi^2}[ik\pi P(-1)^k + P(-1)^k + ik\pi P(-1)^k - P(-1)^k]$$

$$= \frac{1}{2k^2\pi^2}[2ik\pi P(-1)^k]$$

$$= \frac{iP(-1)^k}{k\pi}.$$

For $k = 0$, we have

$$\widehat{f}(0) = \frac{1}{2P}\int_{-P}^{P} x\, dx = \frac{1}{2P}\left(\frac{x^2}{2}\right)\Bigg|_{x=-P}^{P} = 0.$$

Then

$$\mathsf{FS}_{\mathbb{C}}[f](x) = \frac{iP}{\pi}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k}{k} e^{ik\pi x/P}.$$ ▲

2.4.8 Example.

Let

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x \le \pi. \end{cases}$$

Compute the real Fourier coefficients of f on $[-\pi, \pi]$.

Solution. We compute

$$\mathsf{a}_0[f] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} x\, dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}\right)\Bigg|_{x=0}^{x=\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2}$$

$$\begin{aligned}
\mathsf{a}_k[f] &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{k\pi x}{\pi}\right)\ dx,\ \ k \geq 1 \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(kx)\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\ dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\cos(kx)\ dx \\
&= \left.\frac{\cos(kx)+kx\sin(kx)}{k^2\pi}\right|_{x=0}^{\pi} \\
&= \frac{\cos(k\pi)+k\pi\sin(k\pi)-\cos(0)-0}{k^2\pi} \\
&= \frac{(-1)^k-1}{k^2\pi}
\end{aligned}$$

$$\begin{aligned}
\mathsf{b}_k[f] &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{k\pi x}{\pi}\right)\ dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\sin(kx)\ dx \\
&= \left.\frac{\sin(kx)-kx\cos(kx)}{k^2\pi}\right|_{x=0}^{x=\pi} \\
&= \frac{\sin(k\pi)-k\pi\cos(k\pi)-[0-0]}{k^2\pi} \\
&= -\frac{k\pi(-1)^k}{k^2\pi} \\
&= \frac{(-1)^{k+1}}{k}.
\end{aligned}$$

Thus

$$\mathsf{FS}[f](x) = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{(-1)^k-1}{k^2\pi}\cos(kx) + \frac{(-1)^{k+1}}{k}\sin(kx).$$ ▲

2.4.9 Example.

Find the complex Fourier coefficients of $f(x) = e^x$ on $[-\pi, \pi]$.

Solution. We know

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ik\pi x/\pi}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ik)x}\,dx.$$

We antidifferentiate:

$$\int e^{(1-ik)x}\,dx = \frac{e^{(1-ik)x}}{1-ik}.$$

This is permissible since $1-ik \neq 0$ for all $k \in \mathbb{R}$. Thus

$$\widehat{f}(k) = \frac{e^{(1-ik)x}}{2\pi(1-ik)}\bigg|_{x=-\pi}^{x=\pi} = \frac{e^{(1-ik)\pi} - e^{-(1-ik)\pi}}{2\pi(1-ik)} = \frac{e^{\pi}e^{-ik\pi} - e^{-\pi}e^{ik\pi}}{2\pi(1-ik)} = \frac{e^{\pi}(-1)^k - e^{-\pi}(-1)^k}{2\pi(1-ik)}$$

$$= (-1)^k\frac{e^{\pi}-e^{-\pi}}{2\pi(1-ik)}.$$

Then

$$\mathsf{FS}_{\mathbb{C}}[f](x) = \sum_{k=-\infty}^{\infty} (-1)^k\frac{e^{\pi}-e^{-\pi}}{2\pi(1-ik)}e^{ikx}.$$ ▲

2.4.10 Remark.

We note that in these examples, we typically had to compute $\mathsf{a}_0[f]$ *or* $\widehat{f}(0)$ *separately from* $\mathsf{a}_k[f]$ *or* $\widehat{f}(k)$ *for* $k \neq 0$*, since the formulas for the* $k \neq 0$ *coefficients often involved division by* k*. One should pay attention to this possibility when computing Fourier coefficients. Moving beyond symbolic manipulations, the zeroth coefficient is a scalar multiple of* $\int_{-P}^{P} f(x)\,dx$*. Such an integral is sometimes called the* **MEAN** *of* f *on* $[-P,P]$*, and since there are no oscillatory trigonometric factors in the integrand, morally it stands to reason that the behavior of the mean of* f *can be "special" or "separate" from the other Fourier coefficients. We discuss means/averages with integrals further in Section 2.4.5, specifically at* (2.4.13)*. Bottom line: be careful with* $k = 0$*.*

2.4.4. Pointwise convergence theory.

We begin with a (hopefully) obvious result.

2.4.11 Lemma.

Let $P > 0$ *and* $f\colon [-P,P] \to \mathbb{C}$ *be integrable.*

(i) *For a given* $x \in [-P,P]$*, the real Fourier series* $\mathsf{FS}[f](x)$ *converges (as a series in* $\mathbb{C}$*) if and only if the complex Fourier series* $\mathsf{FS}_{\mathbb{C}}[f](x)$ *converges. In this case,* $\mathsf{FS}[f](x) = \mathsf{FS}_{\mathbb{C}}[f](x)$*.*

(ii) $\mathsf{FS}[f](P)$ *converges if and only if* $\mathsf{FS}[f](-P)$ *converges, in which case* $\mathsf{FS}[f](P) = \mathsf{FS}[f](-P)$*.*

Proof. **(i)** By Lemmas 2.4.2 and 2.4.3, the nth partial sums $\mathsf{S}_n[f](x)$ of $\mathsf{FS}[f](x)$ and $\mathsf{FS}_{\mathbb{C}}[f](x)$ are equal:

$$\sum_{k=-n}^{n} \widehat{f}(k)e^{ik\pi x/P} = \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{n} \mathsf{a}_k[f]\cos\left(\frac{k\pi x}{P}\right) + \mathsf{b}_k[f]\sin\left(\frac{k\pi x}{P}\right). \tag{2.4.11}$$

Hence, given $x \in [-P, P]$, the limit as $n \to \infty$ of one side exists if and only if the limit as $n \to \infty$ of the other side exists, and so $\mathsf{FS}[f](x)$ converges if and only if $\mathsf{FS}_{\mathbb{C}}[f](x)$ converges. Moreover, by (2.4.11), if they converge, then they converge to the same number.

(ii) Observe that the nth term of $\mathsf{FS}[f](P)$ is

$$a_k \cos\left(\frac{k\pi P}{P}\right) + b_k \sin\left(\frac{k\pi P}{P}\right) = a_k \cos(k\pi) + b_k \sin(k\pi) = (-1)^k a_k,$$

while the nth term of $\mathsf{FS}[f](-P)$ is also

$$a_k \cos\left(\frac{k\pi(-P)}{P}\right) + b_k \sin\left(\frac{k\pi(-P)}{P}\right) = a_k \cos(-k\pi) + b_k \sin(-k\pi) = (-1)^k a_k. \quad \blacksquare$$

From now on we will just refer to the Fourier series by $\mathsf{FS}[f]$, to avoid writing a subscript. Also, we have a *necessary* condition for the Fourier series of f to converge to f on $[-P, P]$: if $\mathsf{FS}[f](x) = f(x)$ for all $x \in [-P, P]$, then f is $2P$-periodic, i.e., $f(P) = f(-P)$. Here, then, are our major convergence questions.

1. Under what conditions on f will $\mathsf{FS}[f](x)$ converge as a series in $\mathbb{C}$ for some (all?) $x \in [-P, P]$?

2. If $\mathsf{FS}[f](x)$ converges as a series in $\mathbb{C}$, what is the relation between $f(x)$ and $\mathsf{FS}[f](x)$? In particular, do we have $f(x) = \mathsf{FS}[f](x)$? In the case that $f(x) = \mathsf{FS}[f](x)$, we say that the series $\mathsf{FS}[f]$ **CONVERGES POINTWISE** to f at x.

Remarkably, under straightforward regularity[28] conditions on f, we can both ensure the convergence of $\mathsf{FS}[f](x)$ and find a formula for $\mathsf{FS}[f](x)$. From Definition A.6.1, we recall that $\mathcal{C}^1_{\mathrm{pw}}([-P, P])$ is the space of all functions on $[-P, P]$ that are continuous and differentiable at all but finitely many points in $[-P, P]$ and that do not "blow up" at the endpoints of any subinterval of $[-P, P]$ on which they are continuous and differentiable. In particular, such a function has left and right limits at all points in $[-P, P]$ (with right limit only at $-P$ and left limit only at P, of course), and, likewise, its derivative has left and right limits at all points where the derivative is defined.

2.4.12 Theorem.

If $f \in \mathcal{C}^1_{\mathrm{pw}}([-P, P])$, then

$$\mathsf{FS}[f](x) = \begin{cases} \dfrac{f(x^+) + f(x^-)}{2}, & -P < x < P \\[2ex] \dfrac{f(-P^+) + f(P^-)}{2}, & x = \pm P. \end{cases}$$

In particular, if $f \in \mathcal{C}^1_{\mathrm{pw}}([-P, P])$ is also continuous at $x \in (-P, P)$, then $\mathsf{FS}[f](x) = f(x)$. If $f \in \mathcal{C}^1_{\mathrm{pw}}([-P, P])$ is both continuous on $[-P, P]$ and $2P$-periodic, then $\mathsf{FS}[f](x) = f(x)$ for all $x \in [-P, P]$.

This theorem is proved in [9].

[28] This is a fancy synonym for differentiability.

2.4.13 Example.

Let

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x \le 1. \end{cases}$$

Find a formula for $\mathsf{FS}[f]$ *on* $[-1,1]$ *without actually computing any integrals.*

Solution. The only discontinuity of f occurs at $x = 0$, so

$$\frac{f(0^+) + f(0^-)}{2} = \frac{1+0}{2} = \frac{1}{2}.$$

On $(-1,0)$ and $(0,1)$, f is continuous, so for x in those intervals we have

$$\frac{f(x^+) + f(x^-)}{2} = f(x).$$

Last, at the endpoints, we have

$$\frac{f(-1^+) + f(1^-)}{2} = \frac{0+1}{2} = \frac{1}{2}.$$

We put this all together to get

$$\mathsf{FS}[f](x) = \begin{cases} 1/2, & x = -1 \\ 0, & -1 < x < 0 \\ 1/2, & x = 0 \\ 1, & 0 < x < 1 \\ 1/2, & x = 1. \end{cases}$$ ▲

2.4.5. L^2-convergence theory.

Most of the functions that we could possibly meet in this course — indeed, most of the functions that are "easy" to define formulaically — fall under the purview of Theorem 2.4.12. However, if one strays but a little from this theorem's requirements on well-behaved piecewise derivatives, one can induce pathological Fourier behavior. The following example[29] is taken almost verbatim from Remark 8.5.20 in [17].

2.4.14 Example.

Fix $P > 0$.

(i) *There exists a function* $f\colon [-P,P] \to \mathbb{C}$ *such that* $|f|$ *is integrable on* $[-P,P]$ *and* $\mathsf{FS}[f](x)$ *diverges (as a series in* $\mathbb{C}$*) for every* $x \in [-P,P]$.

(ii) *For any countable set* $\{x_k\}_{k=1}^{\infty} \subseteq [-P,P]$*, there exists a function* $f \in \mathcal{C}([-P,P])$ *such that* $\mathsf{FS}[f](x_k)$ *diverges (as a series in* $\mathbb{C}$*) for each* k.

[29] In some sense, this is a "nonexample," since we do not give actual formulas for these functions. Remember, though, that possessing a formula for something is not the same as understanding it, and here the functions' behavior is much more important than a pointwise formula.

So, can one obtain any control over the convergence of a Fourier series when f is not necessarily in $\mathcal{C}^1_{\text{pw}}([-P,P])$? Let us continue to suppose that f is at least integrable on $[-P,P]$, so that $\mathsf{a}_k[f]$, $\mathsf{b}_k[f]$, and $\widehat{f}(k)$ are defined.

As usual, we denote the nth partial sum of $\mathsf{FS}[f](x)$ by

$$\mathsf{S}_n[f](x) := \sum_{k=-n}^{n} \widehat{f}(k) e^{ik\pi x/P} = \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{n} \mathsf{a}_k[f] \cos\left(\frac{k\pi x}{P}\right) + \mathsf{b}_k[f] \sin\left(\frac{k\pi x}{P}\right). \quad (2.4.12)$$

We recall that, per mathematical custom (Definition A.2.1), the symbol $\mathsf{FS}[f](x)$ can refer both to the sequence of partial sums $\big(\mathsf{S}[f]_n(x)\big)$ and to the limit of that sequence, if it converges. In this notation, $\mathsf{FS}[f]$ converges to f pointwise at a particular x if for that x we have $\lim_{n\to\infty} \mathsf{S}_n[f](x) = f(x)$.

But rather than demand that this limit hold, we might stick with n finite but large and ask if the *function* $\mathsf{S}_n[f]$ is, "on average," a good approximation to f. Recall from calculus that the **AVERAGE VALUE** of an integrable function $g\colon [a,b] \to \mathbb{C}$ is

$$\frac{1}{b-a} \int_a^b g(x)\ dx. \quad (2.4.13)$$

And so $\mathsf{S}_n[f]$ approximates, on average, f well on $[-P,P]$ if, on average, $|f - \mathsf{S}_n[f]|$ is small, i.e., if

$$\frac{1}{2P} \int_{-P}^{P} |f(x) - \mathsf{S}_n[f](x)|\ dx \quad (2.4.14)$$

is small. Since we will always work on the same interval $[-P,P]$, and since $1/2P$ is a constant factor, the integral above in (2.4.14) will be small if and only if

$$\|f(x) - \mathsf{S}_n[f]\|_{L^1} := \int_{-P}^{P} |f(x) - \mathsf{S}_n[f](x)|\ dx$$

is small.

We surely cannot expect this integral to be zero even for large n, and so we will have to accept *some* errors. Hopefully it is reasonable that we can allow ourselves to be less concerned with *small* errors and more concerned with *large* errors. Note that if $w \in \mathbb{C}$ with $|w| < 1$, then $|w|^r < |w| < 1$ for any $r > 1$. That is, if $|w|$ is "small," then $|w|^r$ will be "smaller" for $r > 1$. We can measure how well $\mathsf{S}_n[f]$ approximates, on average, f over $[-P,P]$ and give more "weight" to large errors and less to small errors by measuring the difference $\mathsf{S}_n[f] - f$ with respect to the L^r**-NORM**

$$\|g\|_{L^r} := \left(\int_{-P}^{P} |g(x)|^r\ dx\right)^{1/r}. \quad (2.4.15)$$

We take the $(1/r)$th root so that if we scale g by some constant $\alpha \in \mathbb{C}$, then we will scale the right side of (2.4.15) by $|\alpha|$, not an r-dependent factor. Incidentally, it is a property of the Riemann integral, which we take for granted here, that if $g\colon [-P,P] \to \mathbb{C}$ is integrable, then so is $|g|^r$ for any $r > 0$, so the integral in (2.4.15) is always defined (see part (vii) of Theorem A.5.5 for more detail). Also, since $P > 0$ is always fixed, to simplify notation we are not including the interval $[-P,P]$ in the subscript of $\|\cdot\|_{L^r}$, i.e., we do not write $\|\cdot\|_{L^r([-P,P])}$.

So, we think that if the quantity

$$\|f(x) - \mathsf{S}_n[f]\|_{L^r} = \left(\int_{-P}^{P} |f(x) - \mathsf{S}_n[f](x)|^r \, dx\right)^{1/r}$$

is small for some $r > 1$ and all n large, then $\mathsf{S}_n[f]$ is, on average over $[-P, P]$, a good approximation to f. Although there is no particularly obvious *a priori* reason for the following choice, we can make a great deal of progress if we take $r = 2$. Specifically, the following limit holds.

2.4.15 Theorem.

Let $f\colon [-P, P] \to \mathbb{C}$ be integrable. Then

$$\lim_{n\to\infty} \|f - \mathsf{S}_n[f]\|_{L^2} = 0. \tag{2.4.16}$$

The proof of this theorem requires rather more advanced tools from analysis; see [9]. The chief advantage of Theorem 2.4.15 over the pointwise convergence in Theorem 2.4.12 is that we have removed the regularity (= continuity/differentiability) requirements on our functions. An integrable function on $[-P, P]$ can fail to be continuous or differentiable at a countably infinite sequence of points $\{x_k\}_{k=1}^{\infty}$, but still its Fourier series is a "very good approximation on average," per (2.4.16).

The convergence in the limit (2.4.16) often goes by a special name.

2.4.16 Definition.

If (f_k) is a sequence of integrable functions on $[-P, P]$ and $f\colon [-P, P] \to \mathbb{C}$ is also integrable with

$$\lim_{k\to\infty} \|f - f_k\|_{L^2} = 0,$$

then we say that (f_k) CONVERGES TO f IN THE MEAN.

So, per Theorem 2.4.15, the partial sums of the (real or complex) Fourier series of an integrable function f always converge to f in the mean. Nonetheless, convergence in the mean of a Fourier series to its does not imply pointwise convergence. First, it may be the case that the series $\mathsf{FS}[f](x)$ diverges for some x; this is possible by Example 2.4.14. Second, it may be the case that the series $\mathsf{FS}[f](x)$ converges but not to $f(x)$; this certainly happens at $x = \pm P$ whenever $\mathsf{FS}[f](\pm P)$ converge and f is not $2P$-periodic, per Lemma 2.4.11. Here is another example.

2.4.17 Example.

Fix $P > 0$ and let

$$f_0(x) = 0, \ -P \le x \le P \quad \text{and} \quad f_1(x) = \begin{cases} 0, & -P \le x < P \\ 1, & x = P. \end{cases}$$

Clearly $\widehat{f_0}(k) = 0$ for all k, and likewise

$$\widehat{f_1}(k) = \frac{1}{2P}\int_{-P}^{P} f_1(x)e^{-ik\pi x/P} \, dx = \frac{1}{2P}\int_{-P}^{P} 0 \, dx = 0.$$

Thus $\widehat{f_0}(k) = \widehat{f_1}(k)$ for all k, even though $f_0 \neq f_1$.

We see, therefore, that the Fourier coefficients of a function do not uniquely determine that function. So, not only can we lose all control over the behavior of a Fourier series without the assumption of piecewise continuity and differentiability (and even then the Fourier series may not converge pointwise back to the original function at all points), we do not even have a one-to-one and onto pairing of functions and Fourier coefficients.

We conclude on a more optimistic note. Theorem 2.4.15 tells us that, "on average" and "penalizing large errors more than small ones," the nth partial sums of the Fourier series of f are very good approximations to f when n is large. In fact, they are the *best* possible approximations of f by trigonometric polynomials, at least if "best" is measured by "how small the difference is with respect to $\|\cdot\|_{L^2}$."

2.4.18 Theorem.

Let $f\colon [-P, P] \to \mathbb{C}$ be integrable. The nth partial sum $\mathsf{S}_n[f]$ of the Fourier series for f is the best approximation to f by an nth degree trigonometric polynomial in the L^2-norm on $[-P, P]$. That is, if $c_0, c_{\pm 1}, \ldots, c_{\pm n} \in \mathbb{C}$ and $g(x) := \sum_{k=-n}^{n} c_k e^{ik\pi x/P}$, then

$$\|f - \mathsf{S}_n[f]\|_{L^2} \le \|f - g\|.$$

This theorem is ultimately a consequence of Theorem C.5.8.

2.4.19 Linear algebraic viewpoint: Fourier series

We revisit the results of this section from the more abstract perspectives of Appendices C.3, C.4, and C.5.

1. We first (re)introduce some notation. Fix $P > 0$ and denote by $\mathcal{R}([-P, P])$ the space of all integrable functions from $[-P, P]$ to $\mathbb{C}$. If $f, g \in \mathcal{R}([-P, P])$, put

$$\langle f, g \rangle_{L^2} := \int_{-P}^{P} f(x)\overline{g(x)}\, dx \qquad (2.4.17)$$

It is a property of the Riemann integral that if $f, g \in \mathcal{R}([-P, P])$, then so is the product $f\overline{g}$, so $\langle f, g \rangle_{L^2}$ is defined. Observe from (2.4.15) with $r = 2$ that $\|f\|_{L^2} = \sqrt{\langle f, f \rangle_{L^2}}$.

2. Then in the language of Definitions C.4.1 and C.3.1, the map $\langle \cdot, \cdot \rangle_{L^2}$ is a semi-definite inner product on $\mathcal{R}([-P, P])$, and $\|\cdot\|_{L^2}$ is a seminorm on $\mathcal{R}([-P, P])$. If we restrict to the subspace $\mathcal{C}([-P, P]) \subseteq \mathcal{R}([-P, P])$, then $\|\cdot\|_{L^2}$ is a norm on $\mathcal{C}([-P, P])$ and $\langle \cdot, \cdot \rangle_{L^2}$ is positive definite on $\mathcal{C}([-P, P])$. See parts (ii) and (iii) of Example C.3.2.

3. Now define

$$\mathsf{e}_k(x) := \frac{e^{ik\pi x/P}}{\sqrt{2P}}, \quad \mathsf{c}_k(x) := \begin{cases} \dfrac{1}{\sqrt{2P}}, & k = 0 \\ \dfrac{1}{\sqrt{P}} \cos\left(\dfrac{k\pi x}{P}\right), & k \ge 1, \end{cases} \quad \text{and} \quad \mathsf{s}_k(x) := \frac{1}{\sqrt{P}} \sin\left(\frac{k\pi x}{P}\right).$$

4. The sets $\{\mathsf{s}_k\}_{k=1}^{\infty}$, $\{\mathsf{c}_k\}_{k=0}^{\infty}$, $\{\mathsf{c}_k\}_{k=0}^{\infty} \cup \{\mathsf{s}_k\}_{k=1}^{\infty}$, and $\{\mathsf{e}_k\}_{k=-\infty}^{\infty}$ are all **orthonormal** with respect to $\langle\cdot,\cdot\rangle_{L^2}$ in the sense that

$$\langle \mathsf{s}_k, \mathsf{s}_j\rangle_{L^2} = \langle \mathsf{c}_k, \mathsf{c}_j\rangle_{L^2} = \langle \mathsf{e}_k, \mathsf{e}_j\rangle_{L^2} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \tag{2.4.18}$$

and

$$\langle \mathsf{s}_k, \mathsf{c}_j\rangle_{L^2} = 0 \text{ for all } k,\ j. \tag{2.4.19}$$

Furthermore, if $f\colon [-P,P] \to \mathbb{C}$ is integrable, then we can express the nth partial sum of its Fourier series as

$$\mathsf{S}_n[f] = \sum_{k=-n}^{n} \langle f, \mathsf{e}_k\rangle_{L^2}\, \mathsf{e}_k \tag{2.4.20}$$

and

$$\mathsf{S}_n[f] = \langle f, \mathsf{c}_0\rangle_{L^2}\, \mathsf{c}_0 + \sum_{k=1}^{n} \big(\langle f, \mathsf{c}_k\rangle_{L^2}\, \mathsf{c}_k + \langle f, \mathsf{s}_k\rangle_{L^2}\, \mathsf{s}_k \big). \tag{2.4.21}$$

The proofs of (2.4.18) and (2.4.19) are direct, but lengthy, computations; it is probably easier to do the calculations for the complex exponentials first and then adapt to real sines and cosines. The proofs of (2.4.20) and (2.4.21) amount to just a chase through the definition of $\mathsf{S}_n[f]$ from, say, (2.4.12). Note, however, that in general $\mathsf{a}_k[f] \neq \langle f, \mathsf{c}_k\rangle_{L^2}$, $\mathsf{b}_k[f] \neq \langle f, \mathsf{s}_k\rangle_{L^2}$, and $\widehat{f}(k) \neq \langle f, \mathsf{e}_k\rangle_{L^2}$; this is due to the scalings on the functions c_k, s_k, and e_k, which ensures that the L^2-norm of each of these functions is 1.

5. Furthermore, using the expressions for $\mathsf{S}_n[f]$ from (2.4.20) and (2.4.21), we can rephrase Theorem 2.4.15 as saying

$$\lim_{n\to\infty} \left\| f - \sum_{k=-n}^{n} \langle f, \mathsf{e}_k\rangle_{L^2}\, \mathsf{e}_k \right\|_{L^2} = 0$$

or

$$\lim_{n\to\infty} \left\| f - \left(\langle f, \mathsf{c}_0\rangle_{L^2}\, \mathsf{c}_0 + \sum_{k=1}^{n} \big(\langle f, \mathsf{c}_k\rangle_{L^2}\, \mathsf{c}_k + \langle f, \mathsf{s}_k\rangle_{L^2}\, \mathsf{s}_k \right) \right\|_{L^2} = 0.$$

Thus, per Definition C.5.4, the sets $\{\mathsf{c}_k\}_{k=0}^{\infty} \cup \{\mathsf{s}_k\}_{k=1}^{\infty}$ and $\{\mathsf{e}_k\}_{k=-\infty}^{\infty}$ are orthonormal bases $\mathcal{R}([-P,P])$ with respect to $\langle\cdot,\cdot\rangle_{L^2}$. Consequently, once Theorem 2.4.15 is proved, we unlock the many powerful properties of orthonormal bases from Appendix C.5. In particular, we obtain the "least squares approximation" of Theorem C.5.8 also holds, which proves Theorem 2.4.18, and also the Parseval and Plancherel identities (Theorem C.5.7), which we state in the following section in Theorem 2.4.20.

6. There is a very important dichotomy in play here. We can think of an element $f \in \mathcal{R}([-P,P])$ in two ways: it is both a vector in some semi-definite inner product space and a pointwise determined mapping of the interval $[-P,P]$ to $\mathbb{C}$. Suppose we change the values of $f \in \mathcal{R}([-P,P])$ at finitely many points $x_1, \ldots, x_n \in [-P,P]$. Then we obtain a new function $\widetilde{f}$ such that $f(x) = \widetilde{f}(x)$ for all $x \in [-P,P]$ except $x = x_1, \ldots, x_n$. Then $f \neq \widetilde{f}$, but properties of the integral ensure both

$$\|f - \widetilde{f}\|_{L^2} = 0$$

From the point of view of the L^2-seminorm, then, f and $\widetilde{f}$ are the same vector. Additionally,

$$\int_{-P}^{P} f(x)e^{-ik\pi x/P}\ dx = \int_{-P}^{P} \widetilde{f}(x)e^{-ik\pi x/P}\ dx$$

for all k, and so $\widehat{f}(k) = \widehat{\widetilde{f}}(k)$ for all k. Thus merely knowing the Fourier coefficients of an integrable function does not uniquely determine that function.

However, if we restrict to $\mathcal{C}([-P,P])$, on which $\|\cdot\|_{L^2}$ is a norm, not a seminorm, and $\langle\cdot,\cdot\rangle_{L^2}$ is a *definite* inner product, not semi-definite, then Lemma C.5.6 ensures that the Fourier coefficients of $f \in \mathcal{C}([-P,P])$ uniquely determine f. That is, if f, $g \in \mathcal{C}([-P,P])$ satisfy $\widehat{f}(k) = \widehat{g}(k)$ for all $k \in \mathbb{Z}$, then $f = g$ as vectors in $\mathcal{C}([-P,P])$, and so $f(x) = g(x)$ for all $x \in [-P,P]$.

None of the abstract results of Appendix C.5, however, provides any apparent proof of the pointwise convergence of Fourier series. For that, we would have to work much more closely with properties of the orthonormal bases $\{\mathsf{c}_k\}_{k=0}^{\infty} \cup \{\mathsf{s}_k\}_{k=1}^{\infty}$ or $\{\mathsf{e}_k\}_{k=-\infty}^{\infty}$ as sets of *functions*.

2.4.6. Parseval, Plancherel, and L^2.

The following identities provide a remarkable, and efficient, way to calculate quantities involving integrals of functions over $[-P,P]$ using only their Fourier coefficients.

2.4.20 Theorem.

Let f, $g\colon [-P,P] \to \mathbb{C}$ be integrable. Then

(i) [Parseval] $\langle f, g\rangle_{L^2} := \displaystyle\int_{-P}^{P} f(x)\overline{g(x)}\ dx = 2P \sum_{k=-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}.$

(ii) [Plancherel] $\|f\|_{L^2}^2 := \displaystyle\int_{-P}^{P} |f(x)|^2\ dx = 2P \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2.$

This theorem follows at once from Theorem C.5.7, if one believes the material discussed in Linear Algebraic Viewpoint 2.4.19. The Parseval and Plancherel identities can be used to derive interesting identities for certain series whose values are not easy to calculate using basic methods of calculus.

2.4.21 Example.

Let $f(x) = x$ and recall from Example 2.4.7 that on any interval $[-P,P]$,

$$\widehat{f}(k) = \begin{cases} 0, & k = 0 \\ \dfrac{iP(-1)^k}{k\pi}, & k \neq 0 \end{cases}$$

Use this to show $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$

Solution. We recall from the "p-test" for series from calculus that $\sum_{k=1}^{\infty} k^{-2}$ converges. However, as with many convergence tests from calculus, this does not tell us what the sum of the series is.

If we take $P = \pi$, then
$$|\widehat{f}(k)|^2 = \frac{1}{k^2}, \ k \neq 0.$$
Then
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} \frac{1}{k^2} + 0 + \sum_{k=1}^{\infty} \frac{1}{k^2}\right) = \frac{1}{2} \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2.$$
Next, Plancherel's theorem tells us
$$\sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 = \frac{1}{2\pi} \|f\|^2_{L^2([-\pi,\pi])} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{x^3}{6\pi}\bigg|_{x=-\pi}^{x=\pi}$$
$$= \frac{\pi^3 - (-\pi^3)}{6\pi} = \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3}.$$
We conclude
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2} \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 = \frac{\pi^2}{6}.$$
▲

We conclude with two lemmas that govern the size of a function's Fourier coefficients. The proofs of both are rather easy consequences of the definition of the Fourier coefficient and Plancherel's identity, respectively, and so we defer them to the exercises.

2.4.22 Lemma (Sobolev inequality).

If $f \in \mathcal{C}([-P,P])$, then
$$\max_{k \in \mathbb{Z}} |\widehat{f}(k)| = \max_{k \geq 0} \left|\frac{\mathsf{a}_k[f] \pm i\mathsf{b}_k[f]}{2}\right| \leq \max_{-P \leq x \leq P} |f(x)|.$$

2.4.23 Lemma (Riemann-Lebesgue[30]).

Let $f\colon [-P,P] \to \mathbb{C}$ be integrable. Then the Fourier coefficients of f decay in the sense that
$$\lim_{k\to\infty} \mathsf{a}_k[f] = \lim_{k\to\infty} \mathsf{b}_k[f] = \lim_{k\to\pm\infty} \widehat{f}(k) = 0.$$

2.4.7. Fourier sine and cosine series.

We have cast our problems of L^2-based Fourier series so far on symmetric intervals of the form $[-P,P]$. This is a convenient, but artificial, restriction; one work on any arbitrary interval $[a,b]$. The coefficients are simply messier.

A related, but different, issue is the following. Model problems for the heat equation, such as Example 2.3.2 and other problems in the exercises, suggest that it is convenient to

[30]This result is strong enough to be called a theorem, but in the literature the convention is to refer to it as the "Riemann-Lebesgue lemma."

represent an initial temperature distribution on the interval $[0, P]$ as linear combinations of sines or cosines, but not both. So, we want to adapt our existing Fourier theory in two ways: we want to work on the "half" interval $[0, P]$ and not $[-P, P]$, and we want to use only sines, or only cosines, in our representation of a function.

Before proceeding, we need a definition and an easy lemma, whose proof we leave as an exercise in u-substitution.

2.4.24 Definition.

Recall that a function $f\colon [-P, P] \to \mathbb{C}$ is **EVEN** *if $f(-x) = f(x)$ for all x and* **ODD** *if $f(-x) = -f(x)$ for all x.*

2.4.25 Lemma.

(i) *If $g\colon [-P, P] \to \mathbb{C}$ is integrable and even, then*

$$\int_{-P}^{P} g(x)\ dx = 2\int_{0}^{P} g(x)\ dx.$$

(ii) *If $h\colon [-P, P] \to \mathbb{C}$ is integrable and odd, then*

$$\int_{-P}^{P} h(x)\ dx = 0.$$

Suppose that for a function $f\colon [0, P] \to \mathbb{C}$ we have the representation

$$f(x) = \sum_{k=0}^{\infty} A_k \cos\left(\frac{k\pi x}{P}\right),\ 0 \le x \le P.$$

Since each term in the series is even, the series also converges for each $x \in [-P, 0)$, and we have

$$\sum_{k=0}^{\infty} A_k \cos\left(-\frac{k\pi x}{P}\right) = \sum_{k=0}^{\infty} A_k \cos\left(\frac{k\pi x}{P}\right).$$

This suggests that we extend f to an even function defined on $[-P, 0)$.

2.4.26 Lemma.

Let $f\colon [0, P] \to \mathbb{C}$ be a function. The **EVEN EXTENSION** *of f is*

$$f_{\mathsf{e}}(x) := \begin{cases} f(-x), & -P \le x < 0 \\ f(x), & 0 \le x \le P. \end{cases}$$

The function f_{e} is even on $[-P, P]$. If f is integrable on $[0, P]$, then f_{e} is integrable on $[-P, P]$.

If the Fourier series of f_{e} on $[-P, P]$ is a "good approximation" to f_{e} on $[-P, P]$, then since $f(x) = f_{\mathsf{e}}(x)$ for $x \in [0, P]$, it stands to reason that $\mathsf{FS}[f_{\mathsf{e}}]$ will also be a good approximation to f on $[0, P]$. So, let us calculate the Fourier coefficients of f_{e} on $[-P, P]$.

First,

$$\mathsf{b}_k[f_{\mathsf{e}}] = \frac{1}{P}\int_{-P}^{P} f_{\mathsf{e}}(x)\sin\left(\frac{k\pi x}{P}\right)\,dx = 0,$$

since the integrand is odd. Next,

$$\mathsf{a}_k[f_{\mathsf{e}}] = \frac{1}{P}\int_{-P}^{P} f_{\mathsf{e}}(x)\cos\left(\frac{k\pi x}{P}\right)\,dx = \frac{2}{P}\int_{0}^{P} f_{\mathsf{e}}(x)\cos\left(\frac{k\pi x}{P}\right)\,dx$$

$$= \frac{2}{P}\int_{0}^{P} f(x)\cos\left(\frac{k\pi x}{P}\right)\,dx, =: \mathsf{A}_k[f]$$

since the first integrand is even. So, we have

$$\mathsf{FS}[f_{\mathsf{e}}](x) = \frac{\mathsf{a}_0[f_{\mathsf{e}}]}{2}+\sum_{k=1}^{\infty}\mathsf{a}_k[f_{\mathsf{e}}]\cos\left(\frac{k\pi x}{P}\right)+\mathsf{b}_k[f_{\mathsf{e}}]\sin\left(\frac{k\pi x}{P}\right) = \frac{\mathsf{A}_0[f]}{2}+\sum_{k=1}^{\infty}\mathsf{A}_k[f]\cos\left(\frac{k\pi x}{P}\right)$$

$$=: \mathsf{FCS}[f](x).$$

We call this series the **Fourier cosine series** of f on $[0, P]$. In particular, this is a series of "only cosines" that, plausibly, can represent f on $[0, P]$.

Now, how shall we represent $f\colon [0, P] \to \mathbb{C}$ with a series of sines? If

$$f(x) = \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{P}\right), \ 0 \le x \le P,$$

then since each term in the series is odd, the series also converges for each $x \in [-P, 0)$, and we have

$$\sum_{k=1}^{\infty} B_k \sin\left(-\frac{k\pi x}{P}\right) = -\sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{P}\right).$$

This suggests that we extend f to an odd function defined on $[-P, 0)$.

2.4.27 Lemma.

Let $f\colon [0, P] \to \mathbb{C}$ be a function. The **odd extension** *of f is*

$$f_{\mathsf{o}}(x) := \begin{cases} -f(-x), & -P \le x < 0 \\ 0, & x = 0 \\ f(x), & 0 < x \le P. \end{cases}$$

The function f_{o} is odd on $[-P, P]$. If $f\colon [0, P] \to \mathbb{C}$ is integrable, then so is f_{o}.

If the Fourier series of f_{o} on $[-P, P]$ is a "good approximation" to f_{o} on $[-P, P]$, then since $f(x) = f_{\mathsf{e}}(x)$ for $x \in (0, P]$, it stands to reason that $\mathsf{FS}[f_{\mathsf{o}}]$ will also be a good approximation to f on $(0, P]$. First,

$$\mathsf{a}_k[f_{\mathsf{o}}] = \frac{1}{P}\int_{-P}^{P} f_{\mathsf{o}}(x)\cos\left(\frac{k\pi x}{P}\right)\,dx = 0,$$

since the integrand is odd. Next,

$$b_k[f_\circ] = \frac{1}{P}\int_{-P}^{P} f_\circ(x)\sin\left(\frac{k\pi x}{P}\right)\, dx = \frac{2}{P}\int_0^P f_\circ(x)\sin\left(\frac{k\pi x}{P}\right)\, dx,$$

since the first integrand is even. Now, we have $f_\circ(x) = f(x)$ for $0 < x \le P$, but perhaps $f(0) \neq 0 = f_\circ(0)$. This is immaterial from the point of view of the integral, and we have

$$\frac{2}{P}\int_0^P f_\circ(x)\sin\left(\frac{k\pi x}{P}\right)\, dx = \frac{2}{P}\int_0^P f(x)\sin\left(\frac{k\pi x}{P}\right)\, dx =: \mathsf{B}_k[f].$$

Then

$$\mathsf{FS}[f_\circ](x) = \sum_{k=1}^{\infty} \mathsf{b}_k[f_\circ]\sin\left(\frac{k\pi x}{P}\right) =: \mathsf{FSS}[f](x).$$

We call this series the **Fourier sine series** of f on $[0, P]$. This is a series of "only sines" that, putatively, can represent f on $[0, P]$.

We summarize our work.

2.4.28 Definition.

Let $f\colon [0, P] \to \mathbb{C}$ be integrable and define

$$\mathsf{A}_k[f] := \frac{2}{P}\int_0^P f(x)\cos\left(\frac{k\pi x}{P}\right)\, dx \quad \text{and} \quad \mathsf{B}_k[f] := \frac{2}{P}\int_0^P f(x)\sin\left(\frac{k\pi x}{P}\right)\, dx.$$

The **formal Fourier cosine series of** f **on** $[0, P]$ *is*

$$\mathsf{FCS}[f](x) := \frac{\mathsf{A}_0[f]}{2} + \sum_{k=1}^{\infty}\mathsf{A}_k[f]\cos\left(\frac{k\pi x}{P}\right),$$

and the **formal Fourier sine series of** f **on** $[0, P]$ *is*

$$\mathsf{FSS}[f](x) := \sum_{k=1}^{\infty}\mathsf{B}_k[f]\sin\left(\frac{k\pi x}{P}\right).$$

2.4.29 Example.

Find the Fourier sine and cosine series for $f(x) = x$ on $[0, 1]$. Compare this to what we know about the Fourier series for f.

Solution. We compute $\mathsf{A}_0[f] = 2\int_0^1 x\, dx = 1$ and, for $k \ge 1$,

$$\begin{aligned}\mathsf{A}_k[f] &= 2\int_0^1 x\cos(k\pi x)\, dx \\ &= \frac{2}{k^2\pi^2}\big[\cos(k\pi x) + k\pi x\sin(k\pi x)\big]\Big|_{x=0}^{x=1} \\ &= \frac{2[(-1)^k - 1]}{k^2\pi^2}\end{aligned}$$

$$= \begin{cases} 0, & k \text{ even} \\ -\frac{4}{k^2\pi^2}, & k \text{ odd,} \end{cases}$$

so

$$\mathsf{FCS}[f](x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((2k+1)\pi x)$$

Next, for $k \geq 1$, we have

$$\mathsf{B}_k[f] = 2\int_0^1 x\sin(k\pi x)\,dx = \frac{2}{k^2\pi^2}\big[\sin(k\pi x) - k\pi x\cos(k\pi x)\big]\Big|_{x=0}^{x=1} = \frac{2(-1)^{k+1}}{k\pi},$$

so

$$\mathsf{FSS}[f](x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(k\pi x).$$

In Example 2.4.7, we computed that the Fourier series of $f(x) = x$ on $[-1,1]$ is

$$\mathsf{FS}[f](x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(k\pi x),$$

and so in this case

$$\mathsf{FSS}[f] = \mathsf{FS}[f].$$

This is not surprising: $f(x) = x$ is odd, so its Fourier series on $[-1,1]$ should contain only terms with a factor of $\sin(k\pi x)$. ▲

2.4.30 Example.

Compute the Fourier cosine series of $f(x) = \sin(x)$ on $[0,\pi]$.

Solution. We have

$$\mathsf{A}_k[f] = \frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(kx)\,dx.$$

We antidifferentiate

$$\int \sin(x)\cos(kx)\,dx = \frac{1}{2}\left(\frac{\cos((k-1)x)}{k-1} - \frac{\cos((k+1)x)}{k+1}\right).$$

This means that we will have to treat $k = 1$ separately from $k = 0$ and $k \geq 2$. For $k \neq 1$, we have

$$\mathsf{A}_k[f] = \frac{1}{\pi}\left[\frac{\cos((k-1)x)}{k-1} - \frac{\cos((k+1)x)}{k+1}\right]\Bigg|_{x=0}^{x=\pi} = \frac{1}{\pi}\left[\frac{(-1)^{k-1}}{k-1} - \frac{(-1)^{k+1}}{k+1} - \frac{1}{k-1} + \frac{1}{k+1}\right].$$

We could try to put this all over a common denominator, but it is simpler (or, no harder), to consider the separate cases of k even and k odd. If k is odd, then $k \pm 1$ are even, so

$$\frac{(-1)^{k-1}}{k-1} + \frac{1}{k+1} - \frac{(-1)^{k+1}}{k+1} - \frac{1}{k-1} = \frac{1}{k-1} + \frac{1}{k+1} - \frac{1}{k+1} - \frac{1}{k-1} = 0,$$

while if k is even, then $k \pm 1$ are odd,

$$\frac{(-1)^{k-1}}{k-1} + \frac{1}{k+1} - \frac{(-1)^{k+1}}{k+1} - \frac{1}{k-1} = \frac{(-1)}{k-1} + \frac{1}{k+1} - \frac{(-1)}{k+1} - \frac{1}{k-1}$$
$$= 2\left(\frac{1}{k+1} - \frac{1}{k-1}\right)$$
$$= \frac{4}{1-k^2}.$$

Last, we calculate the special case of $k = 1$:

$$\mathsf{A}_1[f] = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(x)\ dx = \frac{1}{2}\sin^2(x)\Big|_{x=0}^{x=\pi} = 0.$$

We find what we obtained before for other k odd, except we had to do $k = 1$ separately. Hence

$$\mathsf{A}_k[f] = \begin{cases} 0, & k \text{ odd} \\ \dfrac{4}{\pi(1-k^2)}, & k \text{ even} \end{cases}$$

and

$$\mathsf{FCS}[f](x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^\infty \frac{1}{1-(2k)^2}\cos(2kx). \qquad ▲$$

One can develop a pointwise convergence theory for Fourier sine and cosine series very much along the lines of Theorem 2.4.12; we will explore this in the exercises.

2.4.8. Solving ODEs with Fourier series.

We now take up the issue of solving differential equations with Fourier series, which motivated our original foray into this subject. We do not immediately return to the heat equation but instead start with ODEs. For simplicity, we will work on the interval $[-\pi,\pi]$, and we define a periodic analogue of our favorite function spaces from Definition 1.1.1. Put

$$\mathcal{C}^n_{\text{per}}([-\pi,\pi]) := \{f \in \mathcal{C}^n([-\pi,\pi]) \mid f(\pi) = f(-\pi)\} \quad \text{and} \quad \mathcal{C}_{\text{per}}([-\pi,\pi]) := \mathcal{C}^0_{\text{per}}([-\pi,\pi]).$$

We first observe that taking Fourier coefficients converts a *differential* equation into an *algebraic* equation.

2.4.31 Lemma.

Suppose that $f \in \mathcal{C}^1_{\text{per}}([-\pi,\pi])$*. Then* $\widehat{f'}(k) = ik\widehat{f}(k)$*.*

Proof. First we note that if $f \in \mathcal{C}^1_{\text{per}}([-\pi,\pi])$, then f and f' are continuous on $[-\pi,\pi]$, hence the integrals giving $\widehat{f}(k)$ and $\widehat{f'}(k)$ are defined. Now we integrate by parts with

$$u = e^{-ikx} \qquad dv = f'(x)\ dx$$

$$du = -ike^{-ikx} \qquad\qquad v = f(x)$$

to find

$$\int_{-\pi}^{\pi} f'(x)e^{-ikx}\,dx = ik\big(f(\pi)e^{-ik\pi} - f(-\pi)e^{ik\pi}\big) + ik\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

We factor

$$ik\big(f(\pi)e^{-ik\pi} - f(-\pi)e^{ik\pi}\big) = -(ik)(2i)f(\pi)\sin(k\pi) = 0.$$

Thus

$$\widehat{f'}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-ikx}\,dx = ik\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx\right) = ik\widehat{f}(k).$$ ■

2.4.32 Example.

Suppose $f \in \mathcal{C}^2_{\text{per}}([-\pi,\pi])$ solves $f'' + f = 0$ on $[-\pi,\pi]$. Use Fourier series to show that, as expected, $f = c_1\cos(x) + c_2\sin(x)$ for some constants c_1, $c_2 \in \mathbb{C}$.

Solution. If f is 2π-periodic and $f'' + f = 0$, then set $g = f'' + f$ to find that $\widehat{g}(k) = 0$ for all k. But we can also write

$$0 = \widehat{g}(k) = (ik)^2\widehat{f}(k) + \widehat{f}(k) = (1-k^2)\widehat{f}(k).$$

Thus for $k \neq \pm 1$, we have $\widehat{f}(k) = 0$.

Now, since f'' is defined, f is continuous on $[-\pi,\pi]$, and so $\mathsf{FS}[f](x) = f(x)$ for all $x \in (-\pi,\pi)$, and we also have equality at $x = \pm\pi$ since f is 2π-periodic. Hence, for all x,

$$f(x) = \mathsf{FS}[f](x) = \mathsf{FS}_{\mathbb{C}}[f](x) = \widehat{f}(-1)e^{-ikx} + \widehat{f}(1)e^{ikx}.$$

And this is easy to rearrange into a linear combination of sines and cosines:

$$\widehat{f}(-1)e^{-ikx} + \widehat{f}(1)e^{ikx} = \widehat{f}(-1)\cos(kx) - i\widehat{f}(-1)\sin(kx) + \widehat{f}(1)\cos(kx) + i\widehat{f}(1)\sin(kx)$$

$$= \left(\widehat{f}(-1) + \widehat{f}(1)\right)\cos(kx) + i\left(\widehat{f}(1) - \widehat{f}(-1)\right)\sin(kx).$$ ▲

2.4.33 Example.

Let $g \in \mathcal{C}_{\text{per}}([-\pi,\pi])$.

(i) *Suppose that $f \in \mathcal{C}^2_{\text{per}}([-\pi,\pi])$ solves $f'' + f = g(x)$ on $[-\pi,\pi]$. Show that $\widehat{g}(\pm 1) = 0$. Comment on this in light of the standard existence and uniqueness theorem for ODEs.*

(ii) *Suppose that $\widehat{g}(\pm 1) = 0$. Use Fourier series to find a formal[31] solution to $f'' + f = g(x)$ on $[-\pi,\pi]$.*

[31] "[T]he term 'formal' describes any plausible result or procedure which may be unjustified or unjustifiable" [4, p. 249].

Solution. (i) Suppose that $f'' + f = f(x)$, so that

$$-k^2\widehat{f}(k) + \widehat{f}(k) = \widehat{g}(k),\ \ k \in \mathbb{Z},$$

thus

$$\widehat{g}(k) = (1 - k^2)\widehat{f}(k). \tag{2.4.22}$$

In particular,

$$\widehat{g}(\pm 1) = (1 - (\pm 1)^2)\widehat{f}(\pm 1) = 0.$$

And so we have shown that if we can solve $f'' + f = g(x)$ with $f \in \mathcal{C}^2_{\text{per}}([-\pi, \pi])$, then g must satisfy the "solvability conditions"

$$\int_{-\pi}^{\pi} g(x)e^{\pm ix}\ dx = 0.$$

This seems restrictive, as the general theory of ODEs tells us that we can always solve $f'' + f = g(x)$ on $[-\pi, \pi]$ given $g \in \mathcal{C}([-\pi, \pi])$, which is a larger space than $\mathcal{C}_{\text{per}}([-\pi, \pi])$. But we have added an extra condition to our solution f: we want f to be 2π-periodic, and so f must meet the boundary conditions $f(\pi) = f(-\pi)$. This is not an initial value problem, and so it is unsurprising that we can only solve it for a restricted class of forcing functions g.

(ii) Suppose $f'' + f = g$ with $f \in \mathcal{C}^2_{\text{per}}([-\pi, \pi])$. Then by (2.4.22) we have

$$\widehat{f}(k) = \frac{\widehat{g}(k)}{1 - k^2}$$

for $k \neq \pm 1$. This does not tell us what $\widehat{f}(\pm 1)$ are, but we are working with a second-order ODE, so we expect to have two free parameters. Since f is 2π-periodic and $\mathcal{C}^2$, convergence theory for Fourier series implies

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ikx} = \underbrace{\sum_{\substack{k=-\infty \\ k \neq \pm 1}}^{\infty} \frac{\widehat{g}(k)}{1 - k^2}e^{ikx}}_{f_0(x)} + \widehat{f}(-1)e^{-ix} + \widehat{f}(1)e^{ix}. \tag{2.4.23}$$

Let us be clear about the order of the logic here: we have assumed the existence of a solution $f \in \mathcal{C}^2_{\text{per}}([-\pi, \pi])$ to our ODE, and we have derived a formula for f in terms of Fourier coefficients. If such a solution f exists, then the series f_0 in (2.4.23) converges.

But does this series converge for all x, and, if so, is it differentiable? By the Sobolev inequality, we have

$$\max_{k \in \mathbb{Z}} |\widehat{g}(k)| \leq \max_{|x| \leq P} |g(x)| := C.$$

Thus

$$\left| \frac{\widehat{g}(k)}{1 - k^2}e^{ikx} \right| \leq \frac{C}{1 - k^2},$$

and so the series $f_0(x)$ converges for each x by the comparison test.

To show that the function f_0 is differentiable requires some more advanced tools from the theory of uniform convergence for series of functions, and so we omit the details, but we note that *if* we can differentiate this series term-by-term, then

$$f_0'' + f_0 = \sum_{\substack{k=-\infty \\ k\neq\pm1}}^{\infty} \left[\frac{\widehat{g}(k)}{1-k^2}(-k^2)e^{ikx} + \frac{\widehat{g}(k)}{1-k^2}e^{ikx} \right] = \sum_{\substack{k=-\infty \\ k\neq\pm1}}^{\infty} \widehat{g}(k) \left(\frac{1-k^2}{1-k^2} \right) e^{ikx}$$

$$= \sum_{\substack{k=-\infty \\ k\neq\pm1}}^{\infty} \widehat{g}(k)e^{ikx} = g(x).$$

So, up to some details about uniform convergence, which we elide, we have shown that the formal solution (2.4.23) is a genuinely convergent and differentiable solution. ▲

2.4.34 Method: solve an ODE with Fourier series

0. Assume that the ODE is a constant-coefficient problem of the form

$$\mathcal{A}f := \sum_{j=0}^{n} a_j \partial_x^j[f] = g,$$

and we are interested in 2π-periodic solutions.

1. Calculate the Fourier coefficients $\widehat{g}(k)$ of the forcing function g on $[-\pi, \pi]$.

2. Calculate the Fourier coefficients of $\mathcal{A}f$:

$$\widehat{\mathcal{A}f}(k) = \sum_{j=0}^{n} (ik)^j a_j \widehat{f}(k).$$

Put $\widetilde{\mathcal{A}}(k) := \sum_{j=0}^{n} (ik)^j a_j$, so $\widehat{\mathcal{A}f}(k) = \widetilde{\mathcal{A}}(k)\widehat{f}(k)$.

3. Find any zeros of $\widetilde{\mathcal{A}}$. If $\widetilde{\mathcal{A}}(k) \neq 0$ for all k, then the solution f has Fourier coefficients

$$\widehat{f}(k) = \frac{\widehat{g}(k)}{\widetilde{\mathcal{A}}(k)}$$

and the (formal) solution is

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{\widehat{g}(k)}{\widetilde{\mathcal{A}}(k)} e^{ikx}.$$

If $\widetilde{\mathcal{A}}(k) = 0$ for some k, then the ODE has no solution if $\widehat{g}(k) \neq 0$. If $\widehat{g}(k) = 0$ whenever $\widetilde{\mathcal{A}}(k) = 0$, then the (formal) solution is

$$f(x) = \sum_{\substack{k=-\infty \\ \widetilde{\mathcal{A}}(k)\neq 0}}^{\infty} \frac{\widehat{g}(k)}{\widetilde{\mathcal{A}}(k)} e^{ikx}.$$

In either case, check if the series for f converges pointwise (in the sense of Definition A.2.11).

2.4.9. Solving PDEs with Fourier series.

In this section we will derive formal solutions to PDEs using Fourier series; that is, we do not verify that the "solutions" we concoct actually converge (if defined as series) or are differentiable.

First, we need a theorem that allows us to interchange differentiation and integration in certain circumstances; this is nontrivial, as it involves interchanging two limits (the limit giving the derivative and the limit of Riemann sums giving the integral). This theorem is proved in many sources, including [12].

2.4.35 Theorem (Leibniz's rule for differentiating under the integral).

Let a, $b \in \mathbb{R}$ with $a < b$ and $I \subseteq \mathbb{R}$ be an interval, and let $f = f(x,t)$ be a function with the following properties.

(i) *The function $f(\cdot, t)$ is continuous on $[a,b]$ for each $t \in I$.*

(ii) *The partial derivative $f_t(x,t)$ exists for each $(x,t) \in [a,b] \times I$ and the mapping $[a,b] \times I \to \mathbb{C}\colon (x,t) \mapsto f_t(x,t)$ is continuous.*

Then the mapping $I \to \mathbb{C}\colon t \mapsto \int_a^b f(x,t)\, dx$ is differentiable and

$$\partial_t \left[\int_a^b f(x,t)\, dx \right] = \int_a^b f_t(x,t)\, dx.$$

2.4.36 Example.

Determine all functions $u = u(x,t)$ such that

$$u_x + u_t = 0, \ -\pi \leq x \leq \pi, \ t \in \mathbb{R},$$

where $u(\pi,t) = u(-\pi,t)$ for all t.

Solution. Suppose that we have such a solution u to this transport equation. By the continuity and periodicity hypotheses on u, we can write

$$u(x,t) = \sum_{k=-\infty}^{\infty} \widehat{u}(k,t) e^{ikx},$$

where we define the time-dependent "spatial" Fourier coefficient as

$$\widehat{u}(k,t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x,t) e^{-ikx}\, dx. \tag{2.4.24}$$

Thus

$$0 = \widehat{u_x}(k,t) + \widehat{u_t}(k,t) = ik\widehat{u}(k,t) + \widehat{u_t}(k,t).$$

To evaluate this second Fourier coefficient, we use Leibniz's rule:

$$\widehat{u_t}(k,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_t(x,t) e^{-ikx}\, dx = \partial_t \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} u(x,t) e^{-ikx}\, dx \right] = \partial_t \widehat{u}(k,t).$$

Then the Fourier coefficients satisfy

$$\partial_t \widehat{u}(k,t) + ik\widehat{u}(k,t) = 0.$$

If we fix $k \in \mathbb{Z}$ and write $h(t) := \widehat{u}(k,t)$, then h satisfies the familiar ODE

$$h'(t) = -ikh(t),$$

and so $h(t) = C(k)e^{-ikt}$ for some constant $C(k)$ that depends on k. We then have

$$\widehat{u}(k,t) = C(k)e^{-ikt},$$

and so

$$u(x,t) = \sum_{k=-\infty}^{\infty} \widehat{u}(k,t)e^{ikx} = \sum_{k=-\infty}^{\infty} C(k)e^{ikt}e^{ikx} = \sum_{k=-\infty}^{\infty} C(k)e^{ik(x-t)}.$$

If the coefficients $C(k)$ are chosen such that

$$f(\xi) := \sum_{k=-\infty}^{\infty} C(k)e^{ik\xi}$$

converges — for example, if $\sum_{k=-\infty}^{\infty}|C(k)| < \infty$ — then $u(x,t) = f(x-t)$, which is exactly what we expect for the transport equation. In practice, a PDE would have some more initial "data" that would tell us what $C(k)$ should be, which would allow us to check the convergence of f. ▲

2.4.37 Method: solve a PDE with Fourier series

0. Assume that the problem is posed on $[-\pi, \pi]$ and has the form

$$\mathcal{A}u = f,$$

where $f = g(x,t)$ is 2π-periodic in x.

1. Calculate the Fourier coefficients $\widehat{f}(k,t)$ of $f(x,t)$ on $[-\pi,\pi]$. Evaluate the Fourier integral with respect to x.

2. Calculate the (unknown) Fourier coefficients of $(\mathcal{A}u)(x,t)$ with respect to x. Use the identities

$$\widehat{\partial_t[u]}(k,t) = \partial_t[\widehat{u}](k,t) \quad \text{and} \quad \widehat{\partial_x[u]}(k,t) = ik\widehat{u}(k,t).$$

3. Obtain an ODE for $\widehat{u}(k,t)$ in which the only derivatives are taken with respect to t and in which k is a parameter.

4. Solve this ODE for $\widehat{u}(k,t)$. Any arbitrary constants from ODE methods must now depend on k.

5. If the original PDE also has initial conditions of the form

$$u(x,0) = \phi(x),$$

calculate the Fourier coefficient $\widehat{\phi}(k)$ of ϕ on $[-\pi,\pi]$. Impose the initial condition

$$\widehat{u}(k,0) = \widehat{\phi}(k)$$

on the Fourier-side ODE above. (If there are further initial conditions, e.g., $u_t(x,0) = \psi(x)$, obtain the ODE-type initial condition $\partial_t\widehat{u}(k,0) = \widehat{\psi}(k)$.) Solve for arbitrary constants.

6. The formal solution to the PDE is then

$$u(x,t) = \sum_{k=-\infty}^{\infty} \widehat{u}(k,t)e^{ikx},$$

where $\widehat{u}(k,t)$ was determined from the Fourier-side IVP above.

When we are not working with 2π-periodic (or, more broadly, $2P$-periodic, although, for simplicity, we will not pursue that case) functions, we can still use Fourier series to construct formal solutions to PDEs, but the series may have to be sine or cosine series.

2.4.38 Example.

We saw from Example 2.3.2 that a solution to the IVP-BVP

$$\begin{cases} u_t = u_{xx}, \ 0 \le x \le P, \ t \ge 0 \\ u(0,t) = u(P,t) = 0, \ t \ge 0 \\ u(x,0) = \displaystyle\sum_{k=1}^{n} B_k \sin\left(\frac{k\pi x}{P}\right) \end{cases}$$

is

$$u(x,t) = \sum_{k=1}^{n} B_k e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right).$$

Discuss how one might solve this IVP-BVP when the initial temperature distribution is an arbitrary function f.

Solution. If we represent f by its Fourier sine series

$$\mathsf{FSS}[f](x) = \sum_{k=1}^{\infty} \mathsf{B}_k[f] \sin\left(\frac{k\pi x}{P}\right), \tag{2.4.25}$$

the result above suggests that a solution to the IVP-BVP will be

$$u(x,t) = \sum_{k=1}^{\infty} \mathsf{B}_k[f] e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right). \tag{2.4.26}$$

We call this a FORMAL SOLUTION to the IVP-BVP. But will such a series even converge pointwise for $(x,t) \in [-P,P] \times [0,\infty)$? And, if so, will it be differentiable?

Any positive results will hinge on further properties of f that must be specified. For example, if $f \in \mathcal{C}^1([0,P])$, then one can infer from theory about Fourier sine and cosine series in the exercises that $\mathsf{FSS}[f](x)$ converges to $f(x)$ for $0 < x \le P$, so the Fourier sine series is a justifiable representation for f. Then Plancherel's identity for the coefficients $\mathsf{B}_k[f]$ imply that $\sum_{k=1}^{\infty} |\mathsf{B}_k[f]|^2$ converges, hence by the test for divergence there is $M > 0$ such that $|\mathsf{B}_k[f]| \le M$ for all k. Then one estimates

$$\left| \mathsf{B}_k[f] e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right) \right| \le M e^{-(k\pi/P)^2 t},$$

and the ratio test can be used to show that $\sum_{k=1}^{\infty} e^{-(k\pi/P)^2 t}$ converges for all $t > 0$, thus by the comparison test the series (2.4.26) for u converges. There remains the problem of interchanging the infinite sum and the limit in the definition of the derivative to show that term-by-term differentiation is permitted; if it is, i.e., if

$$\partial_t \left[\sum_{k=1}^{\infty} \mathsf{B}_k[f] e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right) \right] = \sum_{k=1}^{\infty} \partial_t \left[\mathsf{B}_k[f] e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right) \right] \tag{2.4.27}$$

and

$$\partial_x^2 \left[\sum_{k=1}^{\infty} \mathsf{B}_k[f] e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right) \right] = \sum_{k=1}^{\infty} \partial_x^2 \left[\mathsf{B}_k[f] e^{-(k\pi/P)^2 t} \sin\left(\frac{k\pi x}{P}\right) \right], \tag{2.4.28}$$

then it follows from simple calculus that u solves the IVP-BVP (one must also check the boundary conditions, but this is easy).

Ultimately, the hypotheses[32] on the initial condition f for the Fourier sine series (2.4.26) to solve the IVP-BVP are benign: we merely need $f \in \mathcal{C}^1_{\mathrm{pw}}([0, P])$ with $f(0) = f(P)$. But as the proof is rather technical, we mention that a practical approach is to approximate f with an appropriate Fourier series, truncate the approximation to some finite sum, and solve exactly the approximate problem in the style of Example 2.3.2. ▲

2.5. The Fourier transform.

As with Fourier series, this section draws on [4, 9, 17, 20]. Omitted proofs can be found in [9].

2.5.1. Motivation for the Fourier transform.

A Fourier series representation of a function is inherently tied to a closed, bounded interval; if we calculate the Fourier coefficients $\widehat{f}(k)$ of a function f on an interval $[-P, P]$, then the representation

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ik\pi x/P} \tag{2.5.1}$$

will only be valid for $x \in [-P, P]$ if f is not $2P$-periodic on $\mathbb{R}$. (Of course, this representation need not be valid on all of $[-P, P]$ if f is not sufficiently regular.) We have already seen the utility of Fourier series in solving ODEs and PDEs; the relation $\widehat{f'}(k) = ik\widehat{f}(k)$ reduces ODEs to algebraic equations and PDEs to ODEs. So, we seek an extension of the Fourier series representation valid for functions defined on $\mathbb{R}$. This will be the Fourier transform.

To motivate the definition of the transform, suppose that $f \in \mathcal{C}^1(\mathbb{R})$ has **compact support**: there is $P > 0$ such that $f(x) = 0$ for $|x| > P$. Moreover, suppose that P is "large." Then f will be defined on a "broad" subinterval of $\mathbb{R}$, but the representation (2.5.1) still holds for $x \in (-P, P)$. Set

$$g(k) := \int_{-P}^{P} f(\xi) e^{-ik\xi}\, d\xi,$$

[32] See Proposition 8.1 in [2].

so (2.5.1) becomes

$$f(x) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left[g\left(\pi \frac{k}{P}\right) e^{i(k/P)\pi x} \right] \frac{1}{P}.$$

Since P is "large," we can think of this sum as an approximation to the improper integral

$$\frac{1}{2} \int_{-\infty}^{\infty} g(\pi\xi) e^{i\pi x\xi}\, d\xi,$$

which rescales to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{g(k)}{\sqrt{2\pi}} e^{ikx}\, dk.$$

Since $f(x) = 0$ for $|x| > P$, we have

$$\frac{g(k)}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi}\, d\xi =: \mathfrak{F}[f](k).$$

We call $\mathfrak{F}[f](k)$ the **Fourier transform** of f at $k \in \mathbb{R}$, and we have formally motivated the identity

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathfrak{F}[f](k) e^{ikx}\, dk.$$

This is the **Fourier representation** of a function defined on all of $\mathbb{R}$.

This is morally similar to a Fourier series, if we think of an integral as a "continuous superposition" instead of an infinite sum. We will see that the Fourier transform (when it converges as an improper integral) reveals a great deal of information about a function defined on $\mathbb{R}$, just as a Fourier series (whether or not it converges) can shed much light on a function defined on a closed, bounded interval.

2.5.2. Definition and properties of the Fourier transform.

We will need several vector spaces of improperly integrable functions and therefore presume familiarity with Appendix B. To review, for $p > 0$, we denote by $L^p(\mathbb{R})$ the vector space of all locally integrable (Definition B.0.1) functions $f: \mathbb{R} \to \mathbb{C}$ such that $|f|^p$ is improperly integrable on $\mathbb{R}$, and we set

$$\|f\|_{L^p(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx \right)^{1/p}.$$

The most important cases for us will be $p = 1$ and $p = 2$. The functions in $L^1(\mathbb{R})$ are often called **absolutely integrable** on $\mathbb{R}$, while those in $L^2(\mathbb{R})$ are **square integrable**. To be clear, for f to be absolutely integrable on $\mathbb{R}$, the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ must exist, while for g to be square integrable on $\mathbb{R}$, the integral $\int_{-\infty}^{\infty} |g(x)|^2\, dx$ must exist. It is possible for $f: \mathbb{R} \to \mathbb{C}$ to be improperly integrable on $\mathbb{R}$ but not absolutely integrable (Example B.0.5).

2.5.1 Definition.

The **Fourier transform** *of a locally integrable function $f: \mathbb{R} \to \mathbb{C}$ at $k \in \mathbb{R}$ is the value*

$$\widehat{f}(k) = \mathfrak{F}[f](k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \tag{2.5.2}$$

defined for all numbers $k \in \mathbb{R}$ for which this integral converges.

Traditionally, if one works with a function $f = f(x)$, then one uses a different letter, like k or ξ, for the Fourier transform. The original variable x is sometimes called the "space" variable (a term inherited from PDE problems in which a function $u = u(x,t)$ depends on "space" x and "time" t, and the Fourier transform is taken there "in x") and the variable k the "Fourier" variable. Of course the notation $\widehat{f}(k)$ for the Fourier transform at $k \in \mathbb{R}$ is the same as the notation for the complex Fourier series coefficient at $k \in \mathbb{Z}$; context will always make it clear what we mean.

The absolute integrability of f and the comparison principle for improper integrals ensure that the integral in (2.5.2) converges whenever $f \in L^1(\mathbb{R})$, since $|f(x)e^{-ikx}| \le |f(x)|$. The factor $1/\sqrt{2\pi}$ is somewhat arbitrary and may not appear in other definitions of the Fourier transform; some conventions write $e^{-2\pi ik}$ instead in the integrand, as well. Some of the more robust theory will only hold for $f \in L^1(\mathbb{R})$, but nonetheless it may be possible to define $\widehat{f}$ even if f is not absolutely integrable.

2.5.2 Example.

Let $a > 0$ and

$$\mathsf{E}_a(x) := \begin{cases} e^{ax}, & x < 0 \\ 0, & x \ge 0. \end{cases}$$

Compute $\widehat{\mathsf{E}_a}(k)$.

Solution. We check in an exercise that $\mathsf{E}_a = |\mathsf{E}_a|$ is improperly integrable on $\mathbb{R}$. We calculate

$$\widehat{\mathsf{E}_a}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathsf{E}_a(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ax} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-ik)x}\, dx.$$

Now, since $a > 0$, we have $a - ik \ne 0$ for all $k \in \mathbb{R}$, and so the function $x \mapsto e^{(a-ik)x}/(a-ik)$ is an antiderivative of $x \mapsto e^{(a-ik)x}$. Hence

$$\int_{-\infty}^{0} e^{(a-ik)x}\, dx = \lim_{b \to -\infty} \int_{b}^{0} e^{(a-ik)x}\, dx = \lim_{b \to -\infty} \frac{e^{(a-ik)x}}{a-ik}\bigg|_{x=b}^{x=0} = \frac{1}{a-ik}.$$

Thus

$$\widehat{\mathsf{E}_a}(k) = \frac{1}{\sqrt{2\pi}(a-ik)}.$$

Observe that $\widehat{\mathsf{E}_a} \in \mathcal{C}^\infty(\mathbb{R})$ but $\widehat{\mathsf{E}_a} \notin L^1(\mathbb{R})$. We make this precise in the exercises and just note here that $\sqrt{2\pi}|\widehat{\mathsf{E}_a}(k)| = (a^2+k^2)^{-1/2} \approx |k|^{-1}$, and so if $\widehat{\mathsf{E}_a} \in L^1(\mathbb{R})$, then by the comparison test we would expect the divergent integral $\int_1^\infty k^{-1}\, dk$ to converge. ▲

2.5.3 Example.

Fix a number $t > 0$ and let

$$f(x) = \begin{cases} 1, & |x| \le t \\ 0, & |x| > t. \end{cases}$$

Compute $\mathfrak{F}[f]$.

Solution. Clearly $f \in L^1(\mathbb{R})$ since f is identically zero outside a closed, bounded interval. We have

$$\mathfrak{F}[f](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-ikx}\, dx.$$

If $k = 0$, then

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} dx = \frac{2t}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} t.$$

If $k \neq 0$, then

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ikx}}{-ik}\bigg|_{x=-t}^{t} = -\frac{e^{-ikt} - e^{ikt}}{\sqrt{2\pi} ik} = \frac{\sqrt{2}}{k\sqrt{\pi}} \left(\frac{e^{ikt} - e^{-ikt}}{2i} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin(kt)}{k}.$$

L'Hospital's rule implies

$$\lim_{k \to 0} \frac{\sin(kt)}{k} = t,$$

so we may as well write

$$\mathfrak{F}[f](k) = \sqrt{\frac{2}{\pi}} \frac{\sin(kt)}{k}$$

for all k. We note from Example B.0.5 that $\widehat{f} \notin L^1(\mathbb{R})$ although $\widehat{f} \in \mathcal{C}^\infty(\mathbb{R})$. ▲

The next theorem describes some essential algebraic and analytic properties of the Fourier transform.

2.5.4 Theorem.

(i) [Linearity] *The Fourier transform is linear in the sense that if f, $g \in L^1(\mathbb{R})$ and α, $\beta \in \mathbb{C}$, then*

$$\mathfrak{F}[\alpha f + \beta g](k) = \alpha \widehat{f}(k) + \beta \widehat{g}(k), \quad k \in \mathbb{R}.$$

(ii) [Shifts] *Let $f \in L^1(\mathbb{R})$ and $d \in \mathbb{R}$ and denote by $S^d f = f(\cdot + d)$ the mapping $x \mapsto f(x + d)$. Then $S^d f \in L^1(\mathbb{R})$ and*

$$\mathfrak{F}[S^d f](k) = e^{ikd} \widehat{f}(k), \quad k \in \mathbb{R}.$$

(iii) [Dilations/scalings] *Let $f \in L^1(\mathbb{R})$ and $a \in \mathbb{R} \setminus \{0\}$ and denote by $f(a\cdot)$ the mapping $x \mapsto f(ax)$. Then $f(a\cdot) \in L^1(\mathbb{R})$ and*

$$\mathfrak{F}[f(a\cdot)](k) = \frac{1}{|a|} \widehat{f}\left(\frac{k}{a}\right), \quad k \in \mathbb{R}.$$

(iv) [Transform of derivative] *Suppose that $f \in L^1(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$. Then*

$$\mathfrak{F}[f'](k) = ik\widehat{f}(k), \quad k \in \mathbb{R}.$$

(v) [Derivative of transform] *Suppose that $f \in L^1(\mathbb{R})$ and let $\mathbf{1}(x) := x$. If $\mathbf{1}f \in L^1(\mathbb{R})$ as well, then $\widehat{f}$ is differentiable, and*

$$\partial_k \widehat{f}(k) = \mathfrak{F}[f]'(k) = -i\widehat{\mathbf{1}f}(k), \quad k \in \mathbb{R}.$$

Proof. (i) This is obvious from the definition of the Fourier transform as an integral, which is always linear.

(ii) We compute

$$\begin{aligned}\mathfrak{F}[f(\cdot+d)](k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+d)e^{-ikx}\ dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-ik(u-d)}\ du,\ u = x+d \\ &= \frac{e^{ikd}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-iku}\ du \\ &= e^{ikd}\mathfrak{F}[f](k).\end{aligned}$$

(iii) We leave this as a practice problem.

(iv) We claim that if $f \in L^1(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, then the limits $\lim_{x\to\pm\infty} f(x)$ exist and are both 0; the proof[33] is an exercise in the fundamental theorem of calculus and the definition of the improper integral. Assuming this to be true, we proceed to calculate

$$\mathfrak{F}[f'](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-ikx}\ dx = \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\int_{-R}^{R} f'(x)e^{-ikx}\ dx.$$

Since $f' \in L^1(\mathbb{R})$, we are allowed to express the improper integral as this symmetric limit. Then we integrate by parts with

$$u = e^{-ikx} \qquad dv = f'(x)\ dx$$

$$du = -ike^{-ikx} \qquad v = f(x)$$

to find

$$\int_{-R}^{R} f'(x)e^{-ikx}\ dx = \left(f(x)e^{-ikx}\right)\Big|_{x=-R}^{x=R} - (-ik)\int_{-R}^{R} f(x)e^{-ikx}\ dx.$$

The squeeze theorem and the condition $\lim_{x\to\pm\infty} f(x) = 0$ imply

$$\lim_{R\to\infty}|f(\pm R)e^{\mp ikR}| = 0,$$

and so

$$\mathfrak{F}[f'](k) = \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\int_{-R}^{R} f'(x)e^{-ikx}\ dx = ik\left(\frac{1}{\sqrt{2\pi}}\right)\lim_{R\to\infty}\int_{-R}^{R} f(x)e^{-ikx}\ dx = ik\widehat{f}(k).$$

[33]Here is that proof. The fundamental theorem of calculus lets us write $f(x) = f(0) + \int_0^x f'(\xi)\ d\xi$. Since $f' \in L^1(\mathbb{R})$, the limits $\lim_{x\to\pm\infty}\int_0^x f'(\xi)\ d\xi$ exist, and so the limits $L_\pm := \lim_{x\to\pm\infty} f(x)$ also exist. Suppose $L_+ \neq 0$. Then by the definition of a limit, there is $M > 0$ such that if $x \geq M$, then $|f(x)| \geq |L_+|/2$. But then $\lim_{b\to\infty}\int_M^b |f(x)|\ dx = \infty$, which contradicts $f \in L^1(\mathbb{R})$. So, $L_+ = 0$, and likewise $L_- = 0$.

(v) One formally sees this by differentiating under the integral

$$\sqrt{2\pi}\partial_k\widehat{f}(k) = \partial_k \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \int_{-\infty}^{\infty} \partial_k[f(x)e^{-ikx}]\,dx = -i\int_{-\infty}^{\infty} xf(x)e^{-ikx}\,dx.$$

To make this precise, one would need a version of Leibniz's rule on differentiating under the integral (Theorem 2.4.35) for improper integrals. ■

Examples 2.5.3 and 2.5.2 show that the Fourier transform does not map $L^1(\mathbb{R})$ back to itself: there are functions $f \in L^1(\mathbb{R})$ such that $\widehat{f} \notin L^1(\mathbb{R})$. However, under various other hypotheses, the Fourier transform exhibits quite a diverse range of "mapping behaviors."

2.5.5 Theorem.

(i) *The Fourier transform is a linear functional on $L^1(\mathbb{R})$ in the sense that for k fixed, the mapping $L^1(\mathbb{R}) \to \mathbb{C}\colon f \mapsto \widehat{f}(k)$ is a linear operator.*

(ii) [Riemann-Lebesgue lemma for Fourier transforms] *The Fourier transform is a linear operator from $L^1(\mathbb{R})$ to $\mathcal{C}_0(\mathbb{R})$, where $\mathcal{C}_0(\mathbb{R})$ is the space of continuous functions that vanish at $\pm\infty$, i.e.,*

$$\mathcal{C}_0(\mathbb{R}) := \left\{ f \in \mathcal{C}(\mathbb{R}) \;\middle|\; \lim_{x\to\pm\infty} f(x) = 0 \right\}.$$

Moreover,

$$\max_{k\in\mathbb{R}} |\widehat{f}(k)| \le \|f\|_{L^1(\mathbb{R})}.$$

(iii) *If $f \in L^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ with*

$$\lim_{x\to\pm\infty} f(x) = 0 = \lim_{x\to\pm\infty} f'(x),$$

then $\widehat{f} \in L^1(\mathbb{R})$.

(iv) *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\widehat{f} \in L^2(\mathbb{R})$.*

Proof. (i) This is just a repackaging of part (i) of Theorem 2.5.4.

(ii) The global bound on $|\widehat{f}(k)|$ follows from the triangle inequality for integrals and the definition of $\|f\|_{L^1(\mathbb{R})}$. The proof that $\widehat{f}$ vanishes at $\pm\infty$ requires an "approximation-by-$\mathcal{C}^\infty$-functions" argument for a general $f \in L^1(\mathbb{R})$, so we only mention that a weaker version can be proved if one assumes that (1) $f \in L^1(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$, (2) $f' \in L^1(\mathbb{R})$, and (3) there is $M > 0$ such that $|f(x)| = 0$ for $|x| \ge M$. In this case, one proves the limit through an integration by parts argument, which yields the estimate $|\widehat{f}(k)| \le C|k|^{-1}$ for some constant $C > 0$. This implies the decay of $\widehat{f}$ for $|k|$ large. We sketch this argument in the exercises, along with the proof of continuity of $\widehat{f}$.

(iii) Again, the idea is to integrate by parts in a clever way, similar to the proof of the preceding property. One obtains $|\widehat{f}(k)| \le Ck^{-2}$. This shows that $\widehat{f}$ is integrable on $[1,\infty)$ and $(-\infty,-1]$ by the comparison test. Since $\widehat{f}$ is continuous by part (ii), $\widehat{f}$ is integrable on $[-1,1]$.

(iv) This requires tools from more advanced analysis (see, e.g., [10]), so we omit the proof. ■

2.5.3. The inverse Fourier transform.

If $f \in L^1(\mathbb{R})$, then the Fourier transform $\widehat{f}$ is also a function on $\mathbb{R}$. But Example 2.5.3 indicates that we may have $f \in L^1(\mathbb{R})$ with $\widehat{f} \notin L^1(\mathbb{R})$, and so the Fourier transform is not a linear operator from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$. Nonetheless, it is possible to invert the Fourier transform in a pointwise sense. The following result is proved in Section 60 of [20]; to state it, we need the Cauchy principal value of an improper integral from Definition B.0.6.

2.5.6 Theorem.

Let $f \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\mathrm{pw}}(\mathbb{R})$. Then the Cauchy principal value $\mathrm{P.V.}\int_{-\infty}^{\infty} \widehat{f}(k)e^{ikx}\, dk$ *exists and*

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}}\,\mathrm{P.V.}\int_{-\infty}^{\infty} \widehat{f}(k)e^{ikx}\, dk, \quad x \in \mathbb{R}. \tag{2.5.3}$$

The improper integral in (2.5.3) deserves its own name in the event that it converges as a genuine improper integral (Definition B.0.2), not a principal value.

2.5.7 Definition.

Let $f \colon \mathbb{R} \to \mathbb{C}$ be locally integrable. The INVERSE FOURIER TRANSFORM *of f at $x \in \mathbb{R}$ is the value*

$$\mathfrak{F}^{-1}[f](x) = \check{f}(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k)e^{ikx}\, dk, \tag{2.5.4}$$

defined for all $x \in \mathbb{R}$ for which this improper integral converges.

There is an obvious relation between the inverse Fourier integral in (2.5.4) and the original Fourier integral in (2.5.2).

2.5.8 Lemma.

Let $f \colon \mathbb{R} \to \mathbb{C}$ be locally integrable and $k \in \mathbb{R}$. If $\widehat{f}(k)$ and $\check{f}(-k)$ are both defined, then $\widehat{f}(k) = \check{f}(-k)$.

We can combine the results above to produce the following more palatable inversion formula.

2.5.9 Theorem.

If $f \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\mathrm{pw}}(\mathbb{R})$ and if $\widehat{f} \in L^1(\mathbb{R})$, then

$$\mathfrak{F}^{-1}[\widehat{f}](x) = \frac{f(x^+) + f(x^-)}{2} \tag{2.5.5}$$

for all $x \in \mathbb{R}$ and

$$\mathfrak{F}[\check{f}](k) = \frac{f(k^+) + f(k^-)}{2} \tag{2.5.6}$$

for all $k \in \mathbb{R}$.

Proof. If $\widehat{f} \in L^1(\mathbb{R})$, then the principal value integral on the right of (2.5.3) is just $\mathfrak{F}^{-1}[\widehat{f}]$, and so (2.5.5) follows from (2.5.3). The identity (2.5.6) is almost as immediate, as we now show. Put $(Rf)(x) := f(-x)$, so R is a "reflection" operator. Since $f \in L^1(\mathbb{R})$, $Rf \in L^1(\mathbb{R})$ as well, and the scaling property of the Fourier transform (part (iii) of Theorem 2.5.4) gives $\widehat{Rf} = R\widehat{f}$. Lemma 2.5.8 now reads $\widehat{f} = R\check{f}$, equivalently, $\check{f} = R\widehat{f}$. Consequently, since we are assuming $\widehat{f} \in L^1(\mathbb{R})$, we also have $\check{f} \in L^1(\mathbb{R})$. Thus

$$\mathfrak{F}[\check{f}](k) = \mathfrak{F}[R\widehat{f}](k) = \mathfrak{F}[\widehat{f}](-k) = \mathfrak{F}^{-1}[\widehat{f}](k) = \frac{f(k^+) + f(k^-)}{2}$$

by (2.5.5). ■

If $f \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R})$ is continuous, then $\mathfrak{F}^{-1}[\widehat{f}] = \mathfrak{F}[\check{f}] = f$ on $\mathbb{R}$. Such a function need not be continuously differentiable (i.e., f' need not exist at all points and/or be continuous everywhere) for this nice result to happen. We might say that such a function f belongs to the baroquely notated space $L^1(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$.

The various properties of the inverse Fourier transform above often help us calculate the actual Fourier transform of a function, with a little trickery here and there.

2.5.10 Example.

Find the Fourier transform of $f(x) = \text{sinc}(x) := \sin(x)/x$.

Solution. Example B.0.5 shows that $\text{sinc}(\cdot) \notin L^1(\mathbb{R})$ but also uses integration by parts to demonstrate that the improper integral $\int_{-\infty}^{\infty} \text{sinc}(x)\, dx$ converges. Similar, but more complicated, integration by parts shows that $\int_{-\infty}^{\infty} \text{sinc}(x) e^{-ikx}\, dx$ converges for all $k \in \mathbb{R}$, and so both $\widehat{f}$ and $\check{f}$ are defined on $\mathbb{R}$. Unfortunately, these integrals are very difficult, if not impossible, to evaluate using standard methods of calculus. However, if we define

$$g(x) := \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1, \end{cases}$$

then Example 2.5.3 yields

$$\widehat{g}(k) = \sqrt{\frac{2}{\pi}} \frac{\sin(k)}{k} = \sqrt{\frac{2}{\pi}} f(k).$$

In particular, by the aforementioned integration by parts, the improper integrals $\int_{-\infty}^{\infty} \widehat{g}(k) e^{ikx}\, dx$ exist for all $x \in \mathbb{R}$. Consequently, both $\mathfrak{F}[\widehat{g}]$ and $\mathfrak{F}^{-1}[\widehat{g}]$ are defined on $\mathbb{R}$, although $\widehat{g} \notin L^1(\mathbb{R})$.

We then have

$$\widehat{f}(k) = \sqrt{\frac{\pi}{2}} \mathfrak{F}[\widehat{g}](k).$$

This looks no better, except we can use Lemma 2.5.8 to rephrase this as

$$\widehat{f}(k) = \check{f}(-k) = \sqrt{\frac{\pi}{2}} \mathfrak{F}^{-1}[\widehat{g}](-k). \tag{2.5.7}$$

Since $g \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R})$ and $\mathfrak{F}^{-1}[\widehat{g}]$ is defined on $\mathbb{R}$, Theorem 2.5.6 tells us

$$\mathfrak{F}^{-1}[\widehat{g}](k) = \frac{g(k^+) + g(k^-)}{2}, \quad x \in \mathbb{R}. \tag{2.5.8}$$

All we have done here is removed the principal value from (2.5.3), since the improper integrals $\int_{-\infty}^{\infty} \widehat{g}(k)e^{ikx}\, dk$ converge by our integration by parts claim above. We continue to use the Fourier variable k in (2.5.8), even though we are calculating an inverse transform, to stay in correspondence with (2.5.7).

Now, since g has discontinuities only at ± 1, the formula (2.5.8) really reads

$$\mathfrak{F}^{-1}[\widehat{g}](k) = \begin{cases} 0, & k < -1 \\ 1/2, & k = -1 \\ 1, & -1 < k < 1 \\ 1/2, & k = 1 \\ 0, & 1 < k. \end{cases} \tag{2.5.9}$$

In particular, $\mathfrak{F}^{-1}[\widehat{g}]$ is even. So, when we calculate $\mathfrak{F}^{-1}[\widehat{g}](-k)$ in (2.5.7), there is nothing special to do.

We combine (2.5.7) and (2.5.9) to conclude

$$\widehat{f}(k) = \begin{cases} \sqrt{\pi/2}, & |k| < 1 \\ \sqrt{\pi}/2\sqrt{2}, & |k| = 1 \\ 0, & |k| > 1. \end{cases}$$

Note, incidentally, that $\widehat{f}$ is not continuous on $\mathbb{R}$. This does not, however, contradict part (ii) of Theorem 2.5.5, since $f \notin L^1(\mathbb{R})$. ▲

2.5.11 Example.

Let $a > 0$. Find the Fourier transform of

$$f_a(x) := \frac{1}{x^2 + a^2}.$$

Solution. By definition,

$$\widehat{f_a}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2 + a^2}\, dx,$$

and by evenness the integral reduces to

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + a^2}\, dx,$$

which still looks difficult to calculate.

Instead, we will be clever and use the inverse transform. First, factor

$$x^2 + a^2 = (x + ia)(x - ia)$$

and then use partial fractions to rewrite

$$f_a(x) = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = -\frac{1}{2ia}\left(\frac{1}{x+ia}\right) + \frac{1}{2ia}\left(\frac{1}{x-ia}\right). \tag{2.5.10}$$

The two functions (of x) on the right side of (2.5.10) do not belong to $L^1(\mathbb{R})$, but nonetheless their products against e^{-ikx} are improperly integrable over $\mathbb{R}$ for all k (integrate by parts). And so their Fourier transforms are still defined. Thus

$$\widehat{f_a}(k) = -\frac{1}{2ia}\mathfrak{F}\left[\frac{1}{x+ia}\right](k) + \frac{1}{2ia}\mathfrak{F}\left[\frac{1}{x-ia}\right](k). \tag{2.5.11}$$

Now, recall from Example 2.5.2 that

$$\mathsf{E}_a(x) := \begin{cases} e^{ax}, & x<0 \\ 0, & x \ge 0 \end{cases} \qquad \Longrightarrow \qquad \widehat{\mathsf{E}_a}(k) = \frac{1}{\sqrt{2\pi}(a-ik)}.$$

Similarly,

$$\mathsf{E}^a(x) := \begin{cases} 0, & x<0 \\ e^{-ax}, & x \ge 0 \end{cases} \qquad \Longrightarrow \qquad \widehat{\mathsf{E}^a}(k) = \frac{1}{\sqrt{2\pi}(a+ik)}.$$

Let us rewrite

$$\frac{1}{x+ia} = \frac{i}{ix+i^2a} = \frac{i}{-a+ix} = -\frac{i}{a-ix} = -i\sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}(a-ix)}\right).$$

Then for $k \neq 0$, Lemma 2.5.8 and Theorem 2.5.9 imply

$$\mathfrak{F}\left[\frac{1}{x+ia}\right](k) = -i\sqrt{2\pi}\mathfrak{F}\left[\frac{1}{\sqrt{2\pi}(a-ix)}\right](k) = -i\sqrt{2\pi}\mathfrak{F}^{-1}\left[\frac{1}{\sqrt{2\pi}(a-ix)}\right](-k)$$

$$= -i\sqrt{2\pi}\mathfrak{F}^{-1}[\widehat{\mathsf{E}_a}](-k) = -i\sqrt{2\pi}\mathsf{E}_a(-k) = -i\sqrt{2\pi}\begin{cases} e^{a(-k)}, & -k<0 \\ 0, & -k>0 \end{cases}$$

$$= \begin{cases} 0, & k<0 \\ -i\sqrt{2\pi}e^{-ak}, & k>0. \end{cases} \tag{2.5.12}$$

We claim that

$$\mathfrak{F}\left[\frac{1}{x-ia}\right](k) = \begin{cases} i\sqrt{2\pi}e^{ak}, & k<0 \\ 0, & k>0. \end{cases}$$

using a calculation similar to (2.5.12).

Now we put everything together using the partial fractions decomposition from back in (2.5.10) and the expansion (2.5.11). We have

$$\widehat{f_a}(k) = \begin{cases} -\frac{1}{2ia}\cdot 0 + \frac{i\sqrt{2\pi}e^{ak}}{2ia}, & k<0 \\ \\ -\frac{(-i)\sqrt{2\pi}e^{-ak}}{2ia} + \frac{1}{2ia}\cdot 0, & k>0 \end{cases} = \begin{cases} \sqrt{\frac{\pi}{2}}\left(\frac{e^{ak}}{a}\right), & k<0 \\ \\ \sqrt{\frac{\pi}{2}}\left(\frac{e^{-ak}}{a}\right), & k>0 \end{cases} = \sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}, \ k \neq 0.$$

By part (ii) of Theorem 2.5.5, we know that since $f \in L^1(\mathbb{R})$, $\widehat{f}$ is continuous, and so

$$\widehat{f}_a(0) = \lim_{k \to 0} \widehat{f}_a(k) = \frac{1}{a}\sqrt{\frac{\pi}{2}}.$$

And so

$$\widehat{f}_a(k) = \sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}$$

for all $k \in \mathbb{R}$.

We will also calculate this transform for $a = 1$ in Example 3.10.18 using residue theory. Note that $\widehat{f}_a \in \mathcal{C}^1_{\text{pw}}(\mathbb{R})$ even though $\int_{-\infty}^{\infty} |xf_a(x)| \, dx$ diverges; compare this to part (v) of Theorem 2.5.4. ▲

2.5.12 Remark.

In situations like (2.5.12)*, we will sometimes denote a Fourier transform as*

$$\mathfrak{F}[\textit{expression involving the independent variable } x](k),$$

without giving a name to that expression (like f). Our understanding will always be that x is the "transformed variable," i.e., we integrate with respect to x. This will just save us time and space by not requiring every single function to have a name.

2.5.13 Example.

Find the Fourier transform of $f(x) = e^{-x^2}$.

Solution. Since $x^2 \ge 1$ for $|x| \ge 1$, we have $|f(x)| \le e^{-|x|}$ for $|x| \ge 1$. This shows $f \in L^1(\mathbb{R})$, and so $\widehat{f}$ is defined (as is $\check{f}$). Of course,

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} \, dx,$$

but the integrand has no apparent antiderivative. We might note that f is even and, since the Fourier integral is taken over a symmetric interval, reduce to

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} \, dx = \int_{-\infty}^{\infty} \cos(kx) e^{-x^2} \, dx,$$

but this is no better.

There are, however, several not very obvious "tricks" one can use to calculate the improper integral. Example 3.10.17 uses residue theory; here we will use a host of properties of the Fourier transform from Theorem 2.5.4. First, we calculate

$$f'(x) = -2xe^{-x^2} = -2xf(x).$$

With $\mathbf{1}(x) := x$, we note that $\mathbf{1}f$ is absolutely integrable on $\mathbb{R}$; this is a quick u-substitution. Then parts (iv) and (v) of Theorem 2.5.4 imply

$$ik\widehat{f}(k) = \widehat{f'}(k) = -2\widehat{\mathbf{1}f}(k) = \frac{2}{i}\big(-i\widehat{\mathbf{1}f}(k)\big) = -2i\partial_k \widehat{f}(k).$$

And so $\widehat{f}$ satisfies the ODE

$$\partial_k \widehat{f}(k) = -\frac{k}{2}\widehat{f}(k).$$

This is a first-order linear ODE, which might look more familiar when cast in the variables

$$y' = -\frac{x}{2}y \iff y' + \frac{x}{2}y = 0.$$

The general solution to this ODE is $y = Ce^{-x^2/4}$, where C is a constant, so $\widehat{f}$ must satisfy

$$\widehat{f}(k) = Ce^{-k^2/4}, \tag{2.5.13}$$

for some constant C to be determined.

Of course, $C = \widehat{f}(0)$, and we have

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{1}{\sqrt{2}}, \tag{2.5.14}$$

using from calculus the fact that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$. We conclude

$$\widehat{f}(k) = \frac{e^{-k^2/4}}{\sqrt{2}}.$$

But if we did not already know the value of the integral in (2.5.14), which usually requires a polar coordinates trick to calculate, we could still find C via Fourier properties. Observe that (2.5.13) gives the relation

$$\widehat{f}(k) = Cf\left(\frac{k}{2}\right). \tag{2.5.15}$$

And so $f, \widehat{f} \in L^1(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. Then we have the following intricate calculation:

$$\begin{aligned}
f(-k) &= \mathfrak{F}^{-1}[\widehat{f}](-k) \text{ by Theorem 2.5.9} \\
&= \mathfrak{F}[\widehat{f}](k) \text{ by Lemma 2.5.8} \\
&= \mathfrak{F}\left[Cf\left(\frac{\cdot}{2}\right)\right](k) \text{ by (2.5.15)} \\
&= 2C\widehat{f}(2k) \text{ by part (iii) of Theorem 2.5.4} \\
&= 2C^2 f(k) \text{ by (2.5.15) once again.}
\end{aligned}$$

In particular, $f(0) = 2C^2 f(0)$, and since $f(0) \neq 0$, we have $C^2 = 1/2$. To see which square root C must be, we return to $C = \widehat{f}(0) = \int_{-\infty}^{\infty} e^{-x^2}\, dx > 0$, and so $C = 1/\sqrt{2}$.

This was, by no means, an easy Fourier transform to evaluate. The strategy here used a variety of properties of the abstract transform to avoid evaluating the original integral $\int_{-\infty}^{\infty} e^{-x^2 - ikx}\, dx$ directly, which will be the path of Example 3.10.17. ▲

2.5.4. Convolution.

Theorem 2.5.5 tells us that the Fourier transform is a linear operator from $L^1(\mathbb{R})$ to $\mathcal{C}_0(\mathbb{R})$ and a linear functional in the pointwise sense that for $k \in \mathbb{R}$ fixed, the mapping $L^1(\mathbb{R}) \to \mathbb{C}\colon f \mapsto \widehat{f}(k)$ is linear. But the vectors in $L^1(\mathbb{R})$ are also functions from $\mathbb{R}$ to $\mathbb{C}$, and so they interact in an additional algebraic manner: function multiplication. If f,

$g \in L^1(\mathbb{R})$, then their product fg is defined, although we need not have $fg \in L^1(\mathbb{R})$. It turns out that requiring $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is strong enough to guarantee $fg \in L^1(\mathbb{R})$; this is a consequence of the comparison test for improper integrals, as we sketch in an exercise.

And so, if $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, it makes sense to ask what the value of $\widehat{fg}(k)$ is. One might expect $\widehat{f}(k)\widehat{g}(k)$, but this is not quite the case. However, it pays to start by calculating this product, with the notational sleight-of-hand that we use different dummy variables of integration in the two Fourier integrals:

$$\widehat{f}(k)\widehat{g}(k) = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)e^{-iky}\ dy\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx\right) g(y)e^{-iky}\ dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-ik(x+y)}\ dx\ dy.$$

Fixing $y \in \mathbb{R}$ momentarily, we have

$$\int_{-\infty}^{\infty} f(x)g(y)e^{-ik(x+y)}\ dx = g(y)\int_{-\infty}^{\infty} f(x)e^{-ik(x+y)}\ dx = g(y)\int_{-\infty}^{\infty} f(u-y)e^{-iku}\ du.$$

Here we have substituted $u = x + y$, $du = dx$. Then

$$\widehat{f}(k)\widehat{g}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{-iku}\ du\ dy.$$

Suppose for the moment that we can interchange the order of integration to write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{-iku}\ du\ dy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u-y)g(y)\ dy\right)e^{-iku}\ du. \tag{2.5.16}$$

Then if we abbreviate

$$(f * g)(u) := \int_{-\infty}^{\infty} f(u-y)g(y)\ dy,$$

we will have

$$\widehat{f}(k)\widehat{g}(k) = \frac{\widehat{f * g}(k)}{\sqrt{2\pi}}.$$

Let us pause from Fourier analysis for now and study this new expression $f * g$.

2.5.14 Definition.

For functions $f, g\colon \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$, the **CONVOLUTION** *of f and g at x is*

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\ d\xi,$$

defined whenever this improper integral converges (Definition B.0.2).

Our first question should be when $f * g$ is defined, and, if it is defined, what are some of its properties as a function in its own right.

2.5.15 Theorem.

(i) *If* f, $g \in L^2(\mathbb{R})$, *then* $(f * g)(x)$ *is defined for every real number* x.

(ii) *Convolution distributes over addition in the sense that*

$$f * (g + h) = (f * g) + (f * h)$$

for any functions f, g, *and* h *for which these convolutions are defined.*

(iii) *Convolution is commutative in the sense that*

$$f * g = g * f.$$

for any functions f, g *for which these convolutions are defined.*

(iv) *If* f, $g \in L^2(\mathbb{R})$ *and* $f \in \mathcal{C}^n(\mathbb{R})$ *for some* $n \geq 0$, *then* $f * g \in \mathcal{C}^n(\mathbb{R})$. *If* f, $g \in L^2(\mathbb{R})$ *and* $f \in \mathcal{C}^1_{\text{pw}}(\mathbb{R})$, *then* $f * g \in \mathcal{C}^1_{\text{pw}}(\mathbb{R})$.

(v) *If* f, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, *then* $f * g \in L^1(\mathbb{R})$ *and*

$$\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}. \tag{2.5.17}$$

(vi) *If* f, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, *then*

$$\widehat{f}(k)\widehat{g}(k) = \frac{\widehat{f * g}(k)}{\sqrt{2\pi}}, \ k \in \mathbb{R}.$$

Proof. (i) Fix $x \in \mathbb{R}$. The inequality

$$|f(x - \xi)g(\xi)| \leq \frac{|f(x - \xi)|^2}{2} + \frac{|g(\xi)|^2}{2}$$

and the assumption f, $g \in L^2(\mathbb{R})$ shows that the integral $\int_{-\infty}^{\infty} |f(x - \xi)g(\xi)| \ d\xi$, and so the integral $\int_{-\infty}^{\infty} f(x - \xi)g(\xi) \ d\xi$ also converges. Hence the convolution $(f * g)(x)$ is defined.

(ii) Exercise.

(iii) Exercise.

(iv) This is quite technical to prove, so we omit it, but we mention that the general strategy is to differentiate under the integral with a version of Leibniz's rule (Theorem 2.4.35) for improper integrals. Also, since convolution is commutative, we would get the same results for $g \in \mathcal{C}^n(\mathbb{R}) \cup \mathcal{C}^1_{\text{pw}}(\mathbb{R})$.

(v) We want to show that the integral

$$\int_{-\infty}^{\infty} |(f * g)(x)| \ dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x - \xi)g(\xi) \ d\xi \right| \ dx$$

converges. By the triangle inequality for integrals and the comparison test, it suffices to show that the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x - \xi)g(\xi)| \ d\xi \ dx \tag{2.5.18}$$

converges. We will look at the "reversed" integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-\xi)g(\xi)| \, dx \, d\xi \tag{2.5.19}$$

instead and show that this double improper integral converges. Then Fubini's theorem[34] will allow us to interchange the order of integration in (2.5.19) and conclude that the double integral (2.5.18) converges.

For $\xi \in \mathbb{R}$ fixed, we have

$$\int_{-\infty}^{\infty} |f(x-\xi)g(\xi)| \, dx = |g(\xi)| \int_{-\infty}^{\infty} |f(x-\xi)| \, dx = |g(\xi)| \left\|S^{-\xi}f\right\|_{L^1(\mathbb{R})} = |g(\xi)| \, \|f\|_{L^1(\mathbb{R})},$$

where we abbreviate $(S^{-\xi}f)(x) := f(x-\xi)$. Since $g \in L^1(\mathbb{R})$ as well, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-\xi)g(\xi)| \, dx \, d\xi = \|f\|_{L^1(\mathbb{R})} \int_{-\infty}^{\infty} |g(\xi) \, d\xi = \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})},$$

and so the double integral in (2.5.19) converges, as desired.

(vi) Since f, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, part (v) ensures that $f * g \in L^1(\mathbb{R})$, so $\widehat{f * g}$ is defined. The desired equality will then follow from the work preceding Definition 2.5.14, provided that the interchange of integrals in (2.5.16) is valid. The integrand in (2.5.16) satisfies

$$|f(u-y)g(y)e^{-iku}| = |f(u-y)g(y)|.$$

Because we continue to assume f, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the estimates from the proof of part (v) still hold, and since f, $g \in \mathcal{C}(\mathbb{R})$, we can invoke Fubini's theorem to permit the interchange.

■

2.5.16 Example.

Let

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1. \end{cases}$$

Compute $f * f$.

Solution. We have

$$(f * f)(x) = \int_{-\infty}^{\infty} f(x-\xi)f(\xi) \, d\xi = \int_{-1}^{1} f(x-\xi) \, d\xi.$$

Substitute $u = x - \xi$ (remembering that x is constant and ξ is the variable of integration) to find $du = -d\xi$ and

$$\int_{-1}^{1} f(x-\xi) \, d\xi = -\int_{x+1}^{x-1} f(u) \, du = \int_{x-1}^{x+1} f(u) \, du.$$

[34] The use of Fubini's theorem for double *improper* integrals is a complicated and delicate matter; see [21] for some of the pitfalls in trying to extend Fubination from "definite" double integrals to improper ones. Our hypothesis that f and g are continuous allows us to invoke a version of Fubini's theorem proved in [4]. Restricting to f and g in the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ is stringent, but ultimately all the functions we might "naturally" consider for convolution will belong to this space.

Observe that if $x \le -2$, then $x-1 < x+1 \le -1$, and so $f(u) = 0$ for $x-1 \le u < x+1$. Similarly, if $x \ge 2$, then $1 \le x-1 < x+1$, and so, again, $f(u) = 0$ for $x-1 < u \le x+1$. So, we see that $(f * f)(x) = 0$ for $|x| \ge 2$.

Now restrict $-2 \le x \le 2$. First consider $-2 \le x \le 0$. Then $-3 \le x-1 \le -1$ and $-1 \le x+1 \le 1$, so

$$f(u) = \begin{cases} 0, & x-1 \le u < -1 \\ 1, & -1 \le u \le x+1, \end{cases}$$

thus

$$\int_{x-1}^{x+1} f(u)\, du = \int_{x-1}^{-1} f(u)\, du + \int_{-1}^{x+1} f(u)\, du = \int_{-1}^{x+1} du = x+2.$$

Similarly, if $0 \le x \le 2$, then $-1 \le x-1 \le 1$ and $1 \le x+1 \le 3$, so

$$f(u) = \begin{cases} 1, & x-1 \le u \le 1 \\ 0, & 1 < u \le x+1, \end{cases}$$

thus

$$\int_{x-1}^{x+1} f(u)\, du = \int_{x-1}^{1} f(u)\, du + \int_{1}^{x+1} f(u)\, du = \int_{x-1}^{1} du = 2-x.$$

All together, we have

$$(f * f)(x) = \begin{cases} 0, & x < -2 \\ x+2, & -2 \le x < 0 \\ 2-x, & 0 \le x < 2 \\ 0, & x \ge 2. \end{cases} \qquad \blacktriangle$$

Now we return to our original goal: calculating $\widehat{fg}(k)$ for $f,\ g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Remember, the condition that these functions belong to $L^2(\mathbb{R})$ is merely to ensure that their product is in $L^1(\mathbb{R})$, so the Fourier transform here is actually defined.

In order to obtain a useful expression for $\widehat{fg}$ in terms of $\widehat{f}$ and $\widehat{g}$, however, we need to assume rather more about f and g. Specifically, suppose $f,\ g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ and also assume $\widehat{f},\ \widehat{g} \in L^1(\mathbb{R})$. Then the following equalities hold:

$$\begin{aligned} f(x)g(x) &\overset{(1)}{=} \big(\check{\widehat{f}}(x)\big)\big(\check{\widehat{g}}(x)\big) \\ &\overset{(2)}{=} \big(\widehat{\widehat{f}}(-x)\big)\big(\widehat{\widehat{g}}(-x)\big) \\ &\overset{(3)}{=} \frac{\widehat{\widehat{f} * \widehat{g}}(-x)}{\sqrt{2\pi}} \\ &\overset{(4)}{=} \frac{\check{\widehat{f} * \widehat{g}}(x)}{\sqrt{2\pi}} \end{aligned}$$

To get (1), we used Theorem 2.5.9, which we are allowed to invoke since $f,\ g \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R})$ and $\widehat{f},\ \widehat{g} \in L^1(\mathbb{R})$ with $f,\ g$ continuous on $\mathbb{R}$. To get (2), we used Lemma

2.5.8, which we are allowed to use since $\widehat{f}$, $\widehat{g} \in L^1(\mathbb{R})$. To get (3), we used part (v) of Theorem 2.5.15, which is valid since $\widehat{f}$, $\widehat{g} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. In turn, this holds because we are assuming $\widehat{f}$, $\widehat{g} \in L^1(\mathbb{R})$, while part (ii) from Theorem 2.5.5 implies $\widehat{f}$, $\widehat{g} \in \mathcal{C}(\mathbb{R})$ since f, $g \in L^1(\mathbb{R})$, and part (iv) of that same theorem gives $\widehat{f}$, $\widehat{g} \in L^2(\mathbb{R})$ since f, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Last, to get (4), we used Lemma 2.5.8 again, which is applicable because $\widehat{f} * \widehat{g} \in L^1(\mathbb{R})$ by virtue of $\widehat{f}$, $\widehat{g} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$.

Thus

$$\widehat{fg}(k) = \frac{\widehat{\widehat{f} * \widehat{g}}(k)}{\sqrt{2\pi}}. \tag{2.5.20}$$

Of course, we want to say

$$\widehat{\widehat{f} * \widehat{g}} = \widehat{\widehat{f}} * \widehat{\widehat{g}}. \tag{2.5.21}$$

That would require us to use Theorem 2.5.9, and to use that theorem we need $\widehat{f} * \widehat{g} \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. We already know $\widehat{f} * \widehat{g} \in L^1(\mathbb{R})$, and, since $\widehat{f} \in \mathcal{C}(\mathbb{R})$, we have $\widehat{f} * \widehat{g} \in \mathcal{C}(\mathbb{R})$ by part (iv) of Theorem 2.5.15. But do we have $\widehat{f} \in \mathcal{C}^1_{\text{pw}}(\mathbb{R})$? That would guarantee $\widehat{f} * \widehat{g} \in \mathcal{C}^1_{\text{pw}}(\mathbb{R})$, which is the last inclusion we need. One way to obtain this is to assume $\mathbf{1}f \in L^1(\mathbb{R})$ as in part (v) in Theorem 2.5.4; then $\widehat{f}$ is differentiable with derivative $\partial_k[\widehat{f}] = \widehat{\mathbf{1}f}$, and since $\mathbf{1}f \in L^1(\mathbb{R})$, we also have $\widehat{\mathbf{1}f} \in \mathcal{C}(\mathbb{R})$, thus $\widehat{f} \in \mathcal{C}^1(\mathbb{R})$. This, however, is overkill: we just want $\widehat{f} \in \mathcal{C}^1_{\text{pw}}(\mathbb{R})$, not necessarily $\widehat{f} \in \mathcal{C}^1(\mathbb{R})$.

In words, the Fourier transform turns multiplication into convolution. Or, multiplication on the "space" side is convolution on the "Fourier" side. We have now proved parts (i) and (iii) of the corollary below; the proofs of the other two parts are left as exercises.

2.5.17 Corollary.

Let f, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ with $\widehat{f}$, $\widehat{g} \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R})$. Then

(i) $\widehat{fg} = \dfrac{1}{\sqrt{2\pi}} \widehat{f} * \widehat{g}$;

(ii) $\widecheck{fg} = \dfrac{1}{\sqrt{2\pi}} \widecheck{f} * \widecheck{g}$;

(iii) $\widehat{f}\,\widehat{g} = \dfrac{1}{\sqrt{2\pi}} \widehat{f * g}$;

(iv) $\widecheck{f}\,\widecheck{g} = \dfrac{1}{\sqrt{2\pi}} \widecheck{f * g}$.

2.5.5. *Parseval and Plancherel for the Fourier transform.*

The Parseval and Plancherel identities from Fourier series (Theorem 2.4.20) have analogous versions for the transform.

2.5.18 Theorem.

Suppose f, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then fg, $\widehat{f}\overline{\widehat{g}} \in L^1(\mathbb{R})$ and

(i) [Parseval] $\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\, dk.$

(ii) [Plancherel] $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\widehat{f}(k)|^2\, dk.$

Note the absence of any constant factor in Parseval's and Plancherel's equalities, unlike in the Fourier series case of Theorem 2.4.20. This is due to our normalization of the Fourier transform with the factor $1/\sqrt{2\pi}$. Rephrased in terms of the L^2-norm, the Plancherel identity reads

$$\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})}.$$

Just as we used Plancherel's theorem for Fourier series to evaluate certain infinite series, we can use Plancherel's theorem for the Fourier transform to evaluate certain improper integrals.

2.5.19 Example.

Let

$$f(x) = \begin{cases} 0, & x < -1 \\ 1+x, & -1 \le x < 0 \\ 1-x, & 0 \le x < 1 \\ 0, & x \ge 1. \end{cases}$$

One can show that

$$\widehat{f}(k) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos(k)}{k^2}\right).$$

Use this to compute

$$\int_{-\infty}^{\infty} \frac{(1-\cos(x))^2}{x^4}\, dx.$$

Solution. We have

$$\frac{(1-\cos(k))^2}{k^4} = \frac{\pi|\widehat{f}(k)|^2}{2},$$

and therefore

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{(1-\cos(k))^2}{k^4}\, dk &= \frac{\pi}{2}\int_{-\infty}^{\infty} |\widehat{f}(k)|^2\, dk \\ &= \frac{\pi}{2}\int_{-\infty}^{\infty} |f(k)|^2\, dk \text{ by Plancherel} \\ &= \frac{\pi}{2}\left(\int_{-1}^{0} (1+k)^2\, dk + \int_{0}^{1} (1-k)^2\, dk\right) \\ &= \frac{\pi}{3}. \end{aligned}$$

▲

2.5.6. Solving ODEs and PDEs with Fourier transforms.

One of the chief virtues of the Fourier transform is the relation

$$\mathfrak{F}[f'](k) = ik\widehat{f}(k)$$

from part (iv) of Theorem 2.5.4. As with the corresponding property for Fourier coefficients, this relation can convert ODEs into "algebraic" problems and PDEs into "ODEs with parameters."

2.5.20 Example.

Use the Fourier transform to construct a formal solution to

$$f'' - f = g(x).$$

What hypotheses on g do we need for this to work? How does the formal result compare to what we know from variation of parameters?

Solution. Suppose that $f \in \mathcal{C}^2(\mathbb{R})$ solves this ODE and take the Fourier transform of both sides:

$$\widehat{g}(k) = \widehat{f''}(k) - \widehat{f}(k) = (ik)^2\widehat{f}(k) - \widehat{f}(k) = -(1+k^2)\widehat{f}(k).$$

In particular, we need $\widehat{f}$ to be defined, so we may as well require our solution f not merely to be in $\mathbb{C}^2(\mathbb{R})$ but also to satisfy f, f', $f'' \in L^1(\mathbb{R})$. Likewise, we need $g \in L^1(\mathbb{R})$. Then

$$\widehat{f}(k) = -\frac{\widehat{g}(k)}{1+k^2}. \tag{2.5.22}$$

Let $\phi(k) = 1/(1+k^2)$, so (2.5.22) is

$$\widehat{f}(k) = -\widehat{g}(k)\phi(k).$$

Then we expect

$$f(x) = -\mathfrak{F}^{-1}[\widehat{g}\phi](x) = -\mathfrak{F}[\widehat{g}\phi](-x). \tag{2.5.23}$$

Since $g \in L^1(\mathbb{R})$, we have $|\widehat{g}(k)| \le \|g\|_{L^1(\mathbb{R})}$ for all $k \in \mathbb{R}$, by part (ii) of Theorem 2.5.5. Since $\phi \in L^1(\mathbb{R})$, the comparison test implies that $\widehat{g}\phi \in L^1(\mathbb{R})$, too, and so the (inverse) Fourier transform in (2.5.23) is defined. That is, the right side of (2.5.23) is a good *candidate* for the solution to our ODE.

But is this (inverse) Fourier transform sufficiently differentiable? A close look at part (v) of Theorem 2.5.4 shows that $\mathfrak{F}[\widehat{g}\phi]$ will be twice-differentiable if the improper integral

$$\int_{-\infty}^{\infty} |k^2\widehat{g}(k)\phi(k)|\ dk = \int_{-\infty}^{\infty} \left(\frac{k^2}{1+k^2}\right)|\widehat{g}(k)|\ dk$$

converges. Since

$$\left|\frac{k^2}{1+k^2}\right||\widehat{g}(k)| \le |\widehat{g}(k)|,$$

by the comparison test it suffices to assume $\widehat{g} \in L^1(\mathbb{R})$.

Using the chain rule and part (v) of Theorem 2.5.4 once more, we calculate

$$f''(x) = \partial_x^2\big(-\mathfrak{F}[\widehat{g}\phi](-x)\big) = -(-i)^2(-1)^2\mathfrak{F}\left[\frac{k^2\widehat{g}(k)}{1+k^2}\right](-x) = \mathfrak{F}\left[\frac{k^2\widehat{g}(k)}{1+k^2}\right](-x).$$

Then

$$f''(x)+f(x) = \mathfrak{F}\left[\frac{k^2\widehat{g}(k)}{1+k^2}\right](-x)+\mathfrak{F}\left[\frac{\widehat{g}(k)}{1+k^2}\right](-x) = \mathfrak{F}\left[\frac{(1+k^2)\widehat{g}(k)}{1+k^2}\right](-x) = \mathfrak{F}[\widehat{g}](-x)$$
$$= \mathfrak{F}^{-1}[\widehat{g}](x) = g(x)$$

by Theorem 2.5.9, if we assume further that $g \in \mathcal{C}^1_{\text{pw}}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. ▲

2.5.21 Example.

We are now assuming quite a lot about the nonhomogeneity g in Example2.5.20, especially given that we could solve the ODE $f'' - f = g(x)$ rather quickly using variation of parameters. Indeed, a fundamental solution set for this is $\mathfrak{z}_1(x) = e^x$, $\mathfrak{z}_2(x) = e^{-x}$, and so a particular solution is

$$f_0(x) := \frac{e^x}{2}\int_0^x e^{-\xi}g(\xi)\ d\xi - \frac{e^{-x}}{2}\int_0^x e^{\xi}g(\xi)\ d\xi.$$

Then the general solution is

$$h(x) := c_1e^x + c_2e^{-x} + \frac{e^x}{2}\int_0^x e^{-\xi}g(\xi)\ d\xi - \frac{e^{-x}}{2}\int_0^x e^{\xi}g(\xi)\ d\xi. \tag{2.5.24}$$

(i) *Why bother assuming so much more about g to use the Fourier transform, when we just need $g \in \mathcal{C}(\mathbb{R})$ to use variation of parameters?*

(ii) *What is the value of going through all the work above using the Fourier transform, when we could have solved the ODE using prior techniques?*

(iii) *How can we write our solution in* (2.5.23) *in the form* (2.5.24) *given by variation of parameters?*

Solution. **(i)** Here is what is special: by using the Fourier transform, we are presuming that the solution f to $f'' + f = g(x)$ is not merely a twice-continuously differentiable function but also absolutely integrable, as are its first and second derivatives. This presumes a certain amount of "decay" on f. Indeed, buried in Footnote 33 as part of the proof of part (iv) of Theorem 2.5.4 is the result that if $f \in L^1(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, then $\lim_{x\to\pm\infty} f(x) = 0$. Consequently, if $f \in \mathcal{C}^2(\mathbb{R})$ and f, f', $f'' \in L^1(\mathbb{R})$, then we also need $\lim_{x\to\pm\infty} f'(x) = 0$, too. Thus, if we insist on using the Fourier transform to solve the ODE, we are no longer merely interested in solving $f''+f = g(x)$ but rather the BVP

$$\begin{cases} f'' + f = g(x) \\ \lim\limits_{x\to\pm\infty} f(x) = 0 \\ \lim\limits_{x\to\pm\infty} f'(x) = 0. \end{cases} \tag{2.5.25}$$

Our heuristic experience with BVPs so far teaches us that we should not expect solutions for all nonhomogeneities g but only ones with more stringent properties.

(ii) Our answer to part (i) explains why we need more information about g than just the anodyne condition $g \in \mathcal{C}(\mathbb{R})$ from variation of parameters. But we did not provide a compelling reason for using the elaborate machinery of the Fourier transform — other than its (hopefully) obvious pedagogical value in this simple problem. There may be valid reasons stemming from an application for assuming the decay on f in a more complicated version (2.5.25). For example, f could represent the profile of a wave whose bulk is localized in a "core" and that vanishes far away from the core. Or f could represent the energy of a physical system that also naturally decays and should be measured using some integral norm. In short, for ODEs posed on all of $\mathbb{R}$, "limiting" boundary conditions that facilitate the Fourier transform may be as appropriate as classical initial conditions.

(iii) From (2.5.23), our solution is

$$f(x) = -\mathfrak{F}^{-1}[\widehat{g}\phi](x), \qquad \phi(k) := \frac{1}{1+k^2}.$$

We are assuming $g \in L^1(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ with $\widehat{g} \in L^1(\mathbb{R})$.

Example 2.5.11 (in the $a = 1$ case) tells us that

$$\widehat{\phi}(k) = \sqrt{\frac{\pi}{2}}e^{-|x|},$$

and so $\widehat{\phi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap \mathcal{C}^1_{\text{pw}}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, too. Corollary 2.5.17 then implies

$$f(x) = -\frac{(\check{\widehat{g}} * \check{\phi})(x)}{\sqrt{2\pi}} = -\frac{(g * \check{\phi})(x)}{\sqrt{2\pi}},$$

where

$$\check{\phi}(x) = \widehat{\phi}(-x) = \sqrt{\frac{\pi}{2}}e^{-|x|}$$

Then

$$f(x) = -\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} e^{-|x-\xi|}g(\xi)\ d\xi = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|}g(\xi)\ dx.$$

Now we use properties of absolute value to split up the integral as

$$\begin{aligned}
f(x) &= -\frac{1}{2}\int_{-\infty}^{x} e^{-|x-\xi|}g(\xi)\ d\xi - \frac{1}{2}\int_{x}^{\infty} e^{-|x-\xi|}g(\xi)\ d\xi \\
&= -\frac{1}{2}\int_{-\infty}^{x} e^{-(x-\xi)}g(\xi)\ d\xi - \frac{1}{2}\int_{x}^{\infty} e^{-(\xi-x)}g(\xi)\ d\xi \\
&= -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}g(\xi)\ d\xi - \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\xi}g(\xi)\ d\xi \\
&= -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}g(\xi)\ d\xi - \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\xi}g(\xi)\ d\xi \\
&= -\frac{e^{-x}}{2}\int_{-\infty}^{0} e^{\xi}g(\xi)\ d\xi - \frac{e^{-x}}{2}\int_{0}^{x} e^{\xi}g(\xi)\ d\xi - \frac{e^{x}}{2}\int_{x}^{0} e^{-\xi}g(\xi)\ d\xi - \frac{e^{x}}{2}\int_{0}^{\infty} e^{-\xi}g(\xi)\ d\xi
\end{aligned}$$

$$= \frac{e^x}{2}\int_0^x e^{-\xi}g(\xi)\,d\xi - \frac{e^{-x}}{2}\int_0^x e^{\xi}g(\xi)\,d\xi - \left(\int_0^\infty \frac{e^{-\xi}g(\xi)}{2}\,d\xi\right)e^x - \left(\int_{-\infty}^0 \frac{e^{\xi}g(\xi)}{2}\,d\xi\right)e^{-x},$$

and this is exactly the form of the solution prophesied by variation of parameters.

▲

2.5.22 Method: solve an ODE with the Fourier transform

0. This method is very similar to Method 2.4.34 for solving ODEs with Fourier series. Assume that the ODE has the form

$$\mathcal{A}f := \sum_{j=0}^{n} a_j \partial_x^j[f] = g,$$

where g is "sufficiently nice" so that the Fourier transform $\widehat{g}$ is defined. We are interested in solutions $f \in L^1(\mathbb{R})$. This method does not allow us to incorporate initial conditions.

1. Write the ODE on the "Fourier side" using the property $\widehat{f'}(k) = ik\widehat{f}(k)$:

$$\mathcal{A}f = g \iff \sum_{j=0}^{n}(ik)^j a_j \widehat{f}(k) = \widehat{g}(k).$$

Put $\widetilde{\mathcal{A}}(k) := \sum_{j=0}^{n}(ik)^j a_j$, so this reads $\widetilde{\mathcal{A}}(k)\widehat{f}(k) = \widehat{g}(k)$.

2. If $\widetilde{\mathcal{A}}(k) \neq 0$ for all k, solve for $\widehat{f}(k)$:

$$\widehat{f}(k) = \frac{\widehat{g}(k)}{\widetilde{\mathcal{A}}(k)}.$$

If $\widetilde{\mathcal{A}}(k) = 0$ for some k, then the ODE only has a solution in $L^1(\mathbb{R})$ if $\widehat{g}(k) = 0$, too. If this is so, then the formal solution is

$$f(x) = \mathfrak{F}^{-1}\left[\frac{\widehat{g}}{\widetilde{\mathcal{A}}}\right](x).$$

Calculate this inverse Fourier transform explicitly, if possible.

2.5.23 Example.

Use the Fourier transform to construct a formal solution for the heat equation IVP

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

which models heat distribution in an "infinite rod." What hypotheses do we need on f for this to work?

Solution. Suppose that we have a solution u. We will use the Fourier transform to deduce a formula for u, and then we must check that this formula is sufficiently differentiable.

When applying the Fourier transform to a PDE, we need to decide what variable will receive the transform. Since the Fourier transform requires the function to be defined on $(-\infty, \infty)$, we should use the unbounded spatial variable x, as we only know time for $t \geq 0$. That is, we will consider the transforms

$$\mathfrak{F}[u_t(\cdot, t)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t) e^{-ikx}\ dx =: \widehat{u_t}(k,t)$$

and

$$\mathfrak{F}[u_{xx}(\cdot, t)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx}(x,t) e^{-ikx}\ dx =: \widehat{u_{xx}}(k,t).$$

Here we first use the elaborate notation $\mathfrak{F}[u_t(\cdot, t)](k)$ and $\mathfrak{F}[u_{xx}(\cdot, t)](k)$ to indicate that we are applying Fourier transform in the x-variable and leaving t as a parameter; subsequently, we will just use $\widehat{\ }$. This is similar to what we did in Example 2.4.36. Likewise, we will write

$$\mathfrak{F}[u(\cdot, t)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\ dx. =: \widehat{u}(k,t).$$

Again, all integration occurs with respect to x. Then we have

$$\widehat{u_{xx}}(k,t) = (ik)^2 \widehat{u}(k,t) = -k^2 \widehat{u}(k,t).$$

Next, let us assume that differentiation under the integral with an improper integral version of Leibniz's rule is valid:

$$\widehat{u_t}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \partial_t[u](x,t) e^{-ikx}]\ dx = \partial_t \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\ dx \right] = \partial_t[\widehat{u}](k,t) = \widehat{u}_t(k,t).$$

So, the function $\widehat{u} = \widehat{u}(k,t)$ satisfies the ODE

$$\widehat{u}_t(k,t) = -k^2 \widehat{u}(k,t),\ t \in \mathbb{R},$$

where now $k \in \mathbb{R}$ is a fixed parameter. Using the initial condition $u(x,0) = f(x)$, we find

$$\widehat{u}(k,0) = \widehat{f}(k),$$

and so we really have an IVP for the function $\widehat{u}(k,\cdot)$:

$$\begin{cases} \widehat{u}_t(k,t) + k^2 \widehat{u}(k,t) = 0,\ t \geq 0 \\ \widehat{u}(k,0) = \widehat{f}(k). \end{cases}$$

Using the integrating factor method (or separation of variables), we find

$$\widehat{u}(k,t) = \widehat{f}(k) e^{-k^2 t}.$$

To solve for u, we then need to take the inverse Fourier transform of $\widehat{u}(\cdot, t)$, where now t is once again a parameter. Since $\widehat{u}(\cdot, t)$ is a product of functions, we will use convolution. Let $\phi(\xi) = e^{-\xi^2}$, so

$$\widehat{u}(k,t) = \widehat{f}(k) \phi(k\sqrt{t}),$$

where we can take $\sqrt{t}$ because we assume $t \geq 0$. Taking the inverse transform with respect to k and leaving t as the parameter now, we find

$$u(x,t) = \mathfrak{F}^{-1}[\widehat{f}(k)\phi(\sqrt{t}k)](x) = \frac{1}{\sqrt{2\pi}} \big(\mathfrak{F}^{-1}[\widehat{f}(k)] * \mathfrak{F}^{-1}[\phi(\sqrt{t}k)] \big)(x). \tag{2.5.26}$$

This, of course, presumes that f and ϕ satisfy the myriad of hypotheses of Corollary 2.5.17. It seems that we must impose all these assumptions on f, since this initial temperature distribution is not specified, but, happily, they are all met for ϕ.

Recall that we studied ϕ in Example 2.5.13 and found $\phi \in \mathcal{C}^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with

$$\widehat{\phi}(k) = \frac{e^{-k^2/4}}{\sqrt{2}} = \frac{e^{-(k/2)^2}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\phi\left(\frac{k}{2}\right).$$

Then if $t > 0$, the scaling property of the Fourier transform (part (iii) of Theorem 2.5.4) implies

$$\mathfrak{F}^{-1}[\phi(\sqrt{t}k)](x) = \mathfrak{F}[\phi(\sqrt{t}k)](-x) = \frac{1}{\sqrt{t}}\widehat{\phi}\left(-\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{2t}}\phi\left(-\frac{x}{2\sqrt{t}}\right) = \frac{e^{-(x^2/4t)}}{\sqrt{2t}}.$$

Thus, for $t > 0$,

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\big(f * \mathfrak{F}^{-1}[\phi(\sqrt{t}\cdot)]\big)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-(x-\xi)^2/4t}}{\sqrt{2t}} f(\xi)\,d\xi = \int_{-\infty}^{\infty} \mathcal{H}(x-\xi,t)f(y)\,d\xi,$$

where

$$\mathcal{H}(\zeta,t) := \frac{e^{-\zeta^2/4t}}{\sqrt{4\pi t}}, \tag{2.5.27}$$

is the **HEAT KERNEL**[35] or the **FUNDAMENTAL SOLUTION**[36] for the heat equation.

Incorporating the $t = 0$ initial temperature distribution, the full solution is

$$u(x,t) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \mathcal{H}(x-\xi,t)f(\xi)\,d\xi, \ x \in \mathbb{R}, \ t > 0 \\ \\ f(x), \ x \in \mathbb{R}, \ t = 0. \end{cases} \tag{2.5.28}$$

Note that

$$\int_{-\infty}^{\infty} \mathcal{H}(x-\xi,t)f(\xi)\,d\xi = \big(\mathcal{H}(\cdot,t) * f\big)(x)$$

is convolution in the first variable of $\mathcal{H}$. This solution of the heat equation obtained by convolution with the heat kernel is definitely not one that arises from separation of variables.

Now, did we really solve the problem? Not quite: we *assumed* there was a solution u to the heat equation's IVP and that this u and the initial temperature distribution f were sufficiently well-behaved to ensure the validity of all the Fourier methods above. One would need to differentiate under the integral to show that u as defined in (2.5.28) does solve the equation. Then it remains to show

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = f(x_0) \tag{2.5.29}$$

for any $x_0 \in \mathbb{R}$, which provides a "continuity condition" linking the two very different pieces of u.

[35] In the sense of integral operators, recall part 3 of Linear Algebraic Viewpoint 1.2.8.

[36] One can check via straightforward differentiation that $\mathcal{H}_t = \mathcal{H}_{\zeta\zeta}$, and so $\mathcal{H}$ itself solves the heat equation.

Observe also from (2.5.27) that, for fixed $t > 0$, the kernel $\mathcal{H}$ decays very rapidly as $\zeta \to \pm\infty$, so rapidly, in fact, that if $f\colon \mathbb{R} \to \mathbb{C}$ is merely bounded and locally integrable, then by the comparison test the convolution $\mathcal{H}(\cdot, t) * f$ is defined for all $t > 0$. A closer examination of Leibniz's rule for differentiating (under) improper integrals and some nontrivial, more technical analysis reveal that if f is continuous and either bounded or absolutely integrable, then (2.5.28) satisfies the heat IVP and the continuity condition (2.5.29). These requirements on f are far less than the wearingly stringent hypotheses of Corollary 2.5.17, which, apparently, we needed to justify (2.5.26). How can this be? Again, note the flow of our work above: *if* the heat equation has a solution that is sufficiently integrable, and if the initial temperature distribution is likewise sufficiently smooth and integrable, then the solution has the form (2.5.28). But this says absolutely nothing about the conditions on f under which (2.5.28) is *a priori* a solution — much to our relief. ▲

2.5.24 Method: solve an PDE with the Fourier transform

0. Assume that the PDE is posed for an unknown function $u = u(x, t)$ for $x \in \mathbb{R}$. This method can incorporate initial conditions of the form $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$, etc., but not boundary conditions. All Fourier transforms involving u are taken with respect to x.

1. Using the identities

$$\mathfrak{F}[\partial_t u(\cdot, t)](k) = \partial_t \widehat{u}(k, t) \quad \text{and} \quad \mathfrak{F}[\partial_x u(\cdot, t)](k) = ik\widehat{u}(k, t),$$

convert the PDE into an ODE in which the only derivatives are with respect to t and k is a parameter.

2. Solve this ODE. Any arbitrary constants from ODE methods must now depend on k.

3. If applicable, incorporate any initial conditions of the form $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$, etc., by rewriting them on the Fourier side, e.g., $\widehat{u}(k, 0) = \widehat{\phi}(k)$ and $\partial_t[\widehat{u}](k, t) = \widehat{\psi}(k)$. Solve for any k-dependent arbitrary coefficients.

4. Attempt to calculate the inverse transform and solve explicitly for u as $u(x, t) = \mathfrak{F}^{-1}[\widehat{u}(\cdot, t)](x)$.

2.6. The wave equation.

The **wave equation** is

$$u_{tt} = c^2 u_{xx}, \ u = u(x, t), \ c \in \mathbb{R} \setminus \{0\}.$$

Unlike the heat equation, in which we took the "diffusivity constant" to be 1, we will want to maintain this extra parameter c in the wave equation. This parameter c will have a natural connection to wave "speed."

2.6.1. D'Alembert's formula and traveling waves.

We consider the wave equation on $\mathbb{R}$, which models an "infinite" string.

2.6.1 Example.

Use the Fourier transform to construct a formal solution to

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ -\infty < x < \infty, \ t \geq 0 \\ u(x,0) = \phi(x), \ -\infty < x < \infty \\ u_t(x,0) = \psi(x), \ -\infty < x < \infty. \end{cases} \tag{2.6.1}$$

Solution. Taking the Fourier transform with respect to the spatial variable x, we get an IVP in the variable t that depends on k (and also c) as a parameter:

$$\begin{cases} \widehat{u}_{tt}(k,t) + c^2k^2\widehat{u}(k,t) = 0 \\ \widehat{u}(k,0) = \widehat{\phi}(k) \\ \widehat{u}_t(k,0) = \widehat{\psi}(k). \end{cases}$$

Here, of course, we need to assume ϕ, $\psi \in L^1(\mathbb{R})$.

If $k \neq 0$, then $c^2k^2 > 0$, and so the general solution to the ODE is

$$\widehat{u}(k,t) = \alpha(k)\cos(ckt) + \beta(k)\sin(ckt). \tag{2.6.2}$$

Then, still assuming $k \neq 0$, we use the initial condition $u(x,0) = \phi(x)$ to find

$$\alpha(k) = \widehat{u}(k,0) = \widehat{\phi}(k).$$

Differentiating (2.6.2) gives

$$\widehat{u}_t(k,t) = -ck\alpha(k)\sin(ckt) + ck\beta(k)\cos(ckt),$$

and so the initial condition $u_t(x,0) = \psi(x)$ implies

$$\widehat{\psi}(k) = \widehat{u}_t(k,0) = ck\beta(k) \Longrightarrow \beta(k) = \frac{\widehat{\psi}(k)}{ck}.$$

So, for $k \neq 0$, we have

$$\widehat{u}(k,t) = \widehat{\phi}(k)\cos(ckt) + \widehat{\psi}(k)\frac{\sin(ckt)}{ck}. \tag{2.6.3}$$

If $k = 0$, then we get the simpler IVP

$$\begin{cases} \widehat{u}_{tt}(0,t) = 0 \\ \widehat{u}(0,0) = \widehat{\phi}(0) \\ \widehat{u}_t(0,0) = \widehat{\psi}(0) \end{cases} \Longrightarrow \widehat{u}(0,t) = \widehat{\phi}(0) + \widehat{\psi}(0)t.$$

Does this agree with the $k \neq 0$ result from (2.6.3)? We use a familiar limit from calculus:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \Longrightarrow \lim_{k \to 0} \frac{\sin(ckt)}{ck} = t \lim_{k \to 0} \frac{\sin(ckt)}{ckt} = t.$$

Using the continuity of $\widehat{\phi}$ and $\widehat{\psi}$ (which holds by part (ii) of Theorem 2.5.5, since ϕ, $\psi \in L^1(\mathbb{R})$), we obtain.

$$\lim_{k \to 0} \widehat{u}(k,t) = \lim_{k \to 0} \left(\widehat{\phi}(k)\cos(ckt) + \widehat{\psi}(k)\frac{\sin(ckt)}{ck}\right) = \widehat{\phi}(0) + \widehat{\psi}(0)t = \widehat{u}(0,t).$$

Then, yes, our work from before agrees with the $k = 0$ case, and so we will only use the formula (2.6.3) for $\widehat{u}(k,t)$,

We now take the inverse Fourier transform of each term in (2.6.3) with respect to k. To calculate

$$\mathfrak{F}^{-1}[\widehat{\phi}(k)\cos(ckt)](x),$$

our first instinct might be to use a convolution identity, but $k \mapsto \cos(ckt)$ is certainly not integrable. Instead, the complex form of the cosine allows us to rewrite

$$\widehat{\phi}(k)\cos(ckt) = \widehat{\phi}(k)\left(\frac{e^{ickt}+e^{-ickt}}{2}\right) = \frac{e^{i(ct)k}\widehat{\phi}(k)+e^{i(-ct)k}\widehat{\phi}(k)}{2}.$$

Now, with $(S^d\phi)(x) := \phi(x+d)$, recall the identity $\widehat{S^d\phi}(k) = e^{ikd}\widehat{\phi}(k)$ to see that

$$e^{i(\pm ct)k}\widehat{\phi}(k) = \widehat{S^{\pm ct}\phi}(k).$$

And so

$$\mathfrak{F}^{-1}[\widehat{\phi}(k)\cos(ckt)](x) = \frac{\mathfrak{F}^{-1}[\widehat{S^{ct}\phi}](x)+\mathfrak{F}^{-1}[\widehat{S^{-ct}\phi}](x)}{2} = \frac{\phi(x+ct)+\phi(x-ct)}{2}. \quad (2.6.4)$$

To compute

$$\mathfrak{F}^{-1}\left[\widehat{\psi}(k)\frac{\sin(ckt)}{ck}\right](x),$$

we do need convolution. First, we recall Example 2.5.3:

$$\chi(x,t) := \begin{cases} 1, & |x| \le t \\ 0, & |x| > t, \end{cases} \Longrightarrow \widehat{\chi}(k,t) = \sqrt{\frac{2}{\pi}}\frac{\sin(tk)}{k} \Longrightarrow \frac{\sin(ckt)}{ck} = \frac{1}{c}\sqrt{\frac{\pi}{2}}\widehat{\chi}(k,ct).$$

As usual for a function of x and t, by $\widehat{\chi}(k,t)$ we mean $\mathfrak{F}[\chi(\cdot,t)](k)$, i.e., the Fourier transform in the x-variable. Then

$$\mathfrak{F}^{-1}\left[\widehat{\psi}(k)\frac{\sin(ckt)}{ck}\right](x) = \frac{1}{c}\sqrt{\frac{\pi}{2}}\mathfrak{F}^{-1}\left[\widehat{\psi}(k)\widehat{\chi}(k,ct)\right](x) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{c}\sqrt{\frac{\pi}{2}}\right)\big(\psi*\chi(\cdot,ct)\big)(x). \quad (2.6.5)$$

By $\psi*\chi(\cdot,ct)$, we mean convolution in the x-variable of χ. That is,

$$\big(\psi*\chi(\cdot,ct)\big)(x) = \int_{-\infty}^{\infty}\psi(x-\xi)\chi(\xi,ct)\ d\xi = \int_{-ct}^{ct}\psi(x-\xi)\ d\xi = -\int_{x+ct}^{x-ct}\psi(s)\ ds$$
$$= \int_{x-ct}^{x+ct}\psi(s)\ ds. \quad (2.6.6)$$

Inspired by the success of our formal Fourier methods for the heat equation in Example (2.5.23), we will not attempt to justify the convolution formula used in (2.6.5) by imposing any of the hypotheses of Corollary 2.5.17. Indeed, we would have some difficulty in doing so, since $\operatorname{sinc}(\cdot) \notin L^1(\mathbb{R})$, by Example B.0.5.

We combine (2.6.4), (2.6.5), and (2.6.6) to conclude

$$u(x,t) = \mathfrak{F}^{-1}[\widehat{\phi}(k)\cos(ckt)](x) + \mathfrak{F}^{-1}\left[\widehat{\psi}(k)\frac{\sin(ckt)}{ck}\right](x)$$
$$= \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\ ds. \quad \blacktriangle \quad (2.6.7)$$

We emphasize that the work above does not prove that the solution to the IVP (2.6.1) is given by (2.6.7). Instead, this works says that *if* a solution to (2.6.1) exists with sufficiently well-behaved initial data ϕ and ψ, then that solution *should* have the form (2.6.7). For u defined via this formula to be differentiable, then, we need ϕ to be differentiable and ψ to be continuous (from the fundamental theorem of calculus). For u to be *twice*-differentiable, we need ψ to be differentiable, as well (again from the FTC). However, we do not need ϕ or ψ to be absolutely integrable; indeed, if ψ is merely continuous, then the integral in (2.6.7) always exists. With this in mind, some straightforward calculus formalizes (2.6.8) as a solution to the wave equation.

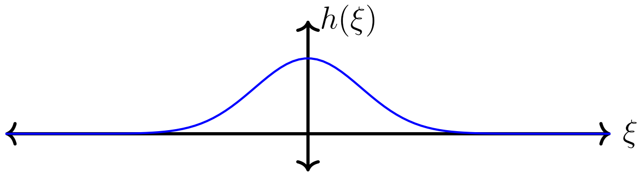
2.6.2 Lemma.

Let $\phi \in \mathcal{C}^2(\mathbb{R})$ and $\psi \in \mathcal{C}^1(\mathbb{R})$. Then the function $u \colon \mathbb{R}^2 \to \mathbb{C}$ defined by (2.6.7) solves the wave equation $u_{tt} = c^2 u_{xx}$.

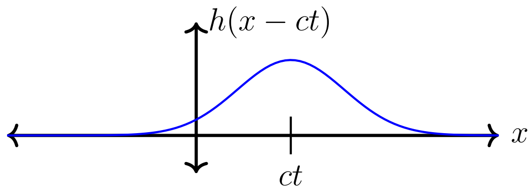
Let us rewrite the solution (2.6.7) a little further as

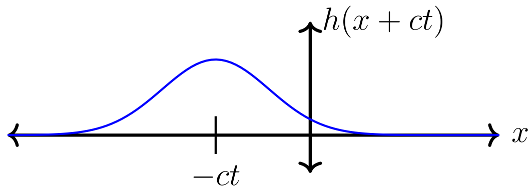
$$u(x,t) = \underbrace{\left(\frac{\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{0} \psi(s)\ ds\right)}_{f(x-ct)} + \underbrace{\left(\frac{\phi(x+ct)}{2} + \frac{1}{2c}\int_{0}^{x+ct} \psi(s)\ ds\right)}_{g(x+ct)}. \quad (2.6.8)$$

The idea now is that f is a "right-moving wave" while g is a "left-moving" wave. More precisely, consider just one function $h = h(\xi)$ graphed below against the independent variable ξ.



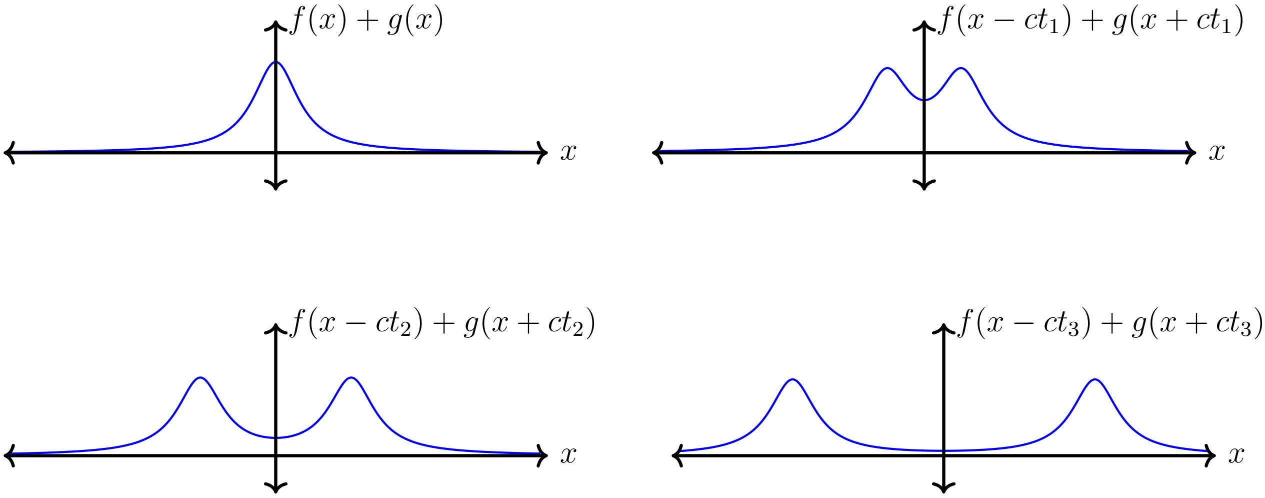
If $ct > 0$, then the graph of $h(\cdot + ct)$ is the graph of h shifted ct units to the left, and the graph of $h(\cdot - ct)$ is the graph of h shifted ct units to the right.





So, if $u(x,t) = h(x+ct)$ with $c > 0$, then at each time t get the graph of u by shifting the graph of h to the left, as though the "profile" given by the function h maintains its shape and translates or "travels" over time. This is exactly how a water wave can behave; on the ocean or on a canal, a wave may pulse forward, maintaining its shape, over a long time, until it disperses or collides with an obstacle.

Returning to the full solution (2.6.8) of the wave equation , $u(x,t) = f(x-ct) + g(x+ct)$, we see that the solution "splits" into two waves, one with profile f that propagates to the right and one with profile g that propagates to the left.



Moving beyond the wave equation, many PDEs involving the unknown function $u = u(x,t)$ may be solved by making a **TRAVELING WAVE ANSATZ** $u(x,t) = f(x-ct)$, where f is a function of one variable, called the **PROFILE**, and c is a constant number called the **WAVE SPEED**. Such an ansatz typically converts the PDE for u into an ODE for f that involves c as a parameter.

Now we return to the wave equation and show that the traveling wave solution predicted by Fourier theory is, in fact, the form the solution always takes.

2.6.3 Theorem.

(i) [D'Alembert's formula] *Suppose that* $u\colon \mathbb{R}^2 \to \mathbb{C}$ *solves the wave equation* $u_{tt} = c^2 u_{xx}$. *Then there exist functions* $f,\, g \in \mathcal{C}^2(\mathbb{R})$ *such that*

$$u(x,t) = f(x-ct) + g(x+ct).$$

We will sometimes refer (idiosyncratically) to the function f *as the* **RIGHT-MOVING PROFILE** *and to* g *as the* **LEFT-MOVING PROFILE**.

(ii) *Let* $\phi \in \mathcal{C}^2(\mathbb{R})$ *and* $\psi \in \mathcal{C}^1(\mathbb{R})$. *The unique solution to the IVP*

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ -\infty < x < \infty, \ t \geq 0 \\ u(x,0) = \phi(x), \ -\infty < x < \infty \\ u_t(x,0) = \psi(x), \ -\infty < x < \infty, \end{cases}$$

is

$$u(x,t) = \frac{\phi(x-ct) + \phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds.$$

Proof. (i) Suppose u is a $\mathcal{C}^2$-function satisfying $u_{tt} = c^2 u_{xx}$. That is, all the partials u_t, u_{tt}, u_x, u_{xx}, and $u_{xt} = u_{tx}$ exist and are continuous. We need to construct twice-differentiable functions f and g such that $u(x,t) = f(x-ct) + g(x+ct)$.

First, we have

$$u_{tt} = c^2 u_{xx} \iff u_{tt} - c^2 u_{xx} = 0 \iff (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0.$$

Set $v = (\partial_t + c\partial_x)u$. Then we must have

$$(\partial_t - c\partial_x)v = 0,$$

and so by our work with the transport equation in Example 2.2.2 we know

$$v(x,t) = F(x+ct)$$

for some differentiable function $F = F(X)$. Then we must have

$$(\partial_t + c\partial_x)u = F(x+ct).$$

This is a nonhomogeneous transport equation, and a practice problem will guide us to its solution:

$$u(x,t) = G(x-ct) + \frac{1}{c}\int_0^t F(x-ct+2cs)\ ds, \tag{2.6.9}$$

where $G \in \mathcal{C}^1(\mathbb{R})$ is arbitrary. Substitute

$$w(s) = x - ct + 2cs, \qquad w(0) = x - ct, \qquad w(t) = x + ct, \qquad dw = 2c\ ds$$

to get

$$\int_0^t F(x-ct+2cs)\ ds = \frac{1}{2c}\int_{x-ct}^{x+ct} F(w)\ dw. \tag{2.6.10}$$

Thus

$$\begin{aligned} u(x,t) &= G(x-ct) + \frac{1}{2c^2}\int_{x-ct}^{x+ct} F(w)\ dw \\ &= \underbrace{\left[G(x-ct) + \frac{1}{2c^2}\int_{x-ct}^{0} F(w)\ dw\right]}_{f(x-ct)} + \underbrace{\frac{1}{2c^2}\int_0^{x+ct} f(w)\ dw}_{g(x+ct)}. \end{aligned} \tag{2.6.11}$$

Since F is differentiable, the integrals $\int_{x-ct}^0 F(w)\ dw$ and $\int_0^{x+ct} F(w)\ dw$ are twice-differentiable with respect to x and t by the fundamental theorem of calculus. And since we are assuming that u is twice-differentiable from the start, we can use (2.6.9) and (2.6.10) to write G as

$$G(x-ct) = u(x,t) - \frac{1}{2c}\int_{x-c}^{x+ct} F(w)\ dw,$$

where the function on the right is twice-differentiable. Hence G is in fact twice-differentiable, not just once differentiable as the nonhomogeneous transport equation result said. We conclude from (2.6.11) that f and g are twice-differentiable, too.

(ii) If u solves the IVP, then $u_{tt} = c^2 u_{xx}$, and so there are twice-differentiable functions f and g such that

$$u(x,t) = f(x-ct) + g(x+ct).$$

Then

$$\phi(x) = u(x,0) = f(x) + g(x). \tag{2.6.12}$$

Next,

$$u_t(x,t) = -cf'(x-ct) + cg'(x+ct),$$

so

$$\psi(x) = u_t(x,0) = cg'(x) - cf'(x).$$

That is, f and g must satisfy the system (of functions)

$$\begin{cases} f + g = \phi \\ cg' - cf' = \psi. \end{cases}$$

Differentiating the first equation, we see that any solution pair (f,g) to this system must also satisfy

$$\begin{cases} f' + g' = \phi' \\ g' - f' = \psi/c. \end{cases}$$

We can write this as a matrix-vector equation:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} \phi' \\ \psi/c \end{pmatrix}.$$

The matrix on the left is invertible, and we find

$$f' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right) \qquad \text{and} \qquad g' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right).$$

We have now decoupled the equations involving f and g into two ODEs, one involving only f, and one involving only g. The right sides of the equations involve ϕ and ψ, which are given to us from the wave equation's initial conditions.

We integrate to find

$$f(s) - f(0) = \int_0^s f'(\xi)\, d\xi = \frac{1}{2}\int_0^s \left(\phi'(\xi) - \frac{\psi(\xi)}{c}\right)\, d\xi$$

and

$$g(s) - g(0) = \int_0^s g'(\xi)\, d\xi = \frac{1}{2}\int_0^s \left(\phi'(\xi) + \frac{\psi(\xi)}{c}\right)\, d\xi.$$

We solve for $f(s)$ and $g(s)$:

$$f(s) = f(0) + \frac{\phi(s) - \phi(0)}{2} - \frac{1}{2c}\int_0^s \psi(\xi)\, d\xi$$

and

$$g(s) = g(0) + \frac{\phi(s) - \phi(0)}{2} + \frac{1}{2c}\int_0^s \psi(\xi)\, d\xi.$$

Now we put everything back together:

$$u(x,t) = f(x-ct) + g(x+ct)$$

$$= f(0) + \frac{\phi(x-ct) - \phi(0)}{2} - \frac{1}{2c}\int_0^{x-ct} \psi(\xi)\, d\xi + g(0) + \frac{\phi(x+ct) - \phi(0)}{2} + \frac{1}{2c}\int_0^{x+ct} \psi(\xi)\, d\xi$$

$$= f(0) + g(0) - \phi(0) + \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\xi)\, d\xi.$$

Recall from (2.6.12) that $f(0) + g(0) = \phi(0)$. We conclude

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\xi)\, d\xi. \qquad \blacksquare$$

2.6.4 Example.

Solve the IVP

$$\begin{cases} u_{tt} = u_{xx}, & -\infty < x < \infty,\ t \geq 0 \\ u(x,0) = \sin(x), & -\infty < x < \infty \\ u_t(x,0) = 0, & -\infty < x < \infty, \end{cases}$$

and discuss.

Solution. By D'Alembert's formula, the solution is

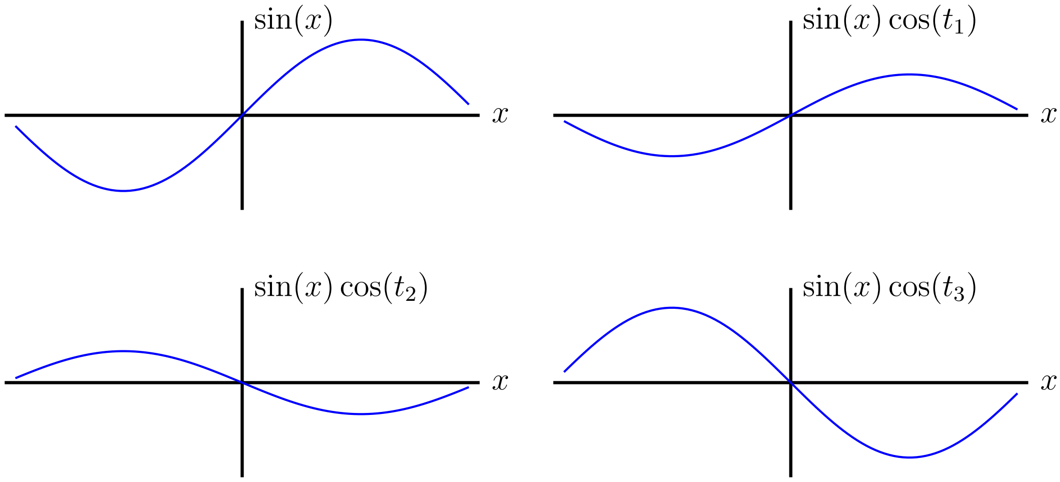
$$u(x,t) = \frac{\sin(x+t) + \sin(x-t)}{2}.$$

The two terms of u both have the same spatial frequency (i.e., their period in the x-variable is the same, 2π) and amplitude (the distance between the maximum and minimum value of each term is the same, namely 1) but they "propagate" in opposite directions: the term $\sin(x+t)$ moves to the left as $t \to +\infty$, while $\sin(x-t)$ moves to the right as $t \to +\infty$.

The situation of two waves with the same spatial frequency and amplitude but traveling in opposite directions is perfectly set up to cancel out some motion. Using the trig addition formulas, we can show

$$u(x,t) = \sin(x)\cos(t).$$

Thus at any point t in time and x in space, the height of the wave is just a multiple of $\sin(x)$, and if we look at a movie of the wave, or at least a series of sketches, it looks like the wave profile is just bouncing up and down over the same region in space; a particle "riding" the wave would not travel to the left or right but just oscillate in place.



In general, a solution $u = u(x,t)$ to a PDE is a **STANDING WAVE** if $u(x,t) = f(x)g(t)$ for functions f and g of one variable. This is precisely the separation of variables ansatz, to which we return shortly. So, separation of variables always yields standing wave solutions, but our habit of using superposition and adding a bunch of solutions formed from separation of variables can destroy the standing wave behavior (we need not be able to write $f_1(x)g_1(t) + f_2(x)g_2(t)$ as $F(x)G(t)$). ▲

2.6.2. Separation of variables for the wave equation.

2.6.5 Example.

Use separation of variables to find a formal solution to the IVP-BVP

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ 0 \le x \le P, \ t \ge 0 \\ u(0,t) = u(P,t) = 0, \ t \ge 0 \\ u(x,0) = \phi(x), \ 0 \le x \le P \\ u_t(x,0) = \psi(x), \ 0 \le x \le P. \end{cases}$$

Solution. With $u(x,t) = X(x)T(t)$, we have

$$X(x)T''(t) = c^2 X''(x)T(t)$$

and thus

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

for some constant λ. Then we have two second-order problems:

$$X''(x) = \lambda X(x) \quad \text{and} \quad T''(t) = c^2 \lambda T(t)$$

Each of these problems has three solutions, depending on the sign of λ, and so nine possible product solutions result!

However, if $\lambda \ge 0$, one can show that the boundary conditions only admit a trivial product solution, and so we assume $\lambda = -\alpha^2 < 0$ for some $\alpha > 0$. Then

$$X''(x) = -\alpha^2 X(x),$$

so

$$X(x) = \beta_1 \cos(\alpha x) + \beta_2 \sin(\alpha x)$$

and

$$T''(t) = -c^2 \alpha^2 T(t),$$

so

$$T(t) = \gamma_1 \cos(\alpha c t) + \gamma_2 \sin(\alpha c t).$$

We put this back together to find

$$u(x,t) = \big(\beta_1 \cos(\alpha x) + \beta_2 \sin(\alpha x)\big)\big(\gamma_1 \cos(\alpha c t) + \gamma_2 \sin(\alpha c t)\big).$$

Now we work with the boundary conditions:

$$0 = u(0,t) = \beta_1 \big(\gamma_1 \cos(\alpha c t) + \gamma_2 \sin(\alpha c t)\big).$$

Since this must hold for all t, by linear independence of $\sin(\cdot)$ and $\cos(\cdot)$, we either have $\beta_1 = 0$ or $\gamma_1 = \gamma_2 = 0$. In the latter case, we reduce to a trivial solution for u, and so we take $\beta_1 = 0$. Then

$$u(x,t) = \beta_2 \sin(\alpha x)\big(\gamma_1 \cos(\alpha ct) + \gamma_2 \sin(\alpha ct)\big),$$

and so using the other boundary condition gives

$$0 = \beta_2 \sin(\alpha P)\big(\gamma_1 \cos(\alpha ct) + \gamma_2 \sin(\alpha ct)\big).$$

So, either $\beta_2 = 0$, or $\sin(\alpha P) = 0$, or a linear combination of $\sin(\cdot)$ and $\cos(\cdot)$ vanishes for all t. The only possibility that avoids the trivial solution is taking $\sin(\alpha P) = 0$, and so

$$\alpha = \frac{k\pi}{P}$$

for some $k \in \mathbb{Z}$.

After relabeling our constants, we conclude that

$$u(x,t) = \left(A_k \cos\left(\frac{k\pi ct}{P}\right) + B_k \sin\left(\frac{k\pi ct}{P}\right)\right) \sin\left(\frac{k\pi x}{P}\right).$$

Superposition then tells us that a solution to the BVP (not yet taking into account initial conditions) is

$$u(x,t) = \sum_{k=1}^{n} \left(A_k \cos\left(\frac{k\pi ct}{P}\right) + B_k \sin\left(\frac{k\pi ct}{P}\right)\right) \sin\left(\frac{k\pi x}{P}\right) \tag{2.6.13}$$

for any integer n. Note that we have no $k = 0$ term because of the factor of $\sin(k\pi x/P)$.

Now we look at the initial conditions. We have

$$\phi(x) = u(x,0) = \sum_{k=1}^{n} A_k \sin\left(\frac{k\pi x}{P}\right)$$

and, since the t-derivative of (2.6.13) is

$$u_t(x,t) = \sum_{k=1}^{n} \left(-\frac{A_k k\pi c}{P} \sin\left(\frac{k\pi ct}{P}\right) + \frac{B_k k\pi}{P} \cos\left(\frac{k\pi ct}{P}\right)\right) \sin\left(\frac{k\pi x}{P}\right),$$

we have

$$\psi(x) = u_t(x,0) = \sum_{k=1}^{n} B_k \left(\frac{k\pi c}{P}\right) \sin\left(\frac{k\pi x}{P}\right).$$

We put all this together to claim that the *formal* solution to the full problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ 0 < x < P \\ u(0,t) = u(P,t) = 0 \\ u(x,0) = \phi(x), \ 0 \le x \le P \\ u_t(x,0) = \psi(x), \ 0 \le x \le P, \end{cases}$$

where

$$\mathsf{FSS}[\phi](x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{P}\right) \quad \text{and} \quad \mathsf{FSS}[\psi](x) = \sum_{k=1}^{\infty} B_k \left(\frac{k\pi c}{P}\right) \sin\left(\frac{k\pi x}{P}\right)$$

is

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos\left(\frac{k\pi ct}{P}\right) + B_k \sin\left(\frac{k\pi ct}{P}\right)\right) \sin\left(\frac{k\pi x}{P}\right). \tag{2.6.14}$$

It is important to note that, in this phrasing, the kth Fourier sine coefficient of ψ is not B_k but rather $B_k(k\pi c)/P$. So, if we write instead

$$\mathsf{FSS}[\phi](x) = \sum_{k=1}^{\infty} \mathsf{A}_k[\phi] \sin\left(\frac{k\pi x}{P}\right) \quad \text{and} \quad \mathsf{FSS}[\psi](x) = \sum_{k=1}^{\infty} \mathsf{B}_k[\psi] \left(\frac{k\pi c}{P}\right) \sin\left(\frac{k\pi x}{P}\right),$$

then the formal solution u is

$$u(x,t) = \sum_{k=1}^{\infty} \left(\mathsf{A}_k[\phi] \cos\left(\frac{k\pi ct}{P}\right) + \frac{P}{k\pi c}\mathsf{B}_k[\psi] \sin\left(\frac{k\pi ct}{P}\right)\right) \sin\left(\frac{k\pi x}{P}\right).$$

The individual terms

$$\left(A_k \cos\left(\frac{k\pi ct}{P}\right) + B_k \sin\left(\frac{k\pi ct}{P}\right)\right) \sin\left(\frac{k\pi x}{P}\right).$$

of this formal series for u are sometimes called the **HARMONICS** or **OVERTONES** of the plucked string with ends fixed at $x = 0$, P. Now we manipulate an arbitrary term of this sum in a peculiar way. Using the addition formula for sine, we have

$$\left(A_k \cos\left(\frac{k\pi ct}{P}\right) + B_k \sin\left(\frac{k\pi ct}{P}\right)\right) \sin\left(\frac{k\pi x}{P}\right)$$

$$= A_k \underbrace{\cos\left(\frac{k\pi ct}{P}\right) \sin\left(\frac{k\pi x}{P}\right)}_{=\frac{1}{2}\sin\left(\frac{k\pi(x-ct)}{P}\right)} + B_k \underbrace{\sin\left(\frac{k\pi ct}{P}\right) \sin\left(\frac{k\pi x}{P}\right)}_{=\frac{1}{2}\sin\left(\frac{k\pi(x+ct)}{P}\right)}.$$

Thus (formally!)

$$u(x,t) = \frac{1}{2}\sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi(x-ct)}{P}\right) + \frac{1}{2}\sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi(x+ct)}{P}\right).$$

If we put

$$f(X) := \frac{1}{2}\sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi X}{P}\right) \quad \text{and} \quad g(X) := \frac{1}{2}\sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi X}{P}\right),$$

then we can rewrite (2.6.14) in D'Alembert's formula $u(x,t) = f(x-ct) + g(x+ct)$. ▲

2.7. Boundary value problems.

The material in this section comes from [6, 7, 18, 29, 31]. Other than the motivational problem below in Section 2.7.1, the material here does not explicitly use techniques from PDE, but we do use notions from (generalized) Fourier series.

2.7.1. Motivation: separation of variables for a generalized heat equation.

The following equation is a more general model of heat conduction in a rod of length P than our original heat equation (2.3.1):

$$r(x)u_t = \partial_x[p(x)u_x] + G(x,t,u(x,t)).$$

Here $u = u(x,t)$ is the temperature of the rod at position $x \in [0,P]$ and time $t \geq 0$, while r, $p \in \mathcal{C}([0,P])$ are positive-valued functions. If $r = p = 1$ and $G = 0$, then this equation reduces to the heat equation $u_t = u_{xx}$.

The term G represents a source (addition) or sink (removal) of heat throughout the bar. We also assume heat flows through the ends of the rod at a rate proportional to the temperature at the ends and so impose the boundary conditions

$$u_x(0,t) + \alpha_1 u(0,t) = 0 \quad \text{and} \quad u_x(P,t) + \beta_1 u(P,t) = 0. \tag{2.7.1}$$

Let us further assume that G depends linearly on u and that the coefficient of u is independent of t:

$$G(x,t,u) = q(x)u + F(x,t).$$

Here $q \in \mathcal{C}([0,P])$ is assumed to be real-valued. So, our PDE becomes

$$r(x)u_t = \partial_x[p(x)u_x] + q(x)u + F(x,t). \tag{2.7.2}$$

Assume for now that $F = 0$ (we will treat the nonzero case in an exercise) and separate variables with $u(x,t) = X(x)T(t)$. We find

$$r(x)X(x)T'(t) = \partial_x[p(x)X]T(t) + q(x)X(x)T(t) \Longrightarrow \frac{T'(t)}{T(t)} = \frac{\partial_x[p(x)X] + q(x)X(x)}{r(x)X(x)} = \lambda$$

for some $\lambda \in \mathbb{R}$. (We are taking $\lambda \in \mathbb{R}$ since p, q, and r are real-valued.) The T equation is, of course,

$$T'(t) = \lambda T(t) \iff T(t) = Ce^{\lambda t}, \tag{2.7.3}$$

as before, in Example 2.3.1.

The X equation rearranges to

$$\partial_x[p(x)X'] + q(x)X(x) = \lambda r(x)X(x).$$

Now, if we expand the ∂_x term with the product rule and rearrange, this equation reads

$$p(x)X'' + p'(x)X' + (q(x) - \lambda r(x)X = 0.$$

Since $p(x) > 0$ for all x, this is equivalent to

$$X'' + \frac{p'(x)}{p(x)}X' + \frac{q(x) - \lambda r(x)}{p(x)}X = 0.$$

For each fixed $\lambda \in \mathbb{R}$, this is a homogeneous second-order linear ODE with continuous coefficients, per Definition 1.1.6, and so it has two linearly independent solutions (which depend on λ), and any solution to this ODE must be a linear combination of these solutions, per Theorem 1.4.13.

So, for which (if any) λ can we meet the boundary conditions? Using the product ansatz $u(x,t) = X(x)T(t)$, the boundary conditions (2.7.1) read

$$X'(0) + \alpha_1 X(0) = 0 \quad \text{and} \quad X'(P) + \beta_1 X(P) = 0. \tag{2.7.4}$$

(The equations (2.7.1) do not involve any derivatives of T, which is positive anyway by (2.7.3), and so we can divide out by T to obtain (2.7.4).) We should approach this problem with some trepidation. After all, we saw long ago in the very simple Example 1.2.6 that BVPs need not have as consistent behavior as IVPs, and, more recently, Example 2.3.2 suggests that we may only be able to meet (2.7.4) for very particular values of λ.

In Sections 2.7.4 and 2.7.5 we will develop two different, but equivalent, methods for solving the **BOUNDARY VALUE PROBLEM (BVP)**

$$\begin{cases} \partial_x[p(x)X'] + q(x)X(x) = \lambda r(x)X(x) \\ X'(0) + \alpha_1 X(0) = 0 \\ X'(P) + \beta_1 X(P) = 0 \end{cases} \tag{2.7.5}$$

under certain conditions on λ. We will then return to the more general PDE problem (2.7.2) with $F \neq 0$. Let us emphasize now that the chief challenge here is *not* solving the ODE but rather finding the values of λ for which the ODE has a *solution that meets the boundary conditions.*

2.7.2. Eigentheory of homogeneous Sturm-Liouville BVPs.

We presume knowledge of Appendix C.6 on eigenvalues and C.8 on self-adjoint operators.

2.7.1 Definition.

A **HOMOGENEOUS STURM-LIOUVILLE BOUNDARY VALUE PROBLEM** *is a BVP of the form*

$$\begin{cases} \partial_x[p(x)f'] + q(x)f = \lambda r(x)f, \ a \le x \le b \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \beta_0 f(b) + \beta_1 f'(b) = 0. \end{cases} \tag{2.7.6}$$

Here $p \in \mathcal{C}^1([a,b])$ and q, $r \in \mathcal{C}([a,b])$ with p and r positive, $\lambda \in \mathbb{R}$, and q real-valued. The constants α_0 and α_1 are not both zero, and the constants β_0 and β_1 are not both zero[37]*.*

The form of the ODE in (2.7.6) may seem unusually restrictive. We discuss some of its properties here and show in Section 2.7.3 how this is the natural form of an ODE to study in the context of boundary conditions.

2.7.2 Remark.

(i) *The ODE in* (2.7.6) *reads*

$$p(x)f'' + p'(x)f' + \big(q(x) - \lambda r(x)\big)f = 0. \tag{2.7.7}$$

[37] Two more succinct ways of expressing this are $\alpha_0^2 + \alpha_1^2 \neq 0$ and $\beta_0^2 + \beta_1^2 \neq 0$, or $(\alpha_0, \alpha_1) \neq (0,0)$ and $(\beta_0, \beta_1) \neq (0,0)$.

Since $p(x) > 0$ for all x with p', q, and r continuous, this rearranges to the ODE

$$f'' + \frac{p'(x)}{p(x)} + \left(\frac{q(x) - \lambda r(x)}{p(x)}\right) f = 0.$$

This is a second-order linear homogeneous ODE with continuous coefficients, so we know (Lemma 1.4.12) it has two linearly independent solutions. Whether or not these solutions, or a linear combination of them, meet the boundary conditions in (2.7.6) *is, right now, anyone's guess.*

(ii) *We will call an ODE of the form*

$$\partial_x[\rho(x)f] + \theta(x)f = 0 \tag{2.7.8}$$

a **Sturm-Liouville ODE** *whenever ρ is positive and θ is real-valued.*

(iii) *The ODE in a Sturm-Liouville problem is sometimes written instead as*

$$\partial_x[p(x)f'] + \big(q(x) + \lambda r(x)\big)f = 0. \tag{2.7.9}$$

We will see in the exercises that if one can solve (2.7.7)*, then it is easy to solve* (2.7.9)*.*

The next example illustrates all of the abstract properties of Sturm-Liouville BVPs that we will subsequently develop. It is essentially a rehashing of Example 2.3.2.

2.7.3 Example.

Find all real numbers λ and all nonzero functions f that satisfy

$$\begin{cases} f'' + f = \lambda f, \ 0 \le x \le \pi \\ f(0) = f(\pi) = 0. \end{cases}$$

Solution. First, observe that the compactly written boundary conditions above really are

$$\big(1 \cdot f(0)\big) + \big(0 \cdot f'(0)\big) = 0 \quad \text{and} \quad \big(1 \cdot f(\pi)\big) + \big(0 \cdot f'(\pi)\big) = 0,$$

and these are of the appropriate form from Definition 2.7.1. Also, the ODE here is really

$$\partial_x[1 \cdot f'] + (1 \cdot f) = \lambda(1 \cdot f).$$

And so our problem is indeed a Sturm-Liouville BVP, with $p = q = r = 1$.

The ODE rearranges to the homogeneous problem

$$f'' + (1 - \lambda)f = 0. \tag{2.7.10}$$

We consider cases on λ.

(i) $\lambda < 0$. Write $\lambda = -\alpha^2$ for $\alpha > 0$. Then the ODE is

$$f'' + (1 + \alpha^2)f = 0,$$

where $1 + \alpha^2 > 0$. The general solution is

$$f(x) = c_1 \cos(\sqrt{1 + \alpha^2}x) + c_2 \sin(\sqrt{1 + \alpha^2}x).$$

To meet the $f(0) = 0$ boundary condition, we need $c_1 = 0$, and so $f(x) = c_2 \sin(\sqrt{1 + \alpha^2}x)$. To meet the $f(\pi) = 0$ boundary condition with $c_2 \neq 0$, we need $\sin(\sqrt{1 + \alpha^2}\pi) = 0$, hence $\sqrt{1 + \alpha^2}\pi = k\pi$ for some $k \in \mathbb{Z}$. Since $\sqrt{1 + \alpha^2} > 0$, this k must be a *positive* integer. We solve $\sqrt{1 + \alpha^2} = k$ to find $\alpha^2 = k^2 - 1$, thus $\lambda = -\alpha^2 = 1 - k^2$. This further requires $k \geq 2$, since $\lambda < 0$ (equivalently, $\alpha > 0$).

(ii) $\lambda = 0$. Then the ODE is $f'' + f = 0$, which has the general solution

$$f(x) = c_1 \cos(x) + c_2 \sin(x).$$

To have $f(0) = 0$, we need $c_1 = 0$, and so $f(x) = c_2 \sin(x)$. Regardless of what c_2 is, we will then have $f(\pi) = 0$. So, the BVP at $\lambda = 0$ has the nontrivial solution $f(x) = c \sin(x)$ where $c \in \mathbb{R}$ is arbitrary.

(iii) $\lambda > 0$. Write $\lambda = \alpha^2$ for $\alpha > 0$, so the ODE is

$$f'' + (1 - \alpha^2)f = 0.$$

We need to consider three further cases on α.

(i) $0 < \alpha < 1$. Then (2.7.10) has the general solution

$$f(x) = c_1 \cos(\sqrt{1 - \alpha^2}x) + c_2 \sin(\sqrt{1 - \alpha^2}x).$$

The condition $f(0) = 0$ forces $c_1 = 0$, and so $f(x) = c_2 \sin(\sqrt{1 - \alpha^2}x)$. The condition $f(\pi) = 0$ forces either $c_2 = 0$ or $\sin(\sqrt{1 - \alpha^2}\pi) = 0$. The latter happens if and only if $\sqrt{1 - \alpha^2}\pi = k\pi$ for some integer $k \in \mathbb{Z}$. Since $0 < \alpha < 1$, we have $k = \sqrt{1 - \alpha^2}$. This rearranges to $1 - k^2 = \alpha^2$. Since k is an integer, we have $1 - k^2 \leq 0$, which contradicts our assumption of $0 < \alpha < 1$, and so we can only meet both boundary conditions if $c_1 = c_2 = 0$.

(ii) $\alpha = 1$. Then (2.7.10) is just $f'' = 0$, from which $f(x) = c_1 x + c_2$. It is then a quick calculation that if $f(0) = f(\pi) = 0$, then $c_1 = c_2 = 0$.

(iii) $\alpha > 1$. We rewrite (2.7.10) slightly as

$$f'' - (\alpha^2 - 1)f = 0,$$

where now $\alpha^2 - 1 > 0$. All solutions then have the form

$$f(x) = c_1 e^{\sqrt{\alpha^2 - 1}x} + c_2 e^{-\sqrt{\alpha^2 - 1}x}.$$

We try to find c_1 and c_2 to meet the boundary conditions. First, we need

$$0 = f(0) = c_1 + c_2.$$

Next, we need

$$0 = f(\pi) = c_1 e^{\sqrt{\alpha^2 - 1}\pi} + c_2 e^{-\sqrt{\alpha^2 - 1}\pi}.$$

We put these two equations together as a matrix-vector equation:

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{\alpha^2 - 1}\pi} & e^{-\sqrt{\alpha^2 - 1}\pi} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.7.11}$$

The determinant of this matrix is $e^{-\sqrt{\alpha^2 - 1}\pi} - e^{\sqrt{\alpha^2 - 1}\pi}$. Since $\alpha > 1$, we have $-\sqrt{\alpha^2 - 1}\pi \neq \sqrt{\alpha^2 - 1}\pi$, so the determinant is nonzero. Hence there is only the trivial solution $c_1 = c_2 = 0$ to (2.7.11) and therefore to (2.7.10).

In particular, there is only the trivial solution for $\lambda > 0$.

Putting all our work together, the BVP has nontrivial solutions if and only if $\lambda = \lambda_k := 1 - k^2$ for $k \in \mathbb{N}$, and when $\lambda = \lambda_k$, all solutions to the BVP are scalar multiples of $\phi_k(x) := \sin(kx)$. Let us make some further observations on these solutions.

- The scalar parameters λ_k can be written in decreasing order: $\lambda_1 > \lambda_2 > \cdots > \lambda_k > \lambda_{k+1}$, and they satisfy $\lim_{k\to\infty} \lambda_k = -\infty$.
- For different values of k, the solutions ϕ_k are orthogonal in the L^2-inner product:

$$\langle \phi_k, \phi_j \rangle_{L^2} = \int_0^\pi \phi_k(x)\phi_j(x)\ dx = \int_0^\pi \sin(kx)\sin(jx)\ dx = 0 \text{ if } k \neq j.$$

- From the exercises on Fourier sine series, we have the pointwise convergence

$$\frac{f(x^+) + f(x^-)}{2} = \mathsf{FSS}[f](x),\ \ 0 < x < \pi,$$

for any $f \in \mathcal{C}^1_{\text{pw}}([0, \pi])$, where

$$\mathsf{FSS}[f](x) = \sum_{k=1}^\infty \frac{2}{\pi} \left(\int_0^\pi f(\xi)\sin(k\xi)\ d\xi \right) \sin(kx).$$

Since

$$\|\phi_k\|^2_{L^2} = \int_0^\pi \sin^2(kx)\ dx = \frac{\pi}{2},\ \ k \geq 1,$$

we can rewrite this Fourier sine series as

$$\mathsf{FSS}[f](x) = \sum_{k=1}^\infty \frac{\langle f, \phi_k \rangle_{L^2}}{\|\phi_k\|^2_{L^2}} \phi_k(x).$$

That is, the "normalized" functions $\{\phi_k / \|\phi_k\|^2_{L^2}\}_{k=1}^\infty$ form a "basis" for $\mathcal{C}^1_{\text{pw}}([0, \pi])$, from a certain point of view. ▲

We will see that the observations above hold for *all* Sturm-Liouville BVPs. This is the content of our ultimate Theorem 2.7.7.

2.7.4 Remark.

The peculiar form of the Sturm-Liouville ODE, to say nothing of the accompanying boundary conditions in (2.7.6)*, may seem restrictive. Of course, this form is motivated by the BVP that we derived for the generalized heat equation in* (2.7.5)*. But, surely there are other interesting boundary conditions to consider, and there are plenty of reasonable second-order linear ODEs in which the coefficient of f'' has no relation to that of f'. And why should we restrict ourselves to real λ, anyway? We know how to handle ODEs with complex-valued coefficients.*

We discuss these issues in detail in Section 2.7.3. It turns out that (1) any second-order ODE with reasonably well-behaved coefficients is equivalent to the special Sturm-Liouville-type ODE in (2.7.12)*, and (2) the full Sturm-Liouville BVP* (2.7.6) *is effectively the only BVP with sufficiently nice mathematical properties to make its solutions* *accessible to us in this course.*

As we mentioned at the end of Section 2.7.1, the challenge of solving a BVP is less solving the underlying ODE and more fitting solutions of the ODE to the boundary conditions. An extra feature of the Sturm-Liouville BVP (2.7.6) is the parameter λ. We will take an abstract view that will allow us to invoke techniques and results from eigenvalue analysis (Appendix C.6).

First, since $r(x) > 0$ for all x, we may divide to see that the ODE part of (2.7.6) is equivalent to

$$\frac{1}{r(x)}\big(\partial_x[p(x)f] + q(x)f]\big) = \lambda f. \tag{2.7.12}$$

If we put

$$\mathcal{S}f := \partial_x[p(x)f'] + q(x)f \quad \text{and} \quad \mathcal{S}_r f := \frac{1}{r(x)}\mathcal{S}f, \tag{2.7.13}$$

then (2.7.12) simply compresses to $\mathcal{S}_r f = \lambda f$. This is clearly an eigenvalue-eigenvector relationship for the operator $\mathcal{S}_r$, viewed as an operator in $\mathcal{C}([a,b])$ with domain $\mathcal{C}^2([a,b])$. For this reason we say that a function $f \in \mathcal{C}^2([a,b])$ that solves (2.7.6) for a particular $\lambda \in \mathbb{R}$ is an **EIGENFUNCTION** of the BVP and λ is an eigenvalue. We call the ordered pair (λ, f) an **EIGENPAIR** of the system (2.7.6).

If one wants to understand something about the eigenvalues of a linear operator — in this case, $\mathcal{S}_r$ from (2.7.13) — it is often helpful to bring an inner product (Definition C.4.1) into play. Perhaps the most natural inner product for interacting with a linear operator consisting of derivatives is the **L^2-INNER PRODUCT**, which is[38]

$$\langle f, g\rangle_{L^2} := \int_a^b f(x)\overline{g(x)}\,dx.$$

Then the fundamental interaction between the L^2-inner product and derivatives arises from integration by parts:

$$\langle \partial_x[f], g\rangle_{L^2} = f(x)g(x)\big|_{x=a}^{x=b} - \langle f, \partial_x[g]\rangle_{L^2}, \ f, g \in \mathcal{C}^2([a,b]). \tag{2.7.14}$$

If we could show that $\mathcal{S}_r$ is self-adjoint with respect to the L^2-inner product (Appendix C.8), then we would unlock significant information about its eigenvalues and eigenfunctions: the eigenvalues are all real and eigenfunctions corresponding to distinct eigenvalues are orthogonal (Proposition C.8.2). However, it will not quite be the case that $\mathcal{S}_r$ is self-adjoint in this inner product — and the right domain for $\mathcal{S}_r$ will not quite be all of $\mathcal{C}^2([a,b])$.

But we do not want to work on all of $\mathcal{C}^2([a,b])$ anyway; we want solutions to (2.7.6) *that meet the boundary conditions.* So, let us encode the boundary data from (2.7.6): put

$$\mathfrak{D}(\mathcal{S}) = \mathfrak{D}(\mathcal{S}_r) := \big\{f \in \mathcal{C}^2([a,b]) \ \big| \ \alpha_0 f(a) + \alpha_1 f'(a) = \beta_0 f(b) + \beta_0 f'(b) = 0\big\}. \tag{2.7.15}$$

Then we have a self-adjointness result for the simpler operator $\mathcal{S}$.

[38] Since we will only work on a closed, bounded interval $[a,b]$ in this section, we write $\langle\cdot,\cdot\rangle_{L^2}$, not $\langle\cdot,\cdot\rangle_{L^2([a,b])}$.

2.7.5 Lemma.

If $p \in \mathcal{C}^1([a,b])$ and $q \in \mathcal{C}([a,b])$ are real valued, and if

$$\mathcal{S}f := \partial_x[p(x)f'] + q(x)f,$$

as in (2.7.13), then $\mathcal{S}$ is self-adjoint as an operator in $\mathcal{C}^2([a,b])$ with domain $\mathfrak{D}(\mathcal{S})$ from (2.7.15) with respect to the L^2-inner product. That is,

$$\langle \mathcal{S}f, g\rangle_{L^2} = \langle f, \mathcal{S}g\rangle_{L^2}, \quad f, g \in \mathfrak{D}(\mathcal{S}).$$

Proof. We have

$$\langle \mathcal{S}f, g\rangle_{L^2} = \int_a^b (\mathcal{S}f)(x)\overline{g(x)}\, dx = \underbrace{\int_a^b \partial_x[p(x)f'(x)]\overline{g(x)}\, dx}_{\mathcal{I}_1} + \underbrace{\int_a^b q(x)f(x)\overline{g(x)}\, dx}_{\mathcal{I}_2}.$$

The term $\mathcal{I}_2$ is simpler, so we start with that:

$$\mathcal{I}_2 = \int_a^b f(x)\overline{q(x)g(x)}\, dx,$$

since q is real-valued.

For $\mathcal{I}_1$, we integrate by parts with

$$u = \overline{g(x)} \qquad dv = \partial_x[p(x)f'(x)]\, dx$$
$$du = \overline{g'(x)} \qquad v = p(x)f'(x).$$

Here we are using part (iii) of Example A.5.4 to interchange the derivative and the complex conjugate. Then

$$\mathcal{I}_1 = \underbrace{p(x)f'(x)\overline{g(x)}\big|_{x=a}^{x=b}}_{\mathcal{B}_1} - \underbrace{\int_a^b p(x)f'(x)\overline{g'(x)}\, dx}_{\mathcal{I}_3}.$$

We integrate by parts on $\mathcal{I}_3$ again with

$$u = p(x)\overline{g'(x)} \qquad dv = f'(x)\, dx$$
$$du = \partial_x[p(x)\overline{g'(x)}] \qquad v = f(x).$$

Then

$$\mathcal{I}_3 = \underbrace{p(x)\overline{g'(x)}f(x)\big|_{x=a}^{x=b}}_{\mathcal{B}_2} - \underbrace{\int_a^b f(x)\partial_x[p(x)\overline{g'(x)}]\, dx}_{\mathcal{I}_4}.$$

We put this back together:

$$\mathcal{I}_1 = \mathcal{B}_1 - \mathcal{B}_2 + \mathcal{I}_4.$$

Since p is real-valued, we can once again interchange the derivative and the complex conjugate to have

$$\mathcal{I}_4 = \int_a^b f(x)\left(\overline{\partial_x[p(x)g'(x)]}\right)\, dx.$$

Then

$$\langle \mathcal{S}f, g\rangle_{L^2} = \mathcal{I}_1+\mathcal{I}_2 = \mathcal{B}_1-\mathcal{B}_2+\mathcal{I}_2+\mathcal{I}_4 = \mathcal{B}_1-\mathcal{B}_2+\int_a^b f(x)\big(\overline{\partial_x[p(x)g'(x)] + q(x)g(x)}\big)\,dx$$
$$= \mathcal{B}_1 - \mathcal{B}_2 + \langle f, \mathcal{S}g\rangle_{L^2}.$$

So, all we need to do is verify that $\mathcal{B}_1 - \mathcal{B}_2 = 0$. We claim that this follows from the boundary conditions in (2.7.12) which both f *and* g must satisfy, since f, $g \in \mathfrak{D}(\mathcal{S})$; the precise calculation is left as an exercise (one will need to consider cases based on which of α_0, α_1 and β_0, β_1 are nonzero). ■

Our interest, however, is not really the Sturm-Liouville operator $\mathcal{S}$ but rather the scaled operator $\mathcal{S}_r$ from (2.7.13). Does Theorem 2.7.12 imply that $\mathcal{S}_r$ is also self-adjoint in the L^2-inner product? Let us check:

$$\langle \mathcal{S}_r f, g\rangle_{L^2} = \left\langle \frac{1}{r}\mathcal{S}f, g\right\rangle_{L^2} = \int_a^b \frac{1}{r(x)}(\mathcal{S}f)(x)\overline{g(x)}\,dx = \int_a^b (\mathcal{S}f)(x)\overline{\frac{g(x)}{r(x)}}\,dx = \left\langle \mathcal{S}f, \frac{g}{r}\right\rangle_{L^2}$$
$$= \left\langle f, \mathcal{S}\left(\frac{g}{r}\right)\right\rangle_{L^2}. \quad (2.7.16)$$

Since r is real-valued, we absorbed it into the complex conjugate on g. Now the question is if

$$\mathcal{S}\left(\frac{g}{r}\right) = \frac{1}{r}\mathcal{S}g = \mathcal{S}_r g.$$

The first equality will not hold in general because of the product rule for derivatives. And so $\mathcal{S}_r$ is *not* self-adjoint with respect to the L^2-inner product.

But we are asking the wrong question. Instead of working with the plain L^2-inner product, let us use a **WEIGHTED L^2-INNER PRODUCT**: put

$$\langle f, g\rangle_{L^2_r} := \langle rf, g\rangle_{L^2} = \int_a^b r(x)f(x)\overline{g(x)} = \left\langle \sqrt{r}f, \sqrt{r}g\right\rangle. \quad (2.7.17)$$

One can check that since r is real, positive, and continuous on $[a, b]$, the map $\langle\cdot,\cdot\rangle_{L^2_r}$ also satisfies the properties of an inner product (on any subspace of $\mathcal{C}([a, b])$). Moreover, the calculation (2.7.16) shows

$$\langle \mathcal{S}_r f, g\rangle_{L^2_r} = \langle r\mathcal{S}_r f, g\rangle_{L^2} = \left\langle r\left(\frac{1}{r}\mathcal{S}f\right), g\right\rangle_{L^2} = \langle \mathcal{S}f, g\rangle_{L^2}. \quad (2.7.18)$$

Now, Theorem 2.7.12 tells us that $\mathcal{S}$ is self-adjoint with respect to the L^2-inner product, and so, using again that r is real and positive,

$$\langle \mathcal{S}f, g\rangle_{L^2} = \langle f, \mathcal{S}g\rangle_{L^2} = \left\langle f, r\left(\frac{1}{r}\mathcal{S}g\right)\right\rangle_{L^2} = \int_a^b f(x)\overline{\left(r(x)\frac{(\mathcal{S}g)(x)}{r(x)}\right)}\,dx$$
$$= \int_a^b f(x)\overline{r(x)(\mathcal{S}_r g)(x)}\,dx = \int_a^b f(x)r(x)\overline{(\mathcal{S}_r g)(x)}\,dx = \langle rf, \mathcal{S}_r g\rangle_{L^2} = \langle f, \mathcal{S}_r g\rangle_{L^2_r}. \quad (2.7.19)$$

Combining (2.7.18) and (2.7.19), we have the following lemma.

2.7.6 Lemma.

The operator $\mathcal{S}_r$ is self-adjoint with respect to the L^2_r-inner product when the domain $\mathfrak{D}(\mathcal{S}_r)$ is a subspace of the form (2.7.15).

At last, we can invoke Proposition C.8.2 to conclude the following about eigenvalues and eigenvectors for the operator $\mathcal{S}_r$ with domain $\mathfrak{D}(\mathcal{S}_r)$ from (2.7.34) — equivalently, about eigenvalues and eigenfunctions for the Sturm-Liouville BVP (2.7.6). If we glance back to Example 2.7.3, we will see that all the properties of this theorem were illustrated there.

2.7.7 Theorem (Eigentheory for Sturm-Liouville BVPs).

Under the hypotheses of Definition 2.7.1, the Sturm-Liouville BVP (2.7.6) *has the following properties.*

(i) *The eigenvalues of the Sturm-Liouville problem* (2.7.6) *are all real. Moreover, they form a countable set $\{\lambda_k\}_{k=1}^{\infty}$, which we can index in strictly decreasing order*[39] *so that $\lambda_{k+1} < \lambda_k$, and $\lim_{k\to\infty} \lambda_k = -\infty$.*

(ii) *Each eigenvalue of* (2.7.6) *is geometrically simple: for each k, there exists $\phi_k \in \mathfrak{D}(\mathcal{S}_r)$ such that (λ_k, ϕ_k) is an eigenpair for* (2.7.6) *and if (λ_k, f) is another eigenpair for* (2.7.6)*, then there exists $\alpha \in \mathbb{C}$ such that $f = \alpha\phi_k$. Moreover, each ϕ_k is not identically zero and can be chosen to be real-valued.*

(iii) *Eigenfunctions of* (2.7.6) *corresponding to distinct eigenvalues are orthogonal in the L^2_r-inner product: if (λ_k, ϕ_k) and (λ_j, ϕ_j) are eigenpairs of* (2.7.6) *with $\lambda_k \neq \lambda_j$, then*

$$\int_a^b r(x)\phi_k(x)\overline{\phi_j(x)}\,dx = 0.$$

In particular, the set $\{\phi_k\}_{k=1}^{\infty}$ is linearly independent.

(iv) *The normalized eigenfunctions $\{\phi_k/\left\|\phi_k\right\|_{L^2_r}\}_{k=1}^{\infty}$ form an orthonormal basis for $\mathcal{C}([a,b])$ under the L^2-inner product in the sense that if $f \in \mathcal{C}([a,b])$, then*

$$\lim_{n\to\infty}\left\| f - \sum_{k=1}^{n} \frac{\langle f, \phi_k\rangle_{L^2}}{\left\|\phi_k\right\|_{L^2_r}^2}\phi_k \right\|_{L^2_r},$$

where $\|g\|_{L^2_r} := \sqrt{\langle g, g\rangle_{L^2_r}}$ from (2.7.17)*. Moreover, we have the pointwise convergence*

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{k=1}^{\infty} \frac{\langle f, \phi_k\rangle_{L^2_r}}{\left\|\phi_k\right\|_{L^2_r}^2}\phi_k(x), \ a < x < b$$

whenever $f \in \mathcal{C}^1_{\mathrm{pw}}([a,b])$. In particular,

$$f(x) = \sum_{k=1}^{\infty} \frac{\langle f, \phi_k\rangle_{L^2_r}}{\left\|\phi_k\right\|_{L^2_r}^2}\phi_k(x), \ a < x < b \tag{2.7.20}$$

for all $f \in \mathfrak{D}(\mathcal{S}_r)$.

2.7.3. Self-adjointness for more general differential operators.

We remarked several times that the form of the ODE in the Sturm-Liouville BVP (2.7.6) is very particular. It reads

$$\underbrace{p(x)f'' + p'(x)f' + q(x)f}_{\mathcal{S}f} = \lambda r(x)f. \tag{2.7.21}$$

Here $p \in \mathcal{C}^1([a,b])$ with $p(x) > 0$ for all x, while q, $r \in \mathcal{C}([a,b])$ with $q(x)$ real for all x and $r(x) > 0$ for all x. These are some very restrictive coefficients! We started in Section 2.7.2 by showing that $\mathcal{S}$ was self-adjoint with respect to the L^2-inner product, from which we deduced that $\mathcal{S}_r := (1/r)\mathcal{S}$ was self-adjoint with respect to the weighted L_r^2-inner product from (2.7.17).

Why did we not try to work with a more general differential operator in lieu of $\mathcal{S}$? What would happen if we tried to solve

$$\underbrace{\Phi(x)f'' + \Psi(x)f' + \Theta(x)f}_{\mathcal{A}f} = \lambda r(x)f \tag{2.7.22}$$

instead? Here let us take Φ, Ψ, and Θ continuous with Φ positive (so this remains a second-order ODE) but perhaps Ψ and Θ complex-valued.

First, we claim it is actually not at all restrictive to work with an ODE of the Sturm-Liouville form (2.7.21) instead of (2.7.21).

2.7.8 Lemma.

Let $\Phi \in \mathcal{C}^1([a,b])$ be positive-valued and Ψ, Θ, $g \in \mathcal{C}([a,b])$. There exist functions $p \in \mathcal{C}^1([a,b])$, q, $\mu \in \mathcal{C}([a,b])$ with μ positive such that a function $f \in \mathcal{C}^2([a,b])$ satisfies

$$\Phi(x)f'' + \Psi(x)f' + \Theta(x)f = g(x) \tag{2.7.23}$$

if and only if f also solves

$$\partial_x[p(x)f'] + q(x)f = \mu(x)g(x). \tag{2.7.24}$$

Proof. At first glance, it would suffice to construct the functions p, q to satisfy

$$p(x)f'' + p'(x)f + q(x)f = \Phi(x)f'' + \Psi(x)f' + \Theta(x)f.$$

But then we would need $p = \Phi$ and $p' = \Psi$, thus $\Phi' = \Psi$. And that would require (2.7.23) to be in Sturm-Liouville form already.

Instead, we take inspiration from the integrating factor method for first-order ODEs (Section 1.2.2). If μ is a positive function on $[a,b]$, then the ODEs (2.7.23) and

$$\mu(x)\Phi(x)f'' + \mu(x)\Psi(x)f' + \mu(x)\Theta(x)f = \mu(x)g(x) \tag{2.7.25}$$

are equivalent; a function f solves (2.7.23) if and only if f solves (2.7.25). So, perhaps we can choose μ so that

$$p = \mu\Phi \quad \text{and} \quad p' = \mu\Psi.$$

[39] This is worth specifying: we can index the integers $\mathbb{Z}$ as $\mathbb{Z} = \{\mu_k\}_{k=1}^\infty$, but it is not possible to find $\mu_1 \in \mathbb{Z}$ such that $\mu_1 < \mu$ for all other $\mu \in \mathbb{Z}$. Similarly, we can index the rationals $\mathbb{Q}$ as $\mathbb{Q} = \{\nu_k\}_{k=1}^\infty$, but it is not possible to put the ν_k in strictly increasing order with $\nu_k < \nu_{k+1}$.

For this to happen, we need

$$\partial_x[\mu\Phi] = \mu\Psi. \tag{2.7.26}$$

If we rearrange (2.7.26), we see that it is a first-order linear ODE for μ:

$$\mu' + \left(\frac{\Phi'(x) - \Psi(x)}{\Phi(x)}\right)\mu = 0. \tag{2.7.27}$$

Here we used both the positivity and the differentiability of Φ.

A solution to the ODE (2.7.27) is, of course,

$$\mu(x) := \exp\left(-\int_a^x \frac{\Phi'(\xi) - \Psi(\xi)}{\Phi(\xi)}\, d\xi\right).$$

And so if we put

$$p(x) := \mu(x)\Phi(x) \quad \text{and} \quad q(x) := \mu(x)\Theta(x),$$

then (2.7.23) and (2.7.24) are equivalent. ■

So, if we start with a solution f to the "general" problem (2.7.22), then we could use this lemma with $g(x) = \lambda r(x)f(x)$ to show that f solves the Sturm-Liouville ODE (2.7.21) with the weight function r replaced by the (still positive!) weight $r(x)\mu(x)$.

But we can do more. We will demonstrate the surprising fact that, under a few more reasonable hypotheses on its coefficients, the operator $\mathcal{A}$ from (2.7.22) is self-adjoint with respect to the L^2-inner product only if $\mathcal{A}$ is already in Sturm-Liouville form! To see if $\mathcal{A}$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L^2}$, the natural thing to do is to start with the integral $\int_a^b (\mathcal{A}f)(x)g(x)\, dx$ for arbitrary f and g and integrate by parts. Ideally, one will find

$$\int_a^b (\mathcal{A}f)(x)g(x)\, dx = \int_a^b f(x)(\mathcal{B}g)(x)\, dx + \text{"boundary terms."}$$

Here $\mathcal{B}$ will likely be another differential operator, and the boundary terms will arise from evaluating various functions at a and b and subtracting. The need to integrate by parts — to "pop" derivatives from $\mathcal{A}f$ onto g — will require us to assume a little more differentiability of the coefficients Φ and Ψ from (2.7.22).

We begin with Lagrange's identity. The verification of the following lemma is just a long calculation using integration by parts and part (iii) of Example A.5.4 to interchange the derivative and the complex conjugate (i.e, $\overline{f'(x)} = \partial_x[\overline{f}](x)$). The calculations closely resemble those in the proof of Lemma 2.7.12.

2.7.9 Lemma (Lagrange's identity).

Fix real numbers $a < b$ and let $\Phi \in \mathcal{C}^2([a,b])$, $\Psi \in \mathcal{C}^1([a,b])$, and $\Theta \in \mathcal{C}([a,b])$. Define[40]

$$\mathcal{A}f := \Phi(x)f'' + \Psi(x)f' + \Theta(x)f, \tag{2.7.28}$$

as in (2.7.22), and put

$$\mathcal{A}^\dagger f := \partial_x^2[\overline{\Phi(x)}f] - \partial_x[\overline{\Psi(x)}f] + \overline{\Theta(x)}f, \tag{2.7.29}$$

and

$$\mathcal{N}_{\mathcal{A}}[f,g] := \left(\Phi(x)\mathcal{W}[f,\overline{g}](x) + (\Psi(x) - \Phi'(x))f(x)\overline{g(x)}\right)\Big|_{x=a}^{x=b}. \tag{2.7.30}$$

Here $\mathcal{W}$ is the Wronskian from (1.4.8)*. Then for any f, $g \in \mathcal{C}^2([a,b])$,*

$$\langle \mathcal{A}f, g\rangle_{L^2} = \left\langle f, \mathcal{A}^\dagger g\right\rangle_{L^2} + \mathcal{N}_{\mathcal{A}}[f,g]. \tag{2.7.31}$$

Now let us ask two questions about the more general operator $\mathcal{A}$, which we will consider as an operator in $\mathcal{C}([a,b])$ with domain $\mathfrak{D}(\mathcal{A}) \subseteq \mathcal{C}^2([a,b]) \subseteq \mathcal{C}([a,b])$ to be determined.

1. Under what conditions on the coefficients Φ, Ψ, and Θ and the domain $\mathfrak{D}(\mathcal{A})$ does the operator $\mathcal{A}$ have an adjoint in $\mathcal{C}([a,b])$?

2. If $\mathcal{A}$ has an adjoint, under what (further?) conditions is $\mathcal{A}$ self-adjoint?

A glance at Lagrange's identity (2.7.31) shows that $\mathcal{A}^\dagger$ will be the adjoint of $\mathcal{A}$ if the boundary terms $\mathcal{N}_{\mathcal{A}}[f,g]$ vanish for all $f \in \mathfrak{D}(\mathcal{A})$ and $g \in \mathfrak{D}(\mathcal{A}^\dagger)$, once these domains have been properly specified. One way to simplify the boundary terms is to require $\Psi = \Phi'$, in which case we just need

$$\mathcal{N}_{\mathcal{A}}[f,g] = \Phi(x)\mathcal{W}[f,\overline{g}(x)\Big|_{x=a}^{x=b} = 0. \tag{2.7.32}$$

Let us write this out explicitly using the definition of the Wronskian from (1.4.8): we want

$$\Phi(b)\big(f(b)\overline{g'(b)} - f'(b)\overline{g(b)}\big) - \Phi(a)\big(f(a)\overline{g'(a)} - f'(a)\overline{g(a)}\big) = 0. \tag{2.7.33}$$

This seems complicated. Suppose instead we just want to make each term vanish. One way to do this is to assume $\Phi(a) = \Phi(b) = 0$, but this would impose an unnatural restriction[41] on Ψ. Instead, suppose that f and g satisfy the boundary conditions from the prototypical Sturm-Liouville BVP (2.7.6). That is, we assume

$$\alpha_0 f(a) + \alpha_1 f'(a) = \beta_0 f(b) + \beta_1 f'(b) = 0$$

with at least one of α_0, α_1 and one of β_0, β_1 nonzero. We assume the same for g. We leave it as an exercise to check that (2.7.33) then holds. Moreover, the set

$$\big\{f \in \mathcal{C}^2([a,b]) \bigm| \alpha_0 f(a) + \alpha_1 f'(a) = \beta_0 f(b) + \beta_1 f'(b) = 0\big\} \tag{2.7.34}$$

is in fact a subspace of $\mathcal{C}^2([a,b])$.

Now we can answer Question 1.

2.7.10 Lemma (Answer to Question 1).

Assume the notation and hypotheses of Lemma 2.7.9. Suppose $\Psi = \Phi'$. Then the operator $\mathcal{A}^\dagger$ is the adjoint of $\mathcal{A}$ with respect to the L^2-inner product on $\mathcal{C}([a,b])$ when both operators have domain given by (2.7.34)*.*

[40] We use the notation $\mathcal{A}^\dagger$ in (2.7.29) because we have in mind that, under the right circumstances, $\mathcal{A}^\dagger$ will be the adjoint of $\mathcal{A}$. We eschew the perhaps more familiar notation $\mathcal{A}^*$ for adjoints; the uniqueness of adjoints is a delicate issue — see Footnote 79 and Example C.8.1.

[41] Specifically, $\int_a^b \Psi(\xi)\, d\xi = 0$.

Toward the second question, let us see what conditions we need on $\mathcal{A}$ and $\mathcal{A}^\dagger$ to have $\mathcal{A} = \mathcal{A}^\dagger$. If we expand the derivatives in (2.7.29), we find

$$\mathcal{A}^\dagger f = \overline{\Phi(x)} f'' + \big(2\overline{\Phi'(x)} - \overline{\Psi(x)}\big) f' + \big(\overline{\Phi''(x)} - \overline{\Psi'(x)} + \overline{\Theta(x)}\big) f. \tag{2.7.35}$$

(It will be enlightening not to use the condition $\Psi = \Phi'$ just yet.) Now, two differential operators are equal if and only if their coefficients are equal. Immediately, then, we see that we need $\Phi = \overline{\Phi}$, so Φ must be real. Next, we want

$$\Psi = 2\Phi' - \overline{\Psi} \quad \text{and} \quad \Theta = \overline{\Phi''} - \overline{\Psi'} + \overline{\Theta}. \tag{2.7.36}$$

Rearranging (2.7.36), we obtain the following lemma, which answers Question 2.

2.7.11 Lemma (Answer to Question 2).

Assume the notation and hypotheses of Lemma 2.7.9. Suppose as well that

$$\Phi' = \mathrm{Re}[\Psi] \quad \textit{and} \quad \mathrm{Im}[\Theta] = -\frac{\mathrm{Im}[\Psi]}{2}.$$

Then $\mathcal{A}f = \mathcal{A}^\dagger f$ *for all* $f \in \mathcal{C}^2([a,b])$.

In the special case that all our coefficients are real-valued, we see that $\Psi = \Phi'$, and then $\mathcal{A}$ is really a Sturm-Liouville operator.

2.7.12 Theorem.

Suppose $\Phi \in \mathcal{C}^2([a,b])$, $\Psi \in \mathcal{C}^1([a,b])$, *and* $\Theta \in \mathcal{C}([a,b])$ *are real-valued. Let* $\mathcal{A}$ *be defined by* (2.7.28) *and have the domain* (2.7.34). *Then* $\mathcal{A}$ *is self-adjoint with respect to the* L^2*-inner product if and only if* $\Phi' = \Psi$.

In other words, if a differential operator of the form (2.7.28) has real-valued, sufficiently differentiable coefficients, then that operator is self-adjoint in L^2 if and only if it has Sturm-Liouville form. And so, if we want to ensure self-adjointness of our problem's differential operator, it makes sense only to work with Sturm-Liouville problems.

2.7.4. Eigenfunction expansions for nonhomogeneous BVPs.

The natural generalization of the homogeneous Sturm-Liouville BVP is, of course, the nonhomogeneous problem.

2.7.13 Definition.

A **NONHOMOGENEOUS STURM-LIOUVILLE BVP** *is a boundary value problem of the form*[42]

$$\begin{cases} \partial_x[p(x)f'] + q(x)f = \lambda r(x) f + h(x), \ a \le x \le b \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \beta_0 f(b) + \beta_1 f'(b) = 0. \end{cases} \tag{2.7.37}$$

Here $p \in \mathcal{C}^1([a,b])$ *is positive and* q, r, $h \in \mathcal{C}([a,b])$ *with* r *positive. We assume* $\alpha_0^2 + \alpha_1^2 \neq 0$ *and* $\beta_0^2 + \beta_1^2 \neq 0$. *The goal is to find a function* $f \in \mathcal{C}^2([a,b])$ *and* $\lambda \in \mathbb{R}$ *that satisfy* (2.7.37).

The ODE in (2.7.37) is equivalent to

$$(\mathcal{S}_r - \lambda)f = \frac{h}{r}, \tag{2.7.38}$$

where $\mathcal{S}f = \partial_x[p(x)f'] + q(x)f$ and $\mathcal{S}_r = (1/r)\mathcal{S}$. The method of solution that we will deploy here strongly resembles the solution of the matrix resolvent equation in Appendix C.7; the reader is encouraged to view this matrix problem as a prototype for our Sturm-Liouville BVP, although the presentation here is independent of that appendix. These finite-dimensional methods of the matrix case essentially remain true for the *infinite-dimensional* problems (2.7.38) and (2.7.37), although we will have to be more careful in certain stages.

Let $\{(\lambda_k, \phi_k)\}_{k=1}^{\infty}$ be the complete set of eigenpairs of $\mathcal{S}_r$ from Theorem 2.7.7. In particular, $\mathcal{S}_r\phi_k = \lambda_k\phi_k$. Assume further that $h/r \in \mathcal{C}^1([a,b])$; this is a bit more restrictive than Definition 2.7.13 proposes. As in our many Fourier problems, we will work backward and assume we have a solution $f \in \mathcal{C}^2([a,b])$ to the BVP (2.7.37), and we will look for a "formula" for f.

Under these assumptions, both f and h/r have the eigenfunction expansions

$$f(x) = \sum_{k=1}^{\infty} \mu_k\phi_k(x) \quad \text{and} \quad \frac{h(x)}{r(x)} = \sum_{k=1}^{\infty} \theta_k\phi_k(x), \tag{2.7.39}$$

except maybe at the endpoints a and b. To be clear, the series in (2.7.39) converge pointwise to f and h/r at least on (a,b).

Since we do not know f, we will have to determine the sequence (μ_k), but we do know

$$\theta_k = \frac{\langle h/r, \phi_k\rangle_{L_r^2}}{\|\phi_k\|_{L_r^2}^2} = \frac{\langle h, \phi_k\rangle_{L^2}}{\|\phi_k\|_{L_r^2}^2}$$

Then (2.7.38) is equivalent to

$$(\mathcal{S}_r - \lambda)\sum_{k=1}^{\infty} \mu_k\phi_k = \sum_{k=1}^{\infty} \theta_k\phi_k. \tag{2.7.40}$$

Let us also suppose that we can interchange the series and $\mathcal{S}_r$ (this is a delicate analytic problem that we will not pursue further):

$$\mathcal{S}_r \sum_{k=1}^{\infty} \mu_k\phi_k = \sum_{k=1}^{\infty} \mathcal{S}_r(\mu_k\phi_k).$$

Then since $\mathcal{S}_r\phi_k = \lambda_k\phi_k$, (2.7.40) becomes

$$\sum_{k=1}^{\infty} \big(\lambda_k\mu_k\phi_k - \lambda\phi_k\big) = \sum_{k=1}^{\infty} \theta_k\phi_k.$$

[42]It is also possible to encounter the ODE in the form

$$\partial_x[p(x)f'] + q(x)f + \lambda r(x)f = h(x),$$

see part (iii) of Remark 2.7.2. Incidentally, we are departing from our usual custom of writing the homogeneity as g, since we will eventually discuss a concept known as a "Green's function," which practically mandates us to use "g" in a different context.

Collecting everything under one sum, we have

$$\sum_{k=1}^{\infty} \big(\mu_k(\lambda_k - \lambda) - \theta_k\big)\phi_k = 0. \tag{2.7.41}$$

This is an infinite sum, and so we cannot use the linear independence of the set $\{\phi_k\}_{k=1}^{\infty}$ to conclude that each scalar factor on ϕ_k is zero. But we can use its orthogonality. Fix an integer j and assume that we can interchange the series in (2.7.41) and the L^2-inner product (another delicate assumption). Then[43]

$$\begin{aligned} 0 = \langle 0, \phi_j \rangle_{L^2} = \left\langle \sum_{k=1}^{\infty} \big(\mu_k(\lambda_k - \lambda) - \theta_k\big)\phi_k, \phi_j \right\rangle_{L^2} &= \sum_{k=1}^{\infty} \big(\mu_k(\lambda_k - \lambda) - \theta_k\big) \langle \phi_k, \phi_j \rangle_{L^2} \\ &= \big(\mu_j(\lambda_j - \lambda) - \theta_j\big). \end{aligned} \tag{2.7.42}$$

Now, if $\lambda \neq \lambda_k$ for all k, then we can solve (2.7.42) for μ_k:

$$\mu_k = \frac{\theta_k}{\lambda_k - \lambda}.$$

Then

$$f = \sum_{k=1}^{\infty} \frac{\theta_k}{\lambda_k - \lambda}\phi_k. \tag{2.7.43}$$

To be clear, we have worked backward, as is often our custom. We assumed that we had a solution f to the BVP (2.7.37), and we found that the only reasonable candidate for f is

$$f(x) := \sum_{k=1}^{\infty} \frac{\theta_k}{\lambda_k - \lambda}\phi_k(x).$$

We have not proved that this formula is differentiable, nor even that it converges pointwise (this is where we tended to stop in finding formal solutions to differential equations with Fourier series, too). As with Fourier (and Taylor) series, a reasonable approximation of the solution may be found by truncating this formal eigenseries to some finite sum.

If, however, $\lambda = \lambda_j$ for some j, then (2.7.42) forces $\theta_j = 0$. That is, we have the solvability condition

$$\langle h, \phi_j \rangle_{L^2} = 0. \tag{2.7.44}$$

If h does not satisfy this extra condition, then we cannot solve the BVP (2.7.37). If h does, however, then we have infinitely many solutions of the form

$$\sum_{\substack{k=1 \\ k \neq j}}^{\infty} \frac{\theta_k}{\lambda_k - \lambda}\phi_k + c\phi_j, \tag{2.7.45}$$

where $c \in \mathbb{C}$ is arbitrary.

We summarize our findings.

[43] Another way to see this is to appeal to part (i) of Lemma C.5.6.

2.7.14 Theorem.

The nonhomogeneous Sturm-Liouville BVP (2.7.37) *has the unique (formal) solution* (2.7.43) *if λ is not an eigenvalue of the BVP and the infinitely many (formal) solutions* (2.7.45) *if λ is an eigenvalue and if the nonhomogeneity h satisfies the solvability condition* (2.7.44).

2.7.15 Example.

For each $\lambda \in \mathbb{R}$, discuss the (formal) solutions to the BVP

$$\begin{cases} f'' + f = \lambda f + \cos(x), \ 0 \le x \le \pi \\ f(0) = f(\pi) = 0. \end{cases}$$

Solution. We solved the homogeneous version of this BVP in Example 2.7.3, where we saw that the eigenvalues were $\lambda_k = 1-k^2$, $k \in \mathbb{N}$, and the (nonnormalized) eigenfunctions were $\phi_k(x) = \sin(kx)$, $k \ge 1$. Any solution to this problem, then, will depend on whether or not $\lambda = \lambda_j$ for some j and also what the expansion of $\cos(\cdot)$ on $[0, \pi]$ relative to these eigenfunctions is. Since $\cos(\cdot) \in \mathcal{C}^1([0,\pi])$, we know

$$\cos(x) = \sum_{k=1}^{\infty} \beta_k \phi_k(x), \ 0 < x < \pi,$$

where

$$\beta_k = \frac{\langle \cos(\cdot), \phi_k \rangle_{L^2}}{\|\phi_k\|_{L^2}^2}.$$

We compute

$$\|\phi_k\|_{L^2}^2 = \int_0^{\pi} \sin^2(kx) \ dx = \frac{\cos(kx)\sin(kx) + kx}{2k}\bigg|_{x=0}^{x=\pi} = \frac{\pi}{2},$$

and so

$$\beta_k = \frac{2}{\pi}\int_0^{\pi} \cos(x)\overline{\phi_k(x)} \ dx = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(kx) \ dx = \mathsf{B}_k[\cos(\cdot)].$$

That is, the coefficients of $\cos(\cdot)$ relative to this "eigenbasis" are precisely the coefficients of its Fourier sine series on $[0, \pi]$. In an exercise, we calculate that these Fourier sine coefficients are

$$\beta_k = \mathsf{B}_k[\cos(\cdot)] = \begin{cases} 0, \ k \ge 1 \text{ is odd} \\ \\ \dfrac{4k}{\pi(k^2-1)}, \ k \ge 2 \text{ is even}. \end{cases} \tag{2.7.46}$$

So, if $\lambda \ne 1 - k^2$ for any $k \in \mathbb{N}$, then the unique solution to the BVP is

$$f(x) = \sum_{k=1}^{\infty} \frac{\beta_k}{\lambda_k - \lambda}\sin(kx) = \sum_{k=1}^{\infty} \frac{\beta_{2k}}{\lambda_{2k} - \lambda}\sin(2kx)$$

$$= \sum_{k=1}^{\infty} \left(\frac{4(2k)}{\pi((2k)^2-1)}\right)\left(\frac{1}{(1-(2k)^2)-\lambda}\right)\sin(2kx) = \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{k}{(4k^2-1)(1-4k^2-\lambda)}\sin(2kx).$$

Suppose next that $\lambda = 1 - j^2$ for some $j \in \mathbb{N}$. In order for the BVP to have a solution, the solvability condition $\langle \cos(\cdot), \phi_j \rangle_{L^2} = 0$ must be satisfied. That is, we need

$$0 = \int_0^\pi \cos(x)\sin(jx)\,dx = \begin{cases} 0, & j = 1 \\ \dfrac{j(\cos(\pi j) + 1)}{j^2 - 1}, & j \in \mathbb{Z} \setminus \{1\}. \end{cases}$$

Consequently, the solvability condition is met[44] if and only if j is odd. In this case, there are infinitely many solutions of the form

$$f(x) = \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \frac{\beta_k}{\lambda_k - \lambda} \sin(kx) + C\sin(jx) = \frac{8}{\pi} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \frac{k}{(4k^2 - 1)(1 - 4k^2 - \lambda)} \sin(2kx) + C\sin(jx).$$

for $C \in \mathbb{C}$. ▲

2.7.16 Method: solve a nonhomogeneous BVP with eigenfunction series

1. Determine the eigenvalues λ_k and the eigenfunctions ϕ_k of the homogeneous BVP. Remember that brute-force calculations (as in Example 2.7.3) rarely produce normalized eigenfunctions, so calculate $\|\phi_k\|_{L^2}$, too.

2. Calculate the Fourier coefficients of the nonhomogeneity with respect to the normalized eigenfunctions, i.e., if the nonhomogeneity is h, compute

$$\theta_k := \frac{\langle h, \phi_k \rangle_{L^2}}{\|\phi_k\|_{L^2_r}^2}.$$

3. Decide whether or not the scalar λ in the BVP is an eigenvalue of the homogeneous BVP.

4. If λ is not an eigenvalue, the unique solution is

$$\sum_{k=1}^{\infty} \frac{\langle h, \phi_k \rangle_{L^2}}{(\lambda_k - \lambda)\|\phi_k\|_{L^2_r}^2} \phi_k(x)$$

5. If λ is an eigenvalue, say, $\lambda = \lambda_j$, determine whether or not $\langle h, \phi_j \rangle_{L^2} = 0$.

6. If $\langle h, \phi_j \rangle_{L^2} \neq 0$, there is no solution.

7. If $\langle h, \phi_j \rangle_{L^2} = 0$, there are infinitely many solutions of the form

$$\sum_{\substack{k=1 \\ k \neq j}}^{\infty} \frac{\langle h, \phi_k \rangle_{L^2}}{(\lambda_k - \lambda)\|\phi_k\|_{L^2_r}^2} \phi_k(x) + c\phi_j(x).$$

for $c \in \mathbb{C}$ arbitrary.

[44] Another way to see this is to calculate

$$\langle \cos(\cdot), \phi_j \rangle_{L^2} = \langle \cos(\cdot), \sin(j\cdot) \rangle_{L^2} = \frac{\pi}{2} \mathsf{B}_j[\cos(\cdot)]$$

and then use the identity (2.7.46).

2.7.5. Green's functions.

In Appendix C.7, we solved a matrix-vector equation like $(A - \lambda \mathbb{1}_n)\mathbf{x} = \mathbf{y}$ for $\mathbf{x} \in \mathbb{C}^n$ given $\mathbf{y} \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$ and $\mathbb{1}_n$ equal to the $n \times n$ identity matrix, but we did not calculate the matrix inverse $(A - \lambda \mathbb{1}_n)^{-1}$. Instead, we used an "eigenvector expansion" and some special properties of the λ-dependent matrix $A - \lambda \mathbb{1}_n$. However, if we were faced with a more general matrix-vector problem, like the classical $A\mathbf{x} = \mathbf{y}$, an eigenvector expansion may not always be possible.

Similarly, in the previous section we solved a BVP with ODE part $\mathcal{S}_r f - \lambda f = h$ using an eigenseries. This left unresolved the delicate question of the convergence of this formal eigenseries — a question that haunted us in our use of Fourier series to solve ODEs and PDEs. We might wonder if there is another way of solving such a BVP, more akin to a direct "matrix inverse" than an "eigenvector (eigenseries) expansion." In other words, given $\lambda \in \mathbb{R}$, does there exist a linear operator $\mathcal{T}_\lambda$ such that putting $f = \mathcal{T}_\lambda h$ solves (2.7.37)? That is, does the operator $\mathcal{S}_r$ from (2.7.13) have an inverse from (some subspace of) $\mathcal{C}([a, b])$ to the domain (2.7.34)? And can we write this inverse more transparently than the eigenseries expansion (2.7.43)?

Happily, the answer is yes to both questions. In this section it will be advantageous (or, at least, no more difficult), to study BVPs whose ODE component is not necessarily in Sturm-Liouville form (2.7.12) but rather is a general second-order ODE like

$$\Phi(x)f'' + \Psi(x)f' + \Theta(x)f = h(x),$$

with $\Phi(x) \neq 0$ at all x. We can recover results for a Sturm-Liouville BVP, of course, by specializing to $\Psi(x) = \Phi'(x)$ and $\Theta(x) = q(x) - \lambda r(x)$. We discuss some concerns with this specialization further in Remark 2.7.20.

2.7.17 Example.

Solve the BVP

$$\begin{cases} f'' = h(x), \ 0 \leq x \leq 1 \\ f(0) = f(1) = 0. \end{cases}$$

Solution. Integrating twice, we have

$$f(x) = \int_0^x H(s) \, ds + c_1 x + c_2, \qquad H(s) := \int_0^s h(\xi) \, d\xi.$$

We try to find c_1 and c_2 to meet the boundary conditions: we want $0 = f(0) = c_2$, so $f(x) = \int_0^x H(s) \, ds + c_1 x$, and then we want

$$0 = f(1) = \int_0^1 H(s) \, ds + c_1,$$

so

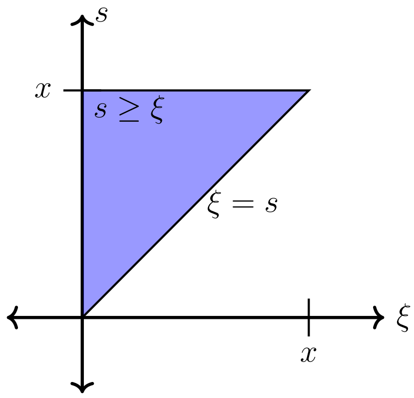
$$c_1 = -\int_0^1 \int_0^s h(\xi) \, d\xi \, ds.$$

Thus

$$f(x) = \int_0^x \int_0^s h(\xi) \, d\xi \, ds - x \int_0^1 \int_0^s h(\xi) \, d\xi \, ds.$$

This is a perfectly valid formula for the solution f, but we can eliminate some of the integrals by changing the order of integration. If we sketch the region $0 \le s \le x$, $0 \le \xi \le s$ as below, we see that

$$\int_0^x \int_0^s h(\xi)\ d\xi\ ds = \int_0^x \int_\xi^x h(\xi)\ ds\ d\xi = \int_0^x h(\xi) \left(\int_\xi^x ds \right)\ d\xi = \int_0^x (x-\xi)h(\xi)\ d\xi.$$



Taking $x = 1$, we likewise compute

$$\int_0^1 \int_0^s h(\xi)\ d\xi\ ds = \int_0^1 (1-\xi)h(\xi)\ d\xi.$$

Thus

$$f(x) = \int_0^x (x-\xi)h(\xi)\ d\xi - x\int_0^1 (1-\xi)h(\xi)\ d\xi.$$

Let us rewrite this as four integrals:

$$f(x) = \underbrace{x\int_0^x \xi h(\xi)\ d\xi}_{\mathcal{I}_1(x)} - \underbrace{\int_0^x \xi h(\xi)\ d\xi}_{\mathcal{I}_2(x)} - \underbrace{x\int_0^1 h(\xi)\ d\xi}_{\mathcal{I}_3(x)} + \underbrace{x\int_0^1 \xi h(\xi)\ d\xi}_{\mathcal{I}_4(x)}. \tag{2.7.47}$$

Observe that

$$\mathcal{I}_1(x) - \mathcal{I}_3(x) = -x\int_x^1 h(\xi)\ d\xi \tag{2.7.48}$$

and

$$\mathcal{I}_4(x) = x\int_0^x \xi h(\xi)\ d\xi + x\int_x^1 \xi h(\xi)\ d\xi. \tag{2.7.49}$$

Substituting (2.7.48) and (2.7.49) into (2.7.47), we find

$$\begin{aligned} f(x) = -x\int_x^1 h(\xi)\ d\xi - \int_0^x \xi h(\xi)\ d\xi + x\int_0^x \xi h(\xi)\ d\xi + x\int_x^1 \xi h(\xi)\ d\xi \\ = \int_0^x \xi(x-1)h(\xi)\ d\xi + \int_x^1 x(\xi-1)h(\xi)\ d\xi. \end{aligned} \tag{2.7.50}$$

Now define

$$\mathcal{G}(x,\xi) := \begin{cases} \xi(x-1), & 0 \le \xi \le x \le 1 \\ x(\xi-1), & 0 \le x \le \xi \le 1. \end{cases}$$

Then (2.7.50) compresses to

$$f(x) = \int_0^1 \mathcal{G}(x,\xi)h(\xi)\,d\xi =: (\mathcal{T}h)(x). \tag{2.7.51}$$

Thus the solution to our BVP is given by the integral operator $\mathcal{T}$, with kernel $\mathcal{G}$, acting on the function h. Here are some properties of this kernel $\mathcal{G}$.

• $\mathcal{G}$ is continuous on $[0,1]\times[0,1]$. This is (hopefully) obvious just from inspection for $x \neq \xi$.

• $\mathcal{G}$ is symmetric in the sense that $\mathcal{G}(x,\xi) = \mathcal{G}(\xi,x)$. This is perhaps best seen by switching to different dummy variables for $\mathcal{G}$ and writing instead

$$\mathcal{G}(u,v) = \begin{cases} v(u-1), & 0 \le v \le u \le 1 \\ u(v-1), & 0 \le u \le v \le 1. \end{cases}$$

This will prevent x and ξ from working overtime. Then for given x, $\xi \in [0,1]$, if $\xi \le x$, then putting $u = x$ and $v = \xi$ gives $G(x,\xi) = \xi(x-1)$, while putting $u = \xi$ and $v = x$ gives $\mathcal{G}(\xi,x) = \xi(x-1)$. Similarly, if $x \le \xi$, then take $u = x$ and $v = \xi$ to find $\mathcal{G}(x,\xi) = x(\xi-1)$, and, last, take $u = \xi$ and $v = x$ fo obtain $\mathcal{G}(\xi,x) = \xi(x-1)$.

• The partial derivative $\partial_x\mathcal{G}(x,\xi)$ exists except along the line $x = \xi$, where, for a fixed $\xi \in [0,1]$, we have

$$\lim_{x\to\xi^-} \partial_x\mathcal{G}(x,\xi) = \xi - 1 \quad \text{while} \quad \lim_{x\to\xi^+} \partial_x\mathcal{G}(x,\xi) = \xi.$$

Thus the "jump" in $\partial_x\mathcal{G}(\cdot,\xi)$ at $x = \xi$ is

$$\lim_{x\to\xi^+} \partial_x\mathcal{G}(x,\xi) - \lim_{x\to\xi^-} \partial_x\mathcal{G}(x,\xi) = 1.$$

• The partial derivative $\partial_x^2\mathcal{G}(x,\xi)$ also exists except for $x = \xi$, in which case $\partial_x^2\mathcal{G}(x,\xi) = 0$. That is, $\mathcal{G}(\cdot,\xi)$ satisfies the homogeneous form $f'' = 0$ of our original (toy) ODE. ▲

The form of the inverse $\mathcal{T}$ in (2.7.51) is unsurprising: we are seeking to solve a differential equation, and integrals tend to be inverse of derivatives. To prove more generally the existence of such an inverse integral operator, first we need a technical lemma. Roughly, this shows that, under certain conditions (which in the Sturm-Liouville case correspond to not having 0 as an eigenvalue of the BVP), there exist solutions to precisely "half" of the BVP with some additional nice properties.

2.7.18 Lemma.

Let Φ, Ψ, $\Theta \in \mathcal{C}([a,b])$ with $\Phi(x) \neq 0$ for all $x \in [a,b]$. Let α_0, α_1, β_0, $\beta_1 \in \mathbb{C}$ with at least one of α_0, α_1 and at least one of β_0, β_1 nonzero. Suppose that the only solution to the BVP

$$\begin{cases} \Phi(x)f'' + \Psi(x)f' + \Theta(x)f = 0, & a \le x \le b \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \beta_0 f(b) + \beta_1 f'(b) = 0, \end{cases} \tag{2.7.52}$$

is $f = 0$. Then there exist functions $\mathfrak{g}_1$, $\mathfrak{g}_2 \in \mathcal{C}^2([a, b])$ that solve the "half" BVPs

$$\begin{cases} \Phi(x)\mathfrak{g}_1'' + \Psi(x)\mathfrak{g}_1' + \Theta(x)\mathfrak{g}_1 = 0, \ a \le x \le b \\ \alpha_0\mathfrak{g}_1(a) + \alpha_1\mathfrak{g}_1'(a) = 0 \\ \beta_0\mathfrak{g}_1(b) + \beta_1\mathfrak{g}_1'(b) \neq 0. \end{cases} \tag{2.7.53}$$

and

$$\begin{cases} \Phi(x)\mathfrak{g}_2'' + \Psi(x)\mathfrak{g}_2' + \Theta(x)\mathfrak{g}_2 = 0, \ a \le x \le b \\ \alpha_0\mathfrak{g}_2(a) + \alpha_1\mathfrak{g}_2'(a) \neq 0 \\ \beta_0\mathfrak{g}_2(b) + \beta_1\mathfrak{g}_2'(b) = 0. \end{cases} \tag{2.7.54}$$

Moreover, any pair of functions satisfying (2.7.53) *and* (2.7.54) *are linearly independent.*

Proof. Let $\mathcal{A}f = \Phi(x)f'' + \Psi(x)f' + \Theta(x)f$. Choose numbers μ_0, μ_1, γ_0, and γ_1 with $\mu_0^2 + \mu_1^2 \neq 0$ and $\gamma_0^2 + \gamma_1^2 \neq 0$ and

$$\alpha_0\mu_0 + \alpha_1\mu_1 = \beta_0\gamma_0 + \beta_1\gamma_1 = 0. \tag{2.7.55}$$

(One can construct these numbers explicitly depending on which of α_0 and α_1 is nonzero, and the same for β_0 and β_1, or one could shoot for overkill and use the fact that $\mathbb{C}$ is a one-dimensional vector space over $\mathbb{C}$.)

Now let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ satisfy the IVPs[45]

$$\begin{cases} \mathcal{A}\mathfrak{g}_1 = 0, \ a \le x \le b \\ \mathfrak{g}_1(a) = \mu_0 \\ \mathfrak{g}_1'(a) = \mu_1 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{A}\mathfrak{g}_2 = 0, \ a \le x \le b \\ \mathfrak{g}_2(b) = \gamma_0 \\ \mathfrak{g}_2'(b) = \gamma_1. \end{cases}$$

This is possible since $\mathcal{A}$ has continuous coefficients on $[a, b]$ and $\Phi(x) \neq 0$ for all x. Since μ_0 and μ_1 are not both zero, we remark that $\mathfrak{g}_1$ is not identically zero, and likewise for $\mathfrak{g}_2$. Now we compute

$$\alpha_0\mathfrak{g}_1(a) + \alpha_1\mathfrak{g}_1'(a) = \alpha_0\mu_0 + \alpha_1\mu_1 = 0 \quad \text{and} \quad \beta_0\mathfrak{g}_2(b) + \beta_1\mathfrak{g}_2'(b) = \beta_0\gamma_0 + \beta_1\gamma_1 = 0.$$

If $\beta_0\mathfrak{g}_1(b) + \beta_1\mathfrak{g}_1'(b) = 0$, then $\mathfrak{g}_1$ is a nontrivial solution to the homogeneous BVP (2.7.52), a contradiction. Likewise, if $\alpha_0\mathfrak{g}_2(a) + \alpha_1\mathfrak{g}_2'(a)$, then $\mathfrak{g}_2$ is a nontrivial solution to this BVP. Last, if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are linearly dependent, then there is a constant λ such that either $\mathfrak{g}_1(x) = \lambda\mathfrak{g}_2(x)$ for all x in $[a, b]$ or $\mathfrak{g}_2(x) = \lambda\mathfrak{g}_1(x)$ for all x. In the first case,

$$\beta_0\mathfrak{g}_1(b) + \beta_1\mathfrak{g}_1'(b) = \lambda\big(\beta_0\mathfrak{g}_2(b) + \beta_1\mathfrak{g}_2'(b)\big) = 0,$$

a contradiction to (2.7.53), and a contradiction likewise arises in the second. ■

Since $\mathfrak{g}_1$ solves the ODE part of the BVP and the boundary conditions at a, but not at b, and since $\mathfrak{g}_2$ also solves the ODE part and the boundary conditions at b, but not at a, we think of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as solving exactly "half" the BVP each.

[45] Recall that it does not matter in an IVP whether the "initial value" point is the left or right endpoint of the interval under consideration, or neither — what *does* matter is that the value of the solution and its derivative are taken at the *same* point in the interval.

2.7.19 Theorem.

Suppose Φ, Ψ,$\Theta \in \mathcal{C}([a,b])$ with $\Phi(x) \neq 0$ for all x. Let α_0, α_1, β_0, $\beta_1 \in \mathbb{C}$ with $(\alpha_0^2+\alpha_1^2)(\beta_0^2+\beta_1^2) \neq 0$. Suppose that the only solution to the BVP

$$\begin{cases} \Phi(x)f''+\Psi(x)f'+\Theta(x)f=0, \ a \le x \le b \\ \alpha_0 f(a)+\alpha_1 f'(a)=0 \\ \beta_0 f(b)+\beta_1 f'(b)=0. \end{cases}$$

is the trivial solution $f=0$. Then for any $h \in \mathcal{C}([a,b])$, the unique solution to the nonhomogeneous BVP

$$\begin{cases} \Phi(x)f''+\Psi(x)f'+\Theta(x)f=h(x), \ a \le x \le b \\ \alpha_0 f(a)+\alpha_1 f'(a)=0 \\ \beta_0 f(b)+\beta_1 f'(b)=0. \end{cases} \tag{2.7.56}$$

is

$$f(x) := \int_a^b \mathcal{G}(x,\xi)h(\xi)\ d\xi, \tag{2.7.57}$$

where

$$\mathcal{G}(x,\xi) := \begin{cases} \dfrac{\mathfrak{g}_1(\xi)\mathfrak{g}_2(x)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}, \ a \le \xi \le x \le b \\[2ex] \dfrac{\mathfrak{g}_1(x)\mathfrak{g}_2(\xi)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}, \ a \le x \le \xi \le b \end{cases} \tag{2.7.58}$$

and $\mathfrak{g}_1$, $\mathfrak{g}_2$ are the functions constructed in Lemma 2.7.18. The function $\mathcal{G}$ is the[46] **Green's function** *for the BVP (2.7.56) and it satisfies the following additional properties.*

(i) *$\mathcal{G}\colon [a,b]\times[a,b] \to \mathbb{R}$ is continuous.*

(ii) *$\partial_x\mathcal{G}$ is continuous on $[a,b]\times[a,b]$ except on the line $x=\xi$, where*

$$\lim_{x\to\xi^+}\partial_x\mathcal{G}(x,\xi) - \lim_{x\to\xi^-}\partial_x\mathcal{G}(x,\xi) = \frac{1}{p(\xi)}.$$

(iii) *$\mathcal{G}(\cdot,\xi)$ satisfies the homogeneous version of the ODE in (2.7.56) for $x \neq \xi$:*

$$\Phi(x)\partial_x^2\mathcal{G}(x,\xi)+\Psi(x)\partial_x\mathcal{G}(x,\xi)+\Theta(x)\mathcal{G}(x,\xi)=0.$$

(iv) *$\mathcal{G}$ is symmetric: $\mathcal{G}(x,\xi)=\mathcal{G}(\xi,x)$ for all $a \le x,\xi \le b$.*

[46]Typically in English one does not use the phrase "(in)definite article + possessive adjective + noun," but rather either "possessive adjective + noun" or "(in)definite article + noun." For example, we say "in complex analysis, Cauchy's integral theorem [Theorem 3.6.13] is a powerful tool" or "in complex analysis, the Cauchy integral theorem is a powerful tool" but not "in complex analysis, the Cauchy's integral theorem is a powerful tool" or "in complex analysis, Cauchy integral theorem is a powerful tool." (Virtually all countable singular nouns in English must be paired with an article.) Our term for the function $\mathcal{G}$ here is something of an exception; [30] contains an amusing statistical analysis of usage trends in publications and argues that keeping the article emphasizes how the function $\mathcal{G}$ varies from

Proof. We defer the proofs of properties (i) through (iv) of $\mathcal{G}$ to the exercises. All of these properties are generalizations of the concrete results of Example 2.7.17.

Here we provide a constructive proof that the solution to (2.7.56) is given by the integral operator (2.7.57) with kernel (2.7.58).

Any solution f to the BVP necessarily solves the ODE

$$f'' + \frac{\Psi(x)}{\Phi(x)} f' + \frac{\Theta(x)}{\Phi(x)} y = \frac{h(x)}{\Phi(x)}, \;\; a < x < b,$$

and therefore has the form

$$f(x) = c_1 \mathfrak{g}_1(x) + c_2 \mathfrak{g}_2(x) + \mathfrak{g}_2(x) \int_a^x \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \mathfrak{g}_1(x) \int_a^x \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi, \tag{2.7.59}$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the linearly independent solutions of

$$f'' + \frac{\Psi(x)}{\Phi(x)} f' + \frac{\Theta(x)}{\Phi(x)} f = 0.$$

from Lemma 2.7.18.

Now we want to choose c_1 and c_2 so that f satisfies the boundary conditions. We compute

$$f(a) = c_1 \mathfrak{g}_1(a) + c_2 \mathfrak{g}_2(a)$$

$$f(b) = c_1 \mathfrak{g}_1(b) + c_2 \mathfrak{g}_2(b) + \mathfrak{g}_2(b) \int_a^b \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \mathfrak{g}_1(b) \int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi$$

$$f'(x) = c_1 \mathfrak{g}_1'(x) + c_2 \mathfrak{g}_2'(x) + \mathfrak{g}_2'(x) \int_a^x \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi + \frac{\mathfrak{g}_2(x) \mathfrak{g}_1(x) f(x)}{\Phi(x) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](x)}$$

$$- \mathfrak{g}_1'(x) \int_a^x \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \frac{\mathfrak{g}_1(x) \mathfrak{g}_2(x) f(x)}{\Phi(x) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](x)}$$

$$= c_1 \mathfrak{g}_1'(x) + c_2 \mathfrak{g}_2'(x) + \mathfrak{g}_2'(x) \int_a^x \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \mathfrak{g}_1'(x) \int_a^x \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi$$

$$f'(a) = c_1 \mathfrak{g}_1'(a) + c_2 \mathfrak{g}_2'(a)$$

$$f'(b) = c_1 \mathfrak{g}_1'(b) + c_2 \mathfrak{g}_2'(b) + \mathfrak{g}_2'(b) \int_a^b \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \mathfrak{g}_1'(b) \int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi.$$

So, we need

$$0 = \alpha_1 f(a) + \alpha_2 f'(a)$$

problem to problem — unlike the more universal properties of the integral to which Cauchy's name is attached.

$$= \alpha_1 c_1 \mathfrak{g}_1(a) + \alpha_1 c_2 \mathfrak{g}_2(a) + \alpha_2 c_1 \mathfrak{g}_1'(a) + \alpha_2 c_2 \mathfrak{g}_2'(a)$$

$$= c_1(\alpha_1 \mathfrak{g}_1(a) + \alpha_2 \mathfrak{g}_1'(a)) + c_2(\alpha_1 \mathfrak{g}_2(a) + \alpha_2 \mathfrak{g}_2'(a))$$

$$= c_2(\alpha_1 \mathfrak{g}_2(a) + \alpha_2 \mathfrak{g}_2'(a))$$

and

$$0 = \beta_1 y(b) + \beta_2 f'(b)$$

$$= \beta_1 c_1 \mathfrak{g}_1(b) + \beta_1 c_2 \mathfrak{g}_2(b) + \beta_1 \mathfrak{g}_2(b) \int_a^b \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \beta_1 \mathfrak{g}_1(b) \int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi$$

$$+ \beta_2 c_1 \mathfrak{g}_1'(b) + \beta_2 c_2 \mathfrak{g}_2'(b) + \beta_2 \mathfrak{g}_2'(b) \int_a^b \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \beta_2 \mathfrak{g}_1'(b) \int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi$$

$$= c_1\big(\beta_1 \mathfrak{g}_1(b) + \beta_2 \mathfrak{g}_1'(b)\big) + c_2\big(\beta_1 \mathfrak{g}_2(b) + \beta_2 \mathfrak{g}_2'(b)\big) + \big(\beta_1 \mathfrak{g}_2(b) + \beta_2 \mathfrak{g}_2'(b)\big) \int_a^b \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi$$

$$- \big(\beta_1 \mathfrak{g}_1(b) + \beta_2 \mathfrak{g}_1'(b)\big) \int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi$$

$$= \big(\beta_1 \mathfrak{g}_1(b) + \beta_2 \mathfrak{g}_1'(b)\big) \left(c_1 - \int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi \right).$$

Lemma 2.7.18 ensures

$$\alpha_1 \mathfrak{g}_2(a) + \alpha_2 \mathfrak{g}_2'(a) \neq 0 \quad \text{and} \quad \beta_1 \mathfrak{g}_1(b) + \beta_2 \mathfrak{g}_1'(b) \neq 0,$$

so we should take

$$c_1 = \int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi \quad \text{and} \quad c_2 = 0.$$

Then the form of f from (2.7.59) implies

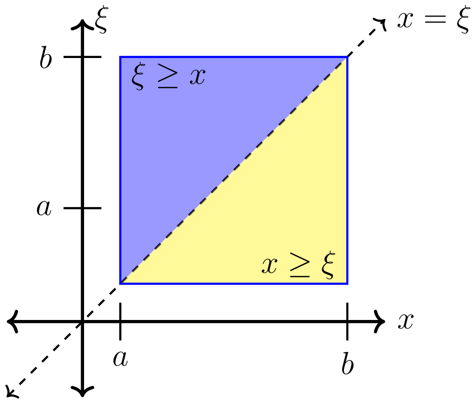
$$\begin{aligned} f(x) = \mathfrak{g}_1(x) &\int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi + \mathfrak{g}_2(x) \int_a^x \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi \\ - \mathfrak{g}_1(x) &\int_a^x \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi \\ = \mathfrak{g}_1(x) &\left(\int_a^b \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi - \int_a^x \frac{\mathfrak{g}_2(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi \right) \\ + \mathfrak{g}_2(x) &\int_a^x \frac{\mathfrak{g}_1(\xi) h(\xi)}{\Phi(\xi) \mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](\xi)} \, d\xi \end{aligned}$$

$$= \mathfrak{g}_1(x)\int_x^b \frac{\mathfrak{g}_2(\xi)h(\xi)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}\,d\xi + \mathfrak{g}_2(x)\int_a^x \frac{\mathfrak{g}_1(\xi)h(\xi)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}\,d\xi$$

$$= \int_a^x \frac{\mathfrak{g}_1(\xi)\mathfrak{g}_2(x)h(\xi)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}\,d\xi + \int_x^b \frac{\mathfrak{g}_1(x)\mathfrak{g}_2(\xi)h(\xi)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}\,d\xi$$

$$= \int_a^b \mathcal{G}(x,\xi)h(\xi)\,d\xi,$$

where for the last line we are using the piecewise definition of $\mathcal{G}$ in (2.7.58). ∎



2.7.20 Remark.

Note carefully that a Green's function depends on the coefficients of the ODE part of the BVP. In particular, if one wishes to find a Green's function for a Sturm-Liouville BVP whose ODE part is given by

$$\partial_x[p(x)f'] + q(x)f = \lambda r(x)f + h(x), \tag{2.7.60}$$

then that Green's function will depend on λ*. Moreover, the assumption in Lemma 2.7.18 and Theorem 2.7.19 that the homogeneous BVP have only the trivial solution manifests itself here by demanding that* λ *not be an eigenvalue of the BVP with ODE part* (2.7.60)*.*

2.7.21 Example.

Find the Green's function for the BVP

$$\begin{cases} f'' + f = h(x), \ 0 \le x \le 1 \\ f(0) = f(1) = 0. \end{cases}$$

Solution. First we check that the homogeneous BVP has no nontrivial solutions. If $f'' + f = 0$, then $f(x) = c_1\cos(x) + c_2\sin(x)$ for some c_1, $c_2 \in \mathbb{C}$. The condition $f(0) = 0$ forces $c_1 = 0$, and then $f(x) = c_2\sin(x)$. Then the condition $f(1) = 0$ forces $c_1\sin(1) = 0$, so $c_1 = 0$.

Next we construct the auxiliary functions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ to satisfy the conditions of Lemma 2.7.18. We want to find $\mathfrak{g}_1$ and $\mathfrak{g}_2$ solving the "half" BVPs

$$\begin{cases} \mathfrak{g}_1'' + \mathfrak{g}_1 = 0 \\ \mathfrak{g}_1(0) = 0, \ \mathfrak{g}_1(1) \neq 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathfrak{g}_2'' + \mathfrak{g}_2 = 0 \\ \mathfrak{g}_2(0) \neq 0, \ \mathfrak{g}_2(1) = 0. \end{cases}$$

For $\mathfrak{g}_1$, we start with the general form $\mathfrak{g}_1(x) = c_1\cos(x) + c_2\sin(x)$ and need $0 = \mathfrak{g}_1(0) = c_1$. Thus $\mathfrak{g}_1(x) = c_2\sin(x)$, and since $\sin(1) \neq 0$, we take $c_2 = 1$, so $\mathfrak{g}_1(x) = \sin(x)$.

For $\mathfrak{g}_2$, we again start with $\mathfrak{g}_2(x) = c_1\cos(x) + c_2\sin(x)$, where

$$c_1\cos(1) + c_2\sin(1) = \mathfrak{g}_2(1) = 0 \quad \text{and} \quad c_1 = \mathfrak{g}_2(0) \neq 0.$$

Since c_2 is not as clearly specified, let us solve for that coefficient:

$$c_2 = -\frac{c_1\cos(1)}{\sin(1)},$$

which we may do, since $\sin(1) \neq 0$. Then we can take c_1 to be any nonzero number, say, $c_1 = -\sin(1)$. This gives $c_2 = \cos(1)$ and

$$\mathfrak{g}_2(x) = -\sin(1)\cos(x) + \cos(1)\sin(x) = \sin(x-1),$$

after a trig addition formula.

Last, the Wronskian of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is

$$\mathcal{W}[\mathfrak{g}_1, \mathfrak{g}_2](x) = \sin(x)\cos(x-1) - \cos(x)\sin(x-1) = \sin(1)$$

after more trig addition. By (2.7.58), the Green's function is

$$\mathcal{G}(x,\xi) = \begin{cases} \dfrac{\sin(\xi)\sin(x-1)}{\sin(1)}, & 0 \le \xi \le x \le 1 \\[2ex] \dfrac{\sin(x)\sin(\xi-1)}{\sin(1)}, & 0 \le x \le \xi \le 1. \end{cases}$$

▲

2.7.22 Method: solve a nonhomogeneous BVP with Green's functions

1. Write the BVP in the form

$$\begin{cases} \mathcal{A}f = h(x), \ a \le x \le b \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \beta_0 f(b) + \beta_1 f'(b) = 0. \end{cases}$$

0. Check that the homogeneous version of the BVP ($h(x) = 0$) has no nontrivial solutions; otherwise, this method will not work.

1. Solve the "half" BVPs

$$\begin{cases} \mathcal{A}\mathfrak{g}_1 = 0, \ a \le x \le b \\ \alpha_0\mathfrak{g}_1(a) + \alpha_1\mathfrak{g}_1'(a) = 0 \\ \beta_0\mathfrak{g}_1(b) + \beta_1\mathfrak{g}_1'(b) \neq 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{A}\mathfrak{g}_2 = 0, \ a \le x \le b \\ \alpha_0\mathfrak{g}_2(a) + \alpha_1\mathfrak{g}_2'(a) \neq 0 \\ \beta_0\mathfrak{g}_2(b) + \beta_1\mathfrak{g}_2'(b) = 0. \end{cases}$$

It is always possible to solve these BVPs; one way to start is to find a fundamental solution set for $\mathcal{A}$ and then select the coefficients to meet/not to meet the boundary conditions.

2. The Green's function is

$$\mathcal{G}(x,\xi) := \begin{cases} \dfrac{\mathfrak{g}_1(\xi)\mathfrak{g}_2(x)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}, \; a \le \xi \le x \le b \\ \\ \dfrac{\mathfrak{g}_1(x)\mathfrak{g}_2(\xi)}{\Phi(\xi)\mathcal{W}[\mathfrak{g}_1,\mathfrak{g}_2](\xi)}, \; a \le x \le \xi \le b, \end{cases}$$

where $\mathcal{W}$ is, as usual, the Wronskian from $(1.4.8)$. The solution to the BVP is

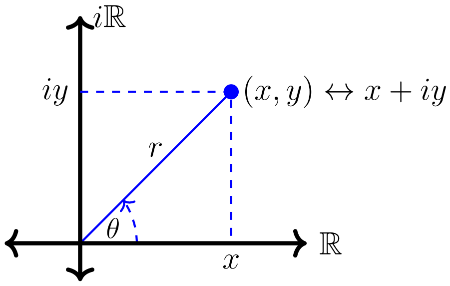
$$f(x) = \int_a^b \mathcal{G}(x,\xi)h(\xi)\; d\xi.$$

3. Complex Analysis

We presume familiarity with the contents of Appendix A. Broadly, this treatment of complex analysis follows [11, 12] with further references to [1, 2, 22]. Most omitted proofs appear in [12].

3.1. Polar coordinates.

Continuing our identification of $\mathbb{C}$ with $\mathbb{R}^2$, we recall that we can write a point $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ in polar coordinates as $x = r\cos(\theta)$ and $y = r\sin(\theta)$, where $r = \sqrt{x^2+y^2}$, and θ, informally, is the "angle that the line segment from $(0,0)$ to (x,y) makes with the positive x-axis." The 2π-periodicity of $\sin(\cdot)$ and $\cos(\cdot)$ means that θ is never unique.



Given $z \in \mathbb{C} \setminus \{0\}$, we find the polar coordinates of $(\text{Re}(z), \text{Im}(z)) \in \mathbb{R}^2 \setminus \{(0,0)\}$ as $\text{Re}(z) = r\cos(\theta)$ and $\text{Im}(z) = r\sin(\theta)$, where

$$r = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2} = |z|.$$

Then

$$z = |z|\cos(\theta) + i|z|\sin(\theta) = |z|(\cos(\theta) + i\sin(\theta)). \tag{3.1.1}$$

3.1.1 Example.

Write $z = 1 + i$ in polar coordinates.

Solution. We recognize $1+i$ as $(1,1)$ in $\mathbb{R}^2$, so $|1+i| = \sqrt{2}$ and trigonometry tells us that the line segment from the origin to $(1,1)$ makes an angle of $\pi/4$ with the positive real axis. Thus

$$1 + i = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right].$$

Of course, we could easily check our work by evaluating the trig functions above at $\theta = \pi/4$. ▲

The angle θ from the polar form of a complex number is, as discussed above in the context of $\mathbb{R}^2$, not unique.

3.1.2 Definition.

Given $z \in \mathbb{C} \setminus \{0\}$, any number $\theta \in \mathbb{R}$ such that

$$z = |z|(\cos(\theta) + i\sin(\theta))$$

is an **ARGUMENT** *of z, and we write*

$$\arg(z) = \{\theta \in \mathbb{R} \mid z = |z|(\cos(\theta) + i\sin(\theta))\}.$$

It is common to abuse notation and **not** *write* $\arg(z)$ *in set-builder notation but just* $\arg(z) = \theta_0 + 2\pi k$*, where* θ_0 *is one fixed argument of* z *and* k *is understood to be an arbitrary integer.*

The symbol $\arg(z)$ is not a single real number but rather a set; the pairing $z \mapsto \arg(z)$ is sometimes called a "multi-valued function" because the output $\arg(z)$ contains infinitely many numbers. In particular, if $\theta \in \arg(z)$, then $\theta + 2\pi k \in \arg(z)$ for any $k \in \mathbb{Z}$. While representing a complex number in its polar form will be very convenient, the ambiguity of $\arg(z)$ requires a certain caution in using this representation. This tension between *utility* and *ambiguity* will resurface whenever we use the argument to define some quantity. We can make the argument unique by requiring it to lie in an interval of width 2π, which is referred to as selecting a **BRANCH** of the argument.

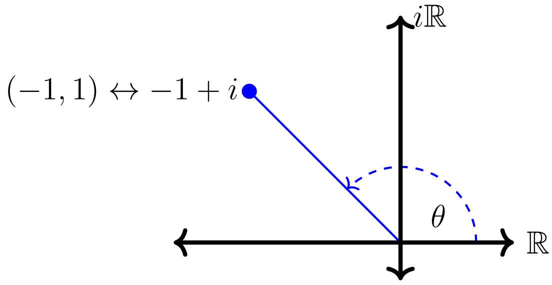
3.1.3 Definition.

The **PRINCIPAL ARGUMENT** *or* **PRINCIPAL BRANCH OF THE ARGUMENT** *of* $z \in \mathbb{C} \setminus \{0\}$ *is the unique number* $\Theta \in (-\pi, \pi]$ *such that* $z = |z|(\cos(\Theta) + i\sin(\Theta))$*. We write* $\Theta = \operatorname{Arg}(z)$*. Note that some treatments require the principal argument to belong to a different interval, such as* $[-\pi, \pi)$*.*

3.1.4 Example.

Calculate $\operatorname{Arg}(-1+i)$ *and* $\arg(-1+i)$*.*

Solution. We plot the point $1 + i \leftrightarrow (-1, 1)$ in $\mathbb{R}^2$.



We see that $\operatorname{Arg}(-1+i) = \pi/2 + \pi/4 = 3\pi/4$, and so $\arg(-1+i) = 3\pi/4 + 2\pi k$. ▲

3.2. Functions of a complex variable.

Let $\mathcal{D} \subseteq \mathbb{C}$. A complex-valued function f on $\mathcal{D}$ is, of course, a pairing of any $z \in \mathcal{D}$ with exactly one $f(z) \in \mathbb{C}$. For example, the pairing $z \mapsto \operatorname{Arg}(z)$ is a well-defined function on $\mathbb{C} \setminus \{0\}$, but, unless we choose a branch of the argument, the assignment $z \mapsto \arg(z)$ is not a function. By writing a function as the sum of its real and imaginary parts, we can consider a function of a complex variable as the linear combination of two functions of real variables:

$$f(x+iy) = \underbrace{\mathrm{Re}[f(x+iy)]}_{u(x,y)} + \underbrace{i\,\mathrm{Im}[f(x+iy)]}_{iv(x,y)}. \tag{3.2.1}$$

We are particularly interested in extending the definitions of familiar real transcendental functions — the exponential, the logarithm, the trigonometric functions — to complex inputs. Broadly, an **EXTENSION** of a function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is a function $\widetilde{f}\colon \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$ for some domain $\mathcal{D} \subseteq \mathbb{C}$, with the property that $\widetilde{f}(x) = f(x)$ for all $x \in I \cap \mathcal{D}$. That is, an extension of f is just f when evaluated on the original domain of f. Of course, there are countless useless ways extend a function — just put $\widetilde{f}(z) = 0$ for $z \in \mathbb{C} \setminus I$, and that extends any function on $I \subseteq \mathbb{R}$ to all of $\mathbb{C}$. Rather, we are interested in *natural* extensions — ones that preserve fundamental mapping and calculus properties of the original functions as much as possible.

3.2.1. Properties of the complex exponential.

We recall from Appendix A.3 the definition of the complex exponential as a power series:

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{3.2.2}$$

and Euler's formula:

$$e^{ix} = \cos(x) + i\sin(x), \ \ x \in \mathbb{R}.$$

An important consequence of Euler's formula is that for $z = x + iy \in \mathbb{C} \setminus \{0\}$, we have

$$z = |z|\big(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))\big) = |z|e^{i\,\mathrm{Arg}(z)}. \tag{3.2.3}$$

We will typically write the polar form of a complex number in this way from now on.

Before stating our next, large theorem on various properties of the complex exponential, we recall some facts about the natural logarithm from calculus.

3.2.1 Lemma (Existence of natural logarithm).

There exists a map $\ln\colon (0,\infty) \to \mathbb{R}$ *such that*

$$\ln(e^x) = x \text{ for all } x \in \mathbb{R} \quad \text{and} \quad e^{\ln(x)} = x \text{ for all } x > 0.$$

This map $\ln(\cdot)$ *is the* **NATURAL LOGARITHM**. *It has the following additional properties.*

(i) $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$ *for all* $x_1,\, x_2 > 0$*;*

(ii) $\ln(a^x) = x\ln(a)$ *for all* $a > 0$ *and* $x \in \mathbb{R}$*;*

(iii) $\ln(1) = 0$ *and* $\ln(e) = 1$*;*

(iv) $\partial_x[\ln(\cdot)](x) = \dfrac{1}{x}$ *and* $\ln(x) = \displaystyle\int_1^x \frac{d\xi}{\xi}$ *for all* $x > 0$.

3.2.2 Theorem (Mapping properties of the complex exponential).

(i) $e^{z+w} = e^z e^w$ *for all* $z, w \in \mathbb{C}$.

(ii) *Let* $z = x + iy \in \mathbb{C}$. *Then*

$$e^z = e^x(\cos(y) + i\sin(y)), \tag{3.2.4}$$

so that

$$|e^z| = e^{\mathrm{Re}(z)} \quad \text{and} \quad \arg(e^z) = \mathrm{Im}(z) + 2\pi k,\ k \in \mathbb{Z}. \tag{3.2.5}$$

(iii) $e^z = 1$ *if and only if* $z = 2\pi ik$ *for some* $k \in \mathbb{Z}$. *In particular, the mapping* $z \mapsto e^z$ *is not one-to-one on* $\mathbb{C}$.

(iv) $e^z = e^w$ *if and only if* $z = w + 2\pi ik$ *for some* $k \in \mathbb{Z}$

(v) $e^z = e^{z+2\pi i}$ *for all* $z \in \mathbb{C}$. *In particular, the complex exponential is not one-to-one on* $\mathbb{C}$.

(vi) *The complex exponential maps* $\mathbb{C}$ *onto* $\mathbb{C} \setminus \{0\}$. *That is, for any* $w \in \mathbb{C} \setminus \{0\}$, *there is* $z \in \mathbb{C}$ *such that* $e^z = w$. *Specifically,*

$$e^z = w \iff z = \ln(|w|) + i\,\mathrm{Arg}(w) + 2\pi ik \text{ for some } k \in \mathbb{Z}.$$

Moreover, $e^z \neq 0$ *for all* $z \in \mathbb{C}$.

(vii) *The complex exponential maps strips of the form* $y_0 < \mathrm{Im}(z) \le y_0 + 2\pi$ *onto* $\mathbb{C} \setminus \{0\}$ *in a one-to-one manner. That is, if we fix* $y_0 \in \mathbb{R}$ *and assume* $z_1, z_2 \in \mathbb{C}$ *with* $y_0 < \mathrm{Im}(z_k) \le y_0 + 2\pi$, $k = 1, 2$ *and* $e^{z_1} = e^{z_2}$, *then* $z_1 = z_2$. *Conversely, if* $w \in \mathbb{C} \setminus \{0\}$, *there is* $z \in \mathbb{C}$ *with* $e^z = w$ *and* $y_0 < \mathrm{Im}(z) \le y_0 + 2\pi$.

Proof. (i) One proof is to use the series definition (3.2.2) and the binomial theorem[47] to expand the powers $(z+w)^k$. After some deft rearrangements, the identity[48] follows; we omit the details, as they are not very enlightening for our course.

(ii) The first equality (3.2.4) follows from Euler's formula and part (i). The second equality (3.2.5) is just a restatement of (3.2.4).

(iii) If $k \in \mathbb{Z}$, then $2\pi k \in \mathbb{R}$, and so Theorem A.3.1 gives

$$e^{2\pi ik} = \cos(2\pi k) + i\sin(2\pi k) = 1 + i \cdot 0 = 1.$$

Conversely, suppose $e^z = 1$ with $z = x + iy$. Then $1 = |e^z| = e^{\mathrm{Re}(z)} = e^x$. Hence $x = 0$ since the *real* exponential is one-to-one. So, $z = iy$, and therefore

$$1 = e^z = e^{iy} = \cos(y) + i\sin(y).$$

Taking the imaginary part of this equation, we have $\sin(y) = 0$, so $y = 2\pi k$ for some $k \in \mathbb{Z}$. That is, $z = 2\pi ik$.

[47] $(z+w)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} z^\ell w^{k-\ell} = \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} z^\ell w^{k-\ell}$.

[48] Involving the Cauchy product formula (1.6.4).

(iv) We have $e^z = e^w$ if and only if $e^{z-w} = 1$. Then part (iii) applies.

(v) By part (i), we have $e^{z+2\pi i} = e^z e^{2\pi i}$, and by part (iii) we know $e^{2\pi i} = 1$.

(vi) First, suppose $e^z = 0$ for some $z = x + iy \in \mathbb{C}$. Then $e^x e^{iy} = 0$. Taking the modulus of both sides, we have

$$0 = |e^z| = |e^x e^{iy}| = e^x,$$

a contradiction, since $e^x \neq 0$ for all real x.

Now suppose $e^z = w$ for some $z \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$. Then

$$e^z = w \Longrightarrow |e^z| = |w| \Longrightarrow e^{\mathrm{Re}(z)} = |w| \Longrightarrow \mathrm{Re}(z) = \ln(|w|).$$

Next, suppose $w = |w|e^{i\phi}$, $\phi = \mathrm{Arg}(w)$. Then

$$e^z = w \Longrightarrow e^{\mathrm{Re}(z)} e^{i\,\mathrm{Im}(z)} = |w|e^{i\phi} \Longrightarrow e^{i\,\mathrm{Im}(z)} = e^{i\phi}.$$

Part (iv) then implies $\mathrm{Im}(z) = \phi + 2\pi k$ for some $k \in \mathbb{Z}$. Thus

$$z = \ln(|w|) + \phi + 2\pi k = \ln(|w|) + i\big(\mathrm{Arg}(w) + 2\pi k\big), \ k \in \mathbb{Z}.$$

Conversely, parts (i) and (iii) show

$$e^{\ln(|w|) + i\,\mathrm{Arg}(w) + 2\pi i k} = e^{\ln(|w|) + i\,\mathrm{Arg}(w)} = |w|e^{i\,\mathrm{Arg}(w)} = w.$$

(vii) For $k = 1, 2$, write $z_k = x_k + iy_k$ where $y_0 < y_k \leq y_0 + 2\pi$. The hypothesis $e^{z_1} = e^{z_2}$ implies $e^{z_1 - z_2} = 1$, and part (iii) then implies $z_1 - z_2 = 2\pi i k$ for some integer k. Taking imaginary parts, we have

$$y_1 - y_2 = \mathrm{Im}(z_1 - z_2) = \mathrm{Im}(2\pi i k) = 2\pi k. \tag{3.2.6}$$

But the condition $y_0 < y_k \leq y_0 + 2\pi$ forces $|y_1 - y_2| < 2\pi$, and (3.2.6) implies $|y_1 - y_2| = 2\pi|k|$. This can only be possible if $k = 0$, in which case $y_1 - y_2 = 0$, so $y_1 = y_2$.

That is, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 = x_2 + iy_1$. The hypothesis $e^{z_1} = e^{z_2}$ implies

$$e^{x_1} e^{iy_1} = e^{x_2} e^{iy_1}.$$

Taking the modulus of both sides, we have $|e^{iy_1}| = 1$, and so

$$e^{x_1} = |e^{x_1} e^{iy_1}| = |e^{x_2} e^{iy_1}| = e^{x_2}.$$

Since the *real* exponential is one-to-one, we have $x_1 = x_2$. Thus

$$z_1 = x_1 + iy_1 = x_2 + iy_2 = z_2,$$

and so the complex exponential is one-to-one on the strip $y_0 < \mathrm{Im}(z) \leq y_0 + 2\pi$.

The proof that the exponential maps the strip $y_0 < \mathrm{Im}(z) \leq y_0 + 2\pi$ onto $\mathbb{C} \setminus \{0\}$ is left as an exercise. ■

The complex exponential therefore differs in (at least) two major respects from the real exponential: the complex exponential is not one-to-one, and the complex exponential can take negative values.

3.2.2. The complex logarithm.

A logarithm of a number w should be some number z such that $e^z = w$. Real-variable theory only permits us to define a logarithm when w is positive, and, in that case, the logarithm is unique. However, part (vi) of Theorem 3.2.2 tells us that if $w \in \mathbb{C} \setminus \{0\}$, then

$$e^z = w \iff z = \ln(|w|) + i\,\mathrm{Arg}(w) + 2\pi ik$$

for some $k \in \mathbb{Z}$. Recall that $\arg(w) = \{\mathrm{Arg}(w) + 2\pi k \mid k \in \mathbb{Z}\}$. This leads us to the following definition.

3.2.3 Definition.

For $w \in \mathbb{C} \setminus \{0\}$, the **LOGARITHM** *of w is the multi-valued "function"*

$$\log(w) := \ln(|w|) + i\arg(w). \tag{3.2.7}$$

The **PRINCIPAL BRANCH** *of the logarithm is*

$$\mathrm{Log}(w) := \ln(|w|) + i\,\mathrm{Arg}(w).$$

More generally, a function f defined on some set $\mathcal{D} \subseteq \mathbb{C} \setminus \{0\}$ is a **BRANCH OF THE LOGARITHM ON** *$\mathcal{D}$ if $e^{f(z)} = z$ for all $z \in \mathcal{D}$. We can always define a branch of the logarithm by specifying a branch of $\arg(\cdot)$, i.e., by requiring the values of $\arg(\cdot)$ to lie in an interval of the form $(y_0, y_0 + 2\pi]$.*

3.2.4 Remark.

We will use the notation $\ln(\cdot)$ to refer exclusively to the "real" natural logarithm. The notation $\log(\cdot)$ will only refer to the complex logarithm (3.2.7). We will not discuss real logarithms to any base other than e.

3.2.5 Example.

Find all values of $\log(-i)$.

Solution. First, $|-i| = 1$ and $\mathrm{Arg}(-i) = -\pi/2$. Then

$$\log(-i) = \ln(|-i|) + i\,\mathrm{Arg}(-i) + 2\pi ik = \ln(1) - i\frac{\pi}{2} + 2\pi ik = -i\frac{\pi}{2} + 2\pi ik. \qquad \blacktriangle$$

3.2.6 Example.

Show that $e^{\log(z)} = z$ but $\log(e^z) = \{z + 2\pi ik \mid k \in \mathbb{Z}\}$.

Proof. Since we are not told to work with a particular branch of the logarithm, we really have

$$e^{\log(z)} = \left\{e^{\ln(|z|)+i\,\mathrm{Arg}(z)+2\pi ik} \;\middle|\; k \in \mathbb{Z}\right\}.$$

But for any $k \in \mathbb{Z}$, we have

$$e^{\ln(|z|)+i\,\mathrm{Arg}(z)+2\pi ik} = e^{\ln(|z|)+i\,\mathrm{Arg}(z)}e^{2\pi ik} = e^{\ln(|z|)+i\,\mathrm{Arg}(z)} = e^{\ln(|z|)}e^{i\,\mathrm{Arg}(z)} = |z|e^{i\,\mathrm{Arg}(z)} = z.$$

By definition of the complex log, we have

$$\log(e^z) = \ln(|e^z|) + i\,\mathrm{Arg}(e^z) + 2\pi i k.$$

We know $|e^z| = e^{\mathrm{Re}(z)}$, and so $\ln(|e^z|) = \ln(e^{\mathrm{Re}(z)}) = \mathrm{Re}(z)$. Then using set notation, we really have

$$\log(e^z) = \{\mathrm{Re}(z) + i\,\mathrm{Arg}(e^z) + 2\pi i k \mid k \in \mathbb{Z}\}.$$

Now, recall from part (ii) of Theorem 3.2.2, that $\arg(e^z) = \{\mathrm{Im}(z) + 2\pi\ell \mid \ell \in \mathbb{Z}\}$. Consequently, there is some $\ell_z \in \mathbb{Z}$ such that $\mathrm{Arg}(e^z) = \mathrm{Im}(z) + 2\pi\ell_z$. Thus

$$\begin{aligned}\log(e^z) &= \{\mathrm{Re}(z) + i\,\mathrm{Im}(z) + 2\pi i\ell_z + 2\pi i k \mid k \in \mathbb{Z}\} = \{z + 2\pi i(\ell_z + k) \mid k \in \mathbb{Z}\} \\ &= \{z + 2\pi i j \mid j \in \mathbb{Z}\}.\end{aligned}$$

We get the last equality since $\{\ell_z + k \mid k \in \mathbb{Z}\} = \mathbb{Z}$ whenever $\ell_z \in \mathbb{Z}$. ■

3.2.7 Example.

Given $\alpha \in \mathbb{C}$, determine all (if any) $\lambda \in \mathbb{C}$ for which there exists a nonzero solution to the BVP

$$\begin{cases} f'(x) = \lambda f(x), \ 0 \le x \le 1 \\ f(0) = \alpha f(1). \end{cases} \tag{3.2.8}$$

Solution. Using Theorem 1.2.4, we have

$$f' = \lambda f \iff f' - \lambda f = 0 \iff f(x) = ce^{\lambda x}$$

for some $c \in \mathbb{C}$. We will assume $c \neq 0$, since otherwise $f = 0$. To have $f \in \mathfrak{D}^\alpha(\mathcal{A})$, we also need

$$f(0) = \alpha f(1) \iff c = c\alpha e^\lambda \iff 1 = \alpha e^\lambda$$

since $c \neq 0$. If $\alpha = 0$, then this last equality cannot hold, and there are no nonzero solutions to the BVP. But when $\alpha = 0$, the BVP is really the IVP

$$\begin{cases} f'(x) - \lambda f(x) = 0, \ 0 \le x \le 1 \\ f(0) = 0, \end{cases}$$

and we know this IVP has the unique solution $f(x) = 0$.

Now, for $\alpha \neq 0$, we may divide to find that $f(x) = e^{\lambda x}$ satisfies both the ODE and the boundary conditions if and only if

$$e^\lambda = \frac{1}{\alpha} \iff \lambda = \log\left(\frac{1}{\alpha}\right) \iff \lambda = \underbrace{\ln\left(\frac{1}{|\alpha|}\right) + i\,\mathrm{Arg}\left(\frac{1}{\alpha}\right) + 2\pi i k}_{\lambda_k(\alpha)}, \ k \in \mathbb{Z}. \tag{3.2.9}$$

So, in the case $\alpha \neq 0$, there exists a nonzero solution to the BVP if and only if $\lambda = \lambda_k(\alpha)$ for some $k \in \mathbb{Z}$. ▲

3.2.8 Linear algebraic viewpoint: eigenvalues and domains

Let $\mathcal{X} = \mathcal{C}([0,1])$ and define the operator $\mathcal{A}$ in $\mathcal{X}$ via $\mathcal{A}f = f'$ with domain

$$\mathfrak{D}^{\alpha}(\mathcal{A}) := \left\{ f \in \mathcal{C}^1([0,1]) \mid f(0) = \alpha f(1) \right\}.$$

Here $\alpha \in \mathbb{C}$ is fixed and the set $\mathfrak{D}^{\alpha}(\mathcal{A})$ is indeed a subspace of $\mathcal{C}([0,1])$. A point $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ as an operator in $\mathcal{X}$ with domain $\mathfrak{D}^{\alpha}(\mathcal{A})$ if and only if there exists a nontrivial solution to the BVP. We see from Example 3.2.7 that if $\alpha = 0$, then $\sigma_{\text{pt}}(\mathcal{A}) = \varnothing$, whereas if $\alpha \neq 0$, then $\sigma_{\text{pt}}(\mathcal{A})$ is the countable set $\{\lambda_k(\alpha)\}_{k\in\mathbb{Z}}$, with $\lambda_k(\alpha)$ defined in (3.2.9). Contrast this with the behavior of the same operator $\mathcal{A}$ given the larger domain $\mathfrak{D}(\mathcal{A}) = \mathcal{C}^1([0,1])$ in Linear Algebra Viewpoint 1.2.8.

3.2.3. Complex powers.

Recall that if $a > 0$ and $x \in \mathbb{R}$, then we define

$$a^x := e^{x\ln(a)}.$$

We can extend this definition for complex bases and exponents using the complex logarithm, but we caution that the result is not, in general, single-valued.

3.2.9 Definition.

Let $a \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$. We define the symbol a^z as the (possibly multi-valued) expression

$$a^z := e^{z\log(a)} = e^{z(\ln(|a|)+i\arg(a))}.$$

If $a = e$, however, we only use the symbol e^z to mean $\exp(z)$ as defined in (3.2.2).

3.2.10 Example.

Find all values of $(-1)^i$.

Solution. By definition,

$$(-1)^i = e^{i\log(-1)},$$

and

$$\log(-1) = \ln(|-1|) + i\arg(-1) = \ln(1) + i(\pi + 2\pi k) = (2k+1)\pi i.$$

Thus

$$(-1)^i = e^{i(2k+1)\pi i} = e^{(2k+1)\pi i^2} = e^{-(2k+1)\pi}.$$

As usual, we interpret the variable k above to represent a set:

$$(-1)^i = \left\{ e^{-(2k+1)\pi} \mid k \in \mathbb{Z} \right\}. \qquad \blacktriangle$$

The definition of a^z above can create some ambiguities when trying to apply familiar rules of exponents from the real-variable theory. For example, if a, b, $z \in \mathbb{C}$, and one defines $(ab)^z$ using the principal branch of the logarithm, it is possible to have $(ab)^z \neq a^z b^z$, even if a^z and b^z are also defined via the principal branches.

3.2.11 Example.

Let $z, w \in \mathbb{C}$. Compare and contrast the (multi-valued?) expressions $(e^z)^w$ and e^{zw}. Are they ever equal?

Solution. To be clear, as usual, we interpret e^z and e^{zw} via the power series definition (3.2.2). In particular, e^{zw} is a single-valued expression. We first use Definition 3.2.9 and Example 3.2.6 to calculate

$$(e^z)^w = e^{w\log(e^z)} = \left\{e^{w(z+2\pi ik)} \;\middle|\; k \in \mathbb{Z}\right\} = \left\{e^{wz}e^{2\pi ikw} \;\middle|\; k \in \mathbb{Z}\right\}.$$

Remember that we interpret each of the factors e^{wz} and $e^{2\pi ikw}$ as the single-valued power series. If $w \in \mathbb{Z}$, then $e^{2\pi ikw} = 1$, and this set reduces to the singleton

$$(e^z)^w = \{e^{zw}\}. \tag{3.2.10}$$

This is, of course, what we expect. Indeed, if z and w are both integers, we could use Definition A.1.5 for integer powers of complex numbers to obtain $(e^z)^w = e^{zw}$. (A precise proof would require us to consider cases on the signs of z and w and induct.) And this is what (3.2.10) says, if we eschew the set-valued semantics.

If, however, w is not integer-valued, then $kw \notin \mathbb{Z}$ for at least one $k \in \mathbb{Z}$ (otherwise, if $kw \in \mathbb{Z}$ for all $k \in \mathbb{Z}$, then $w = 1 \cdot w \in \mathbb{Z}$), and so for this k we have $e^{2\pi ikw} \neq 1$, and so $(e^z)^w$ is necessarily multi-valued. It is possible that $(e^z)^w$ has *finitely-many* multiple values; if $w = 1/2$, then $e^{2\pi ikw} = e^{\pi ik} = (-1)^k \in \{-1, 1\}$, and thus $(e^z)^{1/2} = \pm e^{z/2}$. However, it is also possible that $(e^z)^w$ has *infinitely-many* multiple values; if $w = i$, then $e^{2\pi ikw} = e^{2\pi i^2 k} = e^{-2\pi k}$, and thus $(e^z)^i = e^{iz-2\pi k}$, which has distinct values for all $k \in \mathbb{Z}$.

This leads to such unpleasant, and possibly ridiculous, circumstances. In general, context is key with complex powers. ▲

3.2.12 Example.

Simplify $\exp\left[i\pi\left(\frac{1+i}{\sqrt{2}}\right)^4\right]$.

Solution. We recognize

$$\frac{1+i}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = e^{i\pi/4}.$$

Then

$$\left(\frac{1+i}{\sqrt{2}}\right) = (e^{i\pi/4})^4 = e^{(i\pi/4)4} = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1.$$

Here we have used Example 3.2.11 with $w = 4$, an integer power, and $z = i\pi/4$. Now

$$\exp\left[i\pi\left(\frac{1+i}{\sqrt{2}}\right)^4\right] = e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1.$$ ▲

3.2.13 Example.

Find all $z \in \mathbb{C}$ such that $z^{1+i} = e$.

Solution. First, we reinterpret the symbol z^{1+i} as

$$z^{1+i} = e^{(1+i)\log(z)}.$$

Next, we recall from part (vi) of Theorem 3.2.2 that for $s \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$, we have

$$e^s = w \iff s = \ln(|w|) + i\operatorname{Arg}(w) + 2\pi ik, \ k \in \mathbb{Z}.$$

Then

$$z^{1+i} = e \iff e^{(1+i)\log(z)} = e \iff (1+i)\log(z) = \ln(|e|) + i\operatorname{Arg}(e) + 2\pi ik = 1 + 2\pi ik$$

Hence

$$\log(z) = \frac{1+2\pi ik}{1+i}.$$

Now we use Example 3.2.6 to solve for z:

$$z = e^{\log(z)} = \exp\left(\frac{1+2\pi ik}{1+i}\right).$$

To be clear, we have found a *set* of solutions to the equation $z^{1+i} = e$:

$$z^{1+i} = e \iff z \in \left\{\exp\left(\frac{1+2\pi ik}{1+i}\right) \;\middle|\; k \in \mathbb{Z}\right\}.$$ ▲

3.2.4. Integer roots of complex numbers.

Given $w \in \mathbb{C} \setminus \{0\}$ and a positive integer $n \in \mathbb{N}$, we want to find all $z \in \mathbb{C}$ such that $z^n = w$. Write $w = |w|e^{i\phi}$, where $\phi = \operatorname{Arg}(w)$. Suppose $z = |z|e^{i\theta}$. To be clear, we know the (real) numbers $|w|$ and ϕ, and we want to find the (real) numbers $|z|$ and θ.

As usual, we work backward. We have $z^n = w$ if and only if

$$\left(|z|e^{i\theta}\right)^n = |w|e^{i\phi} \iff |z|^n\left(e^{i\theta}\right)^n = |w|e^{i\phi} \iff |z|^n e^{in\theta} = |w|e^{i\phi}. \tag{3.2.11}$$

Take the modulus of both sides of the last equality above. This gives

$$|z|^n = \left||z|^n e^{in\theta}\right| = \left||w|e^{i\phi}\right| = |w|.$$

Now $|z|$, $|z|^n$, and $|w|$ are positive real numbers, so we may take the real positive nth root and solve for $|z|$:

$$|z|^n = |w| \iff |z| = |w|^{1/n}.$$

Substitute $|z|^n = |w|$ into the last equality in (3.2.11). This implies

$$e^{in\theta} = e^{i\phi}.$$

Part (iv) of Theorem 3.2.2 implies that $in\theta = i\phi + 2\pi ik$ for some $k \in \mathbb{Z}$. We solve for θ:

$$\theta = \frac{\phi + 2\pi k}{n} =: \theta_k.$$

Thus any solution to $z^n = w$ has the form

$$z = |w|^{1/n}\exp\left(\frac{(\phi + 2\pi k)i}{n}\right) =: z_k.$$

We leave it as an exercise to check that $z_k = z_{k+n}$ for all $k \in \mathbb{Z}$, and, conversely, that $z_k \neq z_j$ for $1 \leq k < j \leq n$.

3.2.14 Theorem.

Let $w \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. Suppose $\phi \in \arg(w)$. For any $k \in \mathbb{Z}$, the complex number

$$z_k := |w|^{1/n} \exp\left(\frac{i(\phi + 2\pi k)}{n}\right) = |w|^{1/n}\left[\cos\left(\frac{\phi + 2\pi k}{n}\right) + i \sin\left(\frac{\phi + 2\pi k}{n}\right)\right] \tag{3.2.12}$$

satisfies $z_k^n = w$, and the numbers $z_1, \ldots, z_n$ are all distinct. These numbers are the n**TH ROOTS** *of w.*

When $w = 1$, the solutions to $z^n = 1$ have a special designation.

3.2.15 Definition.

Given $n \in \mathbb{N}$, a number $z \in \mathbb{C}$ such that $z^n = 1$ is an n**TH ROOT OF UNITY**.

3.2.16 Example.

Find all solutions to $z^{12} = 1$.

Solution. We have $\operatorname{Arg}(1) = 0$, so the formula (3.2.12) with $\phi = \operatorname{Arg}(1)$ and $w = 1$ tells us that the solutions are

$$z_k = \cos\left(\frac{2\pi k}{12}\right) + i \sin\left(\frac{2\pi k}{12}\right), \quad k = 1, \ldots, 12. \qquad \blacktriangle$$

3.2.17 Example.

Find the general solution to the ODE $f''' - 8f = 0$.

Solution. The characteristic equation is $\lambda^3 - 8 = 0$, i.e., $\lambda^3 = 8$, so we need to find the third roots of 8. Obviously one of these will be 2, but there are two other distinct roots. Since $\operatorname{Arg}(8) = 0$, the roots are

$$\lambda_k := |8|^{1/3}\left[\cos\left(\frac{2\pi k}{3}\right) + i \sin\left(\frac{2\pi k}{3}\right)\right], \quad k = 1, 2, 3.$$

They simplify to

$$\lambda_1 = -1 + i\sqrt{3}, \qquad \lambda_2 = -1 - i\sqrt{3}, \qquad \text{and} \qquad \lambda_3 = 2.$$

So, the general solution is

$$f(x) = c_1 e^{(-1+i\sqrt{3})x} + c_2 e^{(-1-i\sqrt{3})x} + c_3 e^{2x},$$

or, in terms of real-valued functions,

$$f(x) = d_1 e^{-x} \cos(\sqrt{3}x) + d_2 e^{-x} \sin(\sqrt{3}x) + d_3 e^{2x}. \qquad \blacktriangle$$

3.2.5. Complex trigonometric functions.

Inspired by the real power series for sine and cosine, we define

$$\sin(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}. \tag{3.2.13}$$

More useful formulas for the complex sine and cosine are

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{3.2.14}$$

One develops these formulas by splitting the series for $e^{\pm iz}$ into sums over even and odd indices, rather like the proof of Euler's formula; we omit the details. Conversely, motivated by the real trig formulas in (A.3.4), one could adopt these formulas as the *definitions* of cosine and sine for complex inputs and develop from there the power series (3.2.13).

The next theorem tells us that extending the sine and cosine to complex inputs does not introduce any new periods or roots.

3.2.18 Theorem.

(i) $\sin(z) = 0$ *if and only if* $z = \pi k$ *for some* $k \in \mathbb{Z}$.

(ii) $\cos(z) = 0$ *if and only if* $z = (2k+1)\pi/2$ *for some* $k \in \mathbb{Z}$.

(iii) *If* $P \in \mathbb{C}$ *satisfies* $\sin(z+P) = \sin(z)$ *for all* $z \in \mathbb{C}$, *then* $P = 2\pi k$ *for some* $k \in \mathbb{Z}$. *The same is true if* $\sin(\cdot)$ *is replaced by* $\cos(\cdot)$.

Proof. (i) If $\sin(z) = 0$, then $e^{iz} - e^{-iz} = 0$, and, multiplying through by e^{iz}, we find $e^{2iz} = 1$. Part (iii) of Theorem 3.2.2 implies that $2iz = 2\pi ik$ for some $k \in \mathbb{Z}$, and so $z = \pi k$ for $k \in \mathbb{Z}$.

(ii) Exercise.

(iii) We give the proof for the complex sine. First, we calculate

$$\sin(z + 2\pi k) = \frac{e^{i(z+2\pi k)} - e^{-i(z+2\pi k)}}{2i} = \frac{e^{iz}e^{2\pi ik} - e^{-iz}e^{-2\pi ik}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \sin(z),$$

and so $\sin(\cdot)$ is $2\pi k$-periodic for all $k \in \mathbb{Z}$.

Next, we need to show that the only possible period for the sine is an integer multiple of 2π. Suppose $P \in \mathbb{C}$ is such that $\sin(z+P) = \sin(z)$ for all $z \in \mathbb{C}$. Since z is arbitrary, we are free to take $z = 0$ to find $\sin(P) = \sin(0+P) = \sin(0) = 0$. Consequently, $P = \pi j$ for some $j \in \mathbb{Z}$.

We still need to show, however, that j must be even. Suppose instead that j is odd, $j = 2\ell + 1$ for some $\ell \in \mathbb{Z}$, and so $P = (2\ell+1)\pi$. Then since $\sin(z+P) = \sin(z)$ for all $z \in \mathbb{C}$, we have $\sin(z + 2\ell\pi + \pi) = \sin(z)$ for all $z \in \mathbb{C}$. That is, $\sin(z+\pi) = \sin(z)$ for all $z \in \mathbb{C}$. Take, for instance, $z = \pi/2$. Then we obtain

$$-1 = \sin\left(\frac{3\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \pi\right) = \sin\left(\frac{\pi}{2}\right) = 1,$$

a contradiction. ■

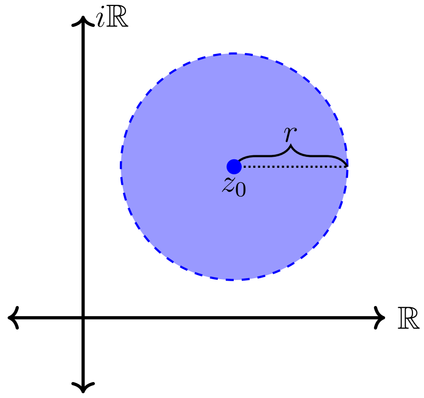
3.3. Limits and continuity.

We begin our study of the calculus of functions of a complex variable with some topology. The following structures are the analogues of bounded intervals (open or closed) in $\mathbb{R}$.

3.3.1 Definition.

Let $z_0 \in \mathbb{C}$ *and* $r > 0$.

(i) *The* **OPEN BALL** *of radius* r *centered at* z_0 *is* $\mathfrak{B}(z_0; r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$.



(ii) *The* **CLOSED BALL** *of radius* r *centered at* z_0 *is* $\overline{\mathfrak{B}}(z_0; r) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$.

(iii) *The* **(OPEN) PUNCTURED BALL** *of radius* r *centered at* z_0 *is* $\mathfrak{B}^*(z_0; r) = \mathfrak{B}(z_0; r) \setminus \{z_0\} = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$.

3.3.2 Remark.

We will not always use set-builder notation when referring to subsets of $\mathbb{C}$ *and instead just use formulas. For example, the circle of radius* r *centered at* z_0 *is the set*

$$\{z \in \mathbb{C} \mid |z - z_0| = r\},$$

but we will often say something like "Consider the circle $|z - z_0| = r$*." Likewise, we will use* $0 < |z - z_0| < r$ *and* $\mathfrak{B}^*(z_0; r)$ *interchangeably.*

3.3.3 Definition.

Suppose z_0, $L \in \mathbb{C}$, $r > 0$, *and* f *is a function defined on* $\mathfrak{B}^*(z_0; r)$. *We say that* $\lim_{z \to z_0} f(z) = L$ *if the values* $f(z)$ *can be made arbitrarily close to* L *by taking* z *sufficiently close to* z_0. *More precisely,* $\lim_{z \to z_0} f(z) = L$ *if and only if for all* $\epsilon > 0$ *there is* $\delta > 0$ *such that if* $0 < |z - z_0| < \delta$, *then* $|f(z) - L| < \epsilon$. *That is,*

$$z \in \mathfrak{B}^*(z_0; \delta) \Longrightarrow f(z) \in \mathfrak{B}(z_0; \epsilon).$$

If f *is defined on* $\mathfrak{B}(z_0; r)$ *and if* $\lim_{z \to z_0} f(z) = f(z_0)$, *then* f *is* **CONTINUOUS** *at* z_0.

Most of the standard limit theorems from real-variable calculus continue to hold in the complex case, e.g.,

$$\lim_{z \to z_0} (f(z) + g(z)) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$$

if both limits on the right exist, and we omit any formal statements of these properties. We do mention one important property that resembles the "componentwise" limits of a function from $\mathbb{R}$ to $\mathbb{R}^2$ in multivariable calculus. Namely, recall that if $\mathbf{f} \colon \mathbb{R} \to \mathbb{R}^2 \colon t \mapsto$

$(f_1(t), f_2(t))$, then for $\mathbf{L} = (L_1, L_2)$, we have

$$\lim_{t\to t_0} \mathbf{f}(t) = \mathbf{L} \iff \lim_{t\to t_0} f_1(t) = L_1 \quad \text{and} \quad \lim_{t\to t_0} f_2(t) = L_2.$$

Then we have the next, similar theorem.

3.3.4 Theorem.

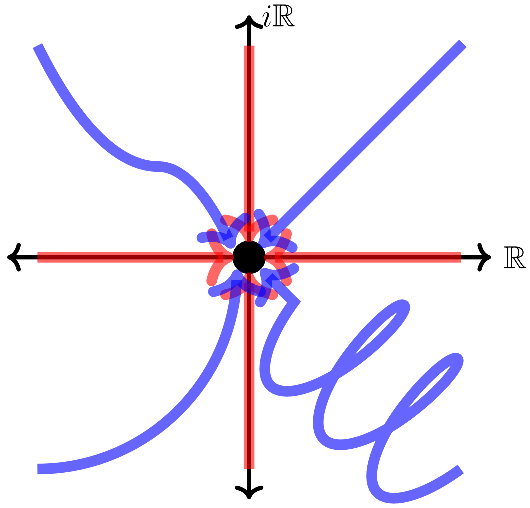
Suppose $z \in \mathbb{C}$ and $r > 0$.

(i) *If f is defined on $\mathfrak{B}^*(z_0; r)$, then*

$$\lim_{z\to z_0} f(z) = L \iff \lim_{z\to z_0} \operatorname{Re}[f(z)] = \operatorname{Re}(L) \quad \textit{and} \quad \lim_{z\to z_0} \operatorname{Im}[f(z)] = \operatorname{Im}(L).$$

(ii) *If f is defined on $\mathfrak{B}(z_0; r)$, then f is continuous at $z = z_0$ if and only both $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are continuous at $z = z_0$.*

We must bear in mind that, in general, it is "harder" for a limit of a function of a complex variable to exist than it is for a limit of a function of a real variable. This occurs for the same reason as when taking limits in $\mathbb{R}^2$ from multivariable calculus: in $\mathbb{C}$ there are many more ways to approach a point than in $\mathbb{R}$, which only has "left/right" directions.



As in multivariable calculus, we may show that a function is *discontinuous* at a certain point by approaching it along two different paths and finding different limits along each path. One can make a rather precise statement of this (which would involve the definition of a "path," something that we otherwise do not need for some time), but a simple, fundamental example will illustrate the challenge just as well.

3.3.5 Example.

Show that $\operatorname{Arg}(\cdot)$ is discontinuous on $\mathbb{R}_- = \{z \in \mathbb{R} \mid z < 0\}$.

Solution. We approach the negative real axis along two semicircular paths, one going counterclockwise and one going clockwise. Specifically, fix $z_0 = -x \in \mathbb{R}_-$ (with $x > 0$) and let

$$\gamma_1(t) = xe^{it},\ t \in \mathbb{R} \quad \text{and} \quad \gamma_2(t) = xe^{-it},\ t \in \mathbb{R}.$$

Then

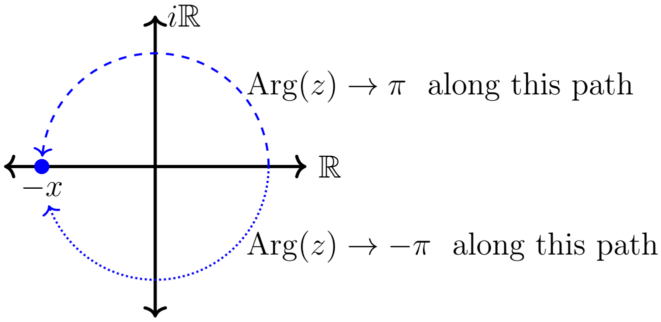
$$\lim_{t\to\pi}\gamma_1(t) = xe^{i\pi} = -x = z_0 \quad \text{and} \quad \lim_{t\to\pi}\gamma_2(t) = xe^{-i\pi} = -x = z_0.$$

However, for $-\pi < t \le \pi$ we have

$$\operatorname{Arg}(\gamma_1(t)) = t \quad \text{and} \quad \operatorname{Arg}(\gamma_2(t)) = -t$$

so

$$\lim_{t\to\pi^-}\operatorname{Arg}(\gamma_1(t)) = \pi \quad \text{but} \quad \lim_{t\to\pi^-}\operatorname{Arg}(\gamma_2(t)) = -\pi.$$



Thus there are points arbitrarily close to z_0 whose principal argument is close to π, but there are also points arbitrarily close to z_0 whose principal argument is $-\pi$. So, $\operatorname{Arg}(\cdot)$ must be discontinuous at z_0.

It is possible to show that $\operatorname{Arg}(\cdot)$ is continuous on $\mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$. However, this involves a lengthy piecewise formula for $\operatorname{Arg}(\cdot)$, largely involving the $\arctan(\cdot)$ function, that depends on what quadrant of $\mathbb{C}$ identified with $\mathbb{R}^2$ we work, and whether or not the point is on the imaginary axis. ▲

We do not state this as a formal theorem, but all the "usual" algebraic rules for limits and continuity still hold for functions of a complex variable, e.g.,

$$\lim_{z\to z_0}\big(f(z) + g(z)\big) = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z)$$

if both of these limits are defined.

3.4. Differentiability.

3.4.1. The definition of the complex derivative.

We define the complex derivative in the same way as the real derivative.

3.4.1 Definition.

If f is a function on $\mathfrak{B}(z_0; r)$ for some $z_0 \in \mathbb{C}$ and $r > 0$, then f is **(COMPLEX) DIFFERENTIABLE** *at z_0 if and only if the limit*

$$\lim_{h\to 0}\frac{f(z_0 + h) - f(z_0)}{h} \tag{3.4.1}$$

exists, in which case this limit is the **DERIVATIVE** *of f at $z = z_0$. We also say that f is* **HOLOMORPHIC** *at z_0 if f is complex differentiable there. If f is holomorphic on $\mathbb{C}$, then f is* **ENTIRE**.

The limit (3.4.1) exists if and only if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{3.4.2}$$

exists, in which case the two limits (3.4.1) and (3.4.2) are equal. The sum, product, and quotient rules all hold verbatim from real-variable calculus.

3.4.2 Theorem.

Every polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \in \mathbb{C}$, is entire. Any rational function $f(z) = p(z)/q(z)$, where p and q are polynomials, is holomorphic on the set $\{z \in \mathbb{C} \mid q(z) \neq 0\}$.

3.4.3 Example.

Show that $f(z) = \overline{z}$ is not differentiable at any $z \in \mathbb{C}$.

Solution. For $h \neq 0$, we compute the difference quotient

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{z} + \overline{h} - \overline{z}}{h} = \frac{\overline{h}}{h}.$$

So, it suffices to show that the limit

$$\lim_{h \to 0} \frac{\overline{h}}{h} \tag{3.4.3}$$

does not exist. We do this by allowing h to approach 0 along two different paths: the real axis and the imaginary axis. For the limit along the real axis, suppose k is real, so $\overline{k} = k$. Then (still understanding k to be real) we have

$$\lim_{k \to 0} \frac{\overline{k}}{k} = \lim_{k \to 0} \frac{k}{k} = 1.$$

For the limit along the imaginary axis, if ℓ is real, then $\overline{i\ell} = -i\ell$, so

$$\lim_{\ell \to 0} \frac{\overline{i\ell}}{\ell} = \lim_{\ell \to 0} -\frac{i\ell}{\ell} = -i.$$

Since these two limits are different, the full limit (3.4.3) does not exist. ▲

3.4.2. The Cauchy-Riemann equations.

We have seen that $\mathbb{C}$ is *topologically* like $\mathbb{R}^2$ in the sense that the existence of a limit is more difficult than in $\mathbb{R}$: the limit has to hold along all possible "paths of approach." But $\mathbb{C}$ is *algebraically* somewhat unlike $\mathbb{R}^2$: it is possible to multiply[49] and divide in $\mathbb{C}$. In particular, if f is differentiable at a point, one can let h approach 0 along different paths in the limit definition (3.4.1) of the derivative *and* manipulate the algebraic structure of the different quotient to obtain a powerful relationship between the real and imaginary parts of a complex differentiable function.

[49]Of course, one can take the dot product in $\mathbb{R}^2$, which has many features in common with multiplication of real (and complex) numbers — save that the dot product of two vectors in $\mathbb{R}^2$ is a real number, not a vector in $\mathbb{R}^2$.

3.4.4 Theorem (Cauchy-Riemann equations).

Suppose that $f = u + iv$ *is complex differentiable at the point* $z_0 = x_0 + iy_0$*. Consider* $u = u(x, y)$ *and* $v = v(x, y)$ *as functions of two real variables, as in* (3.2.1)*. Then*

$$\begin{cases} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0). \end{cases} \tag{3.4.4}$$

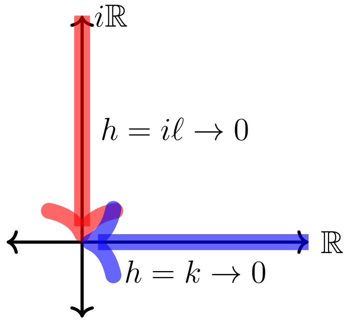
Moreover,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \tag{3.4.5}$$

Proof. Since f is complex differentiable at z_0, we have

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{3.4.6}$$

It is important to remember that h is a *complex* number here. Consequently, this limit (3.4.6) must hold along any path approaching the origin. In particular, it must hold if we approach the origin along the real axis, or along the imaginary axis.



That is, we have

$$f'(z_0) = \lim_{k\to 0} \frac{f(z_0 + k) - f(z_0)}{k}. \tag{3.4.7}$$

We rewrite

$$f(z_0 + k) = f((x_0 + k) + iy_0) = u(x_0 + k, y_0) + iv(x_0 + k, y_0)$$

and

$$f(z_0) = f(x_0 + iy_0) = u(x_0, y_0) + iv(x_0, y_0). \tag{3.4.8}$$

Since k is real, we have

$$\text{Re}\left[\frac{f(z_0 + k) - f(z_0)}{k}\right] = \frac{u(x_0 + k, y_0) - u(x_0, y_0)}{k}$$

and

$$\text{Im}\left[\frac{f(z_0 + k) - f(z_0)}{k}\right] = \frac{v(x_0 + k, y_0) - v(x_0, y_0)}{k}.$$

The existence of the limit (3.4.7) implies the existence of the real and imaginary parts of that limit, namely,

$$\text{Re}[f'(z_0)] = \lim_{k\to 0} \text{Re}\left[\frac{f(z_0 + k) - f(z_0)}{k}\right] = \lim_{k\to 0} \frac{u(x_0 + k, y_0) - u(x_0)}{k} = u_x(x_0, y_0)$$

and

$$\operatorname{Im}[f'(z_0)] = \lim_{k\to 0} \operatorname{Im}\left[\frac{f(z_0+k)-f(z_0)}{k}\right] = \lim_{k\to 0}\frac{v(x_0+k,y_0)-v(x_0,y_0)}{k} = v_x(x_0,y_0).$$

Thus

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0). \tag{3.4.9}$$

Now consider the original limit (3.4.6) again, but approach the origin along the imaginary axis. That is,

$$f'(z_0) = \lim_{\ell\to 0}\frac{f(z_0+i\ell)-f(z_0)}{i\ell}, \tag{3.4.10}$$

where $\ell \in \mathbb{R}$. We have

$$f(z_0+i\ell) = u(x_0,y_0+\ell) + iv(x_0,y_0+\ell),$$

and so, using (3.4.8),

$$\begin{aligned}\frac{f(x_0+i\ell)-f(z_0)}{i\ell} &= \frac{u(x_0,y_0+\ell)+iv(x_0,y_0+\ell)-u(x_0,y_0)-iv(x_0,y_0)}{i\ell}\\ &= \frac{1}{i}\left(\frac{u(x_0,y_0+\ell)-u(x_0,y_0)}{\ell}\right) + \frac{v(x_0,y_0+\ell)-v(x_0,y_0)}{\ell}\\ &= -i\left(\frac{u(x_0,y_0+\ell)-u(x_0,y_0)}{\ell}\right) + \frac{v(x_0,y_0+\ell)-v(x_0,y_0)}{\ell}.\end{aligned}$$

Here we performed the key algebraic step of multiplying the u-term by $i/i = 1$ and then simplified $i^2 = -1$. So, for for $\ell \in \mathbb{R}$ small,

$$\operatorname{Re}\left[\frac{f(z_0+i\ell)-f(z_0)}{i\ell}\right] = \frac{v(x_0,y_0+\ell)-v(x_0,y_0)}{\ell}$$

and

$$\operatorname{Im}\left[\frac{f(z_0+i\ell)-f(z_0)}{i\ell}\right] = -\left(\frac{u(x_0,y_0+\ell)-u(x_0,y_0)}{\ell}\right).$$

Taking the real and imaginary parts of (3.4.10), we find

$$\operatorname{Re}[f'(z_0)] = \lim_{\ell\to 0}\frac{v(x_0,y_0+\ell)-v(x_0,y_0)}{\ell} = v_y(x_0,y_0)$$

and

$$\operatorname{Im}[f'(z_0)] = \lim_{\ell\to 0} -\left(\frac{u(x_0,y_0+\ell)-u(x_0,y_0)}{\ell}\right) = -u_y(x_0,y_0).$$

That is,

$$f'(z_0) = v_y(x_0,y_0) - iu_y(x_0,y_0). \tag{3.4.11}$$

Equating (3.4.9) and (3.4.11), we find

$$u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$$

and thus

$$\begin{cases} u_x(x_0,y_0) &= v_y(x_0,y_0)\\ v_x(x_0,y_0) &= -u_y(x_0,y_0). \end{cases}$$

■

3.4.5 Remark.

Recall that the **Jacobian matrix** *of* $\mathbf{f}: \mathbb{R}^2 \to \mathbb{R}^2: (x, y) \mapsto (u(x, y), v(x, y))$ *is*

$$\begin{bmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y). \end{bmatrix}$$

We can remember the Cauchy-Riemann equations as saying that the diagonal entries of the Jacobian are equal and the off-diagonal entries are the negatives of each other.

The converse of Theorem 3.4.4 is that if the Cauchy-Riemann equations fail to hold at a point, then the function under consideration is not differentiable at that point.

3.4.6 Example.

Use the Cauchy-Riemann equations to verify that $f(z) = \overline{z}$ *is not complex differentiable at any point in* $\mathbb{C}$*.*

Solution. For x, $y \in \mathbb{R}$, we have

$$\overline{x + iy} = x - iy,$$

so we set $u(x, y) = x$ and $v(x, y) = -y$ to have

$$\overline{x + iy} = u(x, y) + iv(x, y).$$

Then

$$u_x = 1 \qquad u_y = 0$$
$$v_x = 0 \qquad v_y = -1,$$

so $u_x \neq v_y$. Thus the Cauchy-Riemann equations never hold, so the conjugate is not complex differentiable. ▲

The proof of the Cauchy-Riemann equations suggested that if f is complex differentiable at $z_0 = x_0 + iy_0$, then the partial derivatives of $\text{Re}(f)$ and $\text{Im}(f)$, considered as functions on $\mathbb{R}^2$, exist at (x_0, y_0). The mere existence, or even continuity, of these partial derivatives is not sufficient to guarantee complex differentiability, as the function $f(z) = \overline{z}$ indicates. Rather, we need both continuity of the partials *and* the Cauchy-Riemann equations to ensure differentiability.

3.4.7 Theorem.

Suppose that $f = u + iv$ *is defined on* $\mathfrak{B}(z_0; r)$ *and continuous at* z_0*. If* u *and* v *satisfy the Cauchy-Riemann equations* (3.4.4) *at* z_0*, and if* u *and* v *and their partial derivatives* u_x*,* u_y*,* v_x*,* v_y *all exist and are continuous on* $\mathfrak{B}(z_0; r)$*, then* f *is differentiable at* z_0*, and* $f'(z_0)$ *is given by* (3.4.5)*.*

3.4.8 Example.

Show that the complex exponential is entire.

Solution. Write

$$e^{x+iy} = e^x(\cos(y) + i\sin(y)) = e^x\cos(y) + ie^x\sin(y),$$

so we set

$$u(x,y) = e^x\cos(y) \quad \text{and} \quad v(x,y) = e^x\sin(y).$$

We compute

$$\begin{aligned} u_x(x,y) &= e^x\cos(y) & \qquad u_y(x,y) &= -e^x\sin(y) \\ v_x(x,y) &= e^x\sin(y) & \qquad v_y(x,y) &= e^x\cos(y), \end{aligned}$$

so

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

The Cauchy-Riemann equations hold; moreover, u, v, u_x, u_y, v_x, and v_y are continuous on $\mathbb{R}^2$ because of a general theorem from multivariable calculus that says that if f and g are continuous functions on $\mathbb{R}$, then $h(x,y) := f(x)g(y)$ is continuous on $\mathbb{R}^2$. So, Theorem 3.4.7 tells us that f is complex differentiable at each point in $\mathbb{C}$, i.e., f is entire. ▲

3.4.9 Example.

Study the differentiability of $\operatorname{Log}(\cdot)$.

Solution. Recall that $\operatorname{Log}(z) = \ln(|z|) + i\operatorname{Arg}(z)$. A practice problem demonstrates that $\operatorname{Log}(\cdot)$ is not continuous on $\mathbb{R}_- := \{z \in \mathbb{C} \mid z \in \mathbb{R}, z < 0\}$, so we know that $\operatorname{Log}(\cdot)$ cannot be differentiable on this ray. To check differentiability elsewhere, we need a better formula for $\operatorname{Arg}(\cdot)$. Any such formula will be piecewise, depending on the location of $(x,y) \in \mathbb{R}^2$, and so what ends up being simplest is just to show that $\operatorname{Log}(\cdot)$ is differentiable on the half-plane $\operatorname{Re}(z) > 0$.

In that case,

$$\operatorname{Arg}(x+iy) = \arctan\left(\frac{y}{x}\right), \ x > 0, \ y \in \mathbb{R}.$$

So, we fix $z = x + iy$ with $x = \operatorname{Re}(z) > 0$. Then

$$\operatorname{Log}(z) = \operatorname{Log}(x+iy) = \ln(|x+iy|) + i\operatorname{Arg}(x+iy) = \ln(\sqrt{x^2+y^2}) + i\arctan\left(\frac{y}{x}\right).$$

Set

$$u(x,y) := \ln(\sqrt{x^2+y^2}) \quad \text{and} \quad v(x,y) := \arctan\left(\frac{y}{x}\right),$$

so that $\operatorname{Log}(x+iy) = u(x,y) + iv(x,y)$. After some elementary calculus, we find

$$\begin{aligned} u_x(x,y) &= \frac{x}{x^2+y^2} & \qquad u_y(x,y) &= \frac{y}{x^2+y^2} \\ v_x(x,y) &= -\frac{y}{x^2+y^2} & \qquad v_y(x,y) &= \frac{x}{x^2+y^2}. \end{aligned}$$

We conclude that u and v satisfy the Cauchy-Riemann equations and that u, v, u_x, u_y, v_x, and v_y are continuous for $x > 0$, $y \in \mathbb{R}$, i.e., on $(0,\infty) \times \mathbb{R}$. Thus $\operatorname{Log}(\cdot)$ is differentiable on $\operatorname{Re}(z) > 0$. Moreover, we have

$$\frac{d}{dz}[\text{Log}(z)] = \frac{d}{dz}[\text{Log}(x+iy)] = u_x(x+iy) + iv_x(x,y) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}.$$

So, the derivative of $\text{Log}(\cdot)$ is what we expect from real-variable calculus. ▲

3.5. Line integrals.

The most natural and effective integral to define on $\mathbb{C}$ is the *line integral* (i.e., we do not consider "double integrals" over two-dimensional regions of $\mathbb{C}$). We will prove that, under suitable hypotheses, a function f defined on a region $\mathcal{D}$ has a *representation formula* of the form

$$f(z) = \int_\gamma \mathcal{K}(z,\xi)f(\xi)\,d\xi$$

for certain curves γ in $\mathcal{D}$ and a special *kernel* function $\mathcal{K}$. Here $\int_\gamma$ denotes a line integral over γ, much like the line integral from vector calculus.

3.5.1. Curves.

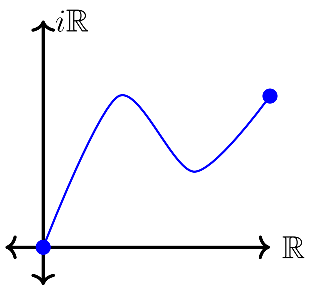
Since line integrals are defined over curves in $\mathbb{C}$, we need a very precise notion of what a curve can be.

3.5.1 Definition.

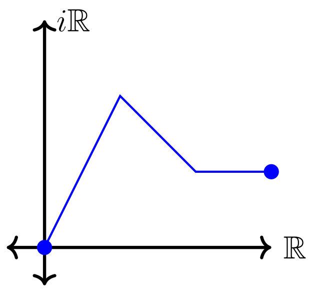
A **PATH**, *or* **CURVE**, *or* **CONTOUR** *in a set $\mathcal{D} \subseteq \mathbb{C}$, is a function $\gamma \in \mathcal{C}^1_{\text{pw}}([a,b]) \cap \mathcal{C}([a,b])$ such that $\gamma(t) \in \mathcal{D}$ for all $a \le t \le b$. The* **INITIAL POINT** *of γ is $\gamma(a)$ and the* **TERMINAL POINT** *of γ is $\gamma(b)$. The* **IMAGE** *of the curve γ is the set $\{\gamma(t) \mid a \le t \le b\} \subseteq \mathcal{D}$. A curve γ is* **CLOSED** *if $\gamma(a) = \gamma(b)$.*

If $\Gamma \subseteq \mathbb{C}$ is a set, and if there is a function $\gamma\colon [a,b] \to \mathbb{C}$ on some interval $[a,b] \subseteq \mathbb{R}$ such that $\Gamma = \{\gamma(t) \mid a \le t \le b\}$, then the function γ is a **PARAMETRIZATION** *of Γ. We will sometimes abuse terminology and call a "one-dimensional" subset Γ of $\mathbb{C}$ a curve even when a parametrization is not specified.*

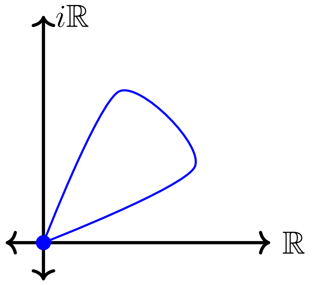
We require curves to be continuous to enforce our intuition that a curve should be "unbroken" in two-dimensional space. However, we permit curves to fail to be differentiable at some points (provided that the left and right limits of the derivatives exist and are finite at those points), since this encompasses curves with "points" or "corners."



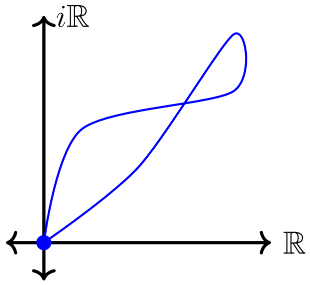
A $\mathcal{C}^1$ curve



A $\mathcal{C} \cap \mathcal{C}^1_{\text{pw}}$ curve that is not $\mathcal{C}^1$



A simple closed curve



A closed curve that is not simple

One can think of curves in $\mathbb{C}$ as an analogue of parametric mappings in $\mathbb{R}^2$, which we recall from calculus to be functions of the form $[a,b] \to \mathbb{R}^2 \colon t \mapsto (x(t), y(t))$ for maps x, $y \colon [a,b] \to \mathbb{R}$. In particular, we will often draw curves in $\mathbb{C} = \mathbb{R}^2$ that are not graphs of functions on $\mathbb{R}$; for example, the curves in the following example fail the vertical line test almost everywhere.

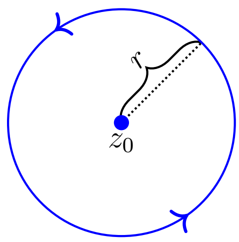
3.5.2 Example.

Parametrize the circle $|z - z_0| = r$, *where* $z_0 \in \mathbb{C}$ *and* $r > 0$.

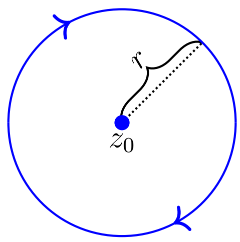
Solution. We need to find an interval $[a,b] \subseteq \mathbb{R}$ and a function $\gamma \colon [a,b] \to \mathbb{C}$ such that $|\gamma(t) - z_0| = r$ for all $a \le t \le b$ and, moreover, if $z \in \mathbb{C}$ such that $|z - z_0| = r$, there is $t \in [a,b]$ such that $\gamma(t) = z$. So, suppose $|z - z_0| = r$. Since $r > 0$, $z - z_0 \neq 0$, and so we may set $t = \operatorname{Arg}(z - z_0)$, so $-\pi < t \le \pi$. Then $z - z_0 = |z - z_0|e^{it} = re^{it}$, thus $z = z_0 + re^{it}$.

That is, any point on the circle $|z - z_0| = r$ has the form $z = z_0 + re^{it}$ for some $t \in (-\pi, \pi]$. This suggests setting $\gamma(t) = z_0 + re^{it}$. We want a curve's domain to be a closed, bounded interval, and since $z_0 + re^{i(-\pi)} = z_0 + re^{i\pi}$, there is no harm in taking the domain of γ to be $[-\pi, \pi]$. Conversely, any such point $\gamma(t)$ clearly satisfies $|\gamma(t) - z_0| = r$.

Another common parametrization of the circle uses the domain $[0, 2\pi]$, which corresponds to how we usually draw the unit circle.



$\gamma(t) = z_0 + re^{it}, \ 0 \le t \le 2\pi$



$\gamma(t) = z_0 + re^{-it}, \ 0 \le t \le 2\pi$

Still another parametrization is $\phi(t) := z_0 + re^{-it}$, $0 \le t \le 2\pi$. The difference between ϕ and γ lies in how the curves "trace out" the unit circle. The curve γ does so "counterclockwise" (which is what precalculus tells us is the "natural" orientation for circles) while ϕ does so "clockwise." ▲

3.5.3 Definition.

Let z_1, $z_2 \in \mathbb{C}$. *The* **LINE SEGMENT FROM** z_1 **TO** z_2 *is the curve*

$$\gamma(t) := (1 - t)z_1 + tz_2, \ 0 \le t \le 1.$$

We will often refer to this curve as $[z_1, z_2]$.

3.5.4 Remark.

We will abuse notation and also denote by $[z_1, z_2]$ *the image of the line segment from* z_1 *to* z_2, *and say things like "Suppose* $\mathcal{D} \subseteq \mathbb{C}$ *and* $z_1, z_2 \in \mathcal{D}$ *with* $[z_1, z_2] \subseteq \mathcal{D}$*" to mean that the set* $\{(1-t)z_1 + tz_2 \mid 0 \le t \le 1\}$ *is a subset of* $\mathcal{D}$ *and and that we are thinking of this set with the "orientation" indicated by Definition 3.5.3.*

Note too that if $a, b \in \mathbb{R}$, *then we define, of course,*

$$[a, b] := \{x \in \mathbb{R} \mid a \le x \le b\}.$$

One can show that

$$\{x \in \mathbb{R} \mid a \le x \le b\} = \{(1-t)a + tb \mid 0 \le t \le 1\} = \{sa + (1-s)b \mid 0 \le s \le 1\}$$

$$= \{\tau_1 a + \tau_2 b \mid 0 \le \tau_1, \tau_2 \le 1, \ \tau_1 + \tau_2 = 1\}. \quad (3.5.1)$$

In fact, the second and third equalities are valid for $a, b \in \mathbb{C}$, *not just in* $\mathbb{R}$. *Given* $a, b \in \mathbb{R}$, *context will make it clear whether we are referring to an ordered interval* $[a, b]$ *or to the directed line segment from* a *to* b; *we would only write something unusual like* $[2, 1]$ *to indicate the line segment "beginning" at 2 and "ending" at 1.*

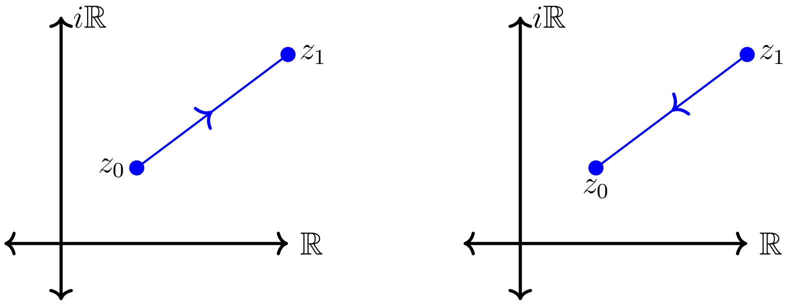
3.5.5 Example.

Let $z_1, z_2 \in \mathbb{C}$ *be distinct points. What is the difference between the functions*

$$\gamma_1(t) := (1-t)z_1 + tz_2 \quad \text{and} \quad \gamma_2(t) := (1-t)z_2 + tz_1$$

defined on $[0, 1]$*?*

Solution. One can check that $\gamma_1(t) = \gamma_2(t)$ if and only if $t = 1/2$, so these functions certainly are not equal. If we sketch the images of γ_1 and γ_2, we see that they produce the same picture: they are line segments connecting the points z_1 and z_2. (One can, and should, show rigorously that γ_1 and γ_2 have the same image.) However, $\gamma_1(0) = z_1$, but $\gamma_2(0) = z_2$; likewise, $\gamma_1(1) = z_2$, but $\gamma_2(1) = z_1$. So, the initial point of γ_1 is the terminal point of γ_2, and vice-versa. It appears, then, that γ_1 and γ_2 both "trace out" the same image but in the "reverse direction."



In fact, a little algebra shows

$$\gamma_1(t) = \gamma_2(1-t), \ 0 \le t \le 1. \tag{3.5.2}$$

Thus, although the image of γ_1 and γ_2 is the same set

$$\{s_1 z_1 + s_2 z_2 \mid 0 \le s_1, s_2 \le 1, \ s_1 + s_2 = 1\}, \tag{3.5.3}$$

each curve has the reverse "orientation" of the other. ▲

We formally define this notion of "reverse."

3.5.6 Definition.

Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a curve. The **REVERSE** *of γ is the curve*

$$\gamma^-(t) := \gamma(a+b-t), \ a \le t \le b.$$

Some books denote this curve by $-\gamma$ instead.

Note that $\gamma^-(a) = \gamma(a+b-a) = \gamma(b)$ and $\gamma^-(b) = \gamma(a+b-b) = \gamma(a)$, so the reverse curve γ^- does indeed reverse the initial and terminal points of γ. This definition shows that the curves γ_1 and γ_2 from Example 3.5.5 are the reverse curves of each other.

3.5.7 Example.

Let

$$\gamma_1(t) = e^{it}, \ 0 \le t \le 2\pi \quad \text{and} \quad \gamma_2(t) = e^{2it}, \ 0 \le t \le \pi.$$

What are the images of γ_1 and γ_2?

Solution. Clearly $|\gamma_1(t)| = |\gamma_2(t)| = 1$, so the images are contained in the unit circle, which is the set $\{z \in \mathbb{C} \mid |z| = 1\}$. In fact, if $|z| = 1$, then $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$, and by taking the appropriate branch of the argument we may assume $0 \le \theta \le 2\pi$. Thus $z = \gamma_1(\theta)$, and so the image of γ_1 is the entire unit circle. Likewise, $z = \gamma_2(\theta/2)$, so the image of γ_2 is also the entire unit circle. Thus γ_1 and γ_2 are both parametrizations of the unit circle. Moreover, we can relate γ_1 and γ_2 by observing that

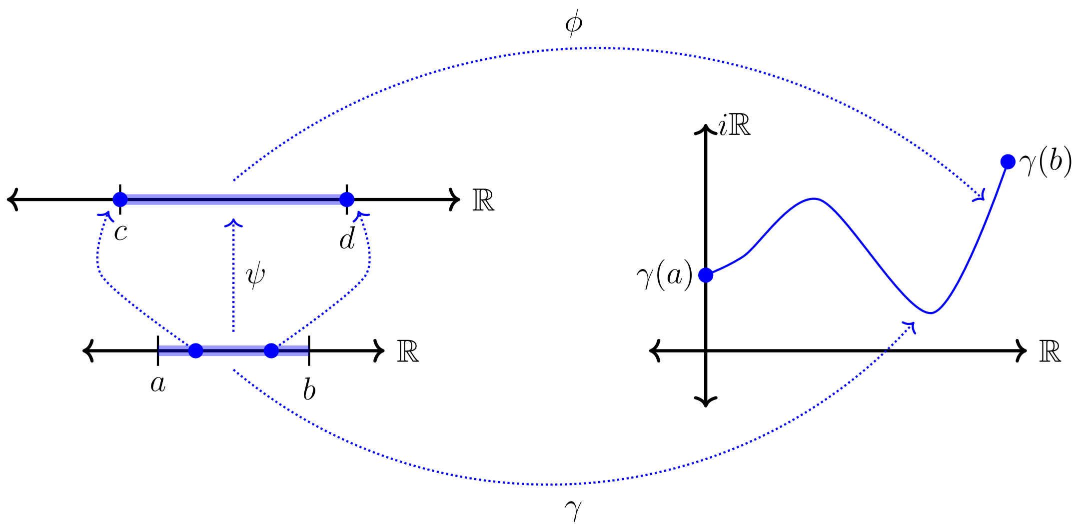
$$\gamma_1(t) = e^{it} = e^{2i(t/2)} = \gamma_2\left(\frac{t}{2}\right), \ 0 \le t \le 2\pi.$$

This corresponds to our graphical notion that γ_2 sketches the unit circle "twice as fast," or "in half the time," as γ_1. ▲

The situation in the preceding example has a precise definition.

3.5.8 Definition.

A curve $\phi\colon [c,d] \to \mathbb{C}$ is a **REPARAMETRIZATION** *of the curve $\gamma\colon [a,b] \to \mathbb{C}$ if there is a $\mathcal{C}^1$-function $\psi\colon [a,b] \to [c,d]$ such that $\psi'(t) > 0$ for $a \le t \le b$ and $\gamma(t) = \phi(\psi(t))$.*



The condition that $\psi'(t) > 0$ in this definition ensures that ψ is strictly increasing on $[a, b]$; it follows that $\psi(a) = c$ and $\psi(b) = d$. In Example 3.5.7, we had $\gamma_1(t) = \gamma_2(\psi(t))$, where $\psi(t) = t/2$ and ψ maps $[0, 2\pi]$ onto $[0, \pi]$ in a one-to-one manner.

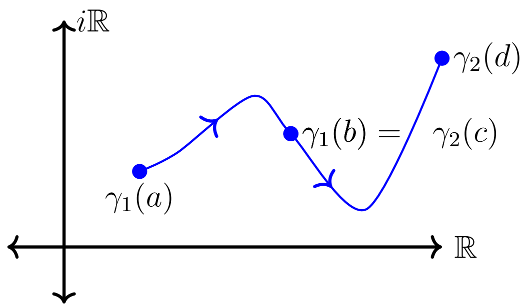
If the terminal point of one curve is the initial point of another curve, then the two curves "join together" in a very natural way.

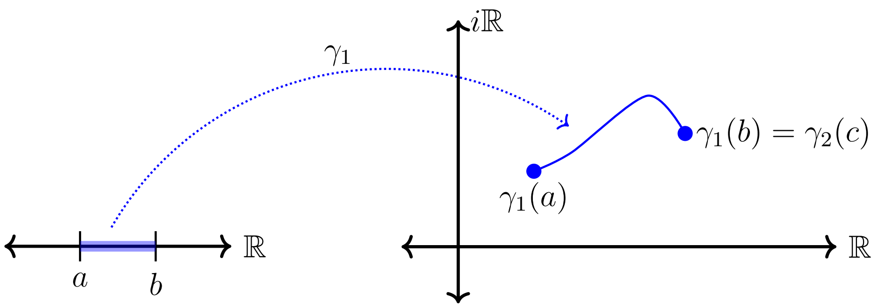
3.5.9 Definition.

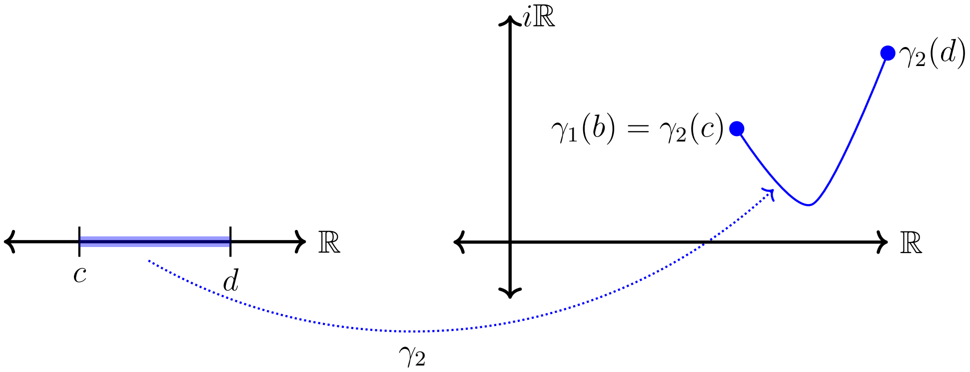
Suppose $\gamma_1 \colon [a, b] \to \mathbb{C}$ and $\gamma_2 \colon [c, d] \to \mathbb{C}$ are two curves with $\gamma_1(b) = \gamma_2(c)$. Then the COMPOSITION *of γ_1 and γ_2 is the curve*

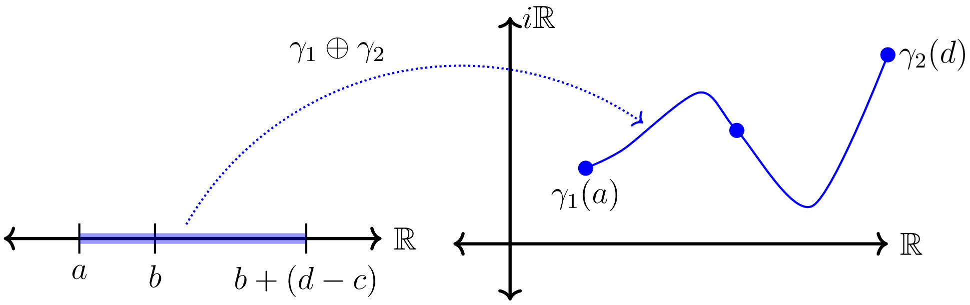
$$\gamma_1 \oplus \gamma_2 \colon [a, b + (d - c)] \to \mathbb{C} \colon t \mapsto \begin{cases} \gamma_1(t), & a \le t \le b \\ \gamma_2(t - b + c), & b \le t \le b + d - c. \end{cases}$$

Sometimes this curve is denoted by $\gamma_1 + \gamma_2$ instead.





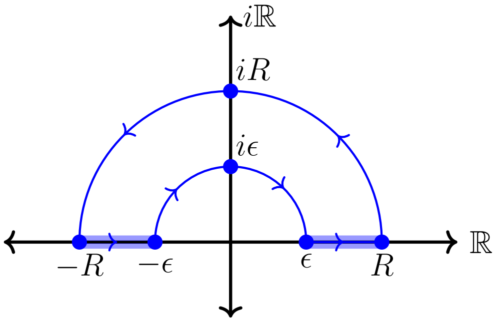




While we will often compose two or more curves, we will rarely need to know what the domain of the resulting composition is; it usually suffices to keep track of the individual domains of the components.

3.5.10 Example.

Find four curves γ_1, γ_2, γ_3, γ_4 *such that the image of* $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ *is the curve* *below* [50].



Solution. The line segment from $z = \epsilon$ to $z = R$ is parametrized by

$$\gamma_1(t) := (1-t)\epsilon + tR = (R-\epsilon)t + \epsilon, \ 0 \leq t \leq 1.$$

The upper half of the circle of radius R with "counterclockwise" orientation is parametrized by

$$\gamma_2(t) := Re^{it}, \ 0 \leq t \leq \pi.$$

The line segment from $z = -R$ to $z = -\epsilon$ is parametrized by

$$\gamma_3(t) := (1-t)(-R) + t(-\epsilon) = (t-1)R - \epsilon t = (R-\epsilon)t - R, \ 0 \leq t \leq 1.$$

And the upper half of the circle of radius ϵ with "clockwise" orientation is parametrized by

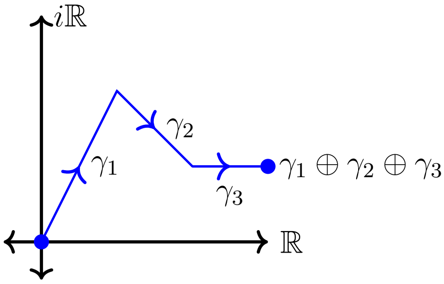
$$\gamma_4(t) := -\epsilon e^{i(\pi-t)}, \ 0 \leq t \leq \pi.$$

(The curve γ_4 needs to be the reverse of the curve $t \mapsto \epsilon e^{it}$, $0 \leq t \leq \pi$.) ▲

In the preceding example, we could write a piecewise formula for $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ over some domain $[0, b]$ for some $b > 0$. However, we will actually never use such a formula when we work with compositions of curves later, and such a formula would only obscure the four individual domains above. Indeed, although a curve need not be continuously differentiable, it can always be expressed as the composition of $\mathcal{C}^1$-curves.

3.5.11 Lemma.

Suppose that $\gamma \colon [a, b] \to \mathbb{C}$ is a curve. Then there exist a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ and curves $\gamma_k \in \mathcal{C}^1([t_{k-1}, t_k])$ such that $\gamma = \gamma_1 \oplus \cdots \oplus \gamma_n$.



A $\mathcal{C} \cap \mathcal{C}^1_{\text{pw}}$ curve that is not $\mathcal{C}^1$

[50] If part of a curve lies along the real or imaginary axis, we draw that part more thickly and translucently, so that the drawing does not blend into the axis.

Proof. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$ such that γ is continuous on each closed subinterval $[t_{k-1}, t_k]$ and continuously differentiable on each open subinterval (t_{k-1}, t_k); see part (ii) of Remark A.6.2. For $k = 1, \ldots, n$, let γ_k be the restriction of γ to $[t_{k-1}, t_k]$, i.e., we define the functions $\gamma_k \colon [t_{k-1}, t_k] \to \mathbb{C}$ by

$$\gamma_k(t) = \gamma(t),\ t_{k-1} \leq k \leq t_k.$$

Since γ is continuous on $[a,b]$, each γ_k is continuous on $[t_{k-1}, t_k]$. In fact, $\gamma_k \in \mathcal{C}^1([t_k, t_{k+1}])$, since the limits

$$\lim_{t \to t_{k-1}^+} \gamma_k'(t) = \lim_{t \to t_{k-1}^+} \gamma'(t) \quad \text{and} \quad \lim_{t \to t_k^-} \gamma_k'(t) = \lim_{t \to t_k^-} \gamma'(t)$$

exist, per Definition A.6.1. Of course, we define $\gamma_k'(t_{k-1})$ and $\gamma_k'(t_k)$ as one-sided derivatives by these limits.

So, $\gamma_k \in \mathcal{C}^1([t_{k-1}, t_k])$, which means that γ_k is a curve according to Definition 3.5.1. Moreover, for $k = 1, \ldots, n-1$, we have

$$\gamma_k(t_k) = \gamma(t_k) = \gamma_{k+1}(t_k).$$

Hence $\gamma = \oplus_{k=1}^n \gamma_k$, where each γ_k is continuously differentiable on its domain. And so we see that a curve, while not necessarily continuously differentiable on its domain, can always be expressed as the composition of continuously differentiable curves. ∎

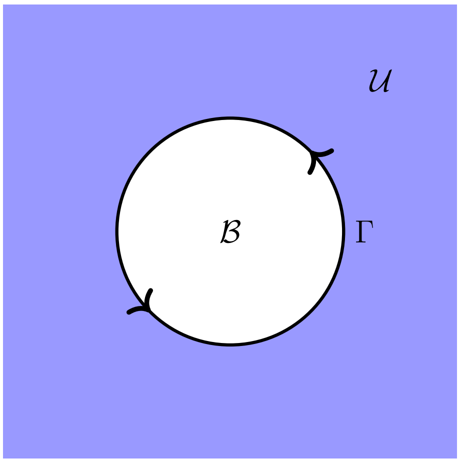
3.5.12 Remark.

The fundamental intuitive difference between a set $\Gamma \subseteq \mathbb{C}$ and a parametrization $\gamma \colon [a,b] \to \mathbb{C}$ of Γ is that Γ is just a set, whereas γ has a "direction" or "orientation." (Of course, γ is also a function!)

Some treatments of complex analysis stress orientation much more than we will. The **Jordan curve theorem** *asserts that if $\gamma \colon [a,b] \to \mathbb{C}$ is a closed curve that is also* **simple** *in the sense that γ is one-to-one or injective (i.e., $\gamma(t_1) \neq \gamma(t_2)$ for $a \leq t_1 < t_2 < b$), and if the image of γ is Γ, then the complex plane can be partitioned disjointly into three sets:*

$$\mathbb{C} = \mathcal{B} \cup \Gamma \cup \mathcal{U}.$$

The set $\mathcal{B}$ is bounded in the sense that $\mathcal{B} \subseteq \mathfrak{B}(0; r_0)$ for some $r > 0$, while $\mathcal{U}$ is unbounded in the sense that $\mathcal{U} \nsubseteq \mathfrak{B}(0; r)$ for all $r > 0$. The set $\mathcal{B}$ is called the **interior** *of γ, and the set $\mathcal{U}$ is the* **exterior** *of γ. The proof of the Jordan curve* *theorem is notoriously difficult.*



Then one says that γ is **POSITIVELY ORIENTED** *if the set $\mathcal{B}$ remains "on the left" of $\gamma(t)$ as t increases from a to b. There are several mathematical ways of making the phrase "on the left" precise, but none are particularly intuitive. We will largely avoid these ambiguities of interpretations in two ways: first, by working with very straightforward curves (typically circles) as long as possible, and second, by introducing the simplest notion of orientation (the winding number) only when we are a little more mathematically seasoned in Section 3.10.2.*

3.5.2. The line integral.

We presume familiarity with differentiation and integration of a complex-valued function of a real variable, as outlined in Appendix A.5.

3.5.13 Definition.

Suppose that f is continuous on $\mathcal{D} \subseteq \mathbb{C}$ and $\gamma\colon [a,b] \to \mathbb{C}$ is a curve in $\mathcal{D}$. If $\gamma \in \mathcal{C}^1([a,b])$, then the **LINE INTEGRAL** *of f over γ is*

$$\int_\gamma f(z)\ dz := \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

If $\gamma \in \mathcal{C}^1_{\text{pw}}([a,b]) \cap \mathcal{C}([a,b])$ restricts to be continuously differentiable on the intervals $[t_{k-1}, t_k]$ with $a = t_0 < t_1 < \cdots < t_n = b$, then we define

$$\int_\gamma f(z)\ dz := \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)\ dt,$$

where, on a given interval $[t_{k-1}, t_k]$, we interpret γ' to be the restriction

$$\gamma'(t) = \begin{cases} \lim\limits_{t \to t_{k-1}^+} \gamma'(t), & t = t_{k-1} \\ \gamma'(t), & t_{k-1} < t < t_k \\ \lim\limits_{t \to t_k^-} \gamma'(t), & t = t_k. \end{cases}$$

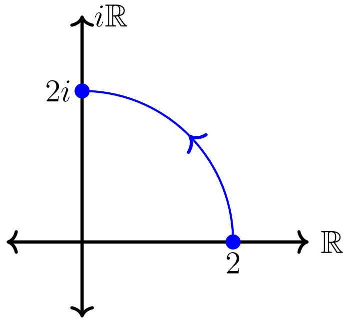
It is possible to define the line integral using Riemann sums for a broader class of functions than just the continuous ones. Under that definition, if f is continuous, its line integral is the same as the one defined above. The restriction to continuous functions in Definition 3.5.13 is really no restriction at all; we will ultimately be most interested in integrating holomorphic functions, which are certainly continuous.

3.5.14 Example.

Let γ be the portion of the circle of radius 2 centered at the origin that runs from $z = 2$ to $z = 2i$. Compute

$$\int_\gamma (z^2 + \operatorname{Im}(z))\ dz.$$

Solution. The portion of the circle under consideration is parametrized by $\gamma(t) := 2e^{it}$, $0 \le t \le \pi/2$, so $\gamma'(t) = 2ie^{it}$.



Here

$$f(z) = z^2 + \operatorname{Im}(z) \Longrightarrow f(\gamma(t)) = (2e^{it})^2 + \operatorname{Im}(2e^{it}) = 4e^{2it} + 2\sin(t).$$

Then

$$\int_\gamma (z^2 + \operatorname{Im}(z))\ dz = \int_0^{\pi/2} (4e^{2it} + 2\sin(t))(2ie^{it})\ dt = \underbrace{8i\int_0^{\pi/2} e^{3it}\ dt}_{8i\mathcal{I}_1} + \underbrace{4i\int_0^{\pi/2} \sin(t)e^{it}\ dt}_{4i\mathcal{I}_2}.$$

The fundamental theorem of calculus for complex-valued functions of a real variable tells us

$$\mathcal{I}_1 = \int_0^{\pi/2} e^{3it}\ dt = \frac{e^{3it}}{3i}\bigg|_{t=0}^{t=\pi/2} = \frac{-i-1}{3i}.$$

Next,

$$\mathcal{I}_2 = \int_0^{\pi/2} \sin(t)e^{it}\ dt = \frac{1}{2i}\int_0^{\pi/2} \left(e^{it} - e^{-it}\right)e^{it}\ dt = \frac{1}{2i}\int_0^{\pi/2} \left(e^{2it} - 1\right)\ dt = \frac{1}{2i}\left(\frac{e^{2it}}{2i} - t\right)\bigg|_{t=0}^{t=\pi/2} = \frac{1}{2} + \frac{i\pi}{4}.$$

We conclude

$$\int_\gamma (z^2 + \operatorname{Im}(z))\ dz = 8i\mathcal{I}_1 + 4i\mathcal{I}_2 = 8i\left(\frac{-i-1}{3i}\right) + 4i\left(\frac{1}{2} + \frac{i\pi}{4}\right) = -\left(\frac{8}{3} + \pi\right) - \frac{2i}{3}. \quad \blacktriangle$$

3.5.15 Remark.

In this course we will frequently evaluate line integrals over circles and line segments. Recall that the circle of radius $r > 0$ and center $z_0 \in \mathbb{C}$ is the set of points $|z - z_0| = r$. Unless otherwise specified, we will assume, when taking a line integral over a circle, that the circle is "positively oriented" in the sense that it is parametrized by

$$\gamma(t) = z_0 + re^{it}, \ 0 \le t \le 2\pi.$$

That is, we define

$$\int_{|z-z_0|=r} f(z)\ dz := ir \int_0^{2\pi} f(z_0 + re^{it})e^{it}\ dt.$$

Likewise, when we consider a line segment $[z_1, z_2]$, we will always assume that the parametrization is

$$\gamma(t) = (1-t)z_1 + tz_2, \ 0 \le t \le 1,$$

and so we define

$$\int_{[z_1,z_2]} f(z)\ dz := (z_2 - z_1) \int_0^1 f((1-t)z_1 + tz_2)\ dt.$$

3.5.16 Example.

Let $z_0 \in \mathbb{C}$, $r > 0$, and $n \in \mathbb{Z}$. Show that

$$\int_{|z-z_0|=r} (z - z_0)^n\ dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \ne -1. \end{cases} \tag{3.5.4}$$

Solution. Per the convention from Remark 3.5.15, we have

$$\int_{|z-z_0|=r} (z-z_0)^n\ dz = ir \int_0^{2\pi} \big((z_0 + re^{it}) - z_0\big)^n e^{it}\ dt = ir \int_0^{2\pi} (re^{it})^n e^{it}\ dt$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\ dt.$$

If $n \ne -1$, then

$$\int_0^{2\pi} e^{i(n+1)t}\ dt = \frac{e^{i(n+1)t}}{i(n+1)}\bigg|_{t=0}^{t=2\pi} = \frac{e^{i(n+1)2\pi} - 1}{i(n+1)} = \frac{1-1}{i(n+1)} = 0.$$

If $n = -1$, then

$$\int_0^{2\pi} e^{i(-1+1)t}\ dt = \int_0^{2\pi} dt = 2\pi.$$

In either case, we conclude the formula (3.5.4). ▲

Now we discuss a number of fundamental properties of the line integral. Before doing so, we need to recall some concepts that we probably met in real-variable calculus.

3.5.17 Definition.

Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ *is piecewise-*$\mathcal{C}^1$ *over the partition* $a = t_0 < t_1 < \cdots < t_n$. *The* **ARC LENGTH** *of* γ *is*

$$\ell(\gamma) := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |\gamma'(t)|\ dt.$$

Many of the properties of line integrals that we will shortly discuss are just the natural analogues for line integrals of properties of the Riemann integral, which we summarize in Definition A.5.3 and Theorem A.5.5.

Now we generalize these familiar properties to line integrals.

3.5.18 Theorem.

Suppose that all integrals below are defined over the given curves.

(i) *If the terminal point of* γ_1 *is the initial point of* γ_2, *then*

$$\int_{\gamma_1 \oplus \gamma_2} f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \int_{\gamma_2} f(z)\ dz.$$

(ii) $\displaystyle\int_{\gamma^-} f(z)\ dz = -\int_{\gamma} f(z)\ dz.$

(iii) [Linearity] $\displaystyle\int_{\gamma} \big(\alpha f(z) + \beta g(z)\big)\ dz = \alpha \int_{\gamma} f(z)\ dz + \beta \int_{\gamma} g(z)\ dz$

(iv) *Suppose that* ϕ *is a reparametrization of the curve* γ. *Then*

$$\int_{\gamma} f(z)\ dz = \int_{\phi} f(z)\ dz.$$

(v) [Fundamental theorem of calculus] *Suppose that the initial point of* γ *is* z_1 *and the terminal point is* z_2. *If* F *is an* **ANTIDERIVATIVE** *of* f, *i.e., if* $F'(z) = f(z)$ *for all* z *in the domain of* F *and* f, *then*

$$\int_{\gamma} f(z)\ dz = F(z_2) - F(z_1).$$

(vi) ["ML estimate" = "maximum × length"] *Suppose that* Γ *is the image of the curve* γ. *Then*

$$\left| \int_{\gamma} f(z)\ dz \right| \leq \left(\max_{z \in \Gamma} f(z) \right) \ell(\gamma).$$

Proof. (i) Exercise.

(ii) Exercise.

(iii) Exercise.

(iv) To be concrete, we need domains for the curves. Suppose $\gamma\colon [a,b] \to \mathbb{C}$ and

$\phi\colon [c,d] \to \mathbb{C}$ and, for simplicity, take both to be $\mathcal{C}^1$. Since ϕ is a reparametrization of ψ, there is a function $\psi\colon [a,b] \to [c,d]$ such that $\gamma(t) = \phi(\psi(t))$ for all t. Then

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt = \int_a^b f\big(\phi(\psi(t))\big)\phi'(\psi(t))\ dt.$$

Substitute $u = \psi$ to conclude

$$\int_a^b f\big(\phi(\psi(t))\big)\phi'(\psi(t))\ dt = \int_{\psi(a)}^{\psi(b)} f(\phi(u))\phi'(u)\ du = \int_c^d f(\phi(u))\phi'(u)\ du = \int_\phi f(z)\ dz.$$

(v) First suppose $\gamma\colon [a,b] \to \mathbb{C}$ is $\mathcal{C}^1$ on $[a,b]$. Then

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt = \int_a^b F'(\gamma(t))\gamma'(t)\ dt = \int_a^b \partial_t[F \circ \gamma](t)\ dt$$

$$= F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1). \quad (3.5.5)$$

If γ is piecewise-$\mathcal{C}^1$, we repeat the calculation above but over each subinterval on which γ is $\mathcal{C}^1$. The sum and difference of $F \circ \gamma$ evaluated at consecutive endpoints cancel until we are left with $F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1)$. More precisely, write $\gamma = \oplus_{k=1}^n \gamma_k$, where γ_k is $\mathcal{C}^1$ on $[t_{k-1}, t_k]$ and $\gamma_k(t_k) = \gamma_{k+1}(t_k)$. Here $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a,b]$. Then the preceding work shows

$$\int_{\gamma_k} f(z)\ dz = F(\gamma_k(t_k)) - F(\gamma_k(t_{k-1})). \quad (3.5.6)$$

Then

$$\int_\gamma f(z)\ dz = \int_{\oplus_{k=1}^n \gamma_k} f(z)\ dz = \sum_{k=1}^n \int_{\gamma_k} f(z)\ dz = \sum_{k=1}^n F(\gamma_k(t_k)) - F(\gamma_k(t_{k-1})).$$

This second sum is telescoping (Lemma A.2.5):

$$\sum_{k=1}^n F(\gamma_k(t_k)) - F(\gamma_k(t_{k-1})) = F(\gamma_n(t_n)) - F(\gamma_1(t_0)).$$

Recall that

$$\gamma_n(t_n) = \gamma(t_n) = \gamma(b) = z_2 \quad \text{and} \quad \gamma_1(t_0) = \gamma(t_0) = \gamma(a) = z_1.$$

Hence

$$F(\gamma_n(t_n)) - F(\gamma_1(t_0)) = F(z_2) - F(z_1).$$

(vi) Suppose that γ is $\mathcal{C}^1$ on $[a,b]$; the piecewise case is similar. First estimate using the definition of the line integral and the triangle inequality for integrals of a complex-valued function of a real variable that

$$\left|\int_\gamma f(z)\ dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\ dt\right| \le \int_a^b |f(\gamma(t))||\gamma'(t)|\ dt.$$

For $a \le t \le b$, we have $\gamma(t) \in \gamma$, so

$$|f(\gamma(t))| \le \max_{z \in \Gamma} |f(z)| =: M.$$

Thus

$$\int_a^b |f(\gamma(t))||\gamma'(t)|\ dt \le M \int_a^b |\gamma'(t)|\ dt. \qquad \blacksquare$$

3.5.19 Example.

Let γ be any curve in $\mathbb{C}$ with initial point 0 and terminal point i. Evaluate

$$\int_\gamma ze^{z^2}\,dz.$$

Solution. Our experience with real-variable calculus tells us that $F(z) = e^{z^2}/2$ is an antiderivative for $f(z) = ze^{z^2}$, so the fundamental theorem of calculus for line integrals gives

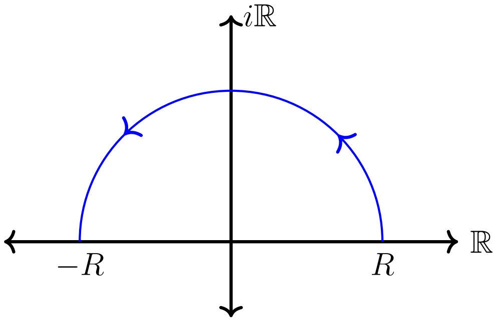
$$\int_\gamma ze^{z^2}\,dz = \frac{e^{z^2}}{2}\bigg|_{z=0}^{z=i} = \frac{e^{i^2}-e^0}{2} = \frac{e^{-1}-1}{2}.$$ ▲

3.5.20 Example.

Let γ_R be the arc of the circle $|z| = R$ in $\operatorname{Im}(z) > 0$ with initial point R and terminal point $-R$. Show that

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{dz}{z^2+4} = 0.$$

Solution. To be clear, we graph the arc.



We will use the ML-inequality and the squeeze theorem to establish this limit. Let

$$F(R) := \int_{\gamma_R}\frac{dz}{z^2+4}.$$

Then we have

$$\lim_{R\to\infty} F(R) = 0 \iff \lim_{R\to\infty}|F(R)| = 0.$$

On the other hand, if there is a function $g\colon [0,\infty) \to \mathbb{R}$ with $0 \le |F(R)| \le g(R)$, and if $\lim_{R\to\infty} g(R) = 0$, then the squeeze theorem tells us $\lim_{R\to\infty}|F(R)| = 0$ as well, and we will be done.

So, let us estimate $|F(R)|$. First, since γ_R is the upper half of the circle of radius R centered at the origin, its length is

$$\ell(\gamma_R) = \frac{2\pi R}{2} = \pi R.$$

Next, let Γ_R be the image of γ_R. We need to find a number $M(R)$, which is allowed to depend on R, such that

$$z \in \Gamma_R \Longrightarrow \left|\frac{1}{z^2+4}\right| \le M(R).$$

Then we can estimate

$$\left|\int_{\gamma_R} \frac{dz}{z^2+4}\right| \le M(R)\pi R.$$

To do this, note that

$$\left|\frac{1}{z^2+4}\right| \le M(R) \iff \frac{1}{M(R)} \le |z^2+4|,$$

so it suffices to find a lower bound for $|z^2+4|$ on Γ_R. One way to do this is to use the dreaded reverse triangle inequality from Theorem A.1.4. We find

$$|z^2+4| = |z^2-(-4)| \ge \big||z^2|-|-4|\big| = \big||z|^2-4\big| \ge |z|^2-4.$$

Now, observe that if $z \in \Gamma_R$, we have $|z| = R$. Thus

$$z \in \Gamma_R \Longrightarrow R^2-4 = |z|^2-4 \le |z^2+4| \Longrightarrow \frac{1}{|z^2+4|} \le \frac{1}{R^2-4} =: M(R).$$

We conclude

$$\left|\int_{\gamma_R} \frac{dz}{z^2+4}\right| \le \frac{\pi R}{R^2-4},$$

where we know

$$\lim_{R\to\infty} \frac{\pi R}{R^2-4} = 0.$$

By the squeeze theorem and the comments above, we are done. ▲

3.5.21 Remark.

Parts (i) and (ii) of Theorem 3.5.18 allow us to avoid, in large part, worrying about piecewise-$\mathcal{C}^1$ curves and finding common domains for the composition of curves. For example, if $\gamma\colon [0,2] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$ and γ is $\mathcal{C}^1$ on $[0,1]$ and $\mathcal{C}^1$ on $[1,2]$, then define two curves

$$\gamma_1(t) := \gamma(t),\ 0 \le t \le 1 \quad \text{and} \quad \gamma_2(t) := \gamma(t),\ 1 \le t \le 2$$

to find $\gamma = \gamma_1 \oplus \gamma_2$, and thus

$$\int_\gamma f(z)\ dz = \int_{\gamma_1\oplus\gamma_2} f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \int_{\gamma_2} f(z)\ dz. \tag{3.5.7}$$

Likewise, if we are given some curves, say, γ_1 and γ_2, we could figure out a piecewise formula for $\gamma_1\oplus\gamma_2$ and then compute $\int_{\gamma_1\oplus\gamma_2} f(z)\,dz$, but it is often easier just to compute (3.5.7) *In general, we should use properties of line integrals as much as we can to "break up" the line integral over convenient curves and not worry about parametrizing the curve all over one interval.*

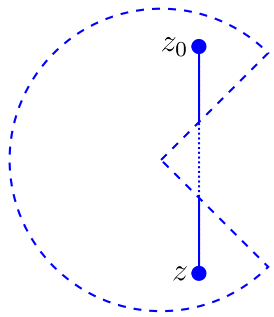
3.5.3. Path independence.

Part (v) of Theorem 3.5.18 gives us an easy way of evaluating a line integral: if the integrand has an antiderivative, the fundamental theorem of calculus carries over, except we replace the "endpoints" from real-variable calculus with the initial and terminal points of the curve over which we integrate. However, it is much more difficult for a function of a complex variable to have an antiderivative than it is for a function of a real variable. Recall that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then $F(x) := \int_a^x f(t)\ dt$ for $a \le x \le b$ is an antiderivative for f, i.e., $F'(x) = f(x)$ for all $x \in [a,b]$.

Suppose we attempt a similar construction of an antiderivative for a continuous function f of a complex variable defined on some subset $\mathcal{D}$ of $\mathbb{C}$. The greater freedom afforded by the two-dimensional geometry of $\mathbb{C}$ complicates this attempt. First, we have no natural analogue of the endpoint a of the real interval $[a,b]$ from before. We could still fix some $z_0 \in \mathcal{D}$ and try to "base" our antiderivative there. We might then try to define an antiderivative as

$$F(z) := \int_{[z_0,z]} f(\xi)\ d\xi,$$

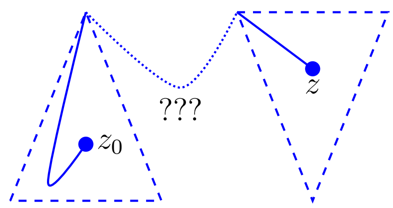
where $[z_0, z]$ is the line segment from z_0 to z. After all, in $\mathbb{R}$ the line segment from a to x for $x \ge a$ is just the interval $[a,x]$. However, depending on the geometry of $\mathcal{D}$, we have no guarantee that $[z_0,z] \subseteq \mathcal{D}$ for all $z \in \mathcal{D}$, and consequently f may not be defined on all of $[z_0,z]$.



The next option would be not to restrict ourselves to line segments. Suppose we take an arbitrary curve γ_z in $\mathcal{D}$ whose initial point is z_0 and whose terminal point is z. Then we could define

$$F(z) := \int_{\gamma_z} f(\xi)\ d\xi, \tag{3.5.8}$$

and perhaps that would be an antiderivative of f. There are, again, problems with this approach. First, we have no guarantee that there is a point $z_0 \in \mathcal{D}$ such that for any $z \in \mathcal{D}$, there is also a curve in $\mathcal{D}$ connecting z_0 and z.



Next, even if a set $\mathcal{D}$ does have this property, how do we know that the function F in (3.5.8) is *well-defined*? That is, perhaps there are curves γ_z and ϕ_z in $\mathcal{D}$ whose initial points are both z_0 and whose terminal points are both z, but for which

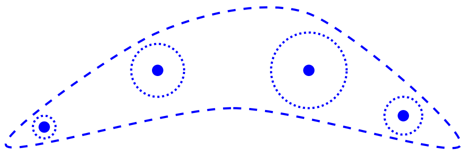
$$\int_{\gamma_z} f(\xi)\ d\xi \ne \int_{\phi_z} f(\xi)\ d\xi.$$

In that case, would the antiderivative depend on which curve we pick? How would we know which one to choose?

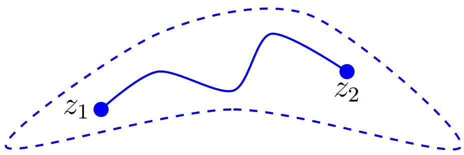
In this section we will build the machinery that determines when a continuous function does have an antiderivative on a given region. This will be the key to evaluating simply and succinctly many line integrals and eliciting deeper properties of functions through, ultimately, rather simple means.

3.5.22 Definition.

(i) *A set $\mathcal{D} \subseteq \mathbb{C}$ is* **OPEN** *if for any $z \in \mathcal{D}$ there is $r > 0$ such that $\mathfrak{B}(z;r) \subseteq \mathcal{D}$.*



(ii) *A set $\mathcal{D} \subseteq \mathbb{C}$ is* **PATH CONNECTED** *if for any z_1, $z_2 \in \mathcal{D}$, there exists a path $\gamma\colon [a,b] \to \mathcal{D}$ such that z_1 is the initial path of γ and z_2 is the terminal point of γ, i.e., $\gamma(a) = z_1$, $\gamma(b) = z_2$, and $\gamma(t) \in \mathcal{D}$ for all $a < t < b$.*



(iii) *A* **DOMAIN** *is a set that is both open and path connected. Of course, the word "domain" is also used to refer to the set of acceptable inputs for a function.*

3.5.23 Example.

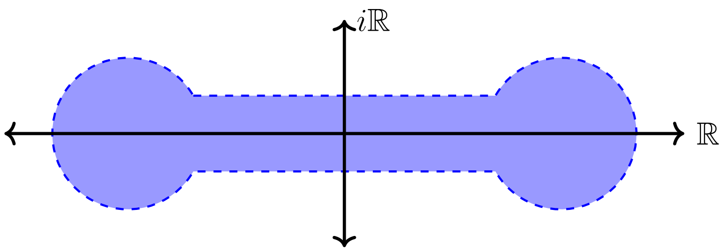
The following sets are all domains; the proofs are left as an exercise and can effectively be accomplished just by drawing a picture. Note that in parts (i) and (iii), the curve connecting two points can be taken to be just the line segment between them. Is that possible in parts (ii) and (iv)?

(i) *The entire complex plane $\mathbb{C}$ is a domain.*

(ii) *The punctured complex plane $\mathbb{C} \setminus \{0\}$ is a domain.*

(iii) *The half-plane $\mathrm{Re}(z) > 0$ is a domain.*

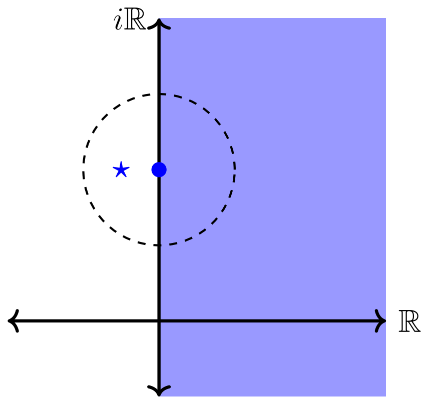
(iv) *The "dumbbell contour" sketched below is a domain.*



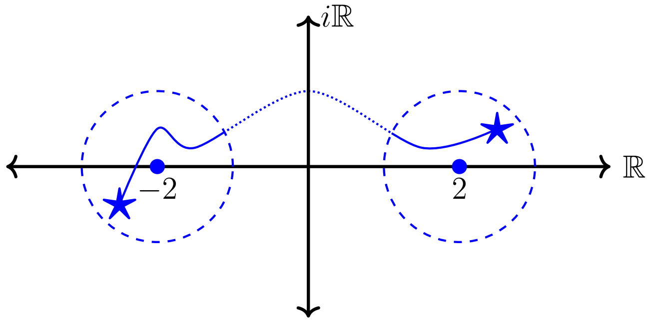
3.5.24 Example.

The following sets are not domains.

(i) *The set* $\mathrm{Re}(z) \geq 0$ *is not open; if* $z \in i\mathbb{R}$*, i.e., if* $z = iy$ *for some* $y \in \mathbb{R}$*, then any ball* $\mathfrak{B}(iy; r)$ *contains points* w *with* $\mathrm{Re}(w) < 0$*.*



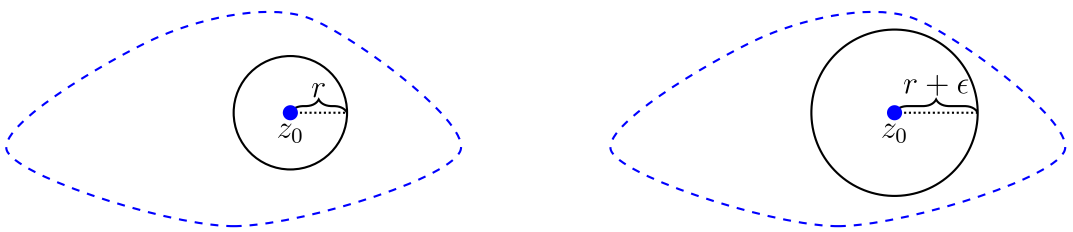
(ii) *The set* $\mathfrak{B}(-2; 1) \cup \mathfrak{B}(2; 1)$ *is not path connected.*



We will use the following "separation" lemma at various points in the future. Informally, it says that if a closed ball is contained in an open set, then a slightly larger closed ball, centered at the same point, is still contained in that set.

3.5.25 Lemma (Separation).

Let $\mathcal{D} \subseteq \mathbb{C}$ *be open and* $z_0 \in \mathcal{D}$*. If* $r > 0$ *is small enough so that* $\overline{\mathfrak{B}}(z_0; r) \subseteq \mathcal{D}$*, then there is* $\epsilon > 0$ *such that* $\overline{\mathfrak{B}}(z_0; r + \epsilon) \subseteq \mathcal{D}$*, too.*



The proof depends on some fundamental topological properties of complex numbers, so we omit it. The next lemma is a technical result that we will use very frequently when evaluating the definition of the derivative; its proof is a rare exercise in ϵ-δ analysis, so we give it in full.

3.5.26 Lemma.

Suppose $z_0 \in \mathbb{C}$, $r > 0$, *and* $f\colon \mathfrak{B}(z_0; r) \to \mathbb{C}$ *is continuous. If* $h \in \mathbb{C}$ *with* $|h| < r$, *then* $[z_0, z_0 + h] \subseteq \mathfrak{B}(z_0; r)$ *and*

$$\lim_{h \to 0} \frac{1}{h} \int_{[z_0, z_0+h]} f(z)\ dz = f(z_0). \tag{3.5.9}$$

Proof. That $[z_0, z_0 + h] \subseteq \mathfrak{B}(z_0; r)$ for $|h| < r$ is left as an exercise. The statement (3.5.9) is true if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |h| < \delta \Longrightarrow \left| \frac{1}{h} \int_{[z_0+h, z_0]} f(z)\ dz - f(z_0) \right| < \epsilon. \tag{3.5.10}$$

First we rewrite the integral. Parametrize the line segment $[z_0, z_0 + h]$ as

$$\gamma(t) = t(z_0 + h) + (1 - t)z_0 = tz_0 + th + z_0 - tz_0 = th + z_0,\ \ 0 \le t \le 1.$$

Then

$$\frac{1}{h} \int_{[z_0, z_0+h]} f(z)\ dz = \frac{1}{h} \int_0^1 f(th + z_0)h\ dt = \int_0^1 f(th + z_0)\ dt.$$

Next, we write

$$f(z_0) = f(z_0)(1 - 0) = \int_0^1 f(z_0)\ dt,$$

so we have

$$\frac{1}{h} \int_{[z_0, z_0+h]} f(z)\ dz - f(z_0) = \int_0^1 f(th + z_0)\ dt - \int_0^1 f(z_0)\ dt = \int_0^1 \big(f(th + z_0) - f(z_0)\big)\ dt.$$

Then

$$\left| \frac{1}{h} \int_{[z_0, z_0+h]} f(z)\ dz - f(z_0) \right| = \left| \int_0^1 \big(f(th + z_0) - f(z_0)\big)\ dt \right| \le (1-0) \max_{0 \le t \le 1} |f(th + z_0) - f(z_0)|. \tag{3.5.11}$$

Now we invoke the continuity of f. Given $\epsilon > 0$, choose $\delta > 0$ such that

$$|z - z_0| < \delta \Longrightarrow |f(z) - f(z_0)| < \epsilon.$$

Take $0 < |h| < \delta$. Then

$$|(th + z_0) - z_0| = |th| \le |h|$$

for $0 \le t \le 1$. Thus

$$\begin{cases} 0 \le t \le 1 \\ 0 < |h| < \delta \end{cases} \Longrightarrow |f(th + z_0) - f(z_0)| < \epsilon \Longrightarrow \max_{0 \le t \le 1} |f(th + z_0) - f(z_0)| < \epsilon.$$

We combine this with (3.5.11) to see that we have achieved our goal (3.5.10). ∎

3.5.27 Theorem.

Let f be continuous on the domain $\mathcal{D}$. The following are equivalent:

(i) *The function f has an antiderivative on $\mathcal{D}$, i.e., there is a holomorphic function F defined on $\mathcal{D}$ with $F'(z) = f(z)$ for all $z \in \mathcal{D}$.*

(ii) *If γ is a closed curve in $\mathcal{D}$, then*

$$\int_\gamma f(z)\, dz = 0.$$

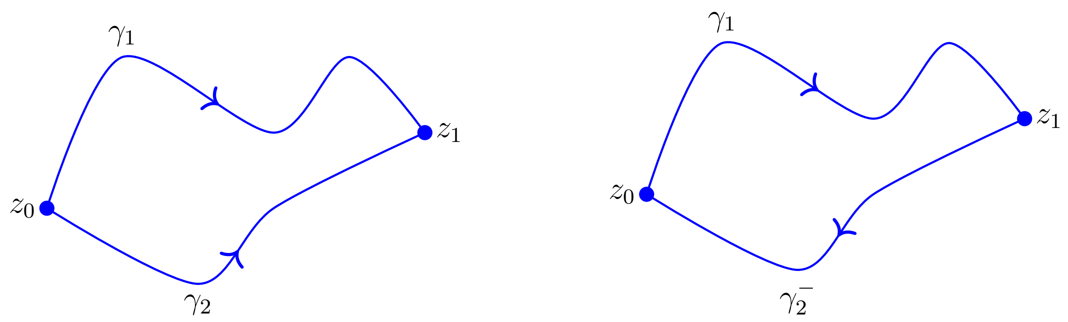
(iii) [Path independence] *Let z_0, $z_1 \in \mathcal{D}$ and let γ_1 and γ_2 be curves in $\mathcal{D}$ such that z_0 is the initial point of both γ_1 and γ_2, and z_1 is the terminal point of both γ_1 and γ_2. Then*

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

Proof. (i) $\Longrightarrow$ (ii) Suppose F is holomorphic on $\mathcal{D}$ with $F'(z) = f(z)$ for all $z \in \mathcal{D}$, and let γ be a closed curve in $\mathcal{D}$, i.e., the initial point and the terminal point of γ are the same point z_0. Then the fundamental theorem of calculus for line integrals implies

$$\int_\gamma f(z)\, dz = \int_\gamma F'(z)\, dz = F(z_0) - F(z_0) = 0.$$

(ii) $\Longrightarrow$ (iii) Suppose that γ_1 and γ_2 are curves in $\mathcal{D}$ with the same initial point z_0 and the same terminal point z_1, as in the sketch below.



Then the curve $\gamma_1 \oplus \gamma_2^-$ is closed, so part (ii) implies

$$0 = \int_{\gamma_1 \oplus \gamma_2^-} f(z)\, dz = \int_{\gamma_1} f(z)\, dz - \int_{\gamma_2} f(z)\, dz \Longrightarrow \int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

(iii) $\Longrightarrow$ (i) Motivated by the discussion at the start of this section and our recollection of antiderivatives on real subintervals, we fix a point $z_\star \in \mathcal{D}$ and, for $z \in \mathcal{D}$, let γ_z be a curve in $\mathcal{D}$ with initial point $z_\star$ and terminal point z. Since $\mathcal{D}$ is a domain, $\mathcal{D}$ is connected, and so such a path γ_z exists, although γ_z need not be the line segment $[z_\star, z]$. Then since f is path-independent, we obtain a well-defined function

$$F(z) := \int_{\gamma_z} f(\xi)\, d\xi.$$

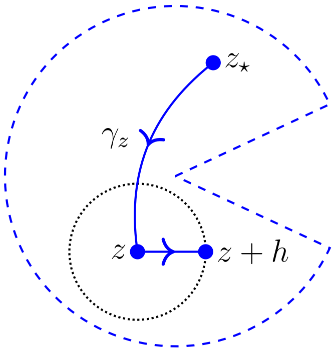
That is, the value of F does not depend on which curve γ_z we select to connect $z_\star$ and z.

Now we need to show that F is differentiable and $F' = f$. We do this by fixing a point[51] $z \in \mathcal{D}$ and studying the difference quotient

$$\frac{F(z+h) - F(z)}{h}.$$

We only need to do this for h small, so assume that h is so small that $[z, z+h] \subseteq \mathcal{D}$. The path $\gamma_z \oplus [z, z+h]$ has initial point $z_\star$ and terminal point $z+h$, so by independence of path

$$F(z+h) = \int_{\gamma_z} f(\xi)\ d\xi + \int_{[z,z+h]} f(\xi)\ d\xi = F(z) + \int_{[z,z+h]} f(\xi)\ d\xi.$$



Thus

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_{[z,z+h]} f(\xi)\ d\xi,$$

so Lemma 3.5.26 shows

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{[z,z+h]} f(\xi)\ d\xi = f(z).$$ ■

3.5.28 Example.

Evaluate

$$\int_\gamma \frac{1}{z+1}\ dz,$$

where γ *is any curve in the domain* $\operatorname{Im}(z) > 0$ *with initial point* $z = -1 + 2i$ *and terminal point* $z = 1 + 2i$.

Solution. The chain rule and Example 3.4.9 tell us

$$\frac{d}{dz}[\operatorname{Log}(z+1)] = \frac{1}{z+1}, \quad z \in \mathbb{C} \setminus (-\infty, -1].$$

The domain $\operatorname{Im}(z) > 0$ is a subset of $\mathbb{C} \setminus (-\infty, -1]$, so this derivative is valid on $\operatorname{Im}(z) > 0$. Then Theorem 3.5.27 gives

$$\int_\gamma \frac{1}{z+1}\ dz = \operatorname{Log}(1 + 2i + 1) - \operatorname{Log}(-1 + 2i + 1) = \operatorname{Log}(2 + 2i) - \operatorname{Log}(2i)$$

[51] In what follows, we will use some evocative pictures that suggest $z \neq z_\star$. The actual calculations that we perform, however, are entirely valid if $z = z_\star$.

$$= \left(\ln(\sqrt{2^2+2^2}) + i\frac{\pi}{4}\right) - \left(\ln(2) + i\frac{\pi}{2}\right) = \ln(\sqrt{2}) - \frac{i\pi}{4}.$$ ▲

3.5.29 Example.

In the exercises we will construct two paths γ_1 and γ_2, each with initial point 0 and terminal point i, such that $\int_{\gamma_1} \overline{z}\, dz \neq \int_{\gamma_2} \overline{z}\, dz$. What does this say about the **complex** *ODE*

$$f'(z) = \overline{z}?$$

Solution. Simply: this seemingly banal ODE does not have a solution, at least in a domain containing both 0 and i. This is wholly unlike our straightforward method of direct integration for an ODE depending on a real variable in Section 1.2.1 and suggests that, at a theoretical level, the existence of solutions to an ODE involving a function of a complex variable will require more stringent hypotheses. ▲

3.6. Cauchy's integral theorem and formula.

3.6.1. Motivation.

The independence of path theorem characterizes those functions of a complex variable that have antiderivatives, and Example 3.5.29 indicates that continuity alone does not guarantee the existence of an antiderivative. Neither does differentiability.

3.6.1 Example.

Show that the function $f(z) = 1/z$ is holomorphic on the domain $\mathbb{C}\setminus\{0\}$ (the "punctured plane"), but f does not have an antiderivative on $\mathcal{D}$.

Solution. The quotient rule proves that f is differentiable at all $z \neq 0$ with derivative $f'(z) = -1/z^2$. The natural choice for an antiderivative of f is $F(z) := \text{Log}(z)$, which is defined on $\mathbb{C} \setminus \{0\}$. However, we saw in Example 3.4.9 that F is not differentiable on $(-\infty, 0)$ and proved that F is differentiable on $\text{Re}(z) > 0$, arguing that a more complicated calculation would establish the differentiability of F on $\mathbb{C} \setminus (-\infty, 0]$.

The failure of the principal logarithm to serve as an antiderivative of f on all of $\mathbb{C}\setminus\{0\}$ does not in and of itself rule out the possibility that f could have *another* antiderivative on $\mathbb{C} \setminus \{0\}$. However, independence of path again helps: let γ_1 be the arc of the unit circle from i to -1 that lies in $\text{Re}(z) < 0$ and γ_2 the arc that lies in $\text{Re}(z) > 0$. We claim $\int_{\gamma_1} f(z)\, dz \neq \int_{\gamma_2} f(z)\, dz$. ▲

The fault in Example 3.6.1 is with our choice of domain (= set of inputs) for f. The set $\mathbb{C} \setminus \{0\}$ is a domain (= open and path connected), but the "puncture" at 0 weakens it topologically. We will ultimately prove, in Theorem 3.7.14, that a function defined on a more specialized (but still fairly ubiquitous) kind of domain $\mathcal{D}$ has an antiderivative if and only if that function is holomorphic on $\mathcal{D}$.

Let us pause our quest for antiderivatives momentarily and enjoy some formal consequences of differentiability. Suppose f is holomorphic on $\mathcal{D}$. For any point $z_0 \in \mathcal{D}$, if $z \approx z_0$, then we expect

$$f(z) \approx f'(z_0)(z - z_0) + f(z_0)$$

since f is "locally linear." Suppose that γ is a closed curve in $\mathcal{D}$ whose image is "close to" z_0; perhaps the image is contained in $\mathfrak{B}(z_0; r)$ for some small r. Then we also expect

$$\int_\gamma f(z)\ dz \approx \int_\gamma \big(f'(z_0)(z - z_0) + f(z_0)\big)\ dz.$$

But

$$\int_\gamma \big(f'(z_0)(z - z_0) + f(z_0)\big)\ dz = 0,$$

since the mapping $z \mapsto f'(z_0)(z - z_0) + f(z_0)$ has the antiderivative

$$z \mapsto \frac{f'(z_0)z^2}{2} - f'(z_0)z + f(z_0)z,$$

and then Theorem 3.5.27 applies.

So, this crude reasoning suggests that "the line integral of a holomorphic function over a closed curve is zero." This is an incorrect conclusion, as it would then imply that every holomorphic function has an antiderivative. As in Example 3.6.1, take $f(z) = 1/z$ on $\mathbb{C} \setminus \{0\}$, so f is holomorphic on $\mathbb{C} \setminus \{0\}$, and Example 3.5.16 gives

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i. \tag{3.6.1}$$

Although f is holomorphic on the open set $\mathbb{C} \setminus \{0\}$ containing the circle $|z| = 1$, f is not holomorphic at $z = 0$, and the point $z = 0$ is "inside" the curve $|z| = 1$. Somehow this causes the integral $\int_{|z|=1} f(z)\ dz$ to be nonzero.

Although it is not obvious, the equality (3.6.1) can be rewritten

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{g(z)}{z - 0}\ dz = g(0), \qquad \text{where} \qquad g(z) := 1.$$

The "representation" of the value $g(0)$ by this integral is no accident! We hasten to add that the calculation of this integral as $2\pi i$ hinged on our convention from Remark 3.5.15 of parametrizing the circle $|z| = 1$ via $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Had we taken the parametrization to be, say, $\mu(t) = e^{-it}$, $0 \le t \le 2\pi$, the integral in (3.6.1) would be $-2\pi i$, reflecting the fact that μ is really the reverse curve γ^-.

This very informal discussion distills into the following crude versions of, arguably, the principal results of complex analysis.

3.6.2 Theorem (Cauchy integral theorem — crude version).

Suppose f is holomorphic on a domain $\mathcal{D}$ and γ is a closed curve in $\mathcal{D}$ whose "inside" or "interior" is contained in $\mathcal{D}$. (In particular, f must be holomorphic on this inside region.) Then

$$\int_\gamma f(z)\ dz = 0.$$

3.6.3 Theorem (Cauchy integral formula — crude version).

Suppose f is holomorphic on a domain $\mathcal{D}$ and γ is a closed curve in $\mathcal{D}$ whose "inside" or "interior" is contained in $\mathcal{D}$. Moreover, suppose γ is "positively oriented" in the sense that the inside of γ "remains on the left" as γ is traversed. If z is a point in the "inside" of γ, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi. \tag{3.6.2}$$

For ξ fixed, the mapping $z \mapsto f(\xi)/(\xi - z)$ is differentiable. Assume that we can differentiate under the integral in (3.6.2). It is possible to make this rigorous for functions of a complex variable and line integrals, in the spirit of Theorem 2.4.35, but we will eschew this. Formally, then,

$$f'(z) = \partial_z\left[\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi\right] = \frac{1}{2\pi i}\int_\gamma \partial_z\left[\frac{f(\xi)}{\xi - z}\right] d\xi = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^2}\, d\xi.$$

Now, for ξ fixed again, the mapping $z \mapsto f(\xi)/(\xi - z)^2$ is differentiable. Differentiate under the integral again:

$$\partial_z\left[\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^2}\, d\xi\right] = \frac{1}{2\pi i}\int_\gamma \partial_z\left[\frac{f(\xi)}{(\xi - z)^2}\right] d\xi = \frac{2}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^3}\, d\xi.$$

This suggests that f is in fact twice-differentiable! If we continue to differentiate under the integral an arbitrary k times, we (formally) obtain the following corollary.

3.6.4 Corollary (Generalized Cauchy integral formula — crude version).

Assume the hypotheses of Theorem 3.6.3. Then f is in fact infinitely differentiable at all points on the "inside" of γ with

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}}\, d\xi.$$

We will first prove Cauchy's integral theorem. By the independence of path theorem, the existence of an antiderivative implies integrals over closed curves are zero, so we will try to construct antiderivatives as much as possible. Then we will derive the integral formula from the integral theorem.

3.6.2. Ancillary results for star-shaped regions involving triangles.

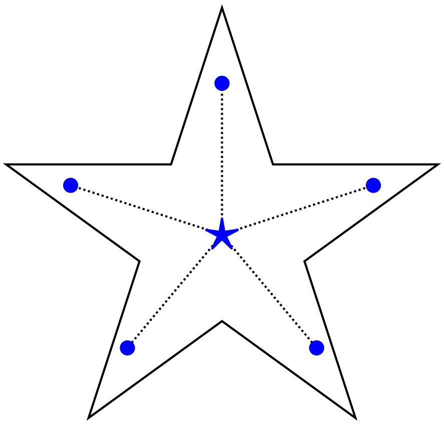
We begin by specifying a type of subset of $\mathbb{C}$ that will play a "starring" role in many of our proofs.

3.6.5 Definition.

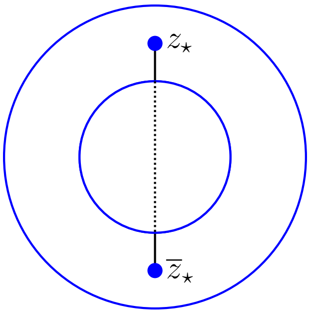
A set $\mathcal{D} \subseteq \mathbb{C}$ is STAR-SHAPED *if there is a point $z_\star \in \mathcal{D}$ such that $[z_\star, z] \subseteq \mathcal{D}$ for all $z \in \mathcal{D}$. The point $z_\star$ is called a* STAR-CENTER *for $\mathcal{D}$. A* STAR-SHAPED DOMAIN *or a* STAR DOMAIN *is a domain that is also star-shaped (recall that a domain is an open, path-connected set).*

3.6.6 Example.

(i) *The set below is (unsurprisingly!) star-shaped, and its star-center is indicated by the symbol* $\star$.



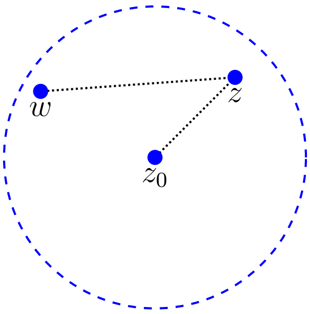
(ii) *The annulus* $\mathcal{A} := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ *is not star-shaped: if* $z \in \mathcal{A}$*, then* $\overline{z} \in \mathcal{A}$*, too, but* $[z, \overline{z}] \not\subseteq \mathcal{A}$*. That is, no matter what point* $z_\star$ *we try to pick for the star-center, we cannot connect* $z_\star$ *to* $\overline{z}_\star$ *by a line segment.*



3.6.7 Example.

For any $z_0 \in \mathbb{C}$ *and* $r > 0$*, the open ball* $\mathfrak{B}(z_0; r)$ *is star-shaped, and any point in* $\mathfrak{B}(z_0; r)$ *is a star-center.*

Proof. Fix $z \in \mathfrak{B}(z_0; r)$. We need to show that for any $w \in \mathfrak{B}(z_0; r)$, we have $[z, w] \subseteq \mathfrak{B}(z_0; r)$.



That is, for any $t \in [0, 1]$, we have $(1 - t)z + tw \in \mathfrak{B}(z_0; r)$. In turn, this amounts to showing

$$|z - z_0| < r, |w - z_0| < r, 0 \leq t \leq 1 \Longrightarrow \big|z_0 - \big((1 - t)z + tw\big)\big| < r.$$

We rewrite $z_0 = z_0 - tz_0 + tz_0 = (1-t)z_0 + tz_0$. Then

$$z_0 - \big((1-t)z + tw\big) = (1-t)z_0 + tz_0 - \big((1-t)z + tw\big) = (1-t)z_0 + tz_0 - (1-t)z - tw$$
$$= (1-t)(z_0 - z) + t(z_0 - w).$$

The triangle inequality then implies

$$\big|z_0 - \big((1-t)z + tw\big)\big| = \big|(1-t)(z_0 - z) + t(z_0 - w)\big| \le |(1-t)(z_0 - z)| + |t(z_0 - w)|$$
$$= (1-t)|z - z_0| + t|z_0 - w| < (1-t)r + tr = r. \quad \blacksquare$$

Next, many steps in the construction of antiderivatives of holomorphic functions will involve triangles.

3.6.8 Definition.

Let z_1, z_2, $z_3 \in \mathbb{C}$.

(i) *The* **TRIANGLE** *spanned by z_1, z_2, and z_3 is the set*

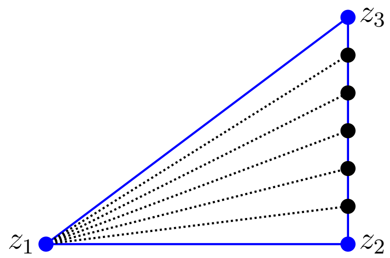
$$\Delta(z_1, z_2, z_3) := \bigcup_{0 \le s \le 1} [z_1, (1-s)z_2 + sz_3]. \tag{3.6.3}$$

(ii) *The* **TRIANGULAR PATH** *spanned by z_1, z_2, and z_3 is the closed curve*

$$\partial\Delta(z_1, z_2, z_3) := [z_1, z_2] \oplus [z_2, z_3] \oplus [z_3, z_1]. \tag{3.6.4}$$

3.6.9 Remark.

(i) *Why is* (3.6.3) *the right definition of a triangle? If we draw a picture and assume that the endpoints z_1, z_2, and z_3 of the triangle are distinct and not collinear (i.e., they do not all belong to the same line segment), we can imagine that the "inside" of this triangle consists of all line segments that have one endpoint at z_1 and the other endpoint at an arbitrary point on the line segment between z_2 and z_3.*



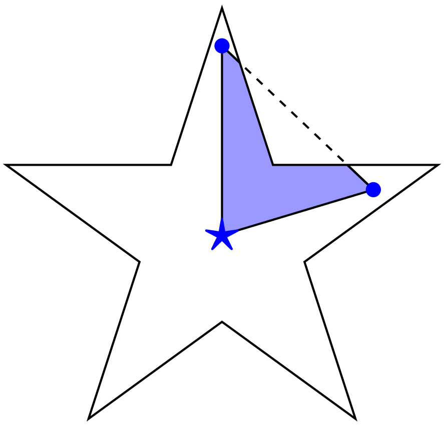
(ii) *If these three points are not all distinct, or if they are collinear, then $\Delta(z_1, z_2, z_3)$ is just a line segment and $\Delta(z_1, z_2, z_3) = \partial\Delta(z_1, z_2, z_3)$, as defined in* (3.6.4)*. Surely this flies in the face of our geometric intuition, and in this case $\Delta(z_1, z_2, z_3)$ is not a "classical" triangle. Remarkably, this turns out not to affect any of our proofs!*

The next lemma tells us that given a point in a star-shaped domain, the triangle whose endpoints are the given point, the star-center, and a third point close to the given point is contained in the domain.

3.6.10 Lemma.

If $\mathcal{D}$ is a star-shaped domain with star-center $z_\star$, then for any $z \in \mathcal{D}$, there is $h_0(z) > 0$ such that if $0 \le |h| < h_0(z)$, then $\Delta(z_\star, z, z+h) \subseteq \mathcal{D}$.

Proof. First, we remark that even if $\mathcal{D}$ is star-shaped with star-center $z_\star$, the triangle $\partial\Delta(z_\star, z_1, z_2)$ need not be wholly contained in $\mathcal{D}$ for *arbitrary* points z_1, $z_2 \in \mathcal{D}$, as the picture below indicates.

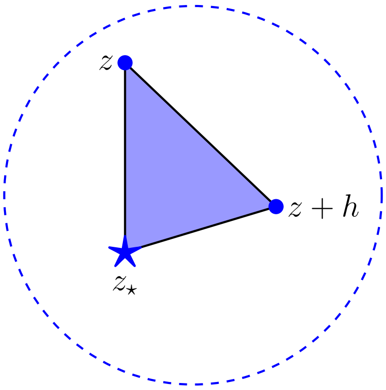


Now fix $z \in \mathcal{D}$. Since $\mathcal{D}$ is a domain, $\mathcal{D}$ is open, and so there is $r > 0$ such that $\mathfrak{B}(z; r) \subseteq \mathcal{D}$. By Lemma 3.5.26, if $h \in \mathbb{C}$ with $0 \le |h| < r$, then $[z, z+h] \subseteq \mathfrak{B}(z; r)$. So, take $h_0(z) = r$, and fix $h \in \mathbb{C}$ with $|h| < r$. To show $\Delta(z_\star, z, z+h) \subseteq \mathcal{D}$, it suffices to show $[z_\star, (1-s)z + s(z+h)] \subseteq \mathcal{D}$ for $0 \le s \le 1$. But observe that $(1-s)z + s(z+h) \in [z, z+h]$, and so $(1-s)z + s(z+h) \in \mathcal{D}$. Since $\mathcal{D}$ is star-shaped with star-center $z_\star$, we have $[z_\star, (1-s)z + s(z+h)] \subseteq \mathcal{D}$, as desired. ■

3.6.11 Lemma.

Suppose that f is continuous on the star-shaped domain $\mathcal{D}$ and let $z_\star$ be a star-center for $\mathcal{D}$. Suppose that for each $z \in \mathcal{D}$, if $h \in \mathbb{C}$ is small enough that $\Delta(z_\star, z, z+h) \subseteq \mathcal{D}$, we have

$$\int_{\partial\Delta(z_\star, z, z+h)} f(\xi)\, d\xi = 0.$$



Then f has an antiderivative on $\mathcal{D}$.

Proof. Our intuition, and the proof of part (iii) of Theorem 3.5.27, suggest that an antiderivative of f has the form

$$F(z) := \int_{[z_\star, z]} f(\xi)\ d\xi.$$

Note that now we have no qualms about the containment $[z_\star, z] \subseteq \mathcal{D}$ since $\mathcal{D}$ is a star-domain. Also, we do not need to require the path independence of f, as in the proof of part (iii) of Theorem 3.5.27, since we are not defining F as the integral over an arbitrary curve connecting $z_\star$ and z but rather the line segment $[z_\star, z]$.

Fix $z \in \mathcal{D}$. We show $F'(z) = f(z)$, i.e.,

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

Take h so small that $\partial\Delta(z_\star, z, z+h) \subseteq \mathcal{D}$; it is entirely possible, and entirely permissible, that the size of h will depend on z. Then

$$0 = \int_{\partial\Delta(z_\star, z, z+h)} f(\xi)\ d\xi = \underbrace{\int_{[z_\star, z]} f(\xi)\ d\xi}_{F(z)} + \int_{[z, z+h]} f(\xi)\ d\xi + \underbrace{\int_{[z+h, z_\star]} f(\xi)\ d\xi}_{-F(z+h)}.$$

This rearranges to

$$F(z+h) - F(z) = \int_{[z, z+h]} f(\xi)\ d\xi.$$

Since this is true for all h small, we have

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{[z, z+h]} f(\xi)\ d\xi = f(z)$$

by Lemma 3.5.26. That is, $F'(z) = f(z)$, as desired. ■

So, when does f integrate to zero over triangles? Whenever f is holomorphic. The proof of the following essential theorem can be found in many sources, including [1, 12].

3.6.12 Theorem (Cauchy-Goursat-Pringsheim).

Suppose that f is holomorphic on an open set $\mathcal{D}$ (which need not be star-shaped or even a domain). Let z_1, z_2, $z_3 \in \mathcal{D}$ such that $\Delta(z_1, z_2, z_3) \subseteq \mathcal{D}$. Then

$$\int_{\partial\Delta(z_1, z_2, z_3)} f(z)\ dz = 0.$$

3.6.3. The Cauchy theorems.

3.6.13 Theorem (Cauchy integral theorem for star-shaped domains).

If f is holomorphic on a star-shaped domain $\mathcal{D}$, then

$$\int_\gamma f(z)\ dz = 0$$

for any closed curve γ in $\mathcal{D}$.

Proof. By Theorem 3.5.27, it suffices to show that f has an antiderivative on $\mathcal{D}$. By Lemma 3.6.11, f will have an antiderivative on $\mathcal{D}$ if $\int_{\partial\Delta(z_\star,z,z+h)} f(\xi)\ d\xi = 0$ for all $z \in \mathcal{D}$ and $h \in \mathbb{C}$ small. This is true by the Cauchy-Goursat-Pringsheim theorem. ■

Why did we prove the Cauchy integral theorem by constructing an antiderivative instead of calculating directly the integral $\int_\gamma f(z)\ dz$? Other than the fact that the proof above is very short, thanks to our diligent preparation, this proof only requires us to do one thing: construct a single antiderivative. A more "direct" proof would require us to compute $\int_\gamma f(z)\ dz$ for **any** possible closed curve γ in $\mathcal{D}$, of which there are infinitely many. So, constructing one antiderivative was more expeditious than evaluating infinitely many integrals!

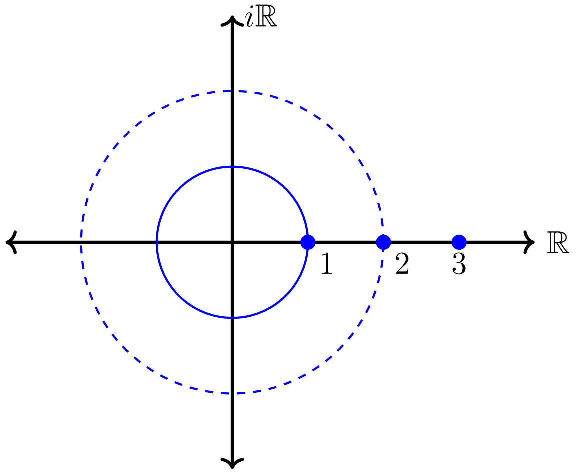
Also, the Cauchy integral theorem subsumes the Cauchy-Goursat-Pringsheim theorem: any triangular path $\partial\Delta(z_1, z_2, z_3)$ is a closed curve. But to prove the Cauchy integral theorem, we specifically needed f to integrate to zero around triangles.

3.6.14 Example.

Evaluate the line integral

$$\int_{|z|=1} \frac{z}{(z-3)^2}\ dz.$$

Solution. One could construct an antiderivative for $f(z) = z/(z-3)^2$ using the method of partial fractions, but that is laborious, and extravagant. Instead, observe that f is a rational functions, and rational functions are holomorphic at all points at which their denominator is nonzero. Thus f is holomorphic on $\mathbb{C} \setminus \{3\}$.



Now, the closed curve $|z| = 1$ is contained in the star-domain $\mathfrak{B}(0; 2)$, and f is holomorphic there. By Cauchy's integral theorem, the integral evaluates to 0. ▲

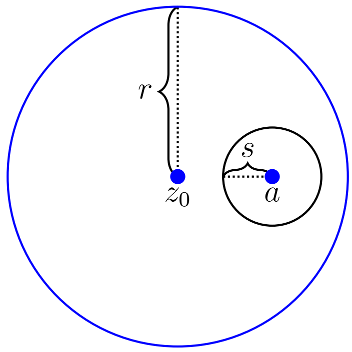
Now we proceed to establish the Cauchy integral formula. We will actually only do so when the curve under consideration is a circle. This will be enough for all of our immediate needs, and later we will obtain the formula for more general curves by means of the residue theorem.

First, we use the Cauchy integral theorem to relate certain line integrals over circles. The proof involves a "deformation of contours" argument: we will show that a line integral over a given circle ϕ_1 is equal to a line integral over a different circle ϕ_2. Very informally, we "deform" or "squeeze" ϕ_1 into ϕ_2 in a "continuous" manner, and the line integral is robust enough that its value does not change during this deformation.

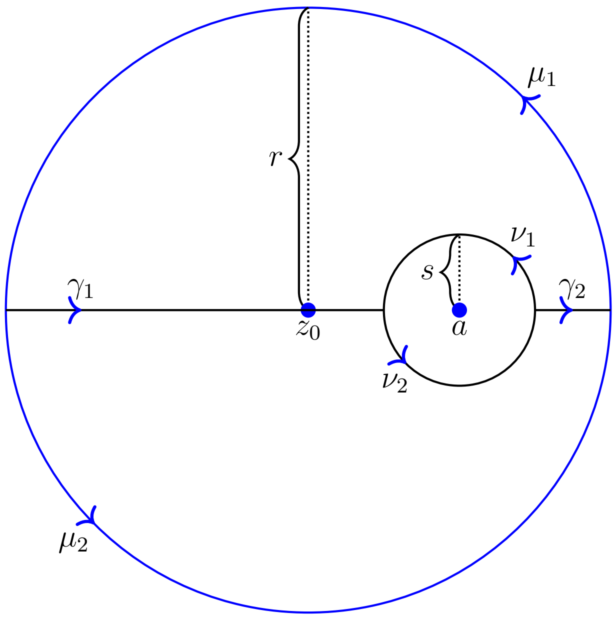
3.6.15 Lemma (Death Star).

Let $\mathcal{D} \subseteq \mathbb{C}$ be open, a, $z_0 \in \mathcal{D}$, and $f\colon \mathcal{D} \setminus \{a\} \to \mathbb{C}$ be holomorphic. Suppose that r, $s > 0$ with $\overline{\mathfrak{B}}(a; s) \subseteq \mathfrak{B}(z_0; r)$ and $\overline{\mathfrak{B}}(z_0; r) \subseteq \mathcal{D}$. Then

$$\int_{|z-z_0|=r} f(z)\ dz = \int_{|z-a|=s} f(z)\ dz.$$



Proof. We define a number of curves by the following sketch; we leave their precise formulas for the exercises.



Observe that

$$\int_{|z-z_0|=r} f(z)\ dz = \int_{\mu_1 \oplus \mu_2} f(z)\ dz \quad \text{and} \quad \int_{|z-a|=s} f(z)\ dz = \int_{\nu_1 \oplus \nu_2} f(z)\ dz$$

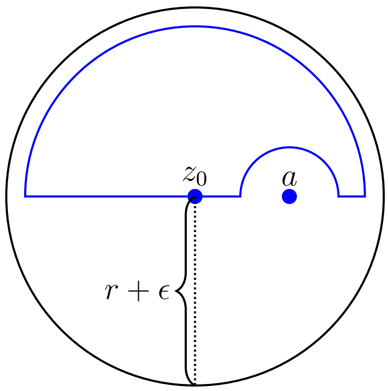
So, our goal is to show $\int_{\mu_1 \oplus \mu_2} f(z)\ dz = \int_{\nu_1 \oplus \nu_2} f(z)\ dz$.

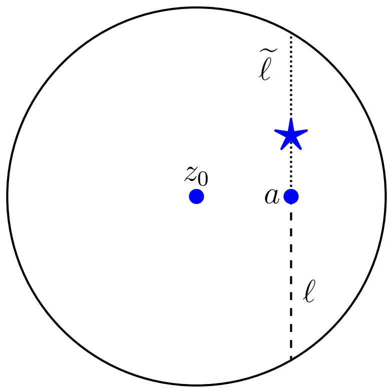
Consider the closed curves

$$\Gamma_1 := \gamma_1 \oplus \nu_1^- \oplus \gamma_2 \oplus \mu_1 \quad \text{and} \quad \Gamma_2 := \gamma_1 \oplus \nu_2^- \oplus \gamma_2 \oplus \mu_2^-.$$

We claim that Γ_1 is contained within a star domain $\mathcal{D}_1 \subseteq \mathcal{D} \setminus \{a\}$. If this is true, then since g is holomorphic on $\mathcal{D} \setminus \{a\}$, the Cauchy integral theorem implies $\int_{\Gamma_1} g(z)\ dz = 0$.

We construct this star domain in the following sketches. Since $\overline{\mathfrak{B}}(z_0; r) \subseteq \mathcal{D}$ and $\mathcal{D}$ is open, the separation lemma (Lemma 3.5.25) provides $\epsilon > 0$ such that $\overline{\mathfrak{B}}(z_0; r+\epsilon) \subseteq \mathcal{D}$, too. Let ℓ be the line segment from a to the boundary of $\mathfrak{B}(z_0; r+\epsilon)$ as sketched below; in particular, $a \in \ell$. Let $\mathcal{D}_1$ be the "slitted" ball formed by deleting ℓ from $\mathfrak{B}(z_0; r+\epsilon)$; that is, $\mathcal{D}_1 := \mathfrak{B}(z_0; r+\epsilon) \setminus \ell$. Then $\mathcal{D}_1$ is a star domain, where any point on the remaining line segment $\widetilde{\ell}$ is a star center. This, hopefully, is fairly obvious from our sketches but rather technical to prove; recall the (perhaps nonintuitive) estimates of Example 3.6.7. Indeed, since, by that example, any point in an open ball is a star center for the ball, we would just have to show that the line segment from any point on $\widetilde{\ell}$ to any point in $\mathfrak{B}(z_0; r+\epsilon) \setminus \ell$ does not intersect ℓ.





Similar reasoning shows $\int_{\Gamma_2} f(z)\ dz = 0$. Thus

$$\int_{\Gamma_1} f(z)\ dz = \int_{\Gamma_2} f(z)\ dz.$$

Upon splitting each of these integrals into a sum of four integrals, this reads

$$\int_{\gamma_1} f(z)\ dz + \int_{\nu_1^-} f(z)\ dz + \int_{\gamma_2} f(z)\ dz + \int_{\mu_1} f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \int_{\nu_2} f(z)\ dz + \int_{\gamma_2} f(z)\ dz + \int_{\mu_2^-} f(z)\ dz.$$

Canceling $\int_{\gamma_1}$ and $\int_{\gamma_2}$ from both sides, we have

$$\int_{\nu_1^-} f(z)\ dz + \int_{\mu_1} f(z)\ dz = \int_{\nu_2} f(z)\ dz + \int_{\mu_2^-} f(z)\ dz$$

$$\Longrightarrow \int_{\mu_1} f(z)\ dz - \int_{\mu_2^-} f(z)\ dz = -\int_{\nu_1^-} f(z)\ dz + \int_{\nu_2} f(z)\ dz$$

$$\Longrightarrow \int_{\mu_1} f(z)\ dz + \int_{\mu_2} f(z)\ dz = \int_{\nu_1} f(z)\ dz + \int_{\nu_2} f(z)\ dz$$

$$\Longrightarrow \int_{\mu_1 \oplus \mu_2} f(z)\ dz = \int_{\nu_1 \oplus \nu_2} f(z)\ dz,$$

as desired.

We now use this technological terror to evaluate what appears to be (and what really is) an anodyne line integral that would be rather difficult to calculate directly by definition.

3.6.16 Example.

Let $z_0 \in \mathbb{C}$, $r > 0$, and $a \in \mathfrak{B}(z_0; r)$. Then

$$\int_{|z-z_0|=r} \frac{dz}{z-a} = 2\pi i.$$

Proof. If we try to calculate this integral directly using the definition of the line integral, we obtain

$$\int_{|z-z_0|=r} \frac{dz}{z-a} = ir \int_0^{2\pi} \frac{e^{it}}{z_0 + re^{it} - a}\ dt,$$

and this does not have an obvious antiderivative. Instead, put $f(z) := 1/(z-a)$, so g is holomorphic on $\mathbb{C} \setminus \{a\}$. Example 3.5.16 told us

$$\int_{|z-a|=s} f(z)\ dz = 2\pi i$$

for any $s > 0$. In particular, since $a \in \mathfrak{B}(z_0; r)$, there is $s > 0$ such that $\overline{\mathfrak{B}}(a; s) \subseteq \mathfrak{B}(z_0; r)$. We have now met all the hypotheses of Lemma 3.6.15, which implies

$$\int_{|z-a|=s} f(z)\ dz = \int_{|z-z_0|=r} f(z)\ dz. \qquad \blacksquare$$

The last lemma that we need before we can prove the Cauchy integral formula shows that we can very slightly relax the requirement that f be holomorphic on all of a star-shaped domain $\mathcal{D}$ in the hypotheses of the Cauchy integral theorem. The proof is somewhat technical and unenlightening, so we omit it (see, once again, [12]).

3.6.17 Lemma ("relaxed" Cauchy integral theorem).

Suppose $\mathcal{D}$ is a star-shaped domain with star center $z_\star$. Let f be holomorphic on $\mathcal{D} \setminus \{z_\star\}$ and continuous on $\mathcal{D}$. Then

$$\int_\gamma f(z)\ dz = 0$$

for any closed curve γ in $\mathcal{D}$. That is, the Cauchy integral theorem is still valid for f, if f fails to be holomorphic only at the star-center.

3.6.18 Theorem (Cauchy integral formula).

Let f be holomorphic on $\mathcal{D}$, where $\mathcal{D}$ is open (but not necessarily a domain or star-

shaped). Suppose $z_0 \in \mathcal{D}$ and $r > 0$ with $\overline{\mathfrak{B}}(z_0; r) \subseteq \mathcal{D}$. Then

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{\xi - z} \, d\xi, \ \ z \in \mathfrak{B}(z_0; r). \tag{3.6.5}$$

Proof. The separation lemma (Lemma 3.5.25) provides $R > 0$ such that $\overline{\mathfrak{B}}(z_0; r) \subseteq \mathfrak{B}(z_0; R) \subseteq \mathcal{D}$. Then $|\xi - z_0| = r$ is a curve in the star-domain $\mathfrak{B}(z_0; R)$, and so we are on our way to invoking Cauchy's integral theorem. Consider the function

$$g(\xi) := \begin{cases} \dfrac{f(\xi) - f(z)}{\xi - z}, \ \xi \neq z \\ \\ f'(z), \ \xi = z. \end{cases}$$

This function g is continuous on $\mathcal{D}$ by definition of the derivative and holomorphic on $\mathcal{D} \setminus \{z\}$. In particular, g is continuous on the star-domain $\mathfrak{B}(z_0; R)$ and holomorphic on $\mathfrak{B}(z_0; R) \setminus \{z\}$. Recall from Example 3.6.7 that any point in $\mathfrak{B}(z_0; R)$ is a star-center for $\mathfrak{B}(z_0; R)$, so we can call upon Lemma 3.6.17 to conclude

$$\int_{|\xi - z_0| = r} g(\xi) \, d\xi = 0.$$

Since $|z - z_0| < r$, we have

$$|\xi - z_0| = r \Longrightarrow g(\xi) = \frac{f(\xi) - f(z)}{\xi - z}.$$

Thus

$$0 = \int_{|\xi - z_0| = r} \frac{f(\xi) - f(z)}{\xi - z} \, d\xi = \int_{|\xi - z_0| = r} \frac{f(\xi)}{\xi - z} \, d\xi - \int_{|\xi - z_0| = r} \frac{f(z)}{\xi - z} \, d\xi.$$

The second integral is

$$\int_{|\xi - z_0| = r} \frac{f(z)}{\xi - z} \, d\xi = f(z) \int_{|\xi - z_0| = r} \frac{d\xi}{\xi - z} = 2\pi i f(z)$$

by Example 3.6.16. ■

3.6.19 Remark.

The power of the Cauchy integral formula is that the behavior of f on the "boundary" of $\mathfrak{B}(z_0; r)$ determines the behavior of f on the interior of $\mathfrak{B}(z_0; r)$, the latter being a much "larger" set. Later we will see that the circle $|\xi - z| = r$ can be replaced with a much more general closed curve, but, for immediate applications, it suffices to have the Cauchy integral formula with the integral taken only over circles.

3.6.20 Example.

Evaluate the line integral $\displaystyle\int_{|z|=1} \frac{\cos(z)}{z} \, dz.$

Solution. With $f(z) = \cos(z)$, the integral is

$$\int_{|z|=1} \frac{f(z)}{z-0}\, dz.$$

Since f is entire, we can invoke the Cauchy integral formula to conclude

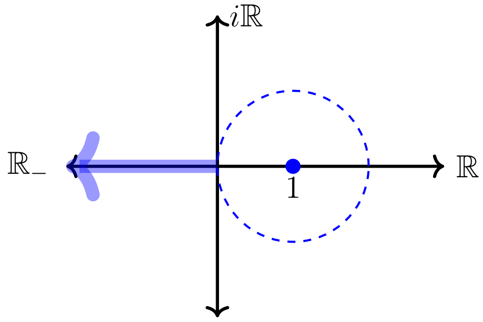
$$\int_{|z|=1} \frac{\cos(z)}{z}\, dz = 2\pi i f(0) = 2\pi i.$$ ▲

3.7. Analytic functions.

We will deploy the Cauchy integral formula to prove the surprising fact that any holomorphic function can be expressed as a power series and, consequently, is infinitely differentiable. We will rely on the conventions and standard results on power series from Appendix A.4.

3.7.1 Example.

The principal logarithm is defined on $\mathbb{C} \setminus \{0\}$ *and differentiable on* $\mathbb{C} \setminus \mathbb{R}_-$. *We know that* $\mathrm{Log}(\cdot)$ *is discontinuous on* $\mathbb{R}_-$ *and hence not differentiable on* $\mathbb{R}_-$. *Therefore, if we have a power series representation* $\mathrm{Log}(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ *valid for* $z \in \mathfrak{B}(z_0; r)$ *for some* $z_0 \in \mathbb{C}$ *and* $r > 0$, *it must be the case that* $\mathfrak{B}(z_0; r) \cap \mathbb{R}_- = \varnothing$. *On the other hand, if we consider instead the function* $f(z) = \mathrm{Log}(1+z)$, *then it is reasonable that* f *might have a power series expansion valid on* $\mathfrak{B}(0; 1)$. *Find it.*



Solution. We develop this power series in a rather roundabout way. The geometric series tells us that if $|z| < 1$, we have

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{k=0}^{\infty} (-z)^k = \sum_{k=0}^{\infty} (-1)^k z^k.$$

Now, since $\mathrm{Log}(\xi)$ is an antiderivative of $1/\xi$ on $\mathbb{C} \setminus \mathbb{R}_-$, it follows that $\mathrm{Log}(1+\xi)$ is an antiderivative of $1/(1+\xi)$ on $\mathbb{C} \setminus (-\infty, -1]$, and so the independence of path theorem implies

$$\int_{[0,z]} \frac{d\xi}{1+\xi} = \mathrm{Log}(1+z) - \mathrm{Log}(1) = \mathrm{Log}(1+z).$$

That is,

$$\mathrm{Log}(1+z) = \int_{[0,z]} \sum_{k=0}^{\infty} (-1)^k \xi^k\, d\xi = \sum_{k=0}^{\infty} \int_{[0,z]} (-1)^k \xi^k\, d\xi = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{k+1}}{k+1}\Bigg|_{\xi=0}^{\xi=z} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{k+1}}{k+1}.$$

Here we have used the convergence of the geometric series on $\mathfrak{B}(0;1)$ and that $[0,z]$ is a curve in $\mathfrak{B}(0;1)$ to justify the interchange of sum and integral. ▲

3.7.1. Real analyticity.

Let $I \subseteq \mathbb{R}$ be an interval. A function $f \colon I \to \mathbb{C}$ is **REAL**[52] **ANALYTIC** if f can be written "locally" as a power series on I. That is, for every $x \in I$, there is $r > 0$ and a sequence (a_k) in $\mathbb{C}$ such that

$$f(t) = \sum_{k=0}^{\infty} a_k (t-x)^k, \ t \in (x-r, x+r) \cap I. \tag{3.7.1}$$

The space of real analytic functions on I is sometimes denoted by $\mathcal{C}^{\omega}(I)$. From Theorem A.4.5, if $f \in \mathcal{C}^{\omega}(I)$, then $f \in \mathcal{C}^{\infty}(I)$, and, given the power series representation (3.7.1), we have

$$f^{(k)}(x) = k! a_k$$

Conversely, if $f \in \mathcal{C}^{\infty}(I)$, then for each $x \in I$ the **TAYLOR SERIES** of f centered at x is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (t-x)^k.$$

By the identity principle for power series (part (iv) of Theorem A.4.5, if $f \in \mathcal{C}^{\infty}(I)$ is real analytic on I, then, for each $x \in I$, the Taylor series for f centered at x must converge to f on a small interval centered at x.

This does not always happen. The classical counterexample is

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{3.7.2}$$

One can show (and this takes a bit of work due to the piecewise definition of f) that $f \in \mathcal{C}^{\infty}(\mathbb{R})$ and $f^{(k)}(0) = 0$ for all k. Thus the Taylor series of f centered at 0 converges to the zero function, whereas f is nonzero for all $x \neq 0$. And so, in general, $\mathcal{C}^{\infty}(I) \subsetneq \mathcal{C}^{\omega}(I)$. We will see that the situation is completely different for functions of a complex variable.

3.7.2. Complex analyticity.

We make a definition analogous to the notion of real analyticity. As for a function defined on a subset of $\mathbb{R}$, we let $\mathcal{C}^{\infty}(\mathcal{D})$ denote the space of all functions $f \colon \mathcal{D} \to C$ that are infinitely differentiable.

3.7.2 Definition.

(i) *Let $\mathcal{D} \subseteq \mathbb{C}$ be open. A function $f \colon \mathcal{D} \to \mathbb{C}$ is* **(COMPLEX) ANALYTIC** *on $\mathcal{D}$ if for all $z \in \mathcal{D}$, there exists $r > 0$ such that f has a power series expansion on $\mathfrak{B}(z;r) \cap \mathcal{D}$, i.e, there are coefficients (a_k) such that*

$$f(\xi) = \sum_{k=0}^{\infty} a_k (\xi - z)^k, \ \xi \in \mathfrak{B}(z;r) \cap \mathcal{D}. \tag{3.7.3}$$

[52] The adjective "real" here refers to the assumption that f is a function of a real variable. This permits f to take complex (nonreal) values. As we have seen throughout this course, the interesting changes in calculus arise when both the input and the output are complex.

(ii) *For $f \in \mathcal{C}^\infty(\mathcal{D})$, the* **(complex) Taylor series** *of f centered at z is the power series*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(\xi - z)^k.$$

Next, we state the complex analogue of Theorem A.4.5.

3.7.3 Theorem.

Let $z_0 \in \mathbb{C}$.

(i) [Term-by-term differentiation] *Suppose that $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ has the radius of convergence $\rho > 0$ and set $f(z) := \sum_{k=0}^{\infty} a_k(z-z_0)^k$ for $z \in \mathfrak{B}(z_0;\rho)$. Then f is a well-defined function and $f \in \mathcal{C}^\infty(\mathfrak{B}(z_0;\rho))$. In particular, f is continuous and*

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1}.$$

More generally,

$$f^{(m)}(z) = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)a_k(z-z_0)^{k-m}.$$

(ii) *Suppose that f is a function defined on $\mathfrak{B}(z_0;r)$ that has the power series expansion $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$, $z \in \mathfrak{B}(z_0;r)$. Then*

$$a_m = \frac{f^{(m)}(z_0)}{m!}.$$

(iii) [Term-by-term integration] *Suppose that $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ has the radius of convergence $r > 0$ and γ is a curve in $\mathfrak{B}(z_0;r)$. Then*

$$\int_\gamma \sum_{k=0}^{\infty} a_k(z-z_0)^k \, dz = \sum_{k=0}^{\infty} a_k \int_\gamma (z-z_0)^k \, dz.$$

(iv) [Identity principle for power series] *Suppose that the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_0)^k$ converge on some ball $\mathfrak{B}(z_0;\delta)$. If*

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} b_k(z-z_0)^k$$

for all $z \in \mathfrak{B}(z_0;\delta)$, then $a_k = b_k$ for all k.

Proof. We prove only part (iv); the proofs of the other parts are identical to their real-variable counterparts in Theorem A.4.5. Set $c_k = a_k - b_k$; our goal is to show $c_k = 0$ for all k. We have $f(z) := \sum_{k=0}^{\infty} c_k(z-z_0)^k = 0$ for all $z \in \mathfrak{B}(z_0;\delta)$. By part (i), we may differentiate to find $f^{(k)}(z) = 0$ for all $k \geq 1$ and $z \in \mathfrak{B}(z_0;\delta)$. Then part (ii) gives $c_k = f^{(k)}(z_0)/k! = 0$. ■

The next theorem then follows at once from Theorem 3.7.3.

3.7.4 Theorem.

If f is complex analytic on $\mathcal{D}$, then $f \in \mathcal{C}^\infty(\mathcal{D})$. Moreover, if for some ball $\mathfrak{B}(z;r) \subseteq \mathcal{D}$, the power series representation (3.7.3) of f holds for all $\xi \in \mathfrak{B}(z;r)$, then

$$a_k = \frac{f^{(k)}(z)}{k!}.$$

That is, if f is complex analytic on $\mathcal{D}$, then the complex Taylor series for f centered at z converges to $f(\xi)$ for all $\xi \in \mathcal{D}$ "close" to z.

Now we come to a striking consequence of the Cauchy integral formula. The next theorem says that if f is merely holomorphic on $\mathcal{D}$ (i.e., f is complex differentiable on $\mathcal{D}$), then f is in fact analytic on $\mathcal{D}$. Thus the existence of the first derivative f' on $\mathcal{D}$ implies both that f is $\mathcal{C}^\infty$ on $\mathcal{D}$ and that the Taylor series for f converges "locally" to f on $\mathcal{D}$.

Neither of these facts are true in the real case. The function

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

is differentiable but not twice differentiable on $\mathbb{R}$, and the function in (3.7.2) is $\mathcal{C}^\infty$ on $\mathbb{R}$ but not real analytic.

3.7.5 Theorem (Generalized Cauchy integral formula).

If f is holomorphic on an open set $\mathcal{D}$ (which need not be a domain or a star-domain), then f is also analytic (and thus $\mathcal{C}^\infty$) on $\mathcal{D}$. Specifically, if $\overline{\mathfrak{B}}(z_0;r) \subseteq \mathcal{D}$, then for $|z - z_0| < r$ we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad a_k := \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z_0)^{k+1}}\, d\xi = \frac{f^{(k)}(z_0)}{k!}. \tag{3.7.4}$$

Proof. The Cauchy integral formula tells us that if $|z - z_0| < r$, then

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{\xi - z}\, d\xi.$$

The fundamental idea of the proof is to rewrite the fraction $1/(\xi - z)$ in a suitable way, factor the result, and then use the geometric series (Example A.2.6) on one of the factors. We start with

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{(\xi - z_0)\left(1 - \dfrac{z - z_0}{\xi - z_0}\right)}.$$

Here we are assuming $|\xi - z_0| = r$ and $|z - z_0| < r$. Thus

$$\left| \frac{z - z_0}{\xi - z_0} \right| = \frac{|z - z_0|}{r} < 1.$$

The geometric series tells us

$$\frac{1}{1-\dfrac{z-z_0}{\xi-z_0}} = \sum_{k=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^k.$$

Now we can rewrite the Cauchy integral formula for $f(z)$ as

$$f(z) = \frac{1}{2\pi i}\int_{|\xi-z_0|=r}\frac{f(\xi)}{\xi-z}\,d\xi = \frac{1}{2\pi i}\int_{|\xi-z_0|=r}\left(\frac{f(\xi)}{\xi-z_0}\right)\frac{1}{1-\dfrac{z-z_0}{\xi-z_0}}\,d\xi$$

$$= \frac{1}{2\pi i}\int_{|\xi-z_0|=r}\frac{f(\xi)}{\xi-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^k d\xi. \quad (3.7.5)$$

Suppose we can "interchange the sum and the integral." That is, suppose

$$\int_{|\xi-z_0|=r}\frac{f(\xi)}{\xi-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^k d\xi = \sum_{k=0}^{\infty}\int_{|\xi-z_0|=r}\frac{f(\xi)}{\xi-z_0}\left(\frac{z-z_0}{\xi-z_0}\right)^k d\xi. \quad (3.7.6)$$

The validity of the interchange (3.7.6) of sum and integral is reminiscent of part (iii) of Theorem 3.7.3, but it does not follow from that part, which deals with power series; in (3.7.6), the powers of of $(\xi - z_0)^k$ are in the denominator. The proof of this validity, while not particularly difficult, is rather technical, so we omit it.

We can factor $(z-z_0)^k$ out of the integral in (3.7.6) to find

$$\sum_{k=0}^{\infty}\int_{|\xi-z_0|=r}\frac{f(\xi)}{\xi-z_0}\left(\frac{z-z_0}{\xi-z_0}\right)^k d\xi = \sum_{k=0}^{\infty}(z-z_0)^k\int_{|z-z_0|=r}\frac{f(\xi)}{(\xi-z_0)^{k+1}}\,d\xi. \quad (3.7.7)$$

We put (3.7.5), (3.7.6), and (3.7.7) together to find

$$f(z) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(\xi)}{(\xi-z_0)^{k+1}}\,d\xi\right)(z-z_0)^k.$$

That is, for $z \in \mathfrak{B}(z_0;r)$ we have expressed $f(z)$ as a power series $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ with

$$a_k = \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(\xi)}{(\xi-z_0)^{k+1}}\,d\xi.$$

We conclude that f is analytic on $\mathcal{D}$, and part (ii) of Theorem 3.7.3 gives the equality $a_k = f^{(k)}(z_0)/k!$. ■

3.7.6 Corollary.

(i) *Suppose that f is holomorphic on a set $\mathcal{D}$ containing the point z_0. The radius of convergence of the Taylor series of f centered at z_0 is the largest $r > 0$ such that f is differentiable on $\mathfrak{B}(z_0;r)$.*

(ii) *If f is entire, then for any $z_0 \in \mathbb{C}$ we have*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

for all $z \in \mathbb{C}$. That is, the Taylor series of f centered at any point z_0 converges to f for all $z \in \mathbb{C}$.

3.7.7 Remark.

As much as possible, we should avoid using the cumbersome definition (3.7.4) *for the coefficients of the power series of the analytic function f centered at z_0. Instead, if we need to calculate a power series, we should try to relate it to one that we already know, or, if we know how the derivatives of the function behave, we should try to calculate derivatives at z_0.*

3.7.8 Remark.

From now on, we will use the words "holomorphic" and "analytic" interchangeably. However, we will never talk about a function being analytic "at a point," only on an open set. The function $f(z) = |z|^2$ is holomorphic only at $z = 0$ but not analytic on any set containing 0 because f is not differentiable at a point other than 0.

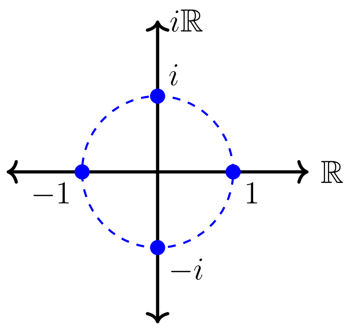
3.7.9 Example.

Let $f(z) = 1/(1 + z^2)$, so f is defined and holomorphic on $\mathbb{C} \setminus \{i, -i\}$. Find the power series for f centered at 0 and the largest $r > 0$ for which the series converges on $\mathfrak{B}(0; r)$.

Solution. Observe that if $|z| < 1$, then $|z|^2 < 1$, and so the geometric series gives

$$f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty}(-z^2)^k = \sum_{k=0}^{\infty}(-1)^k z^{2k}.$$

That is, the series converges on at least $\mathfrak{B}(0; 1)$. If $|z| \geq 1$, then the test for divergence shows that this series diverges, so we know that the series converges at best on $\mathfrak{B}(0; 1)$.



But we know this without the test for divergence: if the series converges on some $\mathfrak{B}(0; r)$ with $r > 1$, then $\pm i \in \mathfrak{B}(0; r)$, yet f is not differentiable, let alone defined, at $\pm i$. ▲

3.7.10 Example.

Recall that a **POLYNOMIAL** *is a function of the form*

$$p(z) = \sum_{k=0}^{n} a_k z^k,$$

where $a_k \in \mathbb{C}$. If $a_n \neq 0$, then the **DEGREE** *of p is the nonnegative integer n. Show that given $z_0 \in \mathbb{C}$, any polynomial p of degree n can be written in the form*

$$p(z) = \sum_{k=0}^{n} b_k (z - z_0)^k, \qquad b_k = \frac{p^{(k)}(z_0)}{k!}.$$

Solution. It is obvious from the power rule for derivatives that p is entire. Then for any $z_0 \in \mathbb{C}$, the function p is holomorphic on an open set containing $\mathfrak{B}(z_0; 1)$; take, for example, the open set to be all of $\mathbb{C}$. Then

$$p(z) = \sum_{k=0}^{\infty} \frac{p^{(k)}(z_0)}{k!}(z - z_0)^k, \ z \in \mathfrak{B}(z_0; 1). \tag{3.7.8}$$

Since p is a polynomial of degree n, we have $p^{(k)}(z) = 0$ for $k \geq n + 1$ and all $z \in \mathbb{C}$, hence

$$\sum_{k=0}^{\infty} \frac{p^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{n} \frac{p^{(k)}(z_0)}{k!}(z - z_0)^k, \ z \in \mathfrak{B}(z_0; 1).$$

The restriction to the ball of radius 1 centered at z_0 is highly artificial. Let $r > 1$, so p is still holomorphic on the open set $\mathbb{C}$ containing the ball $\mathfrak{B}(z_0; r)$, and therefore the power series representation (3.7.8) holds on $\mathfrak{B}(z_0; r)$. Since r is arbitrary, the power series representation holds for all $z \in \mathbb{C}$. ▲

3.7.11 Example.

Evaluate $\displaystyle\int_{|z|=3} \frac{e^{-z}}{z^3}\, dz.$

Solution. Let $f(z) = e^{-z}$, so that

$$\int_{|z|=3} \frac{e^{-z}}{z^3}\, dz = \int_{|z-0|=3} \frac{f(z)}{(z-0)^{2+1}}\, dz =: \mathcal{I}.$$

We recognize this integral as a constant multiple of an integral from the Cauchy integral formula. Specifically, since f is entire, we know

$$f''(0) = \frac{2!}{2\pi i} \int_{|z-0|=3} \frac{f(z)}{(z-0)^{2+1}}\, dz = \frac{\mathcal{I}}{\pi i}.$$

Thus

$$\mathcal{I} = \pi i f''(0),$$

where $f''(z) = e^{-z}$. Thus

$$\mathcal{I} = \pi i e^0 = \pi i.$$ ▲

3.7.12 Example.

What can you say about solutions to the ODE

$$f''(z) + (1 + e^{-z^2})f(z) = |z|^2?$$

Solution. We claim there are no solutions. If a solution f exists on some open set $\mathcal{D}$ in $\mathbb{C}$, then f is twice-differentiable, and therefore holomorphic, and so f'' is holomorphic. The product $z \mapsto (1+e^{-z^2})f(z)$ is also holomorphic, and so the sum $z \mapsto f''(z)+(1+e^{-z^2})f(z)$ is holomorphic. Consequently, $z \mapsto |z|^2$ is holomorphic. But the forcing function $z \mapsto |z|^2$ is real-valued and nonconstant and therefore not holomorphic by the Cauchy-Riemann equations. ▲

The integral coefficient in (3.7.4) requires the factor $(\xi - z_0)^{k+1}$ in the denominator of the integrand to have the same constant z_0 as the center of the circle $|\xi - z_0| = r$ around which we integrate. Contrast this with the Cauchy integral formula (3.6.5), where the denominator is $\xi - z$ and the contour is $|\xi - z_0| = r$. Our formula (3.7.4) turns out to be unnecessarily restrictive, in that we ultimately do not need to use the same z_0 twice.

3.7.13 Corollary (Generalized Cauchy integral formula, again).

Let f be holomorphic on the open set $\mathcal{D}$ and let $z_0 \in \mathcal{D}$, $r > 0$ with $\overline{\mathfrak{B}}(z_0; r) \subseteq \mathcal{D}$. Then

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z)^{k+1}} \, d\xi, \quad |z - z_0| < r.$$

Proof. Let $s > 0$ such that $\overline{\mathfrak{B}}(z; s) \subseteq \mathfrak{B}(z_0; r)$. The generalized Cauchy integral formula implies

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\xi - z| = s} \frac{f(\xi)}{(\xi - z)^{k+1}} \, d\xi.$$

Put $g(\xi) := f(\xi)/(\xi - z)^{k+1}$. Then g is holomorphic on $\mathcal{D} \setminus \{z\}$. Lemma 3.6.15 implies

$$\int_{|\xi - z| = s} g(\xi) \, d\xi = \int_{|\xi - z_0| = r} g(\xi) \, d\xi,$$

and this gives the desired equality. ■

We have previously remarked that differentiability is a more "stringent" property for a function of a complex variable than for a function of a real variable: if f is complex differentiable, then f is both infinitely differentiable and complex analytic. It is also much harder for a function of a complex variable to have an antiderivative. Recall that any continuous function g of a real variable has an antiderivative, even if g itself is not differentiable. The same is not true for functions of a complex variable. We conclude our quest for complex antiderivatives by (partially) characterizing their existence in terms of complex differentiability, instead of independence of path.

3.7.14 Theorem.

Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let f be a function defined on $\mathcal{D}$.

(i) *If f has an antiderivative F on $\mathcal{D}$, then f is holomorphic on $\mathcal{D}$.*

(ii) *If f is holomorphic on $\mathcal{D}$, and if $\mathcal{D}$ is a star domain, then f has an antiderivative on $\mathcal{D}$.*

Proof. **(i)** By definition, F is holomorphic on $\mathcal{D}$, and so F is analytic on $\mathcal{D}$. In particular, F'' exists. Since $F' = f$, the derivative f' must also exist (and equal F''). Hence f is holomorphic, too.

(ii) This is the Cauchy integral theorem. ■

We should be careful with the flow of logic in the hypotheses of part (ii) of Theorem 3.7.14. It says that if $\mathcal{D}$ is a star domain, then f has an antiderivative on $\mathcal{D}$ if and only if f is holomorphic on $\mathcal{D}$. However, it does not say what happens if $\mathcal{D}$ is not a star domain: f could have an antiderivative ($f(z) = z$ and $\mathcal{D} = \mathbb{C} \setminus \{0\}$) or not ($f(z) = 1/z$, still on $\mathcal{D} \setminus \{0\}$). If we return to independence of path, as long as $\mathcal{D}$ is a domain and f is independent of path on $\mathcal{D}$, equivalently, if f integrates to 0 over closed curves in $\mathcal{D}$, then f has an antiderivative on $\mathcal{D}$. A domain $\mathcal{D}$ on which every holomorphic function is independent of path is sometimes called an **ELEMENTARY DOMAIN** or a **SIMPLY CONNECTED** domain; roughly, such domains do not contain any "holes" — so an annulus like $\mathbb{C}\setminus\{0\}$ is not an elementary domain, but a "slitted" domain like $\mathbb{C}\setminus(-\infty, 0]$ is elementary.

3.8. Further consequences of Cauchy's formula.

3.8.1. Liouville's theorem.

First, we have a highly useful estimate that controls the derivatives of a function at a given point.

3.8.1 Lemma (Cauchy estimates).

Suppose that f is analytic on the domain $\mathcal{D}$ and $z_0 \in \mathcal{D}$ with $\overline{\mathfrak{B}}(z_0; r) \subseteq \mathcal{D}$. Then

$$|f^{(k)}(z_0)| \le \frac{k!}{r^k} \max_{|z-z_0|=r} |f(z)|$$

for each $k \in \mathbb{N}$.

3.8.2 Theorem (Liouville).

Every bounded entire function is constant. That is, if f is analytic on $\mathbb{C}$ and if there is some $M > 0$ such that $|f(z)| \le M$ for all $z \in \mathbb{C}$, then there is $c \in \mathbb{C}$ such that $f(z) = c$ for all z.

Proof. It suffices to show that $f'(z) = 0$ for all $z \in \mathbb{C}$, for then, since $\mathbb{C}$ is star-shaped with star-center 0 (or any point in $\mathbb{C}$ that we like) we have

$$f(z) - f(0) = \int_{[0,z]} f'(\xi)\, d\xi = 0.$$

That is, $f(z) = f(0)$ for all z, and so f is constant.

So, assume $|f(z)| \le M$ for all $z \in \mathbb{C}$ and fix $z_0 \in \mathbb{C}$. We will show $f'(z_0) = 0$. Let $r > 0$ be arbitrary. Since f is entire, the Cauchy estimates show

$$|f'(z_0)| \le \frac{1}{r} \max_{|z-z_0|=r} |f(z)| \le \frac{M}{r}.$$

Since r was arbitrary, we have

$$0 \le |f'(z_0)| \le \lim_{r\to\infty} \frac{M}{r} = 0,$$

and so the squeeze theorem implies $|f'(z_0)| = 0$, thus $f'(z_0) = 0$. ■

Conversely, if f is entire and not constant, it is very much unbounded!

3.8.3 Theorem (Picard's little theorem).

If f is entire and not constant, then f assumes all values in $\mathbb{C}$ with at most one exception. That is, either

$$\{f(z) \mid z \in \mathbb{C}\} = \mathbb{C}$$

or there exists $w \in \mathbb{C}$ such that

$$\{f(z) \mid z \in \mathbb{C}\} = \mathbb{C} \setminus \{w\}.$$

3.8.2. The fundamental theorem of algebra.

The fundamental theorem of algebra has several different phrasings and many proofs. Recall that a **POLYNOMIAL** with coefficients in $\mathbb{C}$ is a function p of the form

$$p(z) = \sum_{k=0}^{n} a_k z^k, \tag{3.8.1}$$

where $a_k \in \mathbb{C}$ and $a_n \neq 0$. Also, recall that the **DEGREE** of p is n. Here is one of those phrasings.

3.8.4 Lemma (Fundamental theorem of algebra, preliminary version).

Every polynomial of degree at least 1 with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$. That is, if p is a function of the form (3.8.1), then there exists $z_0 \in \mathbb{C}$ with $p(z_0) = 0$.

Proof. Suppose to the contrary that $p(z) \neq 0$ for all $z \in \mathbb{C}$ and let $q(z) = 1/p(z)$. Then q is a rational function whose denominator is never zero, so the quotient rule implies that q is entire. Suppose for the moment that q is bounded, i.e., there is $M > 0$ such that $|q(z)| \le M$ for all $z \in \mathbb{C}$. Then q is both bounded and entire, so by Liouville's theorem q is constant. That, there exists $c \in \mathbb{C}$ such that $q(z) = c$ for all $z \in \mathbb{C}$. Rearranging, we find $p(z) = 1/c$ for all $z \in \mathbb{C}$. But then p is not a polynomial of degree at least 1, a contradiction.

So, it suffices to show that q is bounded. Our intuition from real-variable calculus tells us this is true; if p has real coefficients, then taking x to be real, we recall

$$\lim_{x\to\pm\infty} \frac{1}{p(x)} = 0.$$

In particular, there is $r_0 > 0$ such that if $|x| \geq r_0$, then $1/|p(x)| \leq 1$. On the other hand, for $|x| \leq r_0$, there is some $m > 0$ such that $|1/p(x)| \leq m$ since $1/p$ is continuous on the closed interval $[-r_0, r_0]$. Effectively the same argument[53] translates over to the complex case to show that q is bounded; we omit the formal details. ■

3.8.5 Lemma.

Suppose that p is a polynomial of degree $n \geq 1$ and $p(z_0) = 0$. Then there is a polynomial q of degree $n-1$ such that $p(z) = (z - z_0)q(z)$.

Proof. Since p is a polynomial of degree n, we have $p(z) = \sum_{k=0}^{n} a_k z^k$ for some coefficients $a_k \in \mathbb{C}$. However, it is more expeditious to expand p as a power series centered at z_0:

$$p(z) = \sum_{k=0}^{n} b_k (z - z_0)^k, \qquad b_k = \frac{p^{(k)}(z_0)}{k!}.$$

Then

$$0 = p(z_0) = b_0,$$

so really

$$p(z) = \sum_{k=1}^{n} b_k (z - z_0)^k = (z - z_0) \sum_{k=1}^{n} b_k (z - z_0)^{k-1} = (z - z_0) \sum_{k=0}^{n-1} b_{k+1} (z - z_0)^k.$$

---

[53]Here is that argument. The reverse triangle inequality implies

$$|p(z)| = \left|\sum_{k=0}^{n} a_k z^k\right| = \left|a_n z^n - \left(\sum_{k=0}^{n-1} a_k z^k\right)\right| \geq \left||a_n z^n| - \left|\sum_{k=0}^{n-1} a_k z^k\right|\right| \geq |a_n||z^n| - \left|\sum_{k=0}^{n-1} a_k z^k\right|.$$

Now, the (ordinary) triangle inequality implies

$$\left|\sum_{k=0}^{n-1} a_k z^k\right| \leq \sum_{k=0}^{n-1} |a_k z^k| = \sum_{k=0}^{n-1} |a_k||z^k| \implies -\sum_{k=0}^{n-1} |a_k||z^k| \leq -\left|\sum_{k=0}^{n-1} a_k z^k\right|.$$

Thus

$$|p(z)| \geq |a_n||z^n| - \left|\sum_{k=0}^{n-1} a_k z^k\right| \geq |a_n||z^n| - \sum_{k=0}^{n-1} |a_k||z^k|. \tag{3.8.2}$$

Observe next that, for $r \neq 0$,

$$|a_n| r^n - \sum_{k=0}^{n-1} |a_k| r^k = r^n \left(|a_n| - \sum_{k=0}^{n-1} |a_k| r^{k-n}\right),$$

and, for r real,

$$\lim_{r \to \infty} \sum_{k=0}^{n-1} |a_k| r^{k-n} = 0.$$

So, there exists $r_0 > 0$ such that if $r > r_0$, then

$$0 < \sum_{k=0}^{n-1} |a_k| r^{k-n} < \frac{|a_n|}{2}, \implies |a_n| r^n - \sum_{k=0}^{n-1} |a_k| r^k \geq \frac{r^n |a_n|}{2}.$$

It follows from (3.8.2) that for $|z| > r_0$,

$$|p(z)| \geq \frac{|z|^n |a_n|}{2} \implies \frac{1}{|p(z)|} \leq \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n| r_0^n}.$$

It is also the case that there exists $m > 0$ with $|1/p(z)| \leq m$ for $|z| \leq r_0$. This is a consequence of the extreme value theorem from real-variable calculus, now extended to continuous complex-valued functions on "closed and bounded" subsets of $\mathbb{C}$, of which the closed ball $\overline{\mathfrak{B}}(0; r_0)$ is a prime example. We did not discuss an extreme value theorem for functions on $\mathbb{C}$, and we will not need one later in the course.

Setting $q(z) = \sum_{k=0}^{n-1} b_{k+1}(z - z_0)^k$, we are done. ■

3.8.6 Theorem (Fundamental theorem of algebra).

Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree* $n \geq 1$ *with coefficients* $a_k \in \mathbb{C}$. *Then there exist* $z_1, \ldots, z_n \in \mathbb{C}$ *such that* $p(z) = 0$ *if and only if* $z = z_n$. *Moreover, if* $a_n = 1$, *then*

$$p(z) = (z - z_1) \cdots (z - z_n). \tag{3.8.3}$$

The numbers z_k *need not be distinct (i.e., it is possible* $z_j = z_\ell$ *for* $j < \ell$*).*

Proof. By Lemma 3.8.4 there is $z_1 \in \mathbb{C}$ such that $p(z_1) = 0$, and by Lemma 3.8.5 we can write $p(z) = (z - z_1)q_1(z)$ for some polynomial q_1 of degree $n - 1$. If $n = 1$, then we are done. Otherwise, q_1 has a zero z_2, which may or may not be equal to z_1, and so $q_1(z) = (z - z_2)q_2(z)$ for some polynomial of degree $n - 2$. If $n = 2$, then we are done; otherwise, we continue this procedure until we have produced n (not necessarily distinct) zeros and factored p in the form (3.8.3). ■

3.8.3. The zeros of an analytic function: the order of a zero.

Analytic functions resemble polynomials in that analytic functions are power series, which are "infinite" polynomials. The behavior of the zeros of an analytic function, unsurprisingly then, rather resembles the behavior of zeros, or roots, of polynomials.

First, we just saw that a polynomial always has a finite number of zeros, and so these zeros are all spaced a minimum distance apart. That is, they are "isolated" from each other. While an analytic function can have infinitely many zeros (for example, $\sin(\cdot)$ certainly does), they too are always isolated — not necessarily in the sense that there is a number $\delta > 0$ such that the zeros are always at least a distance δ apart from each other, but that we can draw a ball of a certain radius around a zero and have no other zeros in that ball.

Next, if $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, and $p(z_0) = 0$, then we can factor $p(z) = (z - z_0)q_{n-1}(z)$, where q_{n-1} is a polynomial of degree $n - 1$. More precisely, there is a positive integer $m \leq n$, called the MULTIPLICITY of z_0 as a root of p, such that $p(z) = (z - z_0)^m Q_m(z)$, where Q_m is a polynomial of degree $n - m$ and $Q_m(z_0) \neq 0$. We will show that an analytic function f with $f(z_0) = 0$ admits a similar factorization, where Q_m is replaced by an analytic function that does not vanish at z_0. Of course, we need to define what the "multiplicity of z_0 as a zero of an analytic function" means.

3.8.7 Definition.

Suppose that f *is a function defined on* $\mathcal{D} \subseteq \mathbb{C}$. *We say that* f *is* IDENTICALLY ZERO *on* $\mathcal{D}$ *if* $f(z) = 0$ *for all* $z \in \mathcal{D}$.

3.8.8 Lemma.

Suppose that f *is analytic on the domain* $\mathcal{D}$ *and there is* $z_0 \in \mathcal{D}$ *such that* $f^{(k)}(z_0) = 0$ *for all* k. *Then* f *is identically zero on* $\mathcal{D}$.

Proof. If $r > 0$ such that $\mathfrak{B}(z_0; r) \subseteq \mathcal{D}$, then, for $z \in \mathfrak{B}(z_0; r)$, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = 0.$$

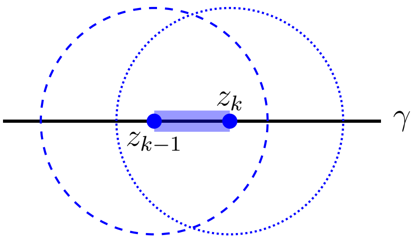
Now we need to show that for $z \in \mathcal{D}$ with z not necessarily in $\mathfrak{B}(z_0; r)$, we still have $f(z) = 0$. Fix $z \in \mathcal{D}$. Since $\mathcal{D}$ is a domain, there is a curve $\gamma\colon [0,1] \to \mathcal{D}$ such that $\gamma(0) = z_0$ and $\gamma(1) = z$.

We will "cover" γ with a finite number of open balls whose centers lie on γ and whose pairwise "overlap" behaves in a special way. We claim there is a number $\delta > 0$ and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ with the following properties.

(1) $\mathfrak{B}(\gamma(t_j); \delta) \subseteq \mathcal{D}$ for $j = 0, \ldots, n$.

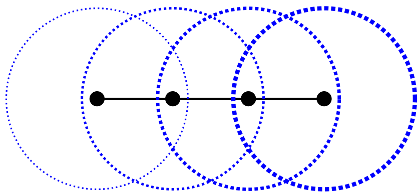
(2) $\{\gamma(t) \mid t_{j-1} \le t \le t_j\} \subseteq \mathfrak{B}(\gamma(t_{j-1}); \delta) \cap \mathfrak{B}(\gamma(t_j); \delta)$ for $j = 1, \ldots, n$.

In other words, the portion of γ parametrized over $[t_{j-1}, t_j]$ lies in both $\mathfrak{B}(\gamma(t_{j-1}); \delta)$ and $\mathfrak{B}(\gamma(t_j); \delta)$. A proof of this claim depends on some fundamental topological properties of the real and complex numbers, so we must omit it. Hopefully, the pictures below illustrate the situation.



$z_{k-1} = \gamma(t_{k-1})$

$z_k = \gamma(t_k)$



Assuming this claim to be true, note that $\gamma(t_0) = \gamma(1) = z_0$, so $f(z) = 0$ for all $z \in \mathfrak{B}(\gamma(t_0); \delta)$. In particular, (2) implies that $\gamma(t_1) \in \mathfrak{B}(\gamma(t_0); \delta)$. By the identity principle, $f^{(k)}(\gamma(t_1)) = 0$ for all k. By the generalized Cauchy integral formula, for any $z \in \mathfrak{B}(\gamma(t_1); \delta)$, we then have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\gamma(t_1))}{k!}(\gamma(t_1) - z) = 0.$$

In particular, $\gamma(t_2) \in \mathfrak{B}(\gamma(t_1); \delta)$, and the same logic shows $f^{(k)}(\gamma(t_2)) = 0$ for all k. Continuing in this manner, we obtain $f^{(k)}(\gamma(t_j)) = 0$ for $j = 0, \ldots, n$ and all k. In particular, $f(\gamma(t_n)) = f(z) = 0$. ■

Now we are ready to extend the concept of multiplicity of a root to analytic functions.

3.8.9 Theorem.

Let f be analytic on the domain $\mathcal{D}$ with $f(z_0) = 0$ for some $z_0 \in \mathcal{D}$. If f is not identically zero on $\mathcal{D}$, then there is an analytic function g defined on $\mathcal{D}$ and a positive integer m such that $f(z) = (z - z_0)^m g(z)$ for all $z \in \mathcal{D}$. The integer m is the smallest integer satisfying $f^{(m)}(z_0) \neq 0$. Moreover, $g(z_0) \neq 0$. In this case, we say that f has a **ZERO OF ORDER** m **AT** z_0.

Proof. Such a function g must satisfy

$$g(z) = \frac{f(z)}{(z-z_0)^m}, \ z \in \mathcal{D} \setminus \{z_0\}, \tag{3.8.4}$$

and the quotient rule tells us that g as defined by (3.8.4) is analytic on $\mathcal{D} \setminus \{z_0\}$. Now we need to (1) define g at z_0 and (2) show that g is analytic at z_0, too.

Since f is not identically zero on $\mathcal{D}$, we cannot have $f^{(k)}(z_0) = 0$ for all k. Let m be the smallest[54] positive integer such that $f^{(m)}(z_0) \neq 0$. Since m is the smallest positive integer with this property, we have $f^{(k)}(z_0) = 0$ for $k = 1, \ldots, m-1$. Let $r > 0$ satisfy $\mathfrak{B}(z_0; r) \subseteq \mathcal{D}$. Then, for $z \in \mathfrak{B}(z_0; r)$, we have

$$f(z) = \sum_{k=m}^{\infty} a_k(z-z_0)^k, \qquad a_k := \frac{f^{(k)}(z_0)}{k!}.$$

We factor

$$f(z) = \sum_{k=m}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_{k+m}(z-z_0)^{k+m} = (z-z_0)^m \sum_{k=0}^{\infty} a_{k+m}(z-z_0)^k. \tag{3.8.5}$$

For $z \in \mathfrak{B}^*(z_0; r)$, we therefore have

$$g(z) = \frac{f(z)}{(z-z_0)^m} = \sum_{k=0}^{\infty} a_{k+m}(z-z_0)^k. \tag{3.8.6}$$

That is, the series $\sum_{k=0}^{\infty} a_{k+m}(z-z_0)^k$ converges on $\mathfrak{B}^*(z_0; r)$ to $f(z)/(z-z_0)^m$. The series of course converges at $z = z_0$ to $a_m = f^{(m)}(z_0)/m!$. So, if we extend our definition (3.8.4) of g to

$$g(z) := \begin{cases} \dfrac{f(z)}{(z-z_0)^m}, \ z \in \mathcal{D} \setminus \{z_0\} \\[2ex] a_m = \dfrac{f^{(m)}(z_0)}{m!}, \ z = z_0, \end{cases}$$

then (3.8.6) tells us that g has a power series representation on $\mathfrak{B}(z_0; r)$ and therefore is analytic at z_0, too. ■

A case worth highlighting for future theory and applications is the situation when a zero has order 1.

3.8.10 Definition.

If z_0 is a zero of order 1 for f, then z_0 is a **SIMPLE ZERO** *of f.*

[54] If $J \subseteq \mathbb{N}$, then the **WELL-ORDERING PROPERTY** of $\mathbb{N}$ states that J has a smallest element, i.e., there exists $m \in J$ such that $m \leq k$ for all $k \in J$. Note that the well-ordering principle is not true if $\mathbb{N}$ is replaced by $\mathbb{Z}$ or $\mathbb{Q}$.

3.8.11 Example.

Find the order of each of the zeros of the following functions.

(i) $f(z) = z^2$

(ii) $g(z) = \sin(z)$

(iii) $h(z) = (e^z - 1)^2$

Solution. (i) We know $f(z) = 0$ if and only if $z = 0$. We calculate $f'(z) = 2z$, so $f'(0) = 0$. Next, $f''(z) = 2$, so $f''(0) = 2 \neq 0$. Hence f has a zero of order 2 at 0.

(ii) Our work with trig functions told us that $\sin(z) = 0$ if and only if $z = k\pi$ for some integer $k \in \mathbb{Z}$. We have $g'(z) = \cos(z)$, and we know $\cos(k\pi) = (-1)^k \neq 0$. So, g has a zero of order 1 at each point $k\pi$.

(iii) We have $h(z) = 0$ if and only if $e^z - 1 = 0$, so we need $e^z = 1$. This happens for $z = 2\pi ik =: z_k$, $k \in \mathbb{Z}$. We calculate

$$h'(z) = 2(e^z - 1)e^z = 2e^{2z} - 2e^z,$$

so

$$h'(z_k) = 2e^{2z_k} - 2e^{z_k} = 2e^{4\pi ik} - 2e^{2\pi ik} = 2 - 2 = 0.$$

However,

$$h''(z) = 4e^{2z} - 2e^z,$$

so

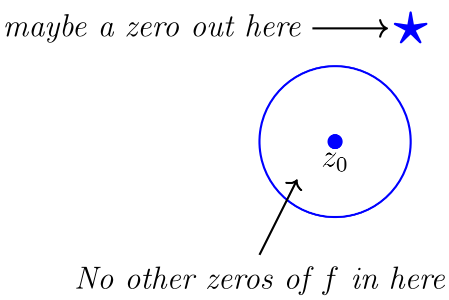
$$h'(z_k) = 4e^{2z_k} - 2e^{z_k} = 4 - 2 = 2 \neq 0.$$

Hence each of these zeros has order 2. ▲

3.8.4. The zeros of an analytic function: isolated zeros.

3.8.12 Definition.

Suppose that f is a function defined on $\mathcal{D} \subseteq \mathbb{C}$ and $f(z_0) = 0$ for some $z_0 \in \mathcal{D}$. The point z_0 is an **ISOLATED ZERO** *of f if there is $r > 0$ such that $f(z) \neq 0$ for $z \in \mathfrak{B}^*(z_0; r)$.*



3.8.13 Theorem.

Suppose that f is analytic on the domain $\mathcal{D}$ and not identically zero. Then every zero of f in $\mathcal{D}$ is isolated in $\mathcal{D}$.

Proof. Suppose that z_0 is a zero of f that is not isolated. Then for every $r > 0$, there is $z \in \mathfrak{B}^*(z_0; r) \cap \mathcal{D}$ such that $f(z) = 0$. In particular, we may take $r = 1/n$ for $n \in \mathbb{N}$ and find $z_n \in \mathfrak{B}^*(z_0; 1/n) \cap \mathcal{D}$ such that $f(z_n) = 0$. Observe that

$$0 < |z_0 - z_n| < \frac{1}{n} \Longrightarrow \lim_{n\to\infty} z_n = z_0.$$

Let m be the order of z_0 as a zero of f, and write $f(z) = (z - z_0)^m g(z)$, where g is analytic on $\mathcal{D}$ and $g(z_0) \neq 0$. Since $z_n \neq z_0$ for all n, we have

$$g(z_n) = \frac{f(z_n)}{(z_n - z_0)^m} = 0.$$

Then since $z_n \to z_0$ and g is analytic, hence continuous, we have

$$g(z_0) = \lim_{n\to\infty} g(z_n) = 0,$$

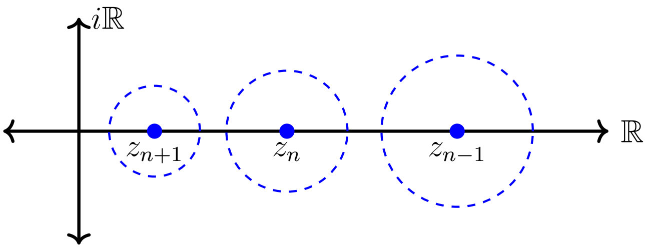
a contradiction. ■

3.8.14 Example.

Let $f(z) = \sin(\pi/z)$ for $z \in \mathcal{D} \setminus \{0\}$. Then $f(z) = 0$ if and only if $z = 1/n =: z_n$ for some $n \in \mathbb{Z} \setminus \{0\}$. Observe that the the points $\{z_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ are isolated in $\mathcal{D}$. For example, if $|n| \geq 2$, take

$$r_n := \min\left\{\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right), \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right)\right\}$$

to see that $z_k \notin \mathfrak{B}(z_n; r_n)$ for $k \neq n$.



Note, though, that $r_n \to 0$ as $|n| \to \infty$, so there is no "minimum distance" separating the zeros of f.

The next corollary says that two analytic functions need only agree on a "small" number of points in $\mathcal{D}$ to be equal everywhere in $\mathcal{D}$.

3.8.15 Corollary.

Suppose that f and g are analytic on the domain $\mathcal{D}$ and there is a sequence of distinct[55] points (z_n) in $\mathcal{D}$ such that $f(z_n) = g(z_n)$ for all n that converges to some $z_0 \in \mathcal{D}$. Then $f(z) = g(z)$ for all $z \in \mathcal{D}$.

Proof. Let $h(z) = f(z) - g(z)$. We will show that $h(z) = 0$ for all $z \in \mathcal{D}$. Since f and g are analytic on $\mathcal{D}$, they are continuous there, and so

$$f(z_0) = \lim_{n\to\infty} f(z_n) \quad \text{and} \quad g(z_0) = \lim_{n\to\infty} g(z_n),$$

thus

$$h(z_0) = \lim_{n\to\infty} h(z_n) = 0.$$

Now, the convergence of (z_n) to $z_0 \in \mathcal{D}$ means that for all $r > 0$, there is $N \in \mathbb{N}$ such that if $n \geq N$, then $z_n \in \mathfrak{B}(z_0; r)$. Since the points z_n are all distinct, for n large we have $z_n \neq z_0$, and so what we really see is that for all $r > 0$, there is $z_n \in \mathfrak{B}^*(z_0; r)$, where $h(z_n) = h(z_0) = 0$. So, the zero z_0 of h is not isolated. By Theorem 3.8.13, it must be the case that $h(z) = 0$ for all $z \in \mathcal{D}$. ■

3.8.16 Remark.

Why do we require the z_n in Corollary 3.8.15 to be distinct? There are plenty of functions f and g for which $f(z_1) = g(z_1)$ for some z_1 but no others, e.g., $f(z) = z$ and $g(z) = 2z$. Thus taking $z_n = z_1$ for all n could produce wrong results!

3.8.17 Example.

Let $f(z) = \sin(\pi/z)$, so f is analytic on the domain $\mathcal{D} := \mathbb{C} \setminus \{0\}$. Let $z_n = 1/n$ for $n \in \mathbb{N}$. Then $z_n \to 0 \notin \mathcal{D}$ and $f(z_n) = 0$. But f is not identically zero on $\mathcal{D}$. So, in Corollary 3.8.15, it is important that $z_0 \in \mathcal{D}$.

3.8.5. Analytic continuation.

We extended the definitions of the familiar transcendental functions (exponential, sine, cosine) to $\mathbb{C}$ simply by using their real Taylor series. One might reasonably ask if there is any other "natural" way to extend, say, the exponential to complex inputs that would create a different function. That is, does there exist some analytic function f defined on $\mathbb{C}$ such that $f(x) = e^x$ for $x \in \mathbb{R}$ but, for some $z \in \mathbb{C}$, $f(z) \neq e^z$? Fortunately, the answer is no, as we now develop.

3.8.18 Definition.

Let $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \mathbb{C}$. A function $\widetilde{f}$ defined on $\mathcal{D}$ is an ANALYTIC CONTINUATION *of an analytic function f on $\mathcal{D}_0$ if $\widetilde{f}$ is analytic and if $f(z) = \widetilde{f}(z)$ for all $z \in \mathcal{D}_0$.*

3.8.19 Theorem.

Suppose that $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \mathbb{C}$ are both domains and the analytic function $f\colon \mathcal{D}_0 \to \mathbb{C}$ has the analytic continuation $\widetilde{f}\colon \mathcal{D} \to \mathbb{C}$. Then $\widetilde{f}$ is unique. That is, if $g\colon \mathcal{D} \to \mathbb{C}$ is another analytic continuation of f, then $g(z) = \widetilde{f}(z)$ for all $z \in \mathcal{D}$.

Proof. Since $\widetilde{f}$ and g are both analytic continuations of f, we have $f(z) = \widetilde{f}(z)$ and $f(z) = g(z)$ for all $z \in \mathcal{D}$. This shows $f^{(k)}(z_0) = \widetilde{f}^{(k)}(z_0) = g^{(k)}(z_0)$ for all $z \in \mathcal{D}$. Now fix some $z_0 \in \mathcal{D}$. Then $\widetilde{f}^{(k)}(z_0) - g^{(k)}(z_0) = 0$ for all k. If we set $h(z) := \widetilde{f}(z) - g(z)$, then $h^{(k)}(z_0) = 0$ for all k. Lemma 3.8.8 then implies that $h(z) = 0$ for all $z \in \mathcal{D}$, and so $\widetilde{f}(z) = g(z)$ for all $z \in \mathcal{D}$. ■

[55] That is, $z_n \neq z_m$ for $n \neq m$.

3.8.20 Corollary.

Suppose that $f\colon I \to \mathbb{R}$ is real analytic on the interval $I \subseteq \mathbb{R}$. Then there exists a domain $\mathcal{D} \subseteq \mathbb{C}$ to which f has an analytic continuation, and this analytic continuation is unique.

Proof. We construct the analytic continuation in the natural way by defining it to be the real-coefficient power series that represents f; this is exactly what we did with the exponential, sine, and cosine.

More precisely, since f is real analytic on I, for each $x \in I$ there is $\epsilon_x > 0$ such that

$$f(\xi) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(\xi - x)^k, \ \xi \in (x - \epsilon_x, x + \epsilon_x) \cap I. \tag{3.8.7}$$

Moreover, by choosing ϵ_x sufficiently small, we can guarantee that the convergence above is absolute. Set

$$\mathcal{D} := \bigcup_{x \in I} \mathfrak{B}(x; \epsilon_x).$$

To be clear, $\mathcal{D}$ is a union of open balls in $\mathbb{C}$ all of which have centers on the real interval I. One can show that since the series (3.8.7) converges for $\xi \in I \subseteq \mathbb{R}$ with $|\xi - x| < \epsilon_x$, it also converges for $z \in \mathbb{C}$ with $|z - x| < \epsilon_x$. So, we want to define the extension of f to be

$$\widetilde{f}(z) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(z - x)^k, \ z \in \mathfrak{B}(x; \epsilon_x). \tag{3.8.8}$$

However, it is entirely possible that $z \in \mathfrak{B}(x_1; \epsilon_{x_1}) \cap \mathfrak{B}(x_2; \epsilon_{x_2})$ for $x_1 \neq x_2$. In this case, for $\widetilde{f}$ to be well-defined, we need to ensure

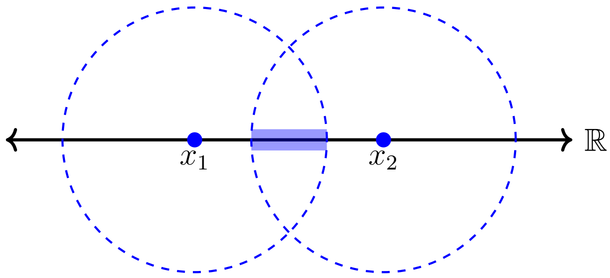
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_1)}{k!}(z - x_1)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_2)}{k!}(z - x_2)^k, \ z \in \mathfrak{B}(x_1; \epsilon_{x_1}) \cap \mathfrak{B}(x_2; \epsilon_{x_2}).$$

Set

$$\phi_1(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_1)}{k!}(z - x_1)^k \quad \text{and} \quad \phi_2(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_2)}{k!}(z - x_2)^k.$$

That is, according to (3.8.8), we have $\phi_1(z) = f(z)$ on $\mathfrak{B}(x_1; \epsilon_{x_1})$ and $\phi_2(z) = f(z)$ on $\mathfrak{B}(x_2; \epsilon_{x_2})$.

Without loss of generality, assume $x_1 < x_2$. Then $(x_2 - \epsilon_{x_2}, x_1 + \epsilon_{x_1}) \subseteq \mathfrak{B}(x_1; \epsilon_{x_1}) \cap \mathfrak{B}(x_2; \epsilon_{x_2})$; we omit the formal proof of this containment and just look at the picture below.



For $x \in (x_2 - \epsilon_{x_2}, x_1 + \epsilon_{x_1})$, we have

$$\phi_1(x) = f(x) = \phi_2(x)$$

since f is real analytic and ϕ_1 and ϕ_2 are just the Taylor series of f centered at x_1 and x_2, respectively. If we fix some point $x_0 \in (x_2 - \epsilon_{x_2}, x_1 + \epsilon_{x_1})$, we can take a sequence of distinct points (z_n) in this interval (say, $z_n := x_0 + 1/(M+n)$ where M is a large positive integer) that converges to x_0, and then we will have $\phi_1(z_n) = \phi_2(z_n)$ for all n. Corollary 3.8.15 then shows that $\phi_1(z) = \phi_2(z)$ for all $z \in \mathfrak{B}(x_1; \epsilon_{x_1}) \cap \mathfrak{B}(x_2; \epsilon_{x_2})$, which is exactly what we wanted to show.

For the uniqueness of the analytic continuation, note that we only proved Theorem 3.8.19 for domains in $\mathbb{C}$, and an interval I is not a domain. (That is, we cannot apply this theorem by taking $\mathcal{D}_0 = I$.) Instead, if g is another analytic continuation of f on $\mathcal{D}$, then of course $\widetilde{f}(x) = f(x) = g(x)$ for all $x \in I$, and thus $\widetilde{f}^{(k)}(x) = g^{(k)}(x)$ for all $x \in I$. Then for $x \in I$ fixed and any $z \in \mathfrak{B}(x; \epsilon_x)$, we have

$$\widetilde{f}(z) = \sum_{k=0}^{\infty} \frac{\widetilde{f}^{(k)}(x)}{k!}(z-x)^k = \sum_{k=0}^{\infty} \frac{g^{(k)}(x)}{k!}(z-x)^k = g(z).$$

By definition of $\mathcal{D}$, we conclude $\widetilde{f}(z) = g(z)$ for all $z \in \mathcal{D}$. ■

3.9. Isolated singularities.

We now have a rich knowledge of the theory of functions of a complex variable that are analytic on some open set. Many "reasonable" functions, however, fail to be analytic in "mild" ways and are otherwise well-behaved. For example, the function $f(z) = 1/z$ is analytic on $\mathbb{C} \setminus \{0\}$ but undefined (and unbounded) at $z = 0$. We now explore the properties of functions of a complex variable that fail to be analytic at a single point in an open set.

3.9.1 Definition.

Let $z_0 \in \mathbb{C}$ and $r > 0$. A function f that is defined on $\mathfrak{B}^*(z_0; r)$ has an **ISOLATED SINGULARITY** *at z_0 if f is analytic on $\mathfrak{B}^*(z_0; r)$.*

3.9.2 Remark.

A function that is defined on $\mathfrak{B}(z_0; r)$ is, of course, defined on $\mathfrak{B}^*(z_0; r)$, and so if f is analytic on $\mathfrak{B}(z_0; r)$, then, trivially, f has an isolated singularity at z_0. We will typically be concerned with the case where f is not defined at z_0, as this is much more interesting.

3.9.3 Example.

The following functions are analytic on $\mathbb{C} \setminus \{0\}$ and have isolated singularities at $z = 0$:

$$f(z) = \frac{\sin(z)}{z}, \qquad g(z) = \frac{1}{z}, \qquad h(z) = e^{1/z}.$$

However, their behavior at $z = 0$ is quite different. Study their limits as $z \to 0$.

Solution. (i) A famous limit[56] from early calculus tells us that for $x \in \mathbb{R}$,

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

We look to see if this carries over to the complex case. We know that $\sin(z) = 0$, so $\sin(z) = zq(z)$, where q is analytic and $q(0) \neq 0$. Thus

$$\lim_{z \to 0} \frac{\sin(z)}{z} = \lim_{z \to 0} \frac{zq(z)}{z} = \lim_{z \to 0} q(z) = q(0).$$

This does not tell us what $q(0)$ is, however, and we do not have a L'Hospital rule valid for complex limits and derivatives. Instead, we go back to power series:

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} = z \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} \right).$$

Hence

$$\lim_{z \to 0} \frac{\sin(z)}{z} = \lim_{z \to 0} \left(1 + \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} \right) = 1.$$

(ii) We want to say that

$$\lim_{z \to 0} \frac{1}{z} = \infty,$$

except we never really discussed what ∞ means in $\mathbb{C}$. A good way to think about this is that the *modulus* $1/|z|$ becomes arbitrarily large as $z \to 0$. More precisely, if $M > 0$, there is $\delta > 0$ such if $|z| < \delta$, then $1/|z| > M$ (just take $\delta = 1/M$).

(iii) On $\mathbb{R}$, we have

$$\lim_{x \to 0+} e^{1/x} \text{ “=” } e^{\infty} = \infty \quad \text{but} \quad \lim_{x \to 0^-} e^{1/x} \text{ “=” } e^{-\infty} = 0.$$

Something similar happens in the complex case. If we approach 0 in one direction, the limit blows up; in another, it vanishes. Specifically, let $z_n = 1/n$. Then $z_n \to 0$ as $n \to \infty$ and

$$\lim_{n \to \infty} e^{1/z_n} = \lim_{n \to \infty} e^n = \infty.$$

Next, if $w_n = -1/n$, so $w_n \to 0$, too, we have

$$\lim_{n \to \infty} e^{1/w_n} = \lim_{n \to \infty} e^{-n} = 0.$$

But we have more options in $\mathbb{C}$. Let $\xi_n = 1/2\pi i n$. Then $\xi_n \to 0$ and

$$\lim_{n \to \infty} e^{1/\xi_n} = \lim_{n \to \infty} e^{2\pi i n} = 1.$$

So, in one direction, the limit is ∞, in another it is zero, and in a third it is 1. In the exercises we will see an even more erratic behavior: given $w \in \mathbb{C} \setminus \{0\}$, there is a sequence (w_n) in $\mathbb{C}$ such that $e^{1/w_n} \to w$. ▲

[56] Depending on how one approaches early calculus, this limit is often used to prove that the derivative of $\sin(\cdot)$ is $\cos(\cdot)$. Moreover, a geometric proof of this limit (which is often the only kind of proof that we have at hand in early calculus) demands that we work in radians, hence our preference for radians forever after to guarantee that this limit holds.

We conclude that a function f can behave in one of three possible ways at an isolated singularity z_0.

(1) The limit $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}$.

(2) The limit $\lim_{z \to z_0} f(z)$ does not exist in $\mathbb{C}$ but $\lim_{z \to z_0} |f(z)| = \infty$.

(3) Neither (1) nor (2) holds.

3.9.1. Removable singularities.

In this section we study the case (1). First, we need a lemma, which serves as a kind of converse to the Cauchy integral theorem. The proof of this lemma is deferred to the exercises.

3.9.4 Lemma (Morera).

Suppose that f is continuous on the open set $\mathcal{D}$ (which need not be star-shaped or even a domain) and that $\int_\gamma f(z)\, dz = 0$ for all closed curves γ in $\mathcal{D}$. Then f is analytic on $\mathcal{D}$.

3.9.5 Theorem (Riemann removability theorem).

Suppose that f is analytic on $\mathfrak{B}^*(z_0; r)$ and $L := \lim_{z \to z_0} f(z)$ exists. Then f has an analytic continuation on $\mathfrak{B}(z_0; r)$. In this case we say that f has a **REMOVABLE SINGULARITY AT** z_0.

Proof. Any analytic continuation $\widetilde{f}$ of f must be continuous at z_0, so necessarily

$$\widetilde{f}(z_0) = \lim_{z \to z_0} f(z) = L.$$

This gives us a way to define $\widetilde{f}$: set

$$\widetilde{f}(z) := \begin{cases} f(z), & z \neq z_0 \\ L, & z = z_0. \end{cases}$$

To see that $\widetilde{f}$ is analytic on $\mathfrak{B}(z_0; r)$, recall the relaxed version of the Cauchy integral theorem from Lemma 3.6.17. The set $\mathfrak{B}(z_0; r)$ is a star-domain with star-center z_0, and $\widetilde{f}$ is continuous on $\mathfrak{B}(z_0; r)$ and analytic on $\mathfrak{B}(z_0; r) \setminus \{z_0\} = \mathfrak{B}^*(z_0; r)$. The lemma then implies that $\int_\gamma \widetilde{f}(z)\, dz = 0$ for any closed curve γ in $\mathfrak{B}(z_0; r)$, and so Morera's lemma implies that $\widetilde{f}$ is analytic on $\mathfrak{B}(z_0; r)$. ■

3.9.6 Example.

Show that

$$f(z) = \frac{z^2}{\sin(z)}$$

has a removable singularity at 0. What value should the analytic continuation of f have at 0?

Solution. First we need to show that

$$\lim_{z\to 0} \frac{z^2}{\sin(z)}$$

exists. We know that $\sin(\cdot)$ has a simple zero at 0, so

$$\sin(z) = zq(z),$$

where q is analytic on some ball $\mathfrak{B}(0;r)$ and $q(0) \neq 0$. By continuity, we may assume[57] that r is so small that $q(z) \neq 0$ for $z \in \mathfrak{B}(0;r)$.

Then

$$f(z) = \frac{z^2}{zq(z)} = \frac{z}{q(z)}, \; z \in \mathfrak{B}^*(0;r),$$

and so

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z}{q(z)} = \frac{0}{q(0)} = 0.$$ ▲

3.9.2. Poles.

Now we study the case (2). Suppose that f has an isolated singularity at z_0 and

$$\lim_{z\to z_0} |f(z)| = \infty. \tag{3.9.1}$$

By Footnote 57, there is $s \in (0,r]$ such that $f(z) \neq 0$ for $z \in \mathfrak{B}^*(z_0;s)$. Next, we observe that

$$\lim_{z\to z_0} |f(z)| = \infty \Longrightarrow \lim_{z\to z_0} \frac{1}{|f(z)|} = 0 \Longrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0,$$

Since f is analytic and nonzero on $\mathfrak{B}^*(z_0;s)$, $1/f$ is also analytic on $\mathfrak{B}^*(z_0;s)$. The Riemann removability theorem then allows us to extend $1/f$ to an analytic function g on $\mathfrak{B}(z_0;s)$. That is,

$$\frac{1}{f(z)} = g(z)$$

for $z \in \mathfrak{B}^*(z_0;s)$. Note that $g(z) \neq 0$ for all $z \in \mathfrak{B}^*(z_0;s)$.

Furthermore,

$$g(z_0) = \lim_{z\to z_0} g(z) = \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

Since g is not identically zero on $\mathfrak{B}(z_0;s)$, we take $m \geq 1$ to be the order of z_0 as a zero of g, and we write

$$g(z) = (z-z_0)^m h(z), \; z \in \mathfrak{B}(z_0;s),$$

for some analytic function h on $\mathfrak{B}(z_0;s)$ with $h(z_0) \neq 0$. By continuity (see, once more, Footnote 57), there is $\rho \in (0,s]$ such that $h(z) \neq 0$ for all $z \in \mathfrak{B}(z_0;\rho)$. Then $H := 1/h$ is analytic on $\mathfrak{B}(z_0;\rho)$ with $H(z_0) = 1/h(z_0) \neq 0$, and we have

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z-z_0)^m}\left(\frac{1}{h(z)}\right) = \frac{H(z)}{(z-z_0)^m}, \; z \in \mathfrak{B}^*(z_0;s).$$

We have proved the following theorem.

[57] Suppose q is continuous on $\mathfrak{B}(0;r)$ with $q(0) \neq 0$, but, for all $0 < s < r$ there exists $z \in \mathfrak{B}(0;s)$ such that $q(z) = 0$. Taking $s = 1/n$, we find a sequence (z_n) such that $z_n \to 0$ and $q(z_n) = 0$ for all n. By continuity, we then have $q(0) = \lim_{n\to\infty} q(z_n) = 0$, a contradiction.

3.9.7 Theorem.

Suppose that f is analytic on $\mathfrak{B}^*(z_0; r)$ and $\lim_{z \to z_0} |f(z)| = \infty$. Then there exist $s \in (0, r]$, an analytic function H on $\mathfrak{B}(z_0; s)$ with $H(z_0) \neq 0$, and an integer $m \geq 1$ such that

$$f(z) = \frac{H(z)}{(z - z_0)^m}, \quad z \in \mathfrak{B}^*(z_0; s).$$

The integer m is the order of the zero of the function

$$g(z) := \begin{cases} \dfrac{1}{f(z)}, & z \in \mathfrak{B}^*(z_0; s) \\ 0, & z = z_0. \end{cases}$$

In this case we say that f has a **POLE OF ORDER** m **AT** z_0.

3.9.8 Example.

Show that

$$f(z) = \frac{z^2}{\sin(z)}$$

has a pole at $z = \pi$ and determine its order.

Solution. We calculate

$$\lim_{z \to \pi} |f(z)| = \lim_{z \to \pi} \left| \frac{z^2}{\sin(z)} \right| = \frac{\pi^2}{0} = \infty,$$

so f has a pole at $z = \pi$. Now we need to find its order. The discussion that proved Theorem 3.9.7 gave us the informal rule "order of pole of f at z_0 = order of zero of $1/f$ at z_0" (where by $1/f$ we really mean its analytic continuation, since f is not defined at z_0). So, consider the function

$$g(z) = \frac{\sin(z)}{z^2}.$$

We know that $\sin(\cdot)$ has a zero of order 1 at π, so we write $\sin(z) = (z - \pi)q(z)$, where q is entire and $q(\pi) \neq 0$. Thus

$$g(z) = (z - \pi) \left(\frac{q(z)}{z^2} \right).$$

The function $h(z) := q(z)/z^2$ is analytic on $\mathbb{C} \setminus \{0\}$ and $h(\pi) = q(\pi)/\pi^2 \neq 0$. Hence g has a zero of order 1 at π, and so f has a pole of order 1 at π. In the style of Theorem 3.9.7, we write

$$f(z) = \frac{1}{z - \pi} \left(\frac{z^2}{q(z)} \right),$$

where $H(z) = z^2/q(z)$ analytic on a ball centered at π and $H(\pi) \neq 0$. ▲

3.9.3. Essential singularities.

Our work on the function $h(z) = e^{1/z}$ in Example 3.9.3 suggests that if neither case (1) nor (2) holds, then a function will have "erratic" or "nervous" behavior at its singularity.

3.9.9 Definition.

Suppose that f has an isolated singularity at z_0 and neither (1) nor (2) holds. Then f has an ESSENTIAL SINGULARITY *at z_0.*

3.9.10 Example.

Show that $f(z) = z\cos(1/z)$ has an essential singularity at $z = 0$.

Solution. First we show that f does not have a removable singularity at $z = 0$. That is, we need to show that the limit $\lim_{z\to 0} z\cos(1/z)$ does not exist. Our intuition from calculus suggests that we first approach 0 along the real axis. If $z = x$ is real and nonzero, then

$$0 \le \left| x\cos\left(\frac{1}{x}\right)\right| \le |x|,$$

and so the squeeze theorem implies

$$0 \le \lim_{x\to 0}\left| x\cos\left(\frac{1}{x}\right)\right| \le \lim_{x\to 0}|x| = 0.$$

Thus

$$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0.$$

This suggests that we next approach 0 along the imaginary axis. If y is real and nonzero, then

$$\cos\left(\frac{1}{iy}\right) = \frac{e^{i(1/iy)} - e^{-i(1/iy)}}{2} = \frac{e^{1/y} - e^{-1/y}}{2},$$

hence

$$\lim_{y\to 0^+} iy\cos\left(\frac{1}{iy}\right) = \lim_{y\to 0^+} iy\left(\frac{e^{1/y} - e^{-1/y}}{2}\right) = \infty$$

and

$$\lim_{y\to 0^-} iy\cos\left(\frac{1}{iy}\right) = \lim_{y\to 0^-} iy\left(\frac{e^{1/y} - e^{-1/y}}{2}\right) = -\infty.$$

The validity of the limits

$$\lim_{y\to 0^+} ye^{1/y} = \infty \quad \text{and} \quad \lim_{y\to 0^-} ye^{1/y} = -\infty$$

can be established using L'Hospital's rule from real-variable calculus. We conclude that $\lim_{z\to 0} z\cos(1/z)$ does not exist, and so f does not have a removable singularity at 0. (In fact, we could have just done the limits along the positive and negative imaginary axes and skipped the real limit entirely).

Next, we already know that f does not have a pole at 0. Otherwise, we would have $\lim_{z\to 0}|f(z)| = \infty$. But, at least along the real axis, this limit is 0. However, if we had used other methods to show that f does not have a removable singularity at 0, then we may have missed this limit. So, here is another way of proceeding: contradiction.

Suppose f does have a pole of order, say, $m \ge 1$, at 0. Then for some $r > 0$, there is an analytic function H on $\mathfrak{B}(0;r)$ with $H(0) \ne 0$ and

$$z\cos\left(\frac{1}{z}\right) = \frac{H(z)}{z^m}, \quad z \in \mathfrak{B}^*(0;r).$$

Then

$$H(z) = z^{m+1} \cos\left(\frac{1}{z}\right), \quad z \in \mathfrak{B}^*(0; r),$$

and so

$$H(0) = \lim_{z \to 0} H(z) = \lim_{z \to 0} z^{m+1} \cos\left(\frac{1}{z}\right).$$

Above we argued that when $m = 0$, this limit does not exist, and so a contradiction should result. Here is another contradiction: recall that $\cos((2k+1)\pi) = 0$ for all $k \in \mathbb{Z}$. Set $z_k = 1/(2k+1)\pi$ for $k \in \mathbb{N}$, so $z_k \to 0$ and $\cos(1/z_k) = 0$. Then

$$H(z_k) = z_k^{m+1} \cos\left(\frac{1}{z_k}\right) = 0 \quad \text{and} \quad H(0) = \lim_{k \to \infty} H(z_k) = 0,$$

a contradiction. ▲

Now we state two theorems that describe the "erratic" behavior of a function near an essential singularity. The proof of the first is outlined in the exercises.

3.9.11 Theorem (Casorati-Weierstrass).

Suppose that f has an essential singularity at z_0. Then on any punctured ball centered at z_0, f becomes arbitrarily close to any point in $\mathbb{C}$. That is, for any $w \in \mathbb{C}$, there is a sequence (z_n) such that $z_n \to z_0$ and $f(z_n) \to w$.

3.9.12 Theorem (Picard's big theorem).

Suppose that f has an essential singularity at z_0 and is analytic on $\mathfrak{B}^*(z_0; r)$. Then in any punctured ball centered at z_0, f takes every complex value infinitely often, with at most one exception. That is, for $0 < s < r$, either

$$\{f(z) \mid 0 < |z - z_0| < s\} = \mathbb{C}$$

or there is $w \in \mathbb{C}$ such that

$$\{f(z) \mid 0 < |z - z_0| < s\} = \mathbb{C} \setminus \{w\}$$

for all $0 < s < r$. Moreover, in either case, if z is in the range of f, then the set $\{\xi \in \mathbb{C} \mid f(\xi) = z\}$ is infinite.

3.9.4. Laurent series.

We now understand the behavior of a function f that is analytic on a punctured ball $\mathfrak{B}^*(z_0; r)$.

- If f has a removable singularity at z_0, then we can write f as a genuine power series at z_0.

- If f has a pole at z_0, we write $f(z) = H(z)/(z - z_0)^m$, $z \neq z_0$, with H analytic, so $H(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$. Then for $z \neq z_0$,

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^{k-m} = \frac{a_0}{(z-z_0)^m} + \frac{a_0}{(z-z_0)^{m-1}} + \cdots + \frac{a_{m-1}}{z-z_0} + \sum_{k=m}^{\infty} a_k (z-z_0)^{k-m}$$

$$= \sum_{k=0}^{m-1} a_k (z-z_0)^{-(m-k)} + \sum_{k=0}^{\infty} a_{k+m} (z-z_0)^k.$$

Thus f "almost" has a power series expansion around a pole z_0, except the "power series" has some negative powers of $z - z_0$.

• We did not develop any firm theory for representing a function around an essential singularity, but we do recall that the "canonical" example of $e^{1/z}$ at 0 has the expansion

$$e^{1/z} = \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right) \frac{1}{z^k} = \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right) z^{-k},$$

and this expansion contains *infinitely* many terms with negative powers of z.

Here then is our idea: although we may not be able to represent a function with a Taylor series centered at an isolated singularity z_0, we may be able to represent the function with a series in $(z - z_0)^k$ where some (possibly many, or even all) of the k are negative. The next example discusses the regions in $\mathbb{C}$ on which we might be able to achieve such a representation.

3.9.13 Example.

Let $f(z) = 1/(z-1)(z-2)$, so f is analytic on $\mathbb{C} \setminus \{1, 2\}$ with isolated singularities at 1 and 2 (these are simple poles). We can of course represent f by a power series centered at any $z_0 \in \mathbb{C} \setminus \{1, 2\}$, but these power series will, of course, only converge on balls of finite radii. Specifically, if $z_0 \neq 1, 2$, then the radius of convergence of the Taylor series for f centered at z_0 is $\min\{|z_0 - 1|, |z_0 - 2|\}$.

But f is not that "bad" a function; it fails to be analytic at only two points. Can we not represent f in a "series" form in a more efficient way? In particular, f is analytic on the **annular** *regions $|z| < 1$, $1 < |z| < 2$, and $2 < |z|$, the last of which is unbounded.*

It turns out that we can find a useful series representation for f if we relax our standards in two ways: if we allow the series to have negative powers of $(z - z_0)^k$, as we observed above, and if we do not work on balls but on annuli.

3.9.14 Definition.

Let $0 \leq r < R \leq \infty$ and $z_0 \in \mathbb{C}$. The ANNULUS *with center z_0, inner radius r, and outer radius R is the set*

$$\mathfrak{A}(z_0; r, R) := \{z \in \mathbb{C} \mid r < |z - z_0| < R\}.$$

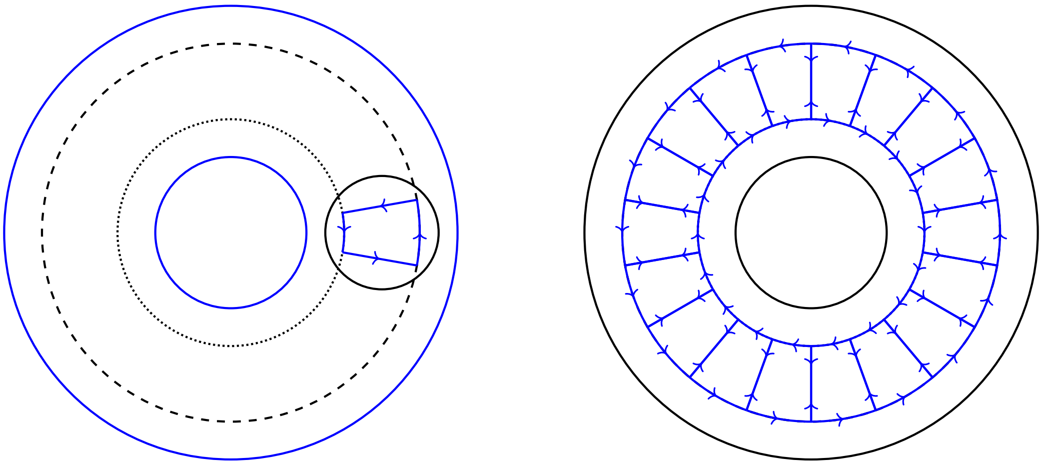
Observe that if $r = 0$ and $R = \infty$, then $\mathfrak{A}(z_0; 0, \infty) = \mathbb{C} \setminus \{z_0\}$ and if $r = 0$ and $R < \infty$, then $\mathfrak{A}(z_0; 0, R) = \mathfrak{B}^*(z_0; R)$. Also, $\mathfrak{A}(z_0; R, \infty) = \mathbb{C} \setminus \overline{\mathfrak{B}}(z_0; R)$.

3.9.15 Lemma.

Suppose that f is analytic on the annulus $\mathfrak{A}(z_0; r, R)$ and $r < \rho < \mathrm{P} < R$. Then

$$\int_{|z-z_0|=\rho} f(z)\ dz = \int_{|z-z_0|=\mathrm{P}} f(z)\ dz.$$

Proof. We "partition" the annulus $\mathfrak{A}(z_0; \rho, \mathrm{P})$ into a family of "rectangles" $\gamma_0, \ldots, \gamma_n$ as in the sketch below.



More precisely, each "rectangle" is of the form

$$\gamma_k := [z_0 + \rho e^{i(k-1)\theta_n}, z_0 + \mathrm{P}e^{i(k+1)\theta_n}] \oplus \lambda_k \oplus [z_0 + \mathrm{P}e^{i(k+1)\theta_n}, z_0 + \rho e^{i(k+1)\theta_n}] \oplus \mu_k^-,$$

where

$$\theta_n = \frac{2\pi}{n}$$

for some positive integer n,

$$\lambda_k(t) = z_0 + \mathrm{P}e^{it},\ \ \theta_k \le t \le \vartheta_k,$$

and

$$\mu_k(t) = z_0 + \rho e^{it},\ \ \theta_k \le t \le \vartheta_k.$$

The integer n is chosen to be large enough that each "rectangle" γ_k is contained in the ball $\mathfrak{B}(z_k; s)$, where

$$z_k = z_0 + \left(\frac{\rho + \mathrm{P}}{2}\right) e^{ik\theta_n} \quad \text{and} \quad s := \frac{\mathrm{P} - \rho}{2} + \min\left\{\frac{R - \mathrm{P}}{2}, \frac{\rho - r}{2}\right\}.$$

This choice of center and radius for $\mathfrak{B}(z_k; s)$ ensures $\mathfrak{B}(z_k; s) \subseteq \mathfrak{A}(z_0; r, R)$, so f is analytic on $\mathfrak{B}(z_k; s)$. Since the ball $\mathfrak{B}(z_k; s)$ is a star-domain, the Cauchy integral theorem implies $\int_{\gamma_k} f(z)\ dz = 0$ for all k.

We then have

$$0 = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\ dz = \sum_{k=1}^{n} \int_{\lambda_k} f(z)\ dz - \sum_{k=1}^{n} \int_{\mu_k} f(z)\ dz$$

$$= \int_{|z-z_0|=\mathrm{P}} f(z)\ dz - \int_{|z-z_0|=\rho} f(z)\ dz. \quad \blacksquare$$

3.9.16 Theorem (Laurent series).

Suppose that f is analytic on the annulus $\mathfrak{A}(z_0; r, R)$. Then there is a set of coefficients $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \;\; z \in \mathfrak{A}(z_0; r, R), \tag{3.9.2}$$

where the series converges in the sense of Definition A.2.9. This series for f is its **Laurent series** *centered at z_0. It has the following properties.*

(i) *The coefficients a_k are*

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z - z_0)^{k+1}}\, dz, \tag{3.9.3}$$

where $r < s < R$; the value of this integral is independent of s.

(ii) [Term-by-term integration] *If γ is a curve in $\mathfrak{A}(z_0; r, R)$, then*

$$\int_\gamma f(z)\, dz = \int_\gamma \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k\, dz = \sum_{k=-\infty}^{\infty} a_k \int_\gamma (z - z_0)^k\, dz.$$

(iii) *The Laurent series for f centered at z_0 is unique in the sense that if $f(z) = \sum_{k=-\infty}^{\infty} b_k (z - z_0)^k$ for all $z \in \mathfrak{A}(z_0; r, R)$ and some family of coefficients $\{b_k\}_{k=-\infty}^{\infty}$, then $a_k = b_k$ for all k.*

(iv) *Write*

$$f_R(z) := \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{and} \quad f_r(z) := \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k}.$$

Then f_R is analytic on $|z - z_0| < R$ and f_r is analytic on $|z - z_0| > r$. In particular, if $r = 0$ (i.e., if z_0 is an isolated singularity of f), then f_r is analytic on $\mathbb{C} \setminus \{z_0\}$. The function f_r is the **principal part** *of the Laurent series of f with center z_0.*

Proof. We give the proof in a number of steps.

1. That the integrals in (3.9.3) are independent of s is an immediate consequence of Lemma 3.9.15.

2. Suppose the result is true for $z_0 = 0$ and let $g(z) = f(z + z_0)$ for $z \in \mathfrak{A}(0; r, R)$. Then g is analytic on this annulus, so we can write

$$g(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \;\; z \in \mathfrak{A}(0; r, R).$$

Then if $z \in \mathfrak{A}(z_0; r, R)$, we have $z - z_0 \in \mathfrak{A}(0; r, R)$, and so

$$f(z) = f((z - z_0) + z_0) = g(z - z_0) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

We have

$$a_k = \frac{1}{2\pi i}\int_{|z|=s} \frac{g(z)}{z^{k+1}}\,dz = \frac{1}{2\pi i}\int_{|z|=s}\frac{f(z+z_0)}{z^{k+1}}\,dz,$$

and the definition of the line integral gives

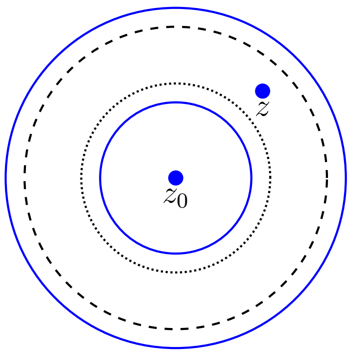
$$\int_{|z|=s}\frac{f(z+z_0)}{z^{k+1}}\,dz = \int_0^{2\pi}\left(\frac{f(se^{it}+z_0)}{(se^{it})^{k+1}}\right)(ise^{it})\,dt = \int_0^{2\pi}\left(\frac{f(se^{it}+z_0)}{\left((se^{it}+z_0)-z_0\right)^{k+1}}\right)(ise^{it})\,dt$$

$$= \int_{|z-z_0|=z}\frac{f(z)}{(z-z_0)^{k+1}}\,dz.$$

3. So, we just prove the result for $z_0 = 0$; this makes much of the notation easier. Fix $z \in \mathfrak{A}(0;r,R)$ and, for $\xi \in \mathfrak{A}(0;r,R)$, define

$$g(\xi) := \begin{cases} \dfrac{f(\xi)-f(z)}{\xi - z}, & \xi \neq z \\ f'(\xi), & \xi = z. \end{cases}$$

Then g is continuous on $\mathfrak{A}(0;r,R)$ and analytic on $\mathfrak{A}(0;r,R)\setminus\{z\}$, so the Riemann removability theorem implies that g is analytic on $\mathfrak{A}(0;r,R)$. Now take $r < \rho < \mathrm{P} < R$ such that $\rho < |z| < \mathrm{P}$.



Lemma 3.9.15 implies that

$$\int_{|\xi|=\rho} g(\xi)\,d\xi = \int_{|\xi|=\mathrm{P}} g(\xi)\,d\xi.$$

For $|\xi| = \rho$, we have $\xi \neq z$, and likewise for $|\xi| = \mathrm{P}$, so we can use the piecewise definition of g to rewrite these integrals as

$$\int_{|\xi|=\rho}\frac{f(\xi)-f(z)}{\xi - z}\,d\xi = \int_{|\xi|=\mathrm{P}}\frac{f(\xi)-f(z)}{\xi-z}\,d\xi,$$

and thus

$$\underbrace{\int_{|\xi|=\rho}\frac{f(\xi)}{\xi-z}\,d\xi}_{\mathcal{I}_1(z,\rho)} - f(z)\underbrace{\int_{|\xi|=\rho}\frac{d\xi}{\xi-z}}_{\mathcal{I}_2(z,\rho)} = \underbrace{\int_{|\xi|=\mathrm{P}}\frac{f(\xi)}{\xi-z}\,d\xi}_{\mathcal{I}_3(z,\mathrm{P})} - f(z)\underbrace{\int_{|\xi|=\mathrm{P}}\frac{d\xi}{\xi-z}}_{\mathcal{I}_4(z,\mathrm{P})} \tag{3.9.4}$$

Note that we have no qualms about the existence of any of these four individual integrals, since each integrand is continuous on the given circle.

We evaluate $\mathcal{I}_2(z,\rho)$ and $\mathcal{I}_4(z,\mathrm{P})$ explicitly. Since $\rho < |z| < \mathrm{P}$, the Cauchy integral theorem implies

$$\mathcal{I}_2(z,\rho) = f(z)\int_{|\xi|=\rho}\frac{d\xi}{\xi - z} = 0,$$

while the Cauchy integral formula implies

$$\mathcal{I}_4(z,\mathrm{P}) = f(z)\int_{|\xi|=\mathrm{P}}\frac{d\xi}{\xi - z} = 2\pi i f(z).$$

Then (3.9.4) reduces to

$$\mathcal{I}_1(z,\rho) = \mathcal{I}_3(z,\mathrm{P}) - \mathcal{I}_4(z,\mathrm{P}) = \mathcal{I}_3(z,\mathrm{P}) - 2\pi i f(z),$$

and therefore

$$f(z) = \frac{1}{2\pi i}\mathcal{I}_3(z,\mathrm{P}) - \frac{1}{2\pi i}\mathcal{I}_1(z,\rho). \tag{3.9.5}$$

We manipulate $\mathcal{I}_3(z,\mathrm{P})$ as we did in the proof of the generalized Cauchy integral formula. Since $|\xi| = \mathrm{P}$ and $|z| < \mathrm{P}$, we have

$$\xi - z = \xi\left(1 - \frac{z}{\xi}\right), \qquad \left|\frac{z}{\xi}\right| = \frac{|z|}{\mathrm{P}} < \frac{\mathrm{P}}{\mathrm{P}} = 1.$$

Then the geometric series implies

$$\frac{1}{\xi - z} = \frac{1}{\xi}\left(\frac{1}{\left(1 - \dfrac{z}{\xi}\right)}\right) = \frac{1}{\xi}\sum_{k=0}^{\infty}\left(\frac{z}{\xi}\right)^k = \sum_{k=0}^{\infty}\frac{z^k}{\xi^{k+1}}.$$

Assuming that we can interchange the sum and the integral, we have

$$\int_{|\xi|=\mathrm{P}}\frac{f(\xi)}{\xi - z}\,d\xi = \int_{|\xi|=\mathrm{P}} f(\xi)\sum_{k=0}^{\infty}\frac{z^k}{\xi^{k+1}}\,d\xi = \sum_{k=0}^{\infty}\left(\int_{|\xi|=\mathrm{P}}\frac{f(\xi)}{\xi^{k+1}}\,d\xi\right)z^k. \tag{3.9.6}$$

We emphasize that this holds for any $z \in \mathfrak{B}(0;\mathrm{P})$.

Now we manipulate $\mathcal{I}_1(z,\rho)$ in a similar way. Here we have $|\xi| = \rho$ and $|z| > \rho$, so

$$\xi - z = -z\left(1 - \frac{\xi}{z}\right), \qquad \left|\frac{\xi}{z}\right| = \frac{\rho}{|z|} < \frac{\rho}{\rho} = 1.$$

The geometric series again tells us

$$\frac{1}{\xi - z} = -\frac{1}{z}\left(\frac{1}{1 - \dfrac{\xi}{z}}\right) = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\xi}{z}\right)^k = -\sum_{k=0}^{\infty}\frac{\xi^k}{z^{k+1}}.$$

Assuming, once again, that we can interchange the sum and the integral, we have

$$\int_{|\xi|=\rho} \frac{f(\xi)}{\xi - z}\, d\xi = -\int_{|\xi|=\rho} f(\xi) \sum_{k=0}^{\infty} \frac{\xi^k}{z^{k+1}}\, d\xi = -\sum_{k=0}^{\infty} \left(\int_{|\xi|=\rho} f(\xi)\xi^k\, d\xi \right) \frac{1}{z^{k+1}}$$

$$= -\sum_{k=1}^{\infty} \left(\int_{|\xi|=\rho} f(\xi)\xi^{k-1}\, d\xi \right) \frac{1}{z^k} = -\sum_{k=1}^{\infty} \left(\int_{|\xi|=\rho} \frac{f(\xi)}{\xi^{-k+1}}\, d\xi \right) z^{-k}. \quad (3.9.7)$$

We emphasize that this holds for any $|z| > \rho$.

Rewriting the expression for f in (3.9.5) using (3.9.6) and (3.9.7), we have

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\xi|=\mathrm{P}} \frac{f(\xi)}{\xi^{k+1}}\, d\xi \right) z^k + \sum_{k=1}^{\infty} \left(\frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{f(\xi)}{\xi^{-k+1}}\, d\xi \right) z^{-k}. \quad (3.9.8)$$

Set

$$a_k = \frac{1}{2\pi i} \int_{|\xi|=s} \frac{f(\xi)}{\xi^{k+1}}\, d\xi \quad (3.9.9)$$

for any fixed $s \in (r, R)$. Then (3.9.8) and Lemma 3.9.15 imply $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$.

This concludes the proof of (3.9.2) and part (i). We will not prove the validity of the interchange of sum and integral in part (ii), but it essentially holds if we accept that the interchanges of sum and integral in (3.9.6) and (3.9.7) are valid.

4. We leave the proof of the uniqueness of the Laurent series in part (iii) as an exercise.

5. Last, we prove part (iv). Recall from (3.9.6) that the series

$$\sum_{k=0}^{\infty} \left(\int_{|\xi|=\mathrm{P}} \frac{f(\xi)}{\xi^{k+1}}\, d\xi \right) z^k$$

converged for $|z| < \mathrm{P}$. The coefficients in this series are independent of P by Lemma 3.9.15, and so

$$\sum_{k=0}^{\infty} \left(\int_{|\xi|=\mathrm{P}} \frac{f(\xi)}{\xi^{k+1}}\, d\xi \right) z^k = \sum_{k=0}^{\infty} a_k z^k,$$

where a_k is from (3.9.9). That is, the series $\sum_{k=0}^{\infty} a_k z^k$ converges for $|z| < \mathrm{P}$. But $\mathrm{P} \in (r, R)$ was arbitrary, so $\sum_{k=0}^{\infty} a_k z^k$ in fact converges for any $|z| < R$. Hence $f_R(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic on $\mathfrak{B}(0; R)$.

Similarly, by (3.9.7) the series

$$\sum_{k=1}^{\infty} \left(\int_{|\xi|=\rho} \frac{f(\xi)}{\xi^{-k+1}}\, d\xi \right) z^{-k}$$

converges for any $|z| > \rho$, and, appealing once more to Lemma 3.9.15, the coefficients in this series are independent of ρ. So, we may write

$$\sum_{k=1}^{\infty} \left(\int_{|\xi|=\rho} \frac{f(\xi)}{\xi^{-k+1}}\, d\xi \right) z^{-k} = \sum_{k=1}^{\infty} a_{-k} z^{-k}, \ |z| > \rho.$$

Now, if $|w| < 1/\rho$, then $1/|w| > \rho$, and so the series

$$\sum_{k=1}^{\infty} a_{-k} w^k = \sum_{k=1}^{\infty} a_{-k} \left(\frac{1}{w} \right)^{-k}$$

converges. Since $\rho \in (r, R)$ was arbitrary, we see that the series

$$g_r(w) := \sum_{k=1}^{\infty} a_{-k} w^k$$

converges for $|w| < \varrho$ for all $\varrho \in (0, 1/r)$. That is, g_r is analytic on $\mathfrak{B}(0; 1/r)$. Since the mapping $z \mapsto 1/z$ is analytic on $\mathbb{C} \setminus \{0\}$, we see that $f_r(z) := g_r(1/z)$ is analytic on the set $r < |z|$. ■

3.9.17 Remark.

(i) *As with the power series expansion from the generalized Cauchy integral formula, we typically try to avoid using the definition* (3.9.3) *of the Laurent coefficients and instead try to work with known power series.*

(ii) *Suppose that f has an isolated singularity at z_0, so that f is analytic on the punctured ball $\mathfrak{B}^*(0; R)$ for some $R > 0$. Then the Laurent series of f converges on the annulus $\mathfrak{A}(z_0; 0, R)$.*

(iii) *If f is analytic at z_0, then the uniqueness of the Laurent series shows that the Laurent series for f at z_0 is just its power series centered at z_0.*

3.9.18 Example.

Find the Laurent series of each function below centered at the given point z_0. Describe the annulus on which it converges. What is the principal part?

(i) $f(z) = \dfrac{\sin(z)}{z^3}$, $z_0 = 0$

(ii) $g(z) = z \cos\left(\dfrac{1}{z}\right)$, $z_0 = 0$

(iii) $h(z) = \dfrac{z^3 + z^2}{(z-1)^2}$, $z_0 = 1$

Solution. (i) The power series for $\sin(\cdot)$ is

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!},$$

so, for $z \neq 0$,

$$\frac{\sin(z)}{z^3} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{z^3(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-2}}{(2k+1)!}.$$

This series converges on the annulus $0 < |z| < \infty$. We write out the first few terms to isolate the principal part:

$$\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-2}}{(2k+1)!} = z^{-2} + \frac{z}{3!} + \sum_{k=2}^{\infty} (-1)^k \frac{z^{2k-2}}{(2k+1)!}.$$

So, the principal part is $h(z) = z^{-2}$.

(ii) The power series for $\cos(\cdot)$ is

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!},$$

so, for $z \neq 0$,

$$z \cos\left(\frac{1}{z}\right) = z \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{z}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{z^{2k-1}}.$$

This series (again) converges on the annulus $0 < |z| < \infty$. We write out the first few terms to isolate the principal part:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{z^{2k-1}} = \frac{1}{z^{-1}} - \frac{1}{2z} + \frac{1}{4!z^3} + \sum_{k=3}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{z^{2k-1}}.$$

The first term is $1/z^{-1} = z$, so the principal part is

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{z^{2k-1}}.$$

Note that the principal part contains infinitely many terms; this is inherently connected to the essential singularity of g at 0.

(iii) The Laurent series will have powers of $z-1$, so a good idea is to expand the polynomial $p(z) := z^3 + z^2$ in powers of $z-1$. We calculate

$$\begin{aligned}
p(1) &= 2 & & \\
p'(z) &= 3z^2 + 2z & p'(1) &= 5 \\
p''(z) &= 6z + 2 & p''(1) &= 8 \\
p'''(z) &= 6 & p'''(1) &= 6 \\
p^{(k)}(z) &= 0, \ k \geq 4 & p^{(k)}(1) &= 0, \ k \geq 4.
\end{aligned}$$

Hence

$$p(z) = \frac{2}{0!} + \frac{5}{1!}(z-1) + \frac{8}{2!}(z-1)^2 + \frac{6}{3!}(z-1)^3 = 2 + 5(z-1) + 4(z-1)^2 + (z-1)^3$$

and therefore, for $z \neq 1$,

$$\frac{z^3 + z^2}{(z-1)^2} = \frac{2}{(z-1)^2} + \frac{5}{(z-1)} + 4 + (z-1).$$

This series clearly converges for $z \in \mathbb{C} \setminus \{1\}$, and so it converges on the annulus $0 < |z-1| < \infty$. The principal part is

$$h(z) = \frac{2}{(z-1)^2} + \frac{5}{(z-1)}. \qquad \blacktriangle$$

3.9.19 Example.

Let

$$f(z) = \frac{1}{z-1} + \frac{1}{z-2}.$$

Find the Laurent series for f centered at $z_0 = 0$ that converges on each of the following annuli.

(i) $\mathfrak{A}(0; 0, 1)$

(ii) $\mathfrak{A}(0; 1, 2)$

(iii) $\mathfrak{A}(0; 2, \infty)$

Solution. Write

$$g(z) := \frac{1}{z-1} \quad \text{and} \quad h(z) := \frac{1}{z-2}.$$

If we find Laurent series for g and h separately, then we will be done. Throughout, we use the geometric series carefully.

(i) When $|z| < 1$, the geometric series gives

$$g(z) = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k$$

and

$$h(z) = -\frac{1}{2-z} = -\frac{1}{2\left(1 - \frac{z}{2}\right)}.$$

Since $|z| < 1$, we also have $|z/2| < 1$, and so

$$h(z) = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k. \tag{3.9.10}$$

Then

$$f(z) = -\sum_{k=0}^{\infty} z^k - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k = \sum_{k=0}^{\infty}(-1)\left(1 + \frac{1}{2^{k+1}}\right)z^k, \ |z| < 1.$$

Note that at the end we were careful to simplify everything so that the series has the correct form $\sum_{k=-\infty}^{\infty} a_k z^k$.

(ii) As above, we still have (3.9.10) for $1 < |z| < 2$ because this series in fact converges for all $|z| < 2$. For g, however, we need to factor in a different way:

$$g(z) = \frac{1}{z-1} = \frac{1}{z\left(1 - \frac{1}{z}\right)} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}}. = \sum_{k=1}^{\infty}\frac{1}{z^k}. \tag{3.9.11}$$

Here we used the fact that $|z| > 1$ implies $|1/z| < 1$ to invoke the geometric series. Also, note that we reindexed to keep our terms in powers of z^k. Thus

$$f(z) = \sum_{k=1}^{\infty}\frac{1}{z^k} - \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}} = \sum_{k=-\infty}^{\infty} a_k z^k, \ 1 < |z| < 2, \qquad a_k := \begin{cases} 1, & k \le -1 \\ -\dfrac{1}{2^{k+1}}, & k \ge 0 \end{cases}$$

(iii) If $|z| > 2$, then $|z| > 1$, and so we can still represent g by (3.9.11). Now we use a similar strategy to represent h:

$$h(z) = \frac{1}{z-2} = \frac{1}{z\left(1 - \dfrac{2}{z}\right)} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k = \sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}} = \sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k}.$$

Here we used the fact that $|z| > 2$ implies $|2/z| < 1$ to invoke the geometric series. Note that we reindexed to keep our terms in powers of z^k. Thus

$$f(z) = \sum_{k=1}^{\infty}\frac{1}{z^k} + \sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k} = \sum_{k=1}^{\infty}\frac{1+2^{k-1}}{z^k}, \quad |z| > 2.$$

▲

3.9.20 Example (Generalized partial fractions decomposition).

Suppose that f is analytic on the domain $\mathcal{D}$ except at the isolated singularities $z_1, \ldots, z_n$. Suppose that at each singularity z_k, f has a pole of order $m_k \geq 1$. Show that there are polynomials $p_1, \ldots, p_n$, where p_k has degree m_k and $p_k(0) = 0$, and an analytic function g on $\mathcal{D}$, such that

$$f(z) = \sum_{k=1}^{n} p_k\left(\frac{1}{z - z_k}\right) + g(z), \quad z \in \mathcal{D} \setminus \{z_1, \ldots, z_n\}.$$

Solution. For simplicity, we prove this when $n = 2$. We expand f as a Laurent series centered at z_1 and use the assumption that z_1 is a pole of order m_1 to separate out its principal part:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_1)^k = \sum_{k=1}^{m_1}\frac{a_{-k}}{(z-z_1)^k} + \sum_{k=0}^{\infty} a_k(z-z_1)^k.$$

Let

$$p_1(z) = \sum_{k=1}^{m_1} a_{-k}z^k,$$

so $p_1(0) = 0$, and set

$$h_1(z) = f(z) - p_1\left(\frac{1}{z - z_1}\right). \tag{3.9.12}$$

The Laurent series for h_1 centered at z_0 is just $\sum_{k=0}^{\infty} a_k(z - z_1)^k$, so h_1 has a removable singularity at z_0. The function $z \mapsto p_1(1/(z - z_1))$ is the principal part of the Laurent series of f at z_0, which is analytic on $\mathbb{C} \setminus \{z_1\}$. Since f is analytic on $\mathcal{D} \setminus \{z_1, z_2\}$, we then see that h_1 is analytic on $\mathcal{D} \setminus \{z_1, z_2\}$. But since h_1 has a removable singularity at z_1, we can find an analytic continuation $\widetilde{h}_1$ for h_1 on $\mathcal{D} \setminus \{z_2\}$.

Next, by (3.9.12), for z close to z_2, we have

$$\widetilde{h}_1(z) = f(z) - p_1\left(\frac{1}{z - z_1}\right).$$

Since $z \mapsto p_1(1/(z - z_1))$ is analytic at z_2 and f has a pole of order m_2 at z_2, it follows that $\widetilde{h}_1$ also has a pole of order m_2 at z_2. Then we can write the Laurent series of $\widetilde{h}_1$ at

z_2 as

$$\widetilde{h}_1(z) = \sum_{k=-\infty}^{\infty} b_k(z - z_2)^k = \sum_{k=1}^{m_2} \frac{b_{-k}}{(z - z_2)^k} + \sum_{k=0}^{\infty} b_k(z - z_0)^k.$$

With

$$p_2(z) = \sum_{k=1}^{m_2} b_{-k} z^k,$$

so $p_2(0) = 0$, we set

$$h_2(z) = \widetilde{h}_1(z) - p_2\left(\frac{1}{z - z_2}\right).$$

Since $\widetilde{h}_1$ and $z \mapsto p_2(1/(z - z_2))$ are analytic on $\mathcal{D} \setminus \{z_2\}$, so is h_2. As before, the Laurent series for h_2 at z_2 is $\sum_{k=0}^{\infty} b_k(z - z_2)^k$, so h_2 has a removable singularity at z_2 and thus an analytic continuation $\widetilde{h}_2$ to all of $\mathcal{D}$.

We conclude that, for $z \in \mathcal{D} \setminus \{z_1, z_2\}$, we have

$$f(z) = p_1\left(\frac{1}{z - z_1}\right) + \widetilde{h}_1(z) = p_1\left(\frac{1}{z - z_1}\right) + p_2\left(\frac{1}{z - z_2}\right) + h_2(z)$$

$$= p_1\left(\frac{1}{z - z_1}\right) + p_2\left(\frac{1}{z - z_2}\right) + \widetilde{h}_2(z),$$

where $g := \widetilde{h}_2$ is analytic on $\mathcal{D}$. ▲

3.10. Residue theory.

3.10.1. Residues.

A significant number of results about a function can be developed from the analysis of one particular coefficient in its Laurent series centered at an isolated singularity.

3.10.1 Definition.

Suppose that f has an isolated singularity at z_0 and the Laurent series expansion $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$ valid on some punctured ball $\mathfrak{B}^*(z_0; R)$. The **RESIDUE** *of f at z_0 is the coefficient a_{-1} from this Laurent series expansion, and we write*

$$\operatorname{Res}(f; z_0) := a_{-1}.$$

One immediately has

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} f(z)\, dz, \ \ 0 < s < R, \tag{3.10.1}$$

from the definition (3.9.3)*, but evaluating this integral is rarely the most expedient way of calculating the residue.*

3.10.2 Example.

Calculate $\operatorname{Res}\left(\frac{z^3+z^2}{(z-1)^2};1\right)$.

Solution. We saw in part (iii) of Example 3.9.18 that the Laurent series for f was

$$f(z) = \frac{2}{(z-1)^2} + \frac{5}{(z-1)} + 4 + (z-1), \; z \neq 1,$$

so $\operatorname{Res}(f;1) = 5$. ▲

In the remainder of this section we develop some simple tools for calculating residues that are sometimes faster than calculating the Laurent series (even when using previously known power series); the applications will follow shortly.

First, the situation in Example 3.10.2 illustrates a general truth. Write $H(z) = z^3+z^2$. Then $H(1) = 5 \neq 0$ and H is entire, hence f has a pole of order 2 at 1. Observe that $H'(z) = 3z^2 + 2z$ and $H'(1) = 5 = \operatorname{Res}(f;1)$. Here is the abstraction of this calculation.

3.10.3 Lemma.

Suppose that f has a pole of order m at z_0. Write $f(z) = H(z)/(z-z_0)^m$ for $z \in \mathfrak{B}^*(z_0;R)$, where H is analytic on $\mathfrak{B}(z_0;R)$ and $H(z_0) \neq 0$. Then

$$\operatorname{Res}(f;z_0) = \frac{H^{(m-1)}(z_0)}{(m-1)!}$$

Proof. Since H is analytic on $\mathfrak{B}(z_0;R)$, we may write $H(z) = \sum_{k=0}^{\infty} b_k(z-z_0)^k$ for $z \in \mathfrak{B}(z_0;R)$. Then

$$f(z) = \sum_{k=0}^{\infty} b_k(z-z_0)^{k-m},$$

hence the coefficient of $(z-z_0)^{-1}$ in this expansion is $b_{m-1} = H^{(m-1)}(z_0)/(m-1)!$. ■

3.10.4 Example.

Calculate $\operatorname{Res}\left(\frac{\cos(z)}{\sin^2(z)};z=0\right)$.

Solution. Since $\sin(\cdot)$ has a simple zero at 0, we can write $\sin(z) = zq(z)$, where q is analytic and $q'(0) \neq 0$. Then

$$f(z) := \frac{\cos(z)}{\sin^2(z)} = \frac{\cos(z)}{z^2 q(z)^2},$$

so f has a pole of order 2 at 0. Write $H(z) = \cos(z)/q(z)^2$. Then

$$\operatorname{Res}\left(\frac{\cos(z)}{\sin^2(z)};z=0\right) = H'(0) = \left.\frac{-q(z)^2\sin(z) + 2\cos(z)q(z)q'(z)}{q(z)^4}\right|_{z=0} = \frac{2q'(0)}{q(0)^3}.$$

Since $\sin(z) = zq(z)$, we have, for $z \neq 0$, $q(z) = \sin(z)/z$, so $q(0) = 1$. Next, we differentiate the equation $\sin(z) = zq(z)$ to find

$$\cos(z) = q(z) + zq'(z) \quad \text{and} \quad -\sin(z) = q'(z) + q'(z) + zq''(z),$$

hence

$$2q'(0) = 0,$$

and so $q'(0) = 0$. We conclude

$$\text{Res}\left(\frac{\cos(z)}{\sin^2(z)}; z = 0\right) = 0. \qquad \blacktriangle$$

A special case of Lemma 3.10.3 worth singling out because of ease and frequency is that of a simple pole.

3.10.5 Lemma.

Suppose that f and g are analytic on $\mathfrak{B}(z_0; R)$ with $f(z_0) \neq 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$ (i.e., g has a simple zero at z_0). Then

$$\text{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. Write $g(z) = (z - z_0)q(z)$, where q is analytic on $\mathfrak{B}(z_0; r)$ and $q(z_0) = g'(z_0) \neq 0$. (That $q(z_0) = g'(z_0)$ follows from the product rule.) Set $H(z) = f(z)/q(z)$, so H is analytic on $\mathfrak{B}(z_0; r)$ and $H(z_0) \neq 0$. Then

$$\frac{f(z)}{g(z)} = \frac{H(z)}{z - z_0}, \; z \neq z_0,$$

so, by Lemma 3.10.3 with $m = 1$,

$$\text{Res}\left(\frac{f}{g}; z_0\right) = H(z_0) = \frac{f(z_0)}{q(z_0)} = \frac{f(z_0)}{g'(z_0)}. \qquad \blacksquare$$

3.10.6 Example.

Let $f(z) = e^{iz}/(z^2 + 1)$. Calculate the residue of f at each of its isolated singularities.

Solution. The function f has simple poles at $\pm i$ and is analytic on $\mathbb{C} \setminus \{i, -i\}$. We have

$$e^{i(\pm i)} = e^{\pm i^2} = e^{\pm(-1)} = e^{\mp 1} \quad \text{and} \quad \frac{d}{dz}[z^2 + 1]\bigg|_{z = \pm i} = \pm 2i.$$

Hence

$$\text{Res}\left(\frac{e^{iz}}{z^2 + 1}; z = \pm i\right) = \pm\frac{e^{\mp 1}}{2i} = \mp\frac{ie^{\mp 1}}{2}. \qquad \blacktriangle$$

3.10.7 Remark.

There are many other formulas and rules for calculating residues, assuming different properties of the function under consideration. Some of these will be presented in the exercises.

If you ever need to calculate a residue for which none of the (admittedly few) methods here are appropriate, consult another book on complex analysis. Frequently there are additional strategies and rules in the "residues" section of the text and the accompanying exercises.

3.10.2. The winding number.

Before we come to our major result on residues, we need a technical concept that allows us to count how many times a closed curve "winds around" a given point.

3.10.8 Example.

Fix $z_0 \in \mathbb{C}$, $r > 0$, *and* $k \in \mathbb{Z}$ *and let*

$$\gamma_n(t) := z_0 + re^{int}, \ 0 \le t \le 2\pi.$$

Intuitively, if $z \in \mathfrak{B}(z_0; r)$, *then* γ_k *"winds around"* z *exactly* n *times, where if* $n < 0$ *we interpret this to mean "*$|n|$ *clockwise times." And if* $z \in \mathbb{C} \setminus \overline{\mathfrak{B}}(z_0; r)$, *then* γ_n *"winds around"* z *zero times.*

To make this precise, show that

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{d\xi}{\xi - z} = \begin{cases} n, & |z - z_0| < r \\ 0, & |z - z_0| > r. \end{cases} \tag{3.10.2}$$

Solution. We do the calculation in a number of cases.

1. $n = 0$. Then γ_n is constant, and so any line integral over γ_n is 0 (because of the factor of $\gamma_n' = 0$ in the definition of the line integral). In this case, the right side of (3.10.2) is 0 in either case, as is the left.

2. $|z - z_0| > r$, $n \in \mathbb{Z} \setminus \{0\}$. Then $f(\xi) := 1/(\xi - z)$ is analytic on a ball containing $\mathfrak{B}(z_0; r)$, so the Cauchy integral formula implies

$$\int_{\gamma_n} \frac{d\xi}{\xi - z} = 0.$$

3. $|z - z_0| < r$, $n \ge 1$. Our instinct should be to use the Cauchy integral theorem. However, the Cauchy integral theorem is only valid for an integral taken over a circle that is traversed exactly once (per our custom in Remark 3.5.15). So, we write γ_n as the direct sum of n curves, each of which parametrizes the circle $|\xi - z_0| = r$ exactly once. Specifically, we set

$$\mu(t) := z_0 + re^{int}, \ 0 \le t \le \frac{2\pi}{n},$$

and then we have

$$\gamma_n = \underbrace{\mu \oplus \cdots \oplus \mu}_{n \text{ times}}.$$

Hence

$$\int_{\gamma_n} \frac{d\xi}{\xi - z} = n \int_{\mu} \frac{d\xi}{\xi - z} = n \int_{|\xi - z_0| = r} \frac{d\xi}{\xi - z} = 2\pi i n$$

by the Cauchy integral formula.

4. $|z - z_0| < r$, $n \le -1$. In this case

$$\gamma_n(t) = z_0 + re^{-i|n|t} = z_0 + re^{i|n|(0 + 2\pi - t)} = \gamma_{|n|}^{-}(t),$$

and so

$$\int_{\gamma_n} \frac{d\xi}{\xi - z} = \int_{\gamma_{|n|}^{-}} \frac{d\xi}{\xi - z} = -\int_{\gamma_{|n|}} \frac{d\xi}{\xi - z} = -2\pi i |n| = 2\pi i n.$$

▲

3.10.9 Definition.

Let γ be a closed curve in $\mathbb{C}$ and let $z \in \mathbb{C}$ be a point that is not in the image of γ. The **WINDING NUMBER** *or* **INDEX** *of γ around z is*

$$\chi(\gamma; z) := \frac{1}{2\pi i} \int_\gamma \frac{d\xi}{\xi - z}. \tag{3.10.3}$$

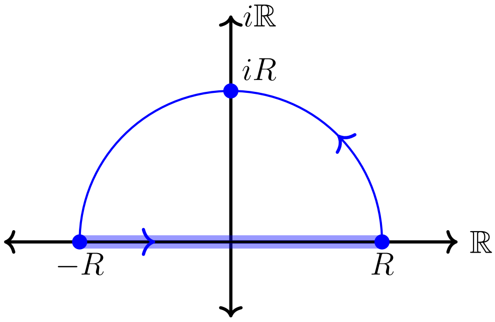
3.10.10 Example.

Fix $R > 0$. Let λ_R be the line segment $[-R, R]$ and let $\mu_R(t) = Re^{it}$, $0 \le t \le \pi$, be the upper half of the circle of radius R centered at the origin. Set $\gamma_R = \lambda_R \oplus \mu_R$. Calculate $\chi(\gamma_R; z)$ for z not in the image of γ_R.

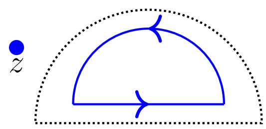
Solution. Intuitively, the "inside" of γ_R is

$$\mathcal{U}_R = \{z \in \mathbb{C} \mid |z| < R,\ \mathrm{Im}(z) > 0\},$$

so we expect $\chi(\gamma_R; z) = 0$ for $z \in \mathbb{C} \setminus (\mathcal{U}_R \cup \Gamma_R)$, where Γ_R is the image of γ_R and $\chi(\gamma_R; z) = 1$ for $z \in \mathcal{U}_R$.



If $z \in \mathbb{C} \setminus (\mathcal{U}_R \cup \Gamma_R)$, then $\xi \mapsto 1/(\xi - z)$ is analytic on a star-domain containing γ_R; see the sketch below for an example of one such star-domain.



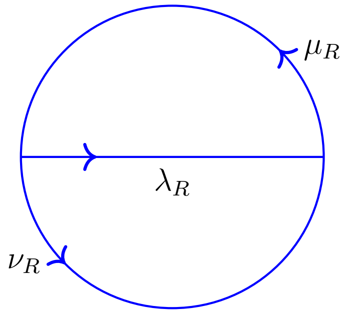
Consequently, Cauchy's integral theorem implies

$$\int_{\gamma_R} \frac{d\xi}{\xi - z} = 0. \tag{3.10.4}$$

If $z \in \mathcal{U}_R$, then Lemma 3.6.15 (or the Cauchy integral formula) implies

$$\int_{\mu_R \oplus \nu_R} \frac{d\xi}{\xi - z} = 2\pi i,$$

where μ_R and ν_R are sketched below.



On the other hand,

$$\int_{\lambda_R \oplus \nu_R^-} \frac{d\xi}{\xi - z} = 0$$

by the Cauchy integral theorem, using the same reasoning that gave (3.10.4). We then find (suppressing the integrand for simplicity)

$$2\pi i = 2\pi i + 0 = \int_{\mu_R \oplus \nu_R} + \int_{\lambda_R \oplus \nu_R^-} = \int_{\mu_R} + \int_{\nu_R} + \int_{\lambda_R} + \int_{\nu_R^-} = \int_{\mu_R} + \int_{\nu_R} + \int_{\lambda_R} - \int_{\nu_R}$$

$$= \int_{\mu_R} + \int_{\lambda_R} = \int_{\mu_R \oplus \lambda_R} = \int_{\gamma_R}.$$

Thus

$$\chi(\gamma_R; z) = \begin{cases} 1, & z \in \mathcal{U}_R \\ 0, & z \in \mathbb{C} \setminus (\mathcal{U}_R \cup \Gamma_R). \end{cases}$$ ▲

The situation in the preceding example indicates a general property of the winding number, which we will not prove: it is always an integer. This, of course, is essential if the winding number is going to "count" something.

3.10.11 Theorem.

Let γ be a closed curve and $z \in \mathbb{C}$ not be a point in the image of γ. Then $\chi(\gamma; z)$ is an integer.

3.10.3. The residue theorem.

3.10.12 Theorem (Residue theorem).

Let $\mathcal{D}$ be a star domain and let f be analytic on $\mathcal{D}$ except for isolated singularities at the points $z_1, \ldots, z_n \in \mathcal{D}$. Let γ be a closed curve in $\mathcal{D}$ whose image does not contain any of the points $z_1, \ldots, z_n$. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^n \operatorname{Res}(f; z_k)\chi(\gamma; z_k). \tag{3.10.5}$$

Proof. For $j = 1, \ldots, n$, let

$$h_j(z) := \sum_{k=1}^{\infty} a_{-k}^{(j)} (z - z_j)^{-k}$$

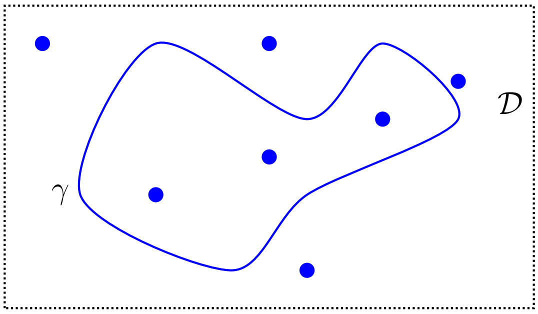
be the principal part of the Laurent series of f centered at z_j. In particular,

$$a_{-1}^{(j)} = \operatorname{Res}(f; z_j).$$

We know that h_j is analytic on $\mathbb{C} \setminus \{z_j\}$, hence

$$g(z) := f(z) - \sum_{j=1}^{n} h_j(z)$$

is analytic on $\mathcal{D} \setminus \{z_1, \ldots, z_n\}$. Moreover, if we compute the Laurent series for g centered at a z_j, we see that it has no powers of $(z - z_j)^k$ with $k < 0$. So, g has removable singularities at each z_j and therefore extends to an analytic function $\widetilde{g}$ on $\mathcal{D}$.



Since $\mathcal{D}$ is star-shaped and γ is closed, we have

$$\int_\gamma g(z)\ dz = 0 \tag{3.10.6}$$

by Cauchy's integral theorem. On the other hand, since none of the points z_j are in the image of γ, the functions f and h_j are analytic on open sets containing the image of γ, and so the integrals $\int_\gamma f(z)\ dz$ and $\int_\gamma h_j(z)\ dz$ all exist. (Of course, these open sets need not be star domains, so the integrals may not vanish.)

Then (3.10.6) translates to

$$\int_\gamma f(z)\ dz = \sum_{j=1}^{n} \int_\gamma h_j(z)\ dz. \tag{3.10.7}$$

For a fixed j, interchanging the sum and the integral, we have

$$\int_\gamma h_j(z)\ dz = \int_\gamma \sum_{k=1}^{\infty} a_{-k}^{(j)} (z - z_k)^{-k}\ dz = \sum_{k=1}^{\infty} a_{-k}^{(j)} \int_\gamma (z - z_j)^{-k}\ dz. \tag{3.10.8}$$

If $k \geq 2$, then $(z-z_j)^{-k}$ has the antiderivative $(z-z_j)^{-k+1}/(1-k)$, and so the independence of path theorem tells us

$$\int_\gamma (z - z_j)^{-k}\ dz = 0,\ \ k \geq 2. \tag{3.10.9}$$

When $k = 1$, we have

$$\int_\gamma (z - z_j)^{-1}\, dz = 2\pi i \chi(\gamma; z_j). \tag{3.10.10}$$

We combine (3.10.9) and (3.10.10) to show that (3.10.8) is

$$\int_\gamma h_j(z)\, dz = 2\pi i \chi(\gamma; z_j) a_{-1}^{(j)} = 2\pi i \chi(\gamma; z_j) \operatorname{Res}(f; z_j),$$

and so (3.10.7) becomes the desired identity (3.10.5). ■

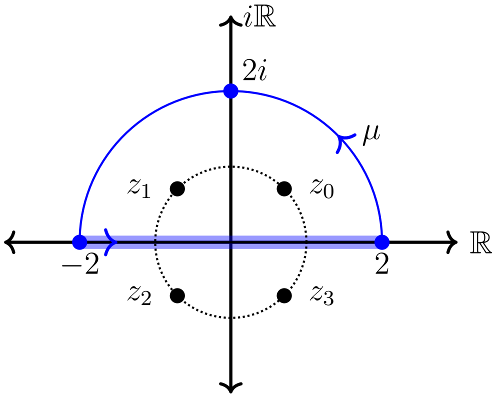
3.10.13 Example.

Let $\mu(t) = 2e^{it}$, $0 \le t \le \pi$. Calculate

$$\int_{[-2,2]\oplus\mu} \frac{dz}{z^4+1}.$$

Solution. The function $f(z) = 1/(z^4+1)$ has simple poles when $z^4 = -1$, i.e., at the fourth roots of unity. We recall from Theorem 3.2.14 that these are

$$z_k = e^{i(\pi+2\pi k)/4}, \quad k = 0, 1, 2, 3.$$



Note that only z_1 and z_2 are "inside" $\gamma := [-2,2] \oplus \mu$. More precisely, we can use Example 3.10.10 to calculate

$$\chi(\gamma; z_0) = \chi(\gamma; z_1) = 1 \quad \text{and} \quad \chi(\gamma; z_2) = \chi(\gamma; z_3) = 0.$$

So,

$$\int_{[-2,2]\oplus\mu} \frac{dz}{z^4+1} = 2\pi i \operatorname{Res}\left(\frac{1}{z^4+1}; z = e^{i\pi/4}\right) + 2\pi i \operatorname{Res}\left(\frac{1}{z^4+1}; z = e^{3i\pi/4}\right).$$

We can calculate these residues using Lemma 3.10.5. With $g(z) = 1$ and $h(z) = z^4 + 1$, we have $h'(z) = 4z^3$, so certainly $h'(z_j) \neq 0$, $j = 0, 1$. Then

$$\operatorname{Res}\left(\frac{g}{h}; z_j\right) = \frac{g(z_j)}{h'(z_j)} = \frac{1}{4z_j^3},$$

so

$$\operatorname{Res}\left(\frac{1}{z^4+1}; z = e^{i\pi/4}\right) = \frac{1}{4e^{3i\pi/4}} = \frac{e^{-3i\pi/4}}{4} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

and

$$\operatorname{Res}\left(\frac{1}{z^4+1}; z = e^{3i\pi/4}\right) = \frac{1}{4e^{9i\pi/4}} = \frac{1}{4e^{i\pi/4}} = \frac{e^{-i\pi/4}}{4} = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right).$$

Hence

$$\int_{[-2,2]\oplus\mu} \frac{dz}{z^4+1} = \frac{2\pi i}{4}\left(-\frac{2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}. \qquad \blacktriangle$$

3.10.4. Definite integrals of rational trigonometric functions.

A **RATIONAL FUNCTION IN TWO VARIABLES** is a function R of the form $R(z,w) = p(z,w)/q(z,w)$, where p and q are polynomials in the *two* variables z and w. For example,

$$p(z,w) = z + w \quad \text{and} \quad q(z,w) = z^2 + 2zw + w^2$$

are polynomials in z and w.

We will study integrals of the form

$$\int_0^{2\pi} R(\sin(\theta), \cos(\theta))\, d\theta,$$

where R is a rational function of two variables.

Our first example actually uses just the Cauchy integral formula, not the residue theorem. But the exercises will show how one can deduce a more general version of the Cauchy integral formula from the residue theorem.

3.10.14 Example.

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin(\theta)}$.

Solution. The key idea is to relate this integral to a line integral. The presence of trigonometric functions in the integrand and the limits of integration 0 and 2π suggest that we try to integrate over a circle. So, we make the formal substitution

$$z = e^{i\theta}, \quad dz = ie^{i\theta}\, d\theta = iz\, d\theta$$

and (formally) find

$$\frac{d\theta}{2+\sin(\theta)} = \frac{dz/iz}{2 + \dfrac{e^{i\theta} - e^{-i\theta}}{2i}} = \frac{dz}{iz\left(2 + \dfrac{z - z^{-1}}{2i}\right)} = \frac{dz}{2iz + \dfrac{z^2}{2} - \dfrac{1}{2}} = \frac{2}{z^2 + 4iz - 1}\, dz.$$

We therefore expect, and can indeed verify using the definition of the line integral, that

$$\int_0^{2\pi} \frac{d}{2+\sin(\theta)} = 2\int_{|z|=1} \frac{dz}{z^2 + 4iz - 1}.$$

To examine the analyticity of the integrand, we use the quadratic formula to find the zeros of the denominator:

$$z^2+4iz-1=0 \iff z=\frac{-4i\pm\sqrt{(4i)^2-4(1)(1)}}{2(1)}=\frac{-4i\pm\sqrt{-16+4}}{2}=\frac{-4i\pm i\sqrt{4}\sqrt{3}}{2}$$

$$=-2i\pm i\sqrt{3}=: z_\pm.$$

The location of these zeros relative to the circle $|z|=1$ will determine how we proceed. Clearly $|-2i-\sqrt{3}|=2+\sqrt{3}>1$, whereas

$$-1=-2+1=-2+\sqrt{1}<-2+\sqrt{3}<-2+\sqrt{4}=0$$

So, we have $z_-\notin\mathfrak{B}(0;1)$ and $z_+\in\mathfrak{B}(0;1)$. Now, we factor

$$\frac{1}{z^2+4iz-1}=\frac{1}{(z-z_-)(z-z_+)}$$

and set

$$g(z):=\frac{1}{z-z_-}.$$

Then the Cauchy integral formula implies

$$\int_{|z|=1}\frac{dz}{z^2+4iz-1}=\int_{|z|=1}\frac{g(z)}{z-z_+}\,dz=2\pi i g(z_+)=\frac{2\pi i}{z_+-z_-}$$

$$=\frac{2\pi i}{(-2i+i\sqrt{3})-(-2i-i\sqrt{3})}=\frac{\pi}{\sqrt{3}}.$$

We conclude

$$\int_0^{2\pi}\frac{d}{2+\sin(\theta)}=2\int_{|z|=1}\frac{dz}{z^2+4iz-1}=\frac{2\pi}{\sqrt{3}}.$$

▲

When evaluating a real definite integral with complex methods, one should always check that the answer is real. Our next example will not permit the simple Cauchy integral formula, and so we will have to use the residue theorem.

3.10.15 Example.

Evaluate $\displaystyle\int_0^{2\pi}\frac{d\theta}{2+\cos^2(\theta)}$.

Solution. Again we convert this integral to a line integral via the substitution

$$z=e^{i\theta},\qquad dz=ie^{i\theta}\,d\theta=iz\,d\theta,\qquad \cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z+z^{-1}}{2}.$$

Then

$$\cos^2(\theta)=\left(\frac{z+z^{-1}}{2}\right)^2=\frac{z^2+2+z^{-2}}{4},$$

and so

$$\frac{d\theta}{2+\cos^2(\theta)} = \frac{dz/iz}{2+\dfrac{z^2+2+z^{-2}}{4}} = \frac{dz}{iz\left(2+\dfrac{z^2+2+z^{-2}}{4}\right)} = \frac{dz}{2iz+\dfrac{iz^3}{4}+\dfrac{iz}{2}+\dfrac{i}{4z}}$$

$$= \left(\frac{4}{i}\right)\frac{z}{z^4+10z^2+1}\,dz.$$

We expect, and can verify using the definition of the line integral, that

$$\int_0^{2\pi}\frac{d\theta}{2+\cos^2(\theta)} = \frac{4}{i}\int_{|z|=1}\frac{z}{z^4+10z^2+1}\,dz.$$

Since the integrand is a rational function, we will know where it fails to be analytic if we know the zeros of its denominator. Setting $u = z^2$, we need to solve

$$u^2+10u+1=0 \iff u = \frac{-10\pm\sqrt{100-4}}{2} = -5\pm 2\sqrt{6}.$$

Thus the integrand has poles at

$$z_1 = \sqrt{-5+2\sqrt{6}}, \qquad z_2 = \sqrt{-5-2\sqrt{6}} = i\sqrt{5+2\sqrt{6}}, \qquad z_3 = -\sqrt{-5+2\sqrt{6}},$$

and

$$z_4 = -i\sqrt{-5-2\sqrt{6}} = \sqrt{5+2\sqrt{6}}.$$

It is clear that $|z_2| = |z_4| > 1$, so we only need to be concerned with the poles at z_1 and z_3. Note that

$$-1 = -5+4 = -5+2\sqrt{4} < -5+2\sqrt{6} < -5+2\sqrt{9} = -5+6 = 1,$$

thus[58]

$$|z_1| = |\sqrt{-5+2\sqrt{6}}| < 1.$$

Likewise,

$$|z_3| = |z_1| < 1.$$

Thus z_1, $z_3 \in \mathfrak{B}(0;1)$ and z_2, $z_4 \notin \mathfrak{B}(0;1)$, and so if γ is the unit circle, then

$$\chi(\gamma; z_1) = \chi(\gamma; z_3) = 1 \quad \text{and} \quad \chi(\gamma; z_2) = \chi(\gamma; z_4) = 0.$$

The residue theorem then implies

$$\int_{|z|=1}\frac{z}{z^4+10z^2+1}\,dz = 2\pi i\left[\operatorname{Res}\left(\frac{z}{z^4+10z^2+1}; z=z_1\right) + \operatorname{Res}\left(\frac{z}{z^4+10z^2+1}; z=z_3\right)\right].$$

We will, once again, use Lemma 3.10.5 to evaluate these residues. Setting $f(z) = z$ and $g(z) = z^4+10z^2+1$, we have

$$g'(z) = 4z^3+20z = 4z(z^2+5),$$

[58] Recall that $|zw| = |z||w|$ for any z, $w \in \mathbb{C}$. Thus $|\sqrt{z}|^2 = |\sqrt{z}||\sqrt{z}| = |\sqrt{z}\sqrt{z}| = |z|$, and so $|\sqrt{z}| = \sqrt{|z|}$.

and so we see that

$$g'(z_1) = 8\sqrt{6}z_1 \neq 0 \quad \text{and} \quad g'(z_3) = 8\sqrt{6}z_3 \neq 0.$$

Thus

$$\text{Res}\left(\frac{f}{g}; z_j\right) = \frac{f(z_j)}{g'(z_j)} = \frac{z_j}{8\sqrt{6}z_j} = \frac{1}{8\sqrt{6}}, \ j = 1, 2,$$

and so

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos^2(\theta)} = \frac{4}{i} \int_{|z|=1} \frac{z}{z^4 + 10z^2 + 1} \ dz = \frac{4}{i}(2\pi i)\left(\frac{2}{8\sqrt{6}}\right) = \frac{2\pi}{\sqrt{6}}.$$

▲

3.10.16 Method: evaluate the integral $\int_0^{2\pi} R(\cos(\theta), \sin(\theta)) \ d\theta$, R rational

1. Make the changes of variables

$$z = e^{i\theta}, \qquad dz = iz \ d\theta, \quad \cos(\theta) = \frac{z + z^{-1}}{2}, \quad \text{and} \quad \sin(\theta) = \frac{z - z^{-1}}{2i}$$

to convert the given integral into

$$\int_{|z|=1} \frac{p(z)}{q(z)} \ dz.$$

Here p and q are polynomials.

2. Find the zeros of q with modulus less than 1; these are the poles of f inside the circle $|z| = 1$.

3. Use the residue theorem to evaluate

$$\int_{|z|=1} \frac{p(z)}{q(z)} \ dz = 2\pi i \sum_{\substack{q(w)=0 \\ |w|<1}} \text{Res}\left(\frac{p}{q}; w\right).$$

Check that your final answer is real, since the original integral is real-valued.

3.10.5. Fourier integrals.

From the definition of the Fourier transform in (2.5.2), we often need to evaluate, or at least understand deeply, integrals of the form

$$\int_{-\infty}^{\infty} f(x)e^{iKx} \ dx$$

for $K \in \mathbb{R}$.

3.10.17 Example.

Let $f(x) = e^{-x^2}$. In Example 2.5.13, we showed that

$$\widehat{f}(k) = \frac{e^{-k^2/4}}{\sqrt{2}}.$$

Rederive this result by using the residue theorem to calculate the Fourier integral defining $\widehat{f}$.

Solution. Since $|x| \leq x^2$ for $|x| \geq 1$, we have $e^{-x^2} \leq e^{-|x|}$ for $|x| \geq 1$, and it follows from the comparison test that $f \in L^1(\mathbb{R})$. We will use some complex analysis techniques to evaluate the Fourier integral

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-ikx}\ dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ikx)}\ dx.$$

However, we previously learned how to find Fourier transforms of rational functions; e^{-x^2} is obviously not rational, so the techniques here will be a little different.

First, completing the square, we find

$$x^2 + ikx = x^2 + ikx - \frac{k^2}{4} + \frac{k^2}{4} = x^2 + ikx + \left(\frac{ik}{2}\right)^2 + \frac{k^2}{4} = \left(x + \frac{ik}{2}\right)^2 + \frac{k^2}{4}.$$

Thus

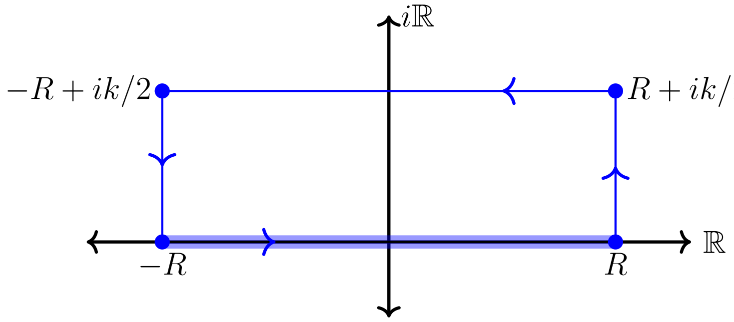
$$\int_{-\infty}^{\infty} e^{-(x^2+ikx)}\ dx = \int_{-\infty}^{\infty} e^{-(x+ik/2)^2 - k^2/4}\ dx = e^{-k^2/4} \int_{-\infty}^{\infty} e^{-(x+ik/2)^2}\ dx =: e^{-k^2/4} \mathcal{I}_k.$$

Our goal is to calculate $\mathcal{I}_k$. The improper integral $\mathcal{I}_k$ must converge, since it is just the original Fourier integral multiplied by the constant $e^{k^2/4}$. This implies

$$\mathcal{I}_k = \lim_{R \to \infty} \int_{-R}^{R} e^{-(x+ik/2)^2}\ dx = \lim_{R \to \infty} \int_{[-R+ik/2, R+ik/2]} e^{-z^2}\ dz.$$

We will relate this integral over a line segment to a line integral over a closed curve. Instead of a semicircle, though, we integrate over

$$\gamma_k^R := [-R, R] \oplus [R, R + ik/2] \oplus [R + ik/2, -R + ik/2] \oplus [-R + ik/2, -R].$$



Since $z \mapsto e^{-z^2}$ is entire, Cauchy's integral theorem implies

$$\int_{\gamma_k^R} e^{-z^2}\ dz = 0.$$

Next, we estimate the integral over the vertical line segments. We parametrize $[R, R + ik/2]$ as

$$\mu_{k,+}^R(t) = (1-t)R + \frac{ikt}{2}.$$

Then

$$\left|e^{-\mu_{k,+}^R(t)^2}\right| = \exp\left[-\operatorname{Re}\left(\mu_{k,+}^R(t)\right)^2\right],$$

where

$$\left(\mu_{k,+}^R(t)\right)^2 = (1-t)^2R^2 + ikt(1-t)R - \frac{k^2t^2}{4}.$$

That is,

$$\left|e^{\mu_{k,+}^R(t)^2}\right| = \exp\left(-(1-t)^2R^2 + \frac{k^2t^2}{4}\right) = \exp\left(-R^2 + 2R - t^2R^2 + \frac{k^2t^2}{4}\right).$$

Since $0 \le t \le 1$, we estimate

$$\left|e^{\mu_{k,+}^R(t)^2}\right| \le e^{k^2/4}e^{2R-R^2},$$

and so

$$\left|\int_{\mu_{k,+}^R} e^{-z^2}\, dz\right| \le \frac{k}{2}e^{k^2/4}e^{2R-R^2} \to 0$$

as $R \to \infty$ (recall that $k \in \mathbb{R}$ is fixed). And so the integral over the right vertical segment vanishes; similar estimates show that the integral over the left vertical segment also vanishes.

Last, we observe that the integral over $[-R, R]$ converges as $R \to \infty$:

$$\lim_{R\to\infty}\int_{[-R,R]} e^{-z^2}\, dz = \lim_{R\to\infty}\int_{-R}^{R} e^{-x^2}\, dx = \sqrt{\pi}$$

by calculus.

We conclude

$$0 = \lim_{R\to\infty}\int_{\gamma_k^R} e^{-z^2}\, dz = \lim_{R\to\infty}\left(\int_{[R+ik/2,-R+ik/2]} e^{-z^2}\, dz + \int_{[-R,R]} e^{-z^2}\, dz\right) = -\mathcal{I}_k + \sqrt{\pi}.$$

Hence $\mathcal{I}_k = \sqrt{\pi}$, and so

$$\widehat{f}(k) = \frac{e^{-k^2/4}\mathcal{I}_k}{\sqrt{2\pi}} = \frac{e^{-k^2/4}}{\sqrt{2}}.$$ ▲

3.10.18 Example.

Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}\, dx.$

Solution. We evaluate the integral in the following steps.

1. The existence of this integral follows from estimating

$$\left|\frac{\cos(x)}{x^2+1}\right| \le \frac{1}{x^2+1}$$

and the existence of the integral

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2},$$

which can be proved by direct calculation (the antiderivative of $1/(1+x^2)$ is $\arctan(x)$). Thus we have

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(x)}{x^2+1}\,dx.$$

2. The classical, if not obvious, idea is to relate the integral over $[-R,R]$ to a line integral (of a different function) over a closed curve formed, in part, by $[-R,R]$. Observe that

$$\int_{-R}^{R} \frac{\cos(x)}{x^2+1}\,dx = \operatorname{Re}\left[\int_{[-R,R]} \frac{e^{iz}}{z^2+1}\,dz\right].$$

(One might think that we should write $\cos(z)$ instead of e^{iz} here. The equality would still be true, but some later steps would not work out; see the exercises for a more detailed discussion of why $\cos(z)$ is a bad idea.) Now let $\mu_R(t) = Re^{it}$, $0 \le t \le \pi$, and consider

$$\int_{[-R,R]\oplus\mu_R} \frac{e^{iz}}{z^2+1}\,dz.$$

3. The denominator z^2+1 has simple poles at $\pm i$, and so, with $\gamma_R = [-R,R]\oplus\mu_R$ and $R > 1$, the residue theorem implies

$$\int_{\gamma_R} \frac{e^{iz}}{z^2+1}\,dz = 2\pi i\left[\operatorname{Res}\left(\frac{e^{iz}}{z^2+1}; z = i\right)\chi(\gamma_R; i) + \operatorname{Res}\left(\frac{e^{iz}}{z^2+1}; z=-i\right)\chi(\gamma_R;-i)\right]$$

$$= 2\pi i\left(\frac{e^{i^2}}{2i}\right) = \pi e^{-1}.$$

4. Now we want to control

$$\lim_{R\to\infty}\int_{\mu_R} \frac{e^{iz}}{z^2+1}\,dz.$$

The best possible circumstance would be for this limit to equal zero, and indeed we can prove this using the ML inequality. We estimate

$$\left|\int_{\mu_R} \frac{e^{iz}}{z^2+1}\,dz\right| \le \pi R \max_{z\in\mu_R}\left|\frac{e^{iz}}{z^2+1}\right| = \pi R \max_{\substack{|z|=R\\ \operatorname{Im}(z)\ge 0}}\left|\frac{e^{iz}}{z^2+1}\right|.$$

If $\operatorname{Im}(z) \ge 0$, then

$$|e^{iz}| = |e^{i\operatorname{Re}(z)-\operatorname{Im}(z)}| = e^{-\operatorname{Im}(z)} \le 1.$$

Next, the reverse triangle inequality gives

$$|z^2+1| = |z^2-(-1)| \ge \big||z^2| - |-1|\big| \ge |z^2| - |-1| = |z|^2 - 1,$$

and so, for $|z| = R > 1$,

$$\frac{1}{|z^2+1|} \le \frac{1}{|z|^2-1} = \frac{1}{R^2-1}.$$

Thus

$$\left|\int_{\mu_R} \frac{e^{iz}}{z^2+1}\,dz\right| \le \frac{\pi R}{R^2-1} \to 0 \text{ as } R\to\infty.$$

5. We conclude

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\, dx = \operatorname{Re}\left[\lim_{R\to\infty} \int_{[-R,R]} \frac{e^{iz}}{z^2+1}\, dz\right]$$

$$= \operatorname{Re}\left[\lim_{R\to\infty} \int_{[-R,R]} \frac{e^{iz}}{z^2+1}\, dz + \lim_{R\to\infty} \int_{\mu_R} \frac{e^{iz}}{z^2+1}\, dz\right]$$

$$= \operatorname{Re}\left[\lim_{R\to\infty} \int_{[-R,R]\oplus\mu_R} \frac{e^{iz}}{z^2+1}\, dz\right] = \operatorname{Re}(\pi e^{-1}) = \pi e^{-1}.$$ ▲

3.10.19 Remark.

We will often integrate over the upper half of a circle of radius R centered at the origin. We parametrize this curve, of course, by $\mu(t) = Re^{it}$, $0 \le t \le \pi$, and we write

$$\int_{\substack{|z|=R \\ \operatorname{Im}(z)\ge 0}} f(z)\, dz := iR \int_0^{\pi} f(Re^{it})e^{it}\, dt.$$

The following lemma helps us control integrals over circular arcs. We omit its proof, which fundamentally depends on some technical inequalities from real-variable calculus (see [1]).

3.10.20 Lemma (Jordan).

(i) *Let $R > 0$. Then*

$$\int_0^{\pi} e^{-R\sin(t)}\, dt \le \frac{\pi}{R}.$$

(ii) *Suppose that f is continuous on $\operatorname{Im}(z) \ge 0$ and*

$$\lim_{R\to\infty} \max_{\substack{|z|=R \\ \operatorname{Im}(z)\ge 0}} |f(z)| = 0.$$

Then

$$\lim_{R\to\infty} \int_{\substack{|z|=R \\ \operatorname{Im}(z)\ge 0}} e^{i\alpha z} f(z)\, dz = 0$$

for any $\alpha > 0$. An analogous result holds for $\alpha < 0$ if we replace $\operatorname{Im}(z) \ge 0$ by $\operatorname{Im}(z) \le 0$ throughout.

3.10.21 Example.

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin(x)}{x^4+16}\, dx.$

Solution. 1. If we attempt to "dominate" the integrand by an improperly integrable function, we will not be successful; we have

$$\left|\frac{x^3\sin(x)}{x^4+16}\right| \leq \frac{|x|^3}{x^4+16},$$

but the function on the right is not improperly integrable on $\mathbb{R}$, as the denominator's degree is only 1 more than the numerator's. Instead, one can integrate by parts to find an antiderivative, still involving another integral, of the original integrand; this new integral turns out to be improperly integrable. We defer the calculation to the exercises and proceed confident that

$$\int_{-\infty}^{\infty} \frac{x^3\sin(x)}{x^4+16}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{x^3\sin(x)}{x^4+16}\,dx.$$

2. We have

$$\int_{-R}^{R} \frac{x^3\sin(x)}{x^4+16}\,dx = \mathrm{Im}\left[\int_{[-R,R]} \frac{z^3e^{iz}}{z^4+16}\,dz\right].$$

As before, let $\mu_R(t) = Re^{it}$, $0 \leq t \leq \pi$, and consider

$$\int_{\gamma_R} \frac{z^3e^{iz}}{z^4+16}\,dz, \qquad \gamma_R = [-R,R] \oplus \mu_R.$$

3. Observe that the integrand above has simple poles when $z^4+16=0$, i.e., when $z^4=-16$. This occurs for

$$z = z_k := |-16|^{1/4}e^{i(\mathrm{Arg}(-16)+2\pi k)/4} = 2e^{i(2k+1)\pi/4},\ k=0,1,2,3.$$

See the sketch in Example 3.10.13 (with 2 replaced by R throughout). We see that when $R > 2$ we have

$$\chi(\gamma_R;z_0) = \chi(\gamma_R;z_1) = 1 \quad\text{and}\quad \chi(\gamma_R;z_2) = \chi(\gamma_R;z_3) = 0.$$

Hence

$$\int_{\gamma_R} \frac{z^3e^{iz}}{z^4+16}\,dz = 2\pi i\left[\mathrm{Res}\left(\frac{z^3e^{iz}}{z^4+16};z=z_0\right) + \mathrm{Res}\left(\frac{z^3e^{iz}}{z^4+16};z=z_1\right)\right].$$

Since the poles z_k are simple, we have

$$\mathrm{Res}\left(\frac{z^3e^{iz}}{z^4+16};z=z_k\right) = \frac{z_k^3e^{iz_k}}{4z_k^3} = \frac{e^{iz_k}}{4},$$

and thus

$$\int_{\gamma_R} \frac{z^3e^{iz}}{z^4+16}\,dz = \frac{\pi i}{2}\left(e^{iz_0}+e^{iz_1}\right).$$

We calculate further

$$z_0 = 2e^{i\pi/4} = 2\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \sqrt{2}(1+i)$$

and

$$z_1 = 2e^{3\pi i/4} = 2\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \sqrt{2}(-1+i),$$

thus

$$e^{iz_0} = e^{i\sqrt{2}-\sqrt{2}} \quad \text{and} \quad e^{iz_1} = e^{-i\sqrt{2}-\sqrt{2}}.$$

Hence

$$e^{iz_0} + e^{iz_1} = e^{-\sqrt{2}}\left(e^{i\sqrt{2}} + e^{-i\sqrt{2}}\right) = 2e^{-\sqrt{2}}\cos(\sqrt{2}),$$

and so

$$\int_{\gamma_R} \frac{z^3 e^{iz}}{z^4+16}\, dz = 2\pi i \left(\frac{2e^{-\sqrt{2}}\cos(\sqrt{2})}{4}\right) = i\pi e^{-\sqrt{2}}\cos(\sqrt{2}).$$

4. Now consider

$$\lim_{R\to\infty} \int_{\mu_R} \frac{z^3 e^{iz}}{z^4+16}\, dz.$$

We will use Jordan's lemma. Observe that

$$\max_{\substack{|z|=R \\ \operatorname{Im}(z)\geq 0}} \left|\frac{z^3}{z^4+16}\right| \leq \frac{R^3}{R^4-16} \to 0 \text{ as } R \to \infty,$$

using the triangle inequality to estimate the denominator via

$$|z^4+16| = |z^4-(-16)| \geq \big||z^4| - |-16|\big| \geq |z^4| - 16 = |z|^4 - 16 = R^4 - 16.$$

Hence the limit is zero.

5. We conclude

$$\int_{-\infty}^{\infty} \frac{x^3\sin(x)}{x^4+16}\, dx = \operatorname{Im}\left[\lim_{R\to\infty}\int_{[-R,R]} \frac{z^3 e^{iz}}{z^4+16}\, dz\right]$$

$$= \operatorname{Im}\left[\lim_{R\to\infty}\int_{[-R,R]} \frac{z^3 e^{iz}}{z^4+16}\, dz + \int_{\mu_R} \frac{z^3 e^{iz}}{z^4+16}\, dz\right] = \operatorname{Im}\left[\lim_{R\to\infty}\int_{[-R,R]\oplus\mu_R} \frac{z^3 e^{iz}}{z^4+16}\, dz\right]$$

$$= \operatorname{Im}\left[i\pi e^{-\sqrt{2}}\cos(\sqrt{2})\right] = \pi e^{-\sqrt{2}}\cos(\sqrt{2}).$$ ▲

3.10.22 Method: evaluate $\int_{-\infty}^{\infty} R(x)\,\mathrm{trig}(\alpha x)\, dx$, R rational

1. Suppose that p and q are polynomials with real coefficients such that $q(x) \neq 0$ for any $x \in \mathbb{R}$ and that the improper integral

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,\mathrm{trig}(\alpha x)\, dx$$

exists, where $\mathrm{trig}(X) = \sin(X)$ or $\mathrm{trig}(X) = \cos(X)$. (Implicitly this will require $\deg(q) \geq \deg(p)+1$.) Assume $\alpha > 0$. This implies

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,\mathrm{trig}(\alpha x)\, dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{p(x)}{q(x)}\,\mathrm{trig}(\alpha x)\, dx.$$

2. Let

$$f(z) = \frac{p(z)}{q(z)} e^{i\alpha z}.$$

Let $\mathcal{U}$ denote the set of isolated singularities of f in $\mathrm{Im}(z) \geq 0$; this set is necessarily finite, since the only singularities of f are the zeros of q. Observe that for R sufficiently large,

$$\int_{[-R,R]\oplus\mu_R} f(z)\ dz = 2\pi i \sum_{w\in\mathcal{U}} \mathrm{Res}(f;w),$$

where $\mu_R(t) = Re^{it}$, $0 \leq t \leq \pi$.

3. Use Jordan's lemma to show

$$\lim_{R\to\infty} \int_{\mu_R} f(z)\ dz = 0.$$

(Implicitly this also requires $\deg(q) \geq \deg(p) + 1$.)

4. Conclude that

$$\lim_{R\to\infty} \int_{[-R,R]} f(z)\ dz = 2\pi i \sum_{w\in\mathcal{U}} \mathrm{Res}(f;w).$$

Take the real or imaginary part of this expression to calculate the original integral.

5. If $\alpha < 0$, replace α with $|\alpha|$ and work instead with

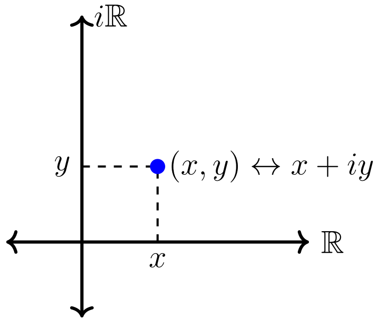
$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \operatorname{trig}(\alpha x)\ dx = \pm \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \operatorname{trig}(|\alpha| x)\ dx.$$

We hasten to add that there are many improper integrals of real-valued functions of a real variable that do not fall into the rather specific method above. If faced with such an integral in the wild, one should consult a standard undergraduate complex analysis book for further techniques ([1] and [22] offer a particularly comprehensive overview of such integrals, as well as the more advanced residue theory needed to evaluate them).

A. Fundamentals of Complex Numbers

A.1. Basic definitions.

A **complex number** is an expression of the form $x+iy$ where x, $y \in \mathbb{R}$ and the symbol i satisfies $i^2 = -1$. We denote the set of all complex numbers by $\mathbb{C}$. We frequently identify $\mathbb{C}$ with $\mathbb{R}^2$, and, in fact, any rigorous "construction" of $\mathbb{C}$ *defines* $\mathbb{C}$ to be $\mathbb{R}^2$, just with a special multiplication structure. We will graph subsets of $\mathbb{C}$ as subsets of $\mathbb{R}^2$ with the x-axis identified with $\mathbb{R}$ and the y-axis identified with $i\mathbb{R} = \{iy \mid y \in \mathbb{R}\}$.



A.1.1 Definition.

Let $z = x + iy \in \mathbb{C}$.

(i) *The* **real part** *of* z *is* $\mathrm{Re}(z) := x$.

(ii) *The* **imaginary part** *of* z *is* $\mathrm{Im}(z) := y$.

(iii) *The* **modulus** *of* z *is* $|z| := \sqrt{x^2+y^2}$. *This corresponds to our notion of* z *as a vector in* $\mathbb{R}^2$*; the length of* (x, y) *is, of course,* $\sqrt{x^2+y^2}$.

(iv) *The* **complex conjugate** *of* z *is* $\overline{z} := x - iy$.

A.1.2 Definition.

Let $z = x_1 + iy_1$, $w = x_2 + iy_2 \in \mathbb{C}$.

(i) *We define* $z = w$ *if and only if* $x_1 = x_2$ *and* $y_1 = y_2$.

(ii) *We define*

$$z + w := (x_1 + x_2) + i(y_1 + y_2).$$

(iii) *We define*

$$zw = (x_1 + iy_1)(x_2 + iy_2) := (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2). \tag{A.1.1}$$

In particular, if $\alpha \in \mathbb{R}$ *is real, then*

$$\alpha z = \alpha x_1 + i\alpha y_1.$$

(iv) *The* **reciprocal**[59] *of* $z \neq 0$ *is*

$$\frac{1}{z} := |z|^{-2}\overline{z} = \frac{\overline{z}}{|z|^2}. \tag{A.1.2}$$

A.1.3 Example.

Calculate the reciprocal of i using the definition in (A.1.2).

Solution. The definition gives

$$\frac{1}{i} = \frac{\bar{i}}{|i|^2} = \frac{-i}{1} = -i.$$

Indeed, $(-i)i = (-1)(i^2) = (-1)(-1) = 1$. ▲

The preceding definitions imply the following essential properties of complex numbers, some of which are just repackaging of the notations above. The proofs are left as exercises.

A.1.4 Theorem.

Let z, $w \in \mathbb{C}$.

(i) $\operatorname{Re}(z+w) = \operatorname{Re}(z) + \operatorname{Re}(w)$ *and* $\operatorname{Im}(z+w) = \operatorname{Im}(z) + \operatorname{Im}(w)$.

(ii) $\overline{z+w} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\,\overline{w}$.

(iii) $|z|^2 = z\overline{z}$.

(iv) [Triangle inequality] $|z+w| \le |z| + |w|$.

(v) [Reverse triangle inequality] $\big||z| - |w|\big| \le |z - w|$.

(vi) $\operatorname{Re}(z) \le |z|$, $\operatorname{Im}(z) \le |z|$, *and* $|z| \le |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$.

(vii) *All the usual rules of arithmetic hold: complex addition and multiplication are commutative and associative, and complex multiplication distributes over addition. That is,*

$$z + w = w + z, \quad zw = wz, \quad (z+w)+u = z+(w+u), \quad (zw)u = z(wu),$$
$$\text{and} \quad z(w+u) = zw + zu.$$

We will find some considerable ambiguities in attempting to pin meaning to the symbol z^w for z, $w \in \mathbb{C}$. For now, we specify what z^n means when n is an integer.

[59] How should we define division for complex numbers? There may not be an apparent numerical algorithm, but we should remember that whatever the "reciprocal" of a number $z \neq 0$ is, it should satisfy the multiplication property

$$\left(\frac{1}{z}\right) z = z\left(\frac{1}{z}\right) = 1.$$

Moreover, from our experience with real numbers, we expect

$$\left(\frac{1}{z}\right)\left(\frac{a}{b}\right) = \frac{a}{zb}.$$

So, *formally*, we expect

$$\frac{1}{z} = \left(\frac{1}{z}\right) \cdot 1 = \left(\frac{1}{z}\right)\left(\frac{\overline{z}}{\overline{z}}\right) = \frac{\overline{z}}{|z|^2}.$$

What should this last symbol mean? Surely it is the same as $|z|^{-2}\overline{z}$. This is the product of the real number $|z|^{-2}$ and the complex number $\overline{z}$ as defined in part (iii) of Definition A.1.2. We can then *check* from the product formula (A.1.1) that $z(|z|^{-2}\overline{z}) = (|z|^{-2}\overline{z})z = 1$.

A.1.5 Definition.

Let $z \in \mathbb{C}$ and $n \in \mathbb{Z}$. Then

$$z^n := \begin{cases} 1, \ n = 0 \\ z, \ n = 1 \\ zz^{n-1}, \ n \geq 2 \\ \dfrac{1}{z^{-n}}, \ n \leq -1 \text{ and } z \neq 0. \end{cases}$$

Note that we do allow $0^0 = 1$; this is chiefly for convenience in power series.

With this definition, we have the familiar relations

$$z^{m+n} = z^m z^n \quad \text{and} \quad (z^m)^n = z^{mn}$$

for m, $n \in \mathbb{Z}$, with $z \neq 0$ if one if these integers is negative.

A.1.6 Example.

Compute $\operatorname{Re}\left(\dfrac{1+i}{1-i}\right)$.

Solution. We have

$$\frac{1+i}{2-i} = \frac{(1+i)(2+i)}{(2-i)(2+i)} = \frac{2+i+2i+i^2}{4+2i-2i-i^2} = \frac{2+3i-1}{4-(-1)} = \frac{1+3i}{5},$$

thus

$$\operatorname{Re}\left(\frac{1+i}{1-i}\right) = \frac{1}{5}. \qquad \blacktriangle$$

A.2. Series.

The basic definitions, properties, and tests for series convergence are the same whether we work with real or complex numbers, so we assume that everything is complex here.

A.2.1. Series convergence.

We will mostly be interested in sequences formed by the partial sums of a *series*.

A.2.1 Definition.

Suppose that (a_k) is a sequence of complex numbers. As we do in the real case, we say that the SERIES $\sum_{k=0}^{\infty} a_k$ CONVERGES *to the sum S if the sequence $\left(\sum_{k=0}^{n} a_k\right)$ of partial sums converges to S, i.e., if*

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k = S.$$

In that case, we write $S = \sum_{k=0}^{\infty} a_k$. *Mathematical parlance uses the symbol* $\sum_{k=0}^{\infty} a_k$ *to refer to both the sequence of partial sums* $(\sum_{k=0}^{n} a_k)$ *and to the limit of that sequence, if it exists.*

The series $\sum_{k=0}^{\infty} a_k$ **CONVERGES ABSOLUTELY** *if the (real, nonnegative) series* $\sum_{k=0}^{\infty} |a_k|$ *converges. If* $m \in \mathbb{N}$, *we define the convergence of the series* $\sum_{k=m}^{\infty} a_k$ *in the same way by replacing* $k = 0$ *with* $k = m$ *above.*

Many of the convergence properties and tests that we learned for real series in calculus apply verbatim to the complex case. We state some useful techniques without proof.

A.2.2 Theorem (Tests for series convergence).

Let (a_k) *be a sequence of complex numbers.*

(i) [Test for divergence] *If* $\lim_{k\to\infty} a_k \neq 0$, *then the series* $\sum_{k=0}^{\infty} a_k$ *diverges. It is possible to have* $\lim_{k\to\infty} a_k = 0$ *and* $\sum_{k=0}^{\infty} a_k$ *still divergent.*

(ii) [Ratio test] *Suppose only finitely many of the terms* a_k *are zero (equivalently, there is* $N \in \mathbb{N}$ *such that* $a_k \neq 0$ *for* $k \geq N$*) and the limit*

$$L := \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

exists. Then $\sum_{k=0}^{\infty} a_k$ *converges absolutely if* $0 \leq L < 1$.

(iii) [Comparison test for series of nonnegative terms] *If* (b_k) *is another sequence in* $\mathbb{C}$, *and* $0 \leq a_k \leq b_k$ *for all* k, *and if* $\sum_{k=0}^{\infty} b_k$ *converges, then* $\sum_{k=0}^{\infty} a_k$ *also converges, and* $\sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k$.

(iv) [Absolute convergence implies "regular" convergence] *If* $\sum_{k=0}^{\infty} |a_k|$ *converges, then* $\sum_{k=0}^{\infty} a_k$ *converges.*

There are many other tests for convergence, but, in practice, this course will only require us to use the test for divergence, the ratio test, and the comparison test. Frequently, the series that we study will be absolutely convergent, so it is often worthwhile to check the stronger condition of absolute convergence first. Most often, the series that interest us will be *power* series, which depend on a variable z.

A.2.3 Example.

Show that the series

$$\sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges absolutely for all $z \in \mathbb{C}$.

Solution. We use the ratio test and compute

$$\left| \frac{\dfrac{z^{k+1}}{(k+1)!}}{\dfrac{z^k}{k!}} \right| = \left| \frac{z^{k+1}}{(k+1)!} \left(\frac{k!}{z^k} \right) \right| = \left| \frac{z^k z}{(k+1)k!} \left(\frac{k!}{z^k} \right) \right| = \frac{|z|}{k+1}.$$

Since

$$\lim_{k\to\infty} \frac{|z|}{k+1} = 0$$

for all $z \in \mathbb{C}$, we have absolute convergence for any z. ▲

An important tool for manipulating series is **REINDEXING**.

A.2.4 Lemma (Reindexing).

Suppose that (a_k) is a sequence of complex numbers and $m \in \mathbb{N}$. Then $\sum_{k=m}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} a_{k+m}$ converges, in which case the two series converge to the same sum.

Proof. The series $\sum_{k=m}^{\infty} a_k$ converges if and only if the sequence of partial sums $(\sum_{k=m}^{n} a_k)$ converges. If $n \geq m$, then $\sum_{k=m}^{n} a_k = \sum_{k=0}^{n-m} a_{k+m}$; a full proof of this identity uses induction on n, which we omit, but one can think of it as analogous to a change of variables in an integral.

Now, given $m \in \mathbb{N}$, a sequence (b_k) converges if and only if (b_{k-m}) converges[60] to the same limit. Thus $(\sum_{k=m}^{n} a_k)$ if and only if $\left(\sum_{k=0}^{n-m} a_{k+m}\right)$ converges, in which case the two sequences of partial sums have the same limit. Last, we have

$$\lim_{n\to\infty} \sum_{k=0}^{n-m} a_{k+m} = \lim_{n\to\infty} \sum_{k=0}^{n} a_{k+m}$$

if one of these limits exists, where, of course, $\lim_{n\to\infty} \sum_{k=0}^{n} a_{k+m} = \sum_{k=0}^{\infty} a_{k+m}$. ■

Another useful, often-overlooked, identity is **TELESCOPING**.

A.2.5 Lemma.

(i) *Let $n \in \mathbb{N}$ and $a_0, \ldots, a_{n+1} \in \mathbb{C}$. Then*

$$\sum_{k=0}^{n+1} \big(a_{k+1} - a_k\big) = a_{n+1} - a_0.$$

(ii) *Let $m \leq n$ be (non) integers and let $a_m, a_{m+1}, \ldots, a_n, a_{n+1} \in \mathbb{C}$. Then*

$$\sum_{k=m}^{n} \big(a_{k+1} - a_k\big) = a_{n+1} - a_m.$$

Proof. Part (ii) follows from part (i) by setting $A_k = a_{k+m}$ for $k = 0, \ldots, n+1-m$. The validity of part (i) is intuitively clear in that successive pairs of terms cancel each other:

$$\sum_{k=0}^{n+1} \big(a_{k+1}-a_k\big) = \big(a_1-a_0\big)+\big(a_2-a_1\big)+\big(a_3-a_2\big)+\cdots+\big(a_{n-1}-a_{n-2}\big)+\big(a_n-a_{n-1}\big)+\big(a_{n+1}-a_n\big)$$
$$= a_{n+1} - a_0.$$

A rigorous proof, however, requires induction on n. ■

[60] If we index (b_k) over $k \in \mathbb{N}$, then we index (b_{k-m}) over $\{k \in \mathbb{N} \mid k > m\}$, since b_j is not defined for $j \leq 0$.

A.2.6 Example.

Let $z \in \mathbb{C}$. Show that the **GEOMETRIC SERIES** $\sum_{k=0}^{\infty} z^k$ *converges if and only if $|z| < 1$, in which case*

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Solution. Suppose $|z| < 1$ and fix $n \in \mathbb{N}$. We develop a formula for the nth partial sum of the geometric series. By telescoping, we have

$$\sum_{k=0}^{n} \left(z^k - z^{k+1}\right) = z^0 - z^{n+1} = 1 - z^{n+1}. \tag{A.2.1}$$

We factor

$$\sum_{k=0}^{n} \left(z^k - z^{k+1}\right) = \sum_{k=0}^{n} z^k - \sum_{k=0}^{n} z^{k+1} = \sum_{k=0}^{n} z^k - z\sum_{k=0}^{n} z^k = (1-z)\sum_{k=0}^{n} z^k. \tag{A.2.2}$$

Equating (A.2.1) and (A.2.2) and dividing by $1-z$, which is permissible since $|z| < 1$, we have

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}.$$

Since $|z| < 1$, we have $\lim_{n\to\infty} z^{n+1} = 0$, and so

$$\sum_{k=0}^{\infty} z^k = \lim_{n\to\infty} \sum_{k=0}^{n} z^k = \lim_{n\to\infty} \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z}.$$

Now suppose $|z| \geq 1$. Then $|z^k| = |z|^k \geq 1$, too, and so the limit $\lim_{k\to\infty} z^k$ does not exist; the test for divergence then applies. ▲

It is sometimes convenient to split a given series into two or more "subseries," say, a sum over all the even terms and a sum over all the odd terms. That is, given a sequence (a_k) in $\mathbb{C}$ with $\sum_{k=0}^{\infty} a_k$ convergent, we would like to have

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}. \tag{A.2.3}$$

This holds at least when the series converges absolutely.

A.2.7 Lemma.

Let (a_k) be a sequence in $\mathbb{C}$.

(i) *The series $\sum_{k=0}^{\infty} a_k$ converges if and only if the series $\sum_{k=0}^{\infty} \left(a_{2k} + a_{2k+1}\right)$ converges, in which case the sums are the same.*

(ii) *Suppose the series $\sum_{k=0}^{\infty} a_k$ converges absolutely. Then the "even" and "odd" series $\sum_{k=0}^{\infty} a_{2k}$ and $\sum_{k=0}^{\infty} a_{2k+1}$ both converge absolutely, and (A.2.3) holds.*

(iii) *More generally, for each $N \geq 2$, the series $\sum_{k=0}^{\infty} a_k$ converges if and only if the* *series*

$$\sum_{j=0}^{\infty}\left(\sum_{r=0}^{N-1} a_{jN+r}\right),$$

converges, in which case the two series have the same sum, and if the original series $\sum_{k=0}^{\infty} a_k$ *converges absolutely, then so do each of the subseries* $\sum_{j=0}^{\infty} a_{jN+r}$.

Proof. (i) Since the limit $\lim_{n\to\infty}\sum_{k=0}^{n} a_k$ exists if and only if the limit $\lim_{n\to\infty}\sum_{k=0}^{2n+1} a_k$ exists, and since $\sum_{k=0}^{2n+1} a_k = \sum_{k=0}^{n}(a_{2k}+a_{2k+1})$, the existence of the two limits $\lim_{n\to\infty}\sum_{k=0}^{n} a_k$ and $\lim_{n\to\infty}\sum_{k=0}^{n}(a_{2k}+a_{2k+1})$ is equivalent.

(ii) We prove only that the even-indexed series converges absolutely. First, we have

$$|a_{2k}| \le |a_{2k}| + |a_{2k+1}| =: b_k. \tag{A.2.4}$$

Next,

$$\sum_{k=0}^{n} b_k = \sum_{k=0}^{2n+1} |a_k|.$$

Since

$$S := \sum_{k=0}^{\infty} |a_k| \lim_{n\to\infty} \sum_{k=0}^{2n+1} |a_k|$$

converges, the limit $\lim_{n\to\infty}\sum_{k=0}^{n} b_k$ exists, too. Then (A.2.4) and the comparison test implies the absolute convergence of $\sum_{k=0}^{\infty} a_{2k}$. In particular, the limit $\lim_{n\to\infty}\sum_{k=0}^{n} a_{2k}$ exists.

Assuming that the odd-indexed series converges (absolutely), we find

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty}\sum_{k=0}^{2n+1} a_k = \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_{2k} + \sum_{k=0}^{n} a_{2k+1}\right) = \lim_{n\to\infty}\sum_{k=0}^{n} a_{2k} + \lim_{n\to\infty}\sum_{k=0}^{n} a_{2k+1}$$
$$= \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}.$$

To obtain the penultimate equality, we needed to know the existence of the separate limits $\lim_{n\to\infty}\sum_{k=0}^{n} a_{2k}$ and $\lim_{n\to\infty}\sum_{k=0}^{n} a_{2k+1}$.

(iii) The proof is similar to the two parts above, so we omit it. ∎

A.2.8 Example.

Discuss the convergence of the even/odd-indexed "subseries" of

$$\sum_{k=2}^{\infty} \ln\left(1 + \frac{(-1)^k}{k}\right).$$

Solution. First, the original series does converge. Put $a_k = \ln(1+(-1)^k/k)$. Then, after some algebraic simplification,

$$a_{2k}+a_{2k+1} = \ln\left(1+\frac{1}{2k}\right)+\ln\left(1-\frac{1}{2k+1}\right) = \ln\left[\left(1+\frac{1}{2k}\right)\left(1-\frac{1}{2k+1}\right)\right] = \ln(1) = 0$$

for all k. That is, $\sum_{k=0}^{\infty}(a_{2k}+a_{2k+1})$ converges to 0, and therefore, by part (i) of Lemma A.2.7, so does $\sum_{k=0}^{\infty}a_k$.

Next, we show that the even subseries $\sum_{k=0}^{\infty}a_{2k}$ diverges. Estimate

$$\ln\left(1+\frac{1}{2k}\right)=\ln\left(\frac{2k+1}{2k}\right)=\ln(2k+1)-\ln(2k)=\int_{2k}^{2k+1}\frac{dx}{x}\geq\frac{(2k+1)-(2k)}{2k+1}=\frac{1}{2k+1}.$$

By comparison with the divergent harmonic series $\sum_{k=1}^{\infty}k^{-1}$, the series $\sum_{k=0}^{\infty}\ln(1+1/2k)$ then diverges. ▲

A.2.2. Doubly infinite series and synchronous convergence.

Suppose that (a_k) is a **DOUBLY INFINITE SEQUENCE** in $\mathbb{C}$, i.e., (a_k) is indexed by integers $k\in\mathbb{Z}$. How should we define the symbol

$$\sum_{k=-\infty}^{\infty}a_k?$$

If we think of series as "discrete" analogues of improper integrals over infinite intervals (a notion crystallized by the integral test for series, which we do not use in this course), then there are at least two natural definitions. One is to recall that the usual way to define the improper integral $\int_{-\infty}^{\infty}f(x)\,dx$ is to fix a "break" point $x_0\in\mathbb{R}$ and put

$$\int_{-\infty}^{\infty}f(x)\,dx:=\int_{-\infty}^{x_0}f(x)\,dx+\int_{x_0}^{\infty}f(x)\,dx, \tag{A.2.5}$$

if each of these two improper integrals on the right converge. We review this in detail in Appendix B, specifically Definition B.0.2 and Theorem B.0.3.

Here, then, is our first definition of convergence for a doubly infinite series.

A.2.9 Definition.

Let (a_k) be a doubly infinite sequence in $\mathbb{C}$. The series $\sum_{k=-\infty}^{\infty}a_k$ **CONVERGES** *to $S\in\mathbb{C}$ if the series $\sum_{k=0}^{\infty}a_k$ converges to $S_1\in\mathbb{C}$ and if the series $\sum_{k=1}^{\infty}a_{-k}$ converges to $S_2\in\mathbb{C}$ (both in the sense of Definition A.2.1) and if $S=S_1+S_2$.*

We use this notion of convergence chiefly for Laurent series in Theorem 3.9.16. All the usual algebraic properties of series carry over to this definition, e.g., if both $\sum_{k=-\infty}^{\infty}a_k$ and $\sum_{k=-\infty}^{\infty}b_k$ converge (in the sense of Definition A.2.9), then so does

$$\sum_{k=-\infty}^{\infty}(\alpha a_k+\beta b_k)$$

for any α, $\beta\in\mathbb{C}$. Also, as with improper integrals (see part (v) of Theorem B.0.3) the election of $k=1$ as the "break point" above is immaterial.

A.2.10 Lemma.

Let (a_k) be a doubly infinite sequence and suppose the series $\sum_{k=-\infty}^{\infty}a_k$ converges (in the sense of Definition A.2.9) to $S\in\mathbb{C}$. Then for any $N\in\mathbb{Z}$, the series $\sum_{k=N}^{\infty}a_k$ and

$\sum_{k=1}^{\infty} a_{N-k}$ *converge and*

$$S = \sum_{k=N}^{\infty} a_k + \sum_{k=1}^{\infty} a_{N-k}.$$

A second definition of convergence for doubly infinite sequences is motivated by the natural way in which complex Fourier series converge, as discussed in Section 2.4.2. This definition resembles the "principal value" improper integral. The principal value of the improper integral of f over $(-\infty, \infty)$ is the limit

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$$

from Definition B.0.6. Taking $f(x) = x$ is an obvious way to see that the principal value improper integral may exist even if the "usual" improper integral does not.

A.2.11 Definition.

Let (a_k) *be a sequence in* $\mathbb{C}$ *indexed by* $\mathbb{Z}$ *and put* $S_n := \sum_{k=-n}^{n} a_k$. *If the limit* $S := \lim_{n\to\infty} S_n$ *exists in* $\mathbb{C}$ *and equals* $S \in \mathbb{C}$, *we write* $\sum_{k=-\infty}^{\infty} a_k := S$ *and say that the series* $\sum_{k=-\infty}^{\infty} a_k$ CONVERGES (SYNCHRONOUSLY)[61] *to* S. *In other words,*

$$\sum_{k=-\infty}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=-n}^{n} a_k,$$

whenever this limit exists. As usual with series, we employ the symbol $\sum_{k=-\infty}^{\infty} a_k$ *to denote both the sequence of partial sums* (S_n) *and the limit of this sequence, if it exists.*

A.2.12 Theorem.

Let (a_k) *be a doubly infinite sequence in* $\mathbb{C}$.

(i) *If* $\sum_{k=-\infty}^{\infty} a_k$ *converges (in the sense of Definition A.2.9), then* $\sum_{k=-\infty}^{\infty} a_k$ *converges synchronously. The reverse is not true.*

(ii) *If* (b_k) *is another doubly infinite sequence in* $\mathbb{C}$, *and if* α, $\beta \in \mathbb{C}$, *and if both* $\sum_{k=-\infty}^{\infty} a_k$ *and* $\sum_{k=-\infty}^{\infty} b_k$ *both converge synchronously, then* $\sum_{k=-\infty}^{\infty} (\alpha a_k + \beta b_k)$ *converges synchronously and*

$$\sum_{k=-\infty}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=-\infty}^{\infty} a_k + \beta \sum_{k=-\infty}^{\infty} b_k.$$

Proof. We prove only part (i). Since $\sum_{k=-\infty}^{\infty} a_k$, the two series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_{-k}$ converge to numbers S_1 and S_2, respectively. Then

$$\left|\sum_{k=-n}^{n} a_k - (S_1 + S_2)\right| = \left|\left(\sum_{k=0}^{n} a_k - S_1\right) + \left(\sum_{k=-n}^{-1} a_k - S_2\right)\right| \le \left|\sum_{k=0}^{n} a_k - S_1\right| + \left|\sum_{k=-n}^{-1} a_k - S_2\right| \to 0$$

as $n \to \infty$. ■

[61] We take this terminology from Definition 4.1.8 in [17].

A.3. The complex exponential.

For any $z \in \mathbb{C}$, we define the **COMPLEX EXPONENTIAL OF** z to be

$$\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}. \tag{A.3.1}$$

By Example A.2.3 we know this series converges on $\mathbb{C}$. We regularly use the familiar symbol e^z for $\exp(z)$.

When z is purely imaginary, there is a special formula for the complex exponential.

A.3.1 Theorem (Euler's formula).

If $x \in \mathbb{R}$, then $e^{ix} = \cos(x) + i\sin(x)$. Moreover, $|e^{ix}| = 1$.

Proof. From the definition of $\exp(\cdot)$, we have

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}, \tag{A.3.2}$$

where

$$i^{2k} = (i^2)^k = (-1)^k \quad \text{and} \quad i^{2k+1} = i^{2k}i = (-1)^k i.$$

Now, recall that the power series for $\sin(\cdot)$ and $\cos(\cdot)$ are

$$\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}.$$

We can rewrite[62] the series in (A.3.2) into two series, one consisting of all terms with k even and the other of all terms with k odd:

$$\sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!}. = \sum_{k=0}^{\infty}(-1)^k + i\sum_{k=0}^{\infty}(-1)^k = \cos(x) + i\sin(x). \tag{A.3.3}$$

The equality $|e^{ix}| = 1$ is a direct calculation using the Pythagorean trig identity:

$$|e^{ix}| = |\cos(x) + i\sin(x)| = \sqrt{\cos^2(x) + i\sin^2(x)} = 1.$$ ■

From Euler's formula, we can solve for $\cos(x)$ and $\sin(x)$ in terms of complex exponentials:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}. \tag{A.3.4}$$

We discuss extensively the properties of e^z when z is complex, and not necessarily real, in Section 3.2.1.

[62] Recall that writing $\sum_{k=0}^{\infty}(a_k + b_k) = \sum_{k=0}^{\infty}a_k + \sum_{k=0}^{\infty}b_k$ presumes both series converge independently. For example, if $a_k = -1$ and $b_k = 1$, then $\sum_{k=0}^{\infty}(a_k + b_k) = 0$ but $\sum_{k=0}^{\infty}a_k$ and $\sum_{k=0}^{\infty}b_k$ diverge. In the case at hand, the ratio test shows that each series after the first equality in (A.3.3) converges absolutely.

A.4. Power series.

We review some fundamental notions and properties of power series. We will state some definitions and results for series with arbitrary complex coefficients and "variable" and other "calculus"-type results only for series of a real "variable."

A.4.1 Definition.

Let (a_k) *be a sequence of complex numbers and* $z_0 \in \mathbb{C}$. *The* **(FORMAL) POWER SERIES** *with coefficients* (a_k) *and center* z_0 *is the series*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

The **VARIABLE** *of this power series is the number* $z \in \mathbb{C}$.

A.4.2 Example.

Determine all points $z \in \mathbb{C}$ *for which the* **GEOMETRIC SERIES**

$$\sum_{k=0}^{\infty} z^k$$

converges.

Solution. First suppose $|z| > 1$. Then $\lim_{k\to\infty} |z|^k = \infty$, and so $\lim_{k\to\infty} z^k \neq 0$. The geometric series then diverges in this case by the test for divergence.

Next, if $|z| = 1$, then $\lim_{k\to\infty} |z|^k = 1 \neq 0$, and so, again, $\lim_{k\to\infty} z^k \neq 0$. The test for divergence applies once more.

Last, suppose $|z| < 1$. Observe that $\lim_{k\to\infty} |z|^k = 0$ and, consequently, $\lim_{k\to\infty} z^k = 0$ as well. This does not, of course, guarantee the convergence of the geometric series for $|z| < 1$, but it is a useful starting point. We could use the ratio test to determine that the series converges in this case:

$$\lim_{k\to\infty} \left| \frac{z^{k+1}}{z^k} \right| = \lim_{k\to\infty} \left| \frac{z^k z}{z^k} \right| = \lim_{k\to\infty} |z| = |z| < 1,$$

but this does not tell us a formula for the sum.

Instead, we use **TELESCOPING**[63] to express the nth partial sum of the geometric series as

$$1 - z^{n+1} = \sum_{k=0}^{n} (z^k - z^{k+1}) = \sum_{k=0}^{n} z^k - \sum_{k=0}^{n} z^{k+1} = \sum_{k=0}^{n} z^k - z \sum_{k=0}^{n} z^k = (1 - z) \sum_{k=0}^{n} z^k.$$

---

[63] Recall that if a_k, $m \le k \le n+1$, are complex numbers, then

$$\sum_{k=m}^{n} (a_k - a_{k+1}) = a_m - a_{n+1}.$$

Since $|z| < 1$, we know $1 - z \neq 0$, and so

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}.$$

Taking the limit as $n \to \infty$, we have

$$|z| < 1 \Longrightarrow \sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}.$$

We conclude that the geometric series converges precisely on $\mathfrak{B}(0;1)$. This is the same conclusion that one reaches when considering the real geometric series; indeed, nothing in this proof required z to be real (or complex). ▲

We summarize without proof the most important properties of power series. All are analogous to familiar properties of power series from real-variable calculus.

A.4.3 Theorem.

Let $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ be a power series. There exists a unique number $\rho \in [0, \infty]$, called the RADIUS OF CONVERGENCE*, such that $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges absolutely for all $z \in \mathbb{C}$ such that $|z - z_0| < \rho$. The series may or may not converge for $|z - z_0| = \rho$.*

A.4.4 Remark.

There are various formulas to determine the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$, most involving the coefficients (a_k). For example, if the limit $L := \lim_{k \to \infty} |a_{k+1}/a_k|$ exists, then $\rho = 1/L$, with $\rho = \infty$ if $L = 0$ and $\rho = 0$ if $L = \infty$.

We review a number of results for power series with complex *coefficients*, real *centers*, and real *variables*. We state the analogous results for power series whose coefficients, centers, and variables are *all* complex in Theorem 3.7.3.

A.4.5 Theorem.

Let $x_0 \in \mathbb{R}$ and let (a_k) be a sequence in $\mathbb{C}$.

(i) [Term-by-term differentiation] *Suppose that $\sum_{k=0}^{\infty} a_k(z - x_0)^k$ has the radius of convergence $\rho > 0$ and set $f(x) := \sum_{k=0}^{\infty} a_k(x - x_0)^k$ for $x \in (x_0 - \rho, x_0 + \rho)$. Then $f \in \mathcal{C}^{\infty}\big((x_0 - \rho, x_0 + \rho)\big)$. In particular, f is continuous and*

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}.$$

More generally,

$$f^{(m)}(x) = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1) a_k (x - x_0)^{k-m}.$$

(ii) *Conversely, suppose that $\rho > 0$ and $f \colon (x_0 - \rho, x_0 + \rho) \to \mathbb{C}$ is a function such that $f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ for all $x \in (x_0 - \rho, x_0 + \rho)$. Then*

$$a_m = \frac{f^{(m)}(x_0)}{m!}.$$

(iii) [Term-by-term integration] *Suppose that* $\sum_{k=0}^{\infty} a_k(z-x_0)^k$ *has the radius of convergence* $\rho > 0$. *Then for any* $x_0 - \rho < \alpha < \beta < x_0 + \rho$,

$$\int_\alpha^\beta \sum_{k=0}^{\infty} a_k(x-x_0)^k \, dx = \sum_{k=0}^{\infty} a_k \int_\alpha^\beta (x-x_0)^k \, dx.$$

(iv) [Identity principle for power series] *Suppose that the power series* $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *and* $\sum_{k=0}^{\infty} b_k(x-x_0)^k$ *converge on some interval* $(x_0-\delta, x_0+\delta)$. *If*

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = \sum_{k=0}^{\infty} b_k(x-x_0)^k$$

for all $x \in (x_0-\delta, x_0+\delta)$, *then* $a_k = b_k$ *for all* k.

A.4.6 Example.

Show that $\partial_x[e^x] = e^x$ *by differentiating the power series for the exponential.*

Solution. We defined

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

and we know this power series converges for all $x \in \mathbb{C}$. We differentiate term-by-term to find

$$\partial_x[e^x] = \frac{d}{dz} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k(k-1)!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

To obtain the last inequality we reindexed (Lemma A.2.4). ▲

A.5. Elementary calculus for complex-valued functions of a real variable.

A.5.1 Definition.

Let $I \subseteq \mathbb{R}$ *be an interval and* $x_0 \in I$. *A function* $f\colon I \to \mathbb{R}$ *is* **DIFFERENTIABLE** *at* x_0 *if either*

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

or

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, in which case both limits exist and are equal and are denoted

$$f'(x_0) = \partial_x[f](x_0).$$

We do not attempt to resurrect the definition of the Riemann integral $\int_a^b f(x)\,dx$ of $f\colon [a,b] \to \mathbb{R}$ but instead give some sufficient conditions for Riemann integrability.

A.5.2 Lemma.

Let $f\colon [a,b] \to \mathbb{R}$ be a function.

(i) *If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

(ii) *If f is integrable on $[a,b]$ and $x_0 \in [a,b]$ and $y_0 \in \mathbb{R}$, and if we put*

$$g(x) := \begin{cases} f(x), & x \in [a,b] \setminus \{x_0\} \\ y_0, & x = x_0, \end{cases}$$

then g is also integrable on $[a,b]$ and $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$.

(iii) [Lebesgue] *If f is continuous at all but countably many points $\{x_k\}_{k=1}^{\infty} \subseteq [a,b]$ and bounded on $[a,b]$ in the sense that there exists $M > 0$ such that $|f(x)| \le M$ for all x, then f is integrable on $[a,b]$.*

Recall from vector calculus that if $\mathbf{f}\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}^2 \colon t \mapsto (f_1(x), f_2(x))$ is integrable, then we define

$$\int_a^b \mathbf{f}(x)\, dx := \left(\int_a^b f_1(x)\, dx, \int_a^b f_2(x)\, dx \right).$$

We make a similar "componentwise" definition to extend many notions of real-valued calculus to functions from $[a,b]$ to $\mathbb{C}$.

A.5.3 Definition.

(i) *Let $I \subseteq \mathbb{R}$ be an interval and $x_0 \in I$. Let $y_1, y_2 \in \mathbb{R}$. For a function $f\colon I \to \mathbb{C}$, we write*

$$\lim_{x \to x_0} f(x) = y_1 + iy_2$$

if

$$\lim_{x \to x_0} \mathrm{Re}[f(x)] = y_1 \quad \text{and} \quad \lim_{x \to x_0} \mathrm{Im}[f(x)] = y_2.$$

(ii) *A function $f\colon I \to \mathbb{C}$ is* **CONTINUOUS** *if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are continuous, and* **DIFFERENTIABLE** *if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are differentiable. If $f(x) = u(x) + iv(x)$ with u and v real-valued, then we set*

$$f'(x) := u'(x) + iv'(x). \tag{A.5.1}$$

(iii) *A function $f\colon [a,b] \to \mathbb{C}$ is* **INTEGRABLE** *if $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are (Riemann) integrable, and we set*

$$\int_a^b f(x)\, dx := \int_a^b u(x)\, dx + i \int_a^b v(x)\, dx. \tag{A.5.2}$$

We also define

$$\int_b^a f(x)\, dx := -\int_a^b f(x)\, dx.$$

(iv) *We denote the space of all Riemann-integrable functions from $[a,b]$ to $\mathbb{C}$ by $\mathcal{R}([a,b])$, and, as in Definition 1.1.1, the space of all n-times differentiable functions on an interval $I \subseteq \mathbb{R}$, whose nth derivative is continuous, by $\mathcal{C}^n(I)$. We put $\mathcal{C}(I) := \mathcal{C}^0(I)$ as the space of (merely) continuous functions on I.*

In particular, it is obvious that

$$\operatorname{Re}\left[\int_a^b f(x)\ dx\right] = \int_a^b \operatorname{Re}[f(x)]\ dx \quad \text{and} \quad \operatorname{Im}\left[\int_a^b f(x)\ dx\right] = \int_a^b \operatorname{Im}[f(x)]\ dx. \tag{A.5.3}$$

A.5.4 Example.

(i) *Fix* $\lambda \in \mathbb{C}$. *Calculate* $\partial_x[e^{\lambda x}]$. *(Do not use Example A.4.6.)*

(ii) *Evaluate* $\displaystyle\int_0^{2\pi} e^{it}\ dx$.

(iii) *Show that if* $f \in \mathcal{C}^1([a,b])$, *then* $\partial_x[\overline{f}] = \overline{\partial_x[f]}$.

Solution. (i) Write $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real. Then

$$e^{\lambda x} = e^{(\lambda_1 + i\lambda_2 x)} = e^{\lambda_1 x}\big(\cos(\lambda_2 x) + i\sin(\lambda_2 x)\big) = e^{\lambda_1 x}\cos(\lambda_2 x) + ie^{\lambda_1 x}\sin(\lambda_2 x).$$

The product rule, Euler's formula, and the definition (A.5.1) give

$$\begin{aligned}
\partial_x[e^{\lambda x}] &= \partial_x\left[e^{\lambda_1 x}\cos(\lambda_2 x)\right] + i\partial_x\left[e^{\lambda_1 x}\sin(\lambda_2 x)\right] \\
&= \lambda_1 e^{\lambda_1 x}\cos(\lambda_2 x) - \lambda_2 e^{\lambda_1 x}\sin(\lambda_2 x) + i\lambda_1 e^{\lambda_1 x}\sin(\lambda_2 x) + i\lambda_2 e^{\lambda_2 x}\cos(\lambda_2 x) \\
&= e^{\lambda_1 x}\left[\lambda_1\cos(\lambda_2 x) + i\lambda_2\cos(\lambda_2 x) + i\lambda_1\sin(\lambda_2 x) + i^2\lambda_2\sin(\lambda_2 x)\right] \\
&= e^{\lambda_1 x}\left[(\lambda_1 + i\lambda_2)\cos(\lambda_2 x) + i(\lambda_1 + i\lambda_2)\sin(\lambda_2 x)\right] \\
&= e^{\lambda_1 x}(\lambda_1 + i\lambda_2)\big(\cos(\lambda_2 x) + i\sin(\lambda_2 x)\big) \\
&= \lambda e^{(\lambda_1 + i\lambda_2)x}.
\end{aligned}$$

(ii) Since $e^{it} = \cos(x) + i\sin(x)$, by the definition (A.5.2) we have

$$\int_0^{2\pi} e^{it}\ dx = \int_0^{2\pi} \cos(x)\ dx + i\int_0^{2\pi} \sin(x)\ dx = \sin(x)\big|_{x=0}^{x=2\pi} - i\cos(x)\big|_{x=0}^{x=2\pi} = 0.$$

(iii) Write $f(x) = f_1(x) + if_2(x)$ with f_1 and f_2 real-valued. We have $f'(x) = f_1'(x) + if_2'(x)$, so $\overline{f'(x)} = f_1'(x) - if_2'(x)$. On the other hand, $\overline{f(x)} = f_1(x) - if_2(x)$, so $\partial_x[\overline{f}](x) = f_1'(x) - if_2'(x)$. ▲

We can import many useful properties of definite integrals of real-valued functions with, in general, little effort.

A.5.5 Theorem.

(i) [Linearity] *If* $f,\ g \in \mathcal{R}([a,b])$ *and* $\alpha,\ \beta \in \mathbb{C}$, *then* $\alpha f + \beta g \in \mathcal{R}([a,b])$ *with*

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\ dx = \alpha\int_a^b f(x)\ dx + \beta\int_a^b g(x)\ dx.$$

(ii) [Fundamental theorem of calculus I] *If* $f \in \mathcal{R}([a,b])$, *then the function* $F := \int_a^x f(\xi)\ d\xi$ *is continuous on* $[a,b]$. *If* $f \in \mathcal{C}([a,b]) \subseteq \mathcal{R}([a,b])$, *then* $F \in \mathcal{C}^1([a,b])$ *with* $F'(x) = f(x)$.

(iii) [Fundamental theorem of calculus II] *If $f \in \mathcal{R}([a,b])$ and $F \in \mathcal{C}^1([a,b])$ is an antiderivative of f on $[a,b]$, i.e., $F'(x) = f(x)$ for all $a \leq t \leq b$, then*

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

(iv) [Change of variables] *If $f \in \mathcal{C}^1([a,b])$ is real-valued and strictly increasing on $[a,b]$, i.e., $f(x_1) < f(x_2)$ for $a \leq x_1 < x_2 \leq b$, and if $g \in \mathcal{R}([f(a), f(b)])$, then*

$$\int_a^b g(f(x))f'(x)\ dx = \int_{f(a)}^{f(b)} g(u)\ du.$$

(v) [Integration by parts] *If $f,\ g \in \mathcal{C}^1([a,b])$, then*

$$\int_a^b f(x)g'(x)\ dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\ dx.$$

(vi) [Composition] *If $f \in \mathcal{R}([a,b])$ with $c \leq f(x) \leq d$ for all $t \in [a,b]$, and if $g \in \mathcal{C}([a,b])$, then $g \circ f \in \mathcal{R}([a,b])$. Here $g \circ f$ is the composition $x \mapsto g(f(x))$.*

(vii) *If $f\colon [a,b] \to \mathbb{C}$ is a function such that $|f| \in \mathcal{R}([a,b])$, then $|f|^r \in \mathcal{R}([a,b])$ for any $r \geq 0$.*

(viii) [Triangle inequality] *If $f\colon [a,b] \to \mathbb{C}$ is integrable, then*

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx.$$

(ix) [Domain additivity] *If $f \in \mathcal{R}([a,b])$ and $a < c < b$, then $f \in \mathcal{R}([a,c])$ and $f \in \mathcal{R}([c,b])$ with*

$$\int_a^b f(t)\ dt = \int_a^c f(t)\ dt + \int_c^b f(t)\ dt.$$

Proof. (i) Exercise: use the definition (A.5.2) of the integral of a complex-valued function and the linear properties of the Riemann integral of real-valued functions.

(ii) Exercise: use (A.5.2) again and also (A.5.1) to separate $F'(x) = f(x)$ into real and imaginary parts. Then use the fundamental theorem of calculus for real-valued functions.

(iii) Exercise: prove in the same manner as part (ii).

(iv) Exercise: use (A.5.2) and u-substitution for integrals of real-valued functions.

(v) Exercise: write $f = f_1 + if_2$ and $g = g_1 + ig_2$, where f_1 and g_1 are the real parts and f_2 and g_2 are the imaginary parts; multiply out (painfully) and use integration by parts for real-valued functions.

(vi) This requires more advanced techniques from analysis, so we omit it.

(vii) Exercise: use part (vi).

(viii) If f is real-valued, this estimate is known from real-variable theory. If $\int_a^b f(x)\ dx = 0$, then the estimate is trivial. So, suppose $\int_a^b f(x)\ dx \neq 0$ and write this integral in its polar form:

$$\int_a^b f(x)\ dx = \left|\int_a^b f(x)\ dx\right| e^{i\theta}, \qquad \theta = \text{Arg}\left(\int_a^b f(x)\ dx\right).$$

That is,

$$\left|\int_a^b f(x)\ dx\right| = e^{-i\theta}\int_a^b f(x)\ dx = \int_a^b e^{-i\theta} f(x)\ dx$$

Since $\left|\int_a^b f(x)\ dx\right| \in \mathbb{R}$, we have

$$\left|\int_a^b f(x)\ dx\right| = \text{Re}\left[\left|\int_a^b f(x)\ dx\right|\right] = \text{Re}\left[\int_a^b e^{-i\theta} f(x)\ dx\right] = \int_a^b \text{Re}[e^{-i\theta} f(x)]\ dx \tag{A.5.4}$$

by (A.5.3).

For any $a \leq t \leq b$, since $\text{Re}[e^{-i\theta} f(x)]$ is (obviously!) real, we have

$$\text{Re}[e^{-i\theta} f(x)] \leq |\,\text{Re}[e^{-i\theta} f(x)]| \leq |e^{-i\theta} f(x)| = |f(x)|. \tag{A.5.5}$$

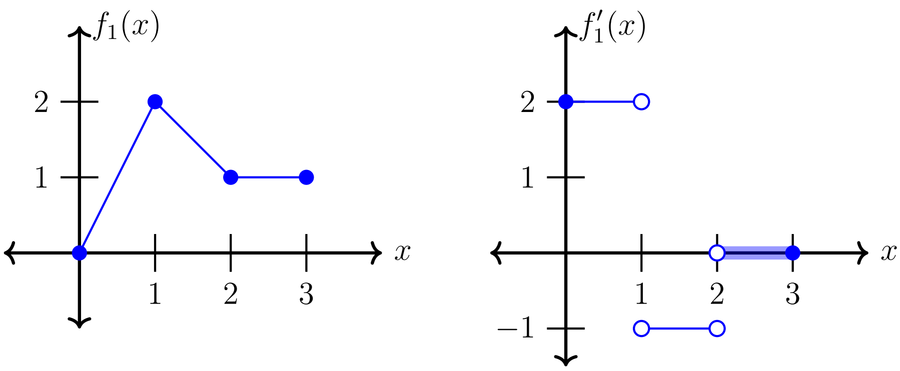
Since the real-valued Riemann integral is monotone[64], we conclude from (A.5.4) and (A.5.5) that

$$\left|\int_a^b f(x)\ dx\right| = \int_a^b \text{Re}[e^{-i\theta} f(x)]\ dx \leq \int_a^b |f(x)|\ dx.$$

(ix) This requires some technical details from analysis, so we omit this proof. ■

A.6. Piecewise continuity and differentiability.

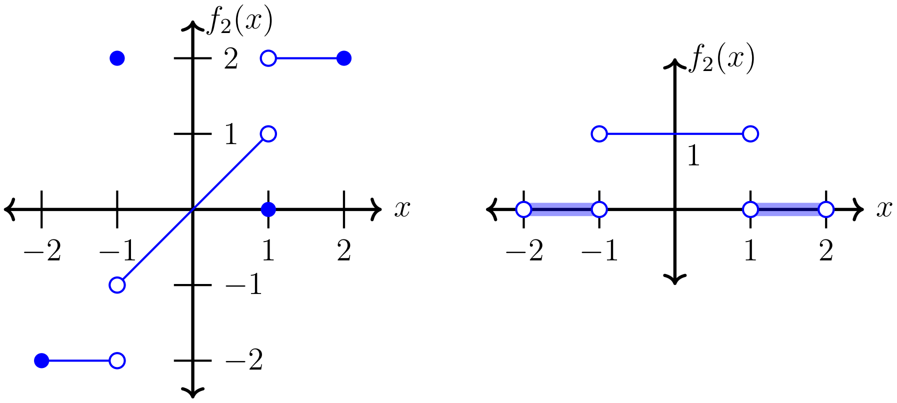
We want a definition of piecewise continuity (and differentiability) broad enough to use in both Fourier theory and complex variable theory. In complex analysis, we will consider functions like the one graphed below, which are continuous on a real interval but whose derivatives may fail to exist at certain points.



(A.6.1)

[64]That is, if g and h are Riemann-integrable on $[a, b]$ and $g(x) \leq h(x)$ for $a \leq t \leq b$, then $\int_a^b g(x)\ dx \leq \int_a^b h(x)\ dx$.

When we study Fourier series, however, we will allow functions that are discontinuous, such as this graph.



(A.6.2)

Both of the functions above have relatively "tame" discontinuities. Namely, (i) there are only finitely many discontinuities (in fact, f_1 is continuous everywhere); (ii) at those discontinuities, the functions have finite left and right limits; and (iii) the functions are differentiable at all but finitely many points, the derivatives are continuous on intervals where they exist, and the left and right limits of the derivatives exist and are finite. We crystalize this notion in the next definition.

A.6.1 Definition.

A function $f\colon I \subseteq \mathbb{R} \to \mathbb{C}$ is **PIECEWISE-CONTINUOUSLY DIFFERENTIABLE** *or* **PIECEWISE-$\mathcal{C}^1$** *on I if*

(i) *f is continuous and differentiable at all but finitely many points of any closed, bounded subinterval $[c,d] \subseteq I$.*

(ii) *The limits*

$$f(x^{\pm}) := \lim_{t \to x^{\pm}} f(t)$$

exist for all $x \in I$. (If I contains one or both of its endpoints, we only demand that the left or right limit above hold, as appropriate.)

(iii) *If f is differentiable on $(c,d) \subseteq I$, and if we set*

$$g(x) := \begin{cases} f(c^-), & x = c \\ f(x), & c < x < d \\ f(d^+), & x = d, \end{cases}$$

then $g \in \mathcal{C}^1([c,d])$. In particular, the left and right derivative limits of g, i.e., the limits

$$\lim_{h \to 0^{\pm}} \frac{g(x+h) - g(x)}{h}, \tag{A.6.3}$$

exist and are finite at all points $x \in I$. (If I contains one or both of its endpoints, we only demand that the left or right limit above exist, as appropriate.) Moreover, the limits (A.6.3) *are equal for all but finitely many points in any closed, bounded interval $[c,d] \subseteq I$.*

We denote the vector space of piecewise-$\mathcal{C}^1$ functions on I by $\mathcal{C}^1_{\text{pw}}(I)$.

The function graphed in (A.6.1) is continuous and piecewise-$\mathcal{C}^1$, while the function graphed in (A.6.2) is piecewise-$\mathcal{C}^1$ but not continuous. Neither function is continuously differentiable in the sense of Definition 1.1.1.

We impose property (i) above because, when I is a closed, bounded interval, we will often want to partition I into a *finite* family of subintervals on which f is continuously differentiable (except, perhaps, at the endpoints of those subintervals). We do this, for instance, in the proof of part (v) of Theorem 3.5.18. Part (ii) makes precise our observations above that the discontinuities of f should be "tame." And part (iii) simply ensures the same for the derivative of f.

A.6.2 Remark.

(i) *Occasionally we will have need to consider continuous functions in* $\mathcal{C}^1_{\text{pw}}(I)$ *that are not differentiable at all points in* I. *The function graphed in* (A.6.1) *is such a function. In that case, such a function belongs to the space* $\mathcal{C}^1_{\text{pw}}(I) \cap \mathcal{C}(I)$; *this notation, while baroque, is not redundant.*

(ii) *Suppose that* $I = [a, b]$ *is a closed, bounded interval in Definition A.6.1 and let* $f \in \mathcal{C}^1_{\text{pw}}([a, b])$. *Then there are finitely many points* $x_1, \ldots, x_n \in [a, b]$ *such that* f *is continuous and differentiable on* $J := [a, b] \setminus \{x_1, \ldots, x_n\}$. *Moreover, the points* $x_1, \ldots, x_n$ *can be chosen so that* f' *is defined and continuous on* J. *Suppose we have ordered these points with* $a \le x_1 < \cdots < x_n \le b$. *Then* f *is continuously differentiable on each open subinterval* (a, x_1), (x_1, x_2), $\ldots$, (x_{n-1}, x_n), (x_n, b). *Moreover, at each left endpoint,* f *and* f' *have right limits, and at each right endpoint,* f *and* f' *have left limits.*

B. Improper Integrals

We give a streamlined treatment of improper integrals, assuming many proofs from calculus, with an eye toward the use of improper integrals in the Fourier transform. First, we need an "intermediate" definition.

B.0.1 Definition.

Let $I \subseteq \mathbb{R}$ be an interval, which need not be closed and/or bounded. A function $f\colon I \to \mathbb{C}$ is LOCALLY INTEGRABLE *on I if f is integrable on every subinterval $[a,b] \subseteq I$ (i.e., per Definition A.5.3, if $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are integrable on all $[a,b] \subseteq I$). We denote by $\mathcal{R}_{\text{loc}}(I)$ the vector space of all functions $f\colon I \to \mathbb{C}$ such that f is locally integrable on I.*

In particular, if $I = [a,b]$ is closed and bounded, then the locally integrable functions on I are exactly the Riemann integrable functions on I. That is, $\mathcal{R}_{\text{loc}}([a,b]) = \mathcal{R}([a,b])$.

Now we define the improper integral over infinite intervals.

B.0.2 Definition.

(i) *Suppose $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$. We set*

$$\mathcal{I}_+[f;x_0] := \lim_{b\to\infty} \int_{x_0}^{b} f(x)\, dx \quad \text{and} \quad \mathcal{I}_-[f;x_0] := \lim_{a\to-\infty} \int_{a}^{x_0} f(x)\, dx \tag{B.0.1}$$

whenever one of these limits exists. If the first limit exists, we say that f is IMPROPERLY INTEGRABLE *on $[x_0,\infty)$, and likewise that f is improperly integrable on $(-\infty,x_0]$ if the second limit exists.*

(ii) *Suppose $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$. If f is improperly integrable over both $(-\infty,0]$ and $[0,\infty)$, then we say that f is* IMPROPERLY INTEGRABLE ON *$\mathbb{R}$ or* ON *$(-\infty,\infty)$, and we set*

$$\int_{-\infty}^{\infty} f(x)\, dx := \mathcal{I}_-[f;0] + \mathcal{I}_+[f;0]. \tag{B.0.2}$$

We will note shortly that choosing $x_0 = 0$ here is ultimately unimportant.

(iii) *For $p > 0$, we set*

$$L^p(\mathbb{R}) = \{f \in \mathcal{R}_{\text{loc}}(\mathbb{R}) \mid |f|^p \text{ is improperly integrable on } \mathbb{R}\}.$$

We emphasize that if $f \in L^p(\mathbb{R})$, then it is not the map $x \mapsto f(x)^p$ that is improperly integrable over $\mathbb{R}$ but the pth power of the modulus of f, i.e., $x \mapsto |f(x)|^p$. From calculus we have a number of useful computational and "comparison" properties for improper integrals.

B.0.3 Theorem.

(i) *If f, $g \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ are improperly integrable on $\mathbb{R}$, then so is $\alpha f + \beta g$ for any α, $\beta \in \mathbb{C}$.*

(ii) *Suppose that f is improperly integrable on $\mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \tag{B.0.3}$$

(iii) *The existence of the limit* (B.0.3) *does not imply that f is improperly integrable on $\mathbb{R}$ (take $f(x) = x$). However, if $f \in \mathcal{R}_{\mathrm{loc}}(\mathbb{R})$ and if $f(x) \geq 0$ for all x, then the existence of* (B.0.3) *does imply that f is improperly integrable.*

(iv) [Comparison test] *Suppose that $f, g \in \mathcal{R}_{\mathrm{loc}}(\mathbb{R})$. If g is improperly integrable on $\mathbb{R}$ with $g(x) \geq 0$ and $|f(x)| \leq g(x)$ for all x, then f and $|f|$ are improperly integrable on $\mathbb{R}$ with*

$$\left|\int_{-\infty}^{\infty} f(x)\,dx\right| \leq \int_{-\infty}^{\infty} |f(x)|\,dx \leq \int_{-\infty}^{\infty} g(x)dx.$$

(v) *Suppose that $f \in \mathcal{R}_{\mathrm{loc}}(\mathbb{R})$. Then for any $x_1, x_2 \in \mathbb{R}$, the limits*

$$\lim_{a\to-\infty}\int_{a}^{x_1} f(x)\,dx \quad \textit{and} \quad \lim_{b\to\infty}\int_{x_1}^{b} f(x)\,dx$$

exist if and only if the limits

$$\lim_{\alpha\to-\infty}\int_{\alpha}^{x_2} f(x)\,dx \quad \textit{and} \quad \lim_{\beta\to\infty}\int_{x_2}^{\beta} f(x)\,dx$$

exist. Moreover, if f is improperly integrable on $\mathbb{R}$, then, with the notation of (B.0.1),

$$\int_{-\infty}^{\infty} f(x)\,dx = \mathcal{I}_+[f;x_0] + \mathcal{I}_-[f;x_0]$$

for any $x_0 \in \mathbb{R}$. That is, the splitting at the particular endpoint 0 in (B.0.2) *is immaterial (but merely convenient) for the existence or nonexistence of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$.*

(vi) *If $p \geq 1$, then $L^p(\mathbb{R})$ is a vector space and*

$$\|f\|_{L^p(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |f(x)|^p\right)^{1/p}$$

is a norm (Definition C.3.1) on $L^p(\mathbb{R})$.

Proof. Most of these properties are proved in calculus, so we do not repeat their proofs here. The proof of part (ii) is virtually identical to that of part (i) of Theorem A.2.12; essentially, one replaces $\sum$ with $\int$. Part (vi) uses the comparison test for improper integrals and the inequality

$$|f(x)+g(x)|^p \leq \big(2\max\{|f(x)|, |g(x)|\}\big)^p = 2^p \max\{|f(x)|^p, |g(x)|^p\} \leq 2^p\big(|f(x)|+|g(x)|^p\big).$$

■

Now we give several concrete examples illustrating these concepts and notation.

B.0.4 Example.

Let

$$f(x) = \frac{1}{1+x^2}.$$

Show that $f \in \cap_{p=1}^{\infty} L^p(\mathbb{R})$.

Solution. Since $f \in \mathcal{C}(\mathbb{R})$, we know f is integrable on any subinterval $[a,b]$ of $\mathbb{R}$, thus $f \in \mathcal{R}_{\mathrm{loc}}(\mathbb{R})$. Next, the fundamental theorem of calculus tells us

$$\lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \big(\arctan(b) - \arctan(0)\big) = \frac{\pi}{2}$$

and

$$\lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \big(0 - \arctan(a)\big) = \frac{\pi}{2},$$

so the integral over $\mathbb{R}$ exists with

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} := \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \pi.$$

In particular, since $|f(x)| = f(x)$ for all x, we have $f \in L^1(\mathbb{R})$.

Next, observe that $1+x^2 \geq 1$ for all x, so $(1+x^2)^p \geq 1+x^2 = f(x)$ for all $p \geq 1$. Then

$$|f(x)|^p = \frac{1}{(1+x^2)^p} \leq \frac{1}{1+x^2} = |f(x)|.$$

Since $|f|$ is absolutely integrable on $\mathbb{R}$, so is $|f|^p$. That is, $f \in L^p(\mathbb{R})$ for any $p \geq 1$. ▲

B.0.5 Example.

Show that the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx$$

exists, but the improper integral

$$\int_{-\infty}^{\infty} \left|\frac{\sin(x)}{x}\right| dx$$

does not.

Solution. Note that the integrand is not defined at $x = 0$, so we really define

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx := \int_{-\infty}^{\infty} f(x)\, dx, \quad \text{where} \quad f(x) := \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Since f is continuous on $\mathbb{R}$, the integral $\int_0^1 f(x)\, dx$ exists, so let us fix $b \geq 1$ and show that

$$\lim_{b\to\infty} \int_1^b \frac{\sin(x)}{x}\, dx$$

exists. Integrating by parts with

$$u = \frac{1}{x} \qquad dv = \sin(x)\ dx$$

$$du = -\frac{1}{x^2} \qquad v = -\cos(x),$$

we have

$$\int_1^b \frac{\sin(x)}{x}\ dx = \frac{\cos(x)}{x}\Big|_{x=1}^{x=b} - \int_1^b \frac{\cos(x)}{x^2}\ dx.$$

Now,

$$\lim_{b\to\infty} \frac{\cos(b)}{b} = 0$$

by the squeeze theorem, and, for $x \geq 1$,

$$\left|\frac{\cos(x)}{x^2}\right| \leq \frac{1}{x^2}.$$

We recall that if $q > 1$, then the mapping $x \mapsto x^q$ is improperly integrable over $[1, \infty)$, and so it follows that

$$\lim_{b\to\infty} \int_1^b \frac{\cos(x)}{x^2}\ dx$$

exists. All together, we have shown that

$$\lim_{b\to\infty} \int_0^b \frac{\sin(x)}{x}\ dx$$

exists. And since the integrand is odd, we have, for any $a < 0$,

$$\int_a^0 \frac{\sin(x)}{x}\ dx = -\int_0^{-a} \frac{\sin(x)}{x}\ dx,$$

thus

$$\lim_{a\to-\infty} \int_a^0 \frac{\sin(x)}{x}\ dx = \lim_{a\to-\infty} \int_0^{-a} \frac{\sin(x)}{x}\ dx = \lim_{b\to\infty} \int_0^b \frac{\sin(x)}{x}\ dx,$$

where we know this third limit exists.

On the other hand, we claim that

$$\lim_{b\to\infty} \int_1^b \left|\frac{\sin(x)}{x}\right|\ dx = \infty.$$

To do so, we cleverly fix an integer $k \geq 2$ and estimate

$$\int_{(k-1)\pi}^{k\pi} \left|\frac{\sin(x)}{x}\right|\ dx \geq \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin(x)|\ dx,$$

since for $\pi \leq (k-1)\pi \leq x \leq k\pi$, we have

$$\frac{1}{k\pi} \leq \frac{1}{x} \leq \frac{1}{(k-1)\pi}.$$

Next, calculus reminds us that $\sin(\cdot)$ is strictly positive or strictly negative on any interval of the form $\big((k-1)\pi, k\pi\big)$, and so

$$\int_{(k-1)\pi}^{k\pi} |\sin(x)|\ dx = 2.$$

Thus

$$\int_{(k-1)\pi}^{k\pi} \left|\frac{\sin(x)}{x}\right|\ dx \geq \frac{2}{k\pi}.$$

Consequently, for any $n \geq 1$,

$$\int_{\pi}^{n\pi} \left|\frac{\sin(x)}{x}\right|\ dx = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin(x)}{x}\right|\ dx \geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k}. \tag{B.0.4}$$

This sum is, of course, a nonzero multiple of the nth partial sum of the harmonic series $\sum_{k=1}^{n} k^{-1}$, which diverges. Now, if

$$\lim_{b\to\infty} \int_{1}^{\infty} \left|\frac{\sin(x)}{x}\right|\ dx$$

exists, then calculus tells us

$$\lim_{n\to\infty} \int_{1}^{n\pi} \left|\frac{\sin(x)}{x}\right|\ dx$$

exists. But (B.0.4) implies

$$\lim_{n\to\infty} \int_{1}^{n\pi} \left|\frac{\sin(x)}{x}\right|\ dx \geq \lim_{n\to\infty} \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k} = \infty.$$

Thus f is improperly integrable on $\mathbb{R}$ but $|f|$ is not, so $f \notin L^1(\mathbb{R})$. ▲

When f is improperly integrable over $\mathbb{R}$, it is often convenient to calculate the integral via the symmetric limit $\int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx$. Of course, to use this limit to calculate the integral, we have to know *first* that f is improperly integrable. Frequently, rather than check that both limits (B.0.1) exist, we can first use the comparison test to deduce the improper integrability of f over $\mathbb{R}$ and then use the symmetric limit.

There is a special situation in Fourier analysis in which an important integral quantity is defined by the symmetric limit, yet for which the two limits (B.0.1) need not exist. We name it here.

B.0.6 Definition.

Let $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$. The CAUCHY PRINCIPAL VALUE *of the improper integral of f over $\mathbb{R}$ is the limit*

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx,$$

if this limit exists.

B.0.7 Example.

(i) *If* $f \in L^1(\mathbb{R})$, *then*

$$\int_{-\infty}^{\infty} f(x)\ dx = \text{P. V.} \int_{-\infty}^{\infty} f(x)\ dx.$$

(ii) *If* $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ *is odd, then*

$$\text{P. V.} \int_{-\infty}^{\infty} f(x)\ dx = 0.$$

Proof. **(i)** This is part (ii) of Theorem B.0.3.

(ii) We have

$$\text{P. V.} \int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx.$$

Since f is odd, $\int_{-R}^{R} f(x)\ dx = 0$ for any $R \in \mathbb{R}$. ∎

C. Linear Algebra

This material can be found in many sources, including [13, 19, 25].

C.1. Vector spaces.

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

C.1.1 Definition.

A vector space over $\mathbb{F}$ is a set $\mathcal{X}$ equipped with[65] *two operations, vector addition and scalar multiplication, that satisfy the following conventions and relations.*

(VS1) *Vector addition is a map $\oplus\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}$, and we write $\oplus(f,g)=f+g$. Scalar multiplication is a map $\odot\colon \mathbb{F}\times\mathcal{X}\to\mathcal{X}$, and we write $\oplus(\alpha,f)=\alpha f$. Formally, the vector space is the 4-tuple $(\mathcal{X},\mathbb{F},\oplus,\odot)$; this is what "over" and "equipped with" means. Of course, this is baroque and pedantic, and in practice we never think of a vector space as a 4-tuple!*

(VS2) *Vector addition is* COMMUTATIVE *and* ASSOCIATIVE*:*

$$f+g=g+f \quad \text{and} \quad (f+g)+h=f+(g+h), \ \ f,g,h\in\mathcal{X}.$$

(VS3) *There exists an* IDENTITY *or* ZERO VECTOR *for vector addition, which we denote by 0 (i.e., the same symbol as the element 0 of $\mathbb{F}$):*

$$f+0=f, \ \ f\in\mathcal{X}.$$

(VS4) *There exists an* INVERSE *for vector addition: for all $f\in\mathcal{X}$, there is a vector $\widetilde{f}\in\mathcal{X}$ such that $f+\widetilde{f}=0$. Of course, we write $\widetilde{f}=-f$, and in fact $\widetilde{f}=(-1)f$.*

(VS5) *Scalar multiplication* DISTRIBUTES *over scalar and vector addition:*

$$(\alpha+\beta)f=\alpha f+\beta f \quad \text{and} \quad \alpha(f+g)=\alpha f+\alpha g, \ \ \alpha,\beta\in\mathbb{F}, \ \ f,g\in\mathcal{X}.$$

(VS6) *Scalar multiplication is* ASSOCIATIVE*:*

$$\alpha(\beta f)=(\alpha\beta)f, \ \ \alpha,\beta\in\mathbb{F}, \ \ f\in\mathcal{X}.$$

(VS7) $1f=f$, $f\in\mathcal{X}$.

C.1.2 Example.

(i) *With $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$, $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ with the usual componentwise operations:*

$$\mathbf{x}+\mathbf{y} := (x_1+y_1,\ldots,x_n+y_n) \quad \text{and} \quad \alpha\mathbf{x} := (\alpha x_1,\ldots,\alpha x_n),$$

[65] By "equipped with," we really mean that a vector space $\mathcal{X}$ over $\mathbb{F}$ with the operations $\odot$ of scalar multiplication and $\oplus$ of vector addition is the 4-tuple $(\mathcal{X},\mathbb{F},\odot,\oplus)$. Of course, no one ever thinks of vector spaces as 4-tuples whose entries consist of sets and functions, but, technically (and maybe uselessly), this is *really* what a vector space is!

with $\mathbf{x} = (x_1, \dots, x_n)$ *and* $\mathbf{y} = (y_1, \dots, y_n)$.

(ii) *If* $I \subseteq \mathbb{R}$ *is an interval, let* $\mathcal{C}^r(I)$ *denote the set of all* $\mathbb{C}$*-valued functions on* I *whose* r*th derivative exists and is continuous on* I. *Then* $\mathcal{C}^r(I)$ *is a vector space under the usual pointwise operations: for* $f, g \in \mathcal{C}^r(I)$ *and* $\alpha \in \mathbb{C}$, *define the vectors* $f + g$ *and* αf *by*

$$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha f(x). \tag{C.1.1}$$

That $\mathcal{C}^r(I)$ *is a vector space is just a consequence of the linearity of limits and derivatives.*

(iii) *If* $I \subseteq \mathbb{R}$ *is an interval, then the set of all Riemann-integrable functions on* I *forms a vector space under, again, the pointwise operations* (C.1.1).

(iv) *Some of the most interesting vector spaces are spaces of functions, like the ones in parts (ii) and (iii), as well as many more exotic flavors. The term* **FUNCTION SPACE** *is often used to refer to a vector space of functions (especially* $L^2(\mathbb{R})$*), but this term is unfortunately ambiguous given the plethora of worthwhile vector spaces of functions.*

C.1.3 Definition.

A set $\mathcal{U} \subseteq \mathcal{X}$ *is* **LINEARLY INDEPENDENT** *if for any finite number of distinct vectors* $f_1, \dots, f_n \in \mathcal{U}$, *whenever* $\sum_{k=1}^n \alpha_k f_k = 0$ *for some* $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, *then* $\alpha_1 = \cdots = \alpha_n = 0$. *If* $\mathcal{U}$ *is not linearly independent, then it is* **LINEARLY DEPENDENT**.

C.1.4 Example.

(i) *Let* $\mathbf{e}_1 := (1, 0)$ *and* $\mathbf{e}_2 := (0, 1)$. *Then* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *is linearly independent in* $\mathbb{C}^2$.

(ii) *For* $\lambda \in \mathbb{C}$, *let* $f_\lambda(x) := e^{\lambda x}$. *Then the set* $\{f_\lambda\}_{\lambda \in \mathbb{C}}$ *is linearly independent in* $\mathcal{C}^r(I)$ *for any interval* $I \subseteq \mathbb{R}$ *and* $r \geq 0$.

C.2. Linear operators.

Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces over $\mathbb{F}$. A map $T \colon \mathcal{X} \to \mathcal{Y}$ is a **LINEAR OPERATOR FROM** $\mathcal{X}$ **TO** $\mathcal{Y}$ if $T(\alpha f + \beta g) = \alpha T f + \beta T g$ for all $\alpha, \beta \in \mathbb{F}$ and $f, g \in \mathcal{X}$. We will denote the identity operator on a space $\mathcal{X}$ by $\mathbb{1}_{\mathcal{X}}$, i.e., $\mathbb{1}_{\mathcal{X}} f = f$ for all $f \in \mathcal{X}$. A **LINEAR FUNCTIONAL** on $\mathcal{X}$ is a linear operator from $\mathcal{X}$ to $\mathbb{F}$.

C.2.1 Example.

(i) *Let* $A \in \mathbb{C}^{m \times n}$ *be an* $m \times n$ *matrix. Then we may define a linear operator* $\mathcal{M}_A \colon \mathbb{C}^n \to \mathbb{C}^m$ *by* $\mathcal{M}_A \mathbf{x} := A\mathbf{x}$, *where* $A\mathbf{x}$ *is the usual matrix-vector product. Of course, we identify* $\mathcal{M}_A$ *with* A, *and usually refer to* A *as both a matrix and a linear operator.*

(ii) *Let* $\mathcal{X} = \mathcal{C}^1([0,1])$ *and* $\mathcal{Y} = \mathcal{C}([0,1])$. *Set* $(Tf)(x) := f'(x)$. *Then* $T \colon \mathcal{X} \to \mathcal{Y}$ *is linear, by the linear properties of the derivative.*

(iii) *Let $\mathcal{X} = \mathcal{C}([a,b])$ and set $Tf := \int_a^b f(x)\,dx$. Then T is a linear functional on $\mathcal{X}$.*

The **KERNEL** of a linear operator $T\colon \mathcal{X} \to \mathcal{Y}$ is the subspace

$$\ker(T) := \{f \in \mathcal{X} \mid Tf = 0\}\,. \tag{C.2.1}$$

We will use special terminology for a linear operator that maps a subspace of a larger space back into that space. Namely, given a vector space $\mathcal{X}$ and a subspace $\mathfrak{D} \subseteq \mathcal{X}$, a **LINEAR OPERATOR IN** $\mathcal{X}$ is a map $T\colon \mathfrak{D} \to \mathcal{X}$. Then, for example, we can think of the linear operator $f \mapsto f'$ either as a linear operator from $\mathcal{C}^1(I)$ to $\mathcal{C}(I)$, or as a linear operator in $\mathcal{C}(I)$ with domain $\mathcal{C}^1(I)$. This distinction, while seemingly academic here, will be useful when we discuss eigenvalues; see Remark C.6.3.

C.3. Normed spaces.

C.3.1 Definition.

Let $\mathcal{X}$ be a vector space over $\mathbb{F}$, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a **NORM** *on $\mathcal{X}$ if*

(i) [Positivity][66] $\|f\| \geq 0$ *for all* $f \in \mathcal{X}$

(ii) [Definiteness] $\|f\| = 0$ *if and only if* $f = 0$;

(iii) [Absolute homogeneity] $\|\alpha f\| = |\alpha|\,\|f\|$ *for all* $\alpha \in \mathbb{F}$ *and* $f \in X$;

(iv) [Triangle inequality] $\|f + g\| \leq \|f\| + \|g\|$ *for all* f, $g \in \mathcal{X}$.

A **NORMED SPACE** *is a vector space $\mathcal{X}$ equipped with*[67] *a norm. If it is possible that $\|f\| = 0$ for some $f \neq 0$ but the other properties above hold, then we call $\|\cdot\|$ a* **SEMINORM** *on $\mathcal{X}$.*

We give examples of norms and seminorms below. Unless otherwise stated, the verification of the defining properties of the norm are very easy to check.

C.3.2 Example.

(i) *There are a plethora of "p"-norms on $\mathbb{C}^n$: for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $1 \leq p < \infty$, set*

$$\|\mathbf{x}\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \tag{C.3.1}$$

and

$$\|\mathbf{x}\|_\infty := \max_{1 \leq k \leq n} |x_n|.$$

We exclude $p < 1$, as the triangle inequality fails in these cases. The proof of the triangle inequality is nontrivial when $p \neq 1$.

[67]This looks like more of a "nonnegativity" condition, but the cultural custom, nonetheless, is to call it positive.

[67]Recall Footnote 65 for a formal (baroque) definition of "equipped with."

(ii) *Similarly, we can define integral p-norms on $\mathcal{C}([a,b])$. For historical and cultural reasons, we call these L^p-norms:*

$$\|f\|_{L^p([a,b])} := \begin{cases} \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}, & 1 \le p < \infty \\ \max_{a \le x \le b} |f(x)|, & p = \infty. \end{cases}$$

To see that $\|f\|_{L^p([a,b])} = 0$ if and only if $f(x) = 0$ for all x, recall from calculus that if $g \in \mathcal{C}([a,b])$ is nonnegative, then

$$\int_a^b g(x) \, dx = 0 \iff g(x) = 0 \text{ for all } a \le x \le b.$$

(iii) *Let $\mathcal{R}([a,b])$ denote the set of Riemann-integrable functions on $[a,b]$. The integral $\|f\|_{L^p([a,b])}$ is defined for any $f \in \mathcal{R}([a,b])$, but $\|\cdot\|_{L^p([a,b])}$ is not a norm on $\mathcal{R}([a,b])$ for $1 \le p < \infty$. Namely, it is only a seminorm.*

To see this, consider the function

$$f_0(x) := \begin{cases} 1, & x = a \\ 0, & a < x \le b. \end{cases}$$

Clearly $f_0 \ne 0$. But $|f_0(x)|^p = |f_0(x)| = f_0(x)$ for all $x \in [a,b]$ and $p \in [1,\infty)$, and $\int_a^b f_0(x) \, dx = 0$. Hence $\|f_0\|_{L^p([a,b])} = 0$ for all $p \ne \infty$.

C.3.3 Remark.

The uncomfortable reality about a seminorm $\|\cdot\|$ on a vector space $\mathcal{X}$ is that one can have $\|f\| = 0$ but $f \ne 0$. Then it may be possible to find two elements $f, g \in \mathcal{X}$ with $\|f - g\| = 0$, but this does not guarantee $f = g$. Such a situation occurs frequently with the L^p-norms on spaces of Riemann-integrable functions; see Example 2.4.17.

C.4. Inner product spaces.

C.4.1 Definition.

An INNER PRODUCT *on the vector space $\mathcal{X}$ (over[68] $\mathbb{C}$) is a function $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ with the following properties.*

(i) [Linearity in the first variable I] $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$ *for all* $f, g, h \in \mathcal{X}$;

(ii) [Linearity in the first variable II] $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$ *for all* $\alpha \in \mathbb{C}$, $f, g \in \mathcal{X}$;

(iii) [Hermitian property] $\langle f, g \rangle = \overline{\langle g, f \rangle}$ *for all* $f, g \in \mathcal{X}$;

(iv) [Positivity][69] $\langle f, f \rangle \ge 0$.

(v) [Definiteness/faithfulness] $\langle f, f \rangle = 0$ *if and only if*[70] $f = 0$.

If $\langle \cdot, \cdot \rangle$ satisfies properties (i) through (iv) but not necessarily (v), we will take the nonstandard route of calling $\langle \cdot, \cdot \rangle$ a **SEMI-DEFINITE** *inner product. An* **(SEMI-DEFINITE) INNER PRODUCT SPACE** *is a vector space $\mathcal{X}$ equipped with an (semi-definite) inner product.*

Easy algebraic consequences of the properties of an inner product include the **SESQUILINEAR**[71] or **CONJUGATE-LINEAR** identities

$$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle \quad \text{and} \quad \langle f, \alpha g \rangle = \overline{\alpha} \langle f, g \rangle .$$

C.4.2 Example.

(i) *The* **DOT PRODUCT** *on $\mathbb{C}^n$,*

$$\mathbf{x} \cdot \mathbf{y} := \sum_{k=1}^{n} x_k \overline{y_k}, \qquad \mathbf{x} = (x_1, \ldots, x_n),\ \mathbf{y} = (y_1, \ldots, y_n), \tag{C.4.1}$$

is an inner product on $\mathbb{C}^n$.

(ii) *The mapping*

$$\langle f, g \rangle_{L^2([a,b])} := \int_a^b f(x)\overline{g(x)}\, dx$$

is an inner product on $\mathcal{C}([a, b])$. This inner product is, for historical and cultural reasons, called the L^2-inner product.

(iii) *The mapping $\langle \cdot, \cdot \rangle_{L^2([a,b])}$ is only semi-definite on the larger space of Riemann integrable functions; see part (iii) of Example C.3.2.*

(iv) *In parts (ii) and (iii) above, we can replace $[a, b]$ with any interval $I \subseteq \mathbb{R}$ if we use the improper integral over I. The (semi-definite) inner product space $L^2(\mathbb{R})$ is sometimes (erroneously!) called "Hilbert space." This is a gross oversimplification, as a Hilbert space is a special kind of inner product space, not necessarily one of square-integrable functions.*

C.4.3 Remark.

"Where the conjugate goes" is something of a cultural and discipline-specific[72] *choice. When defining the dot product and the L^2-inner product, we always put the conjugate on the second input. If we put it on the first, i.e., setting $\mathbf{x} \cdot \mathbf{y} := \sum_{k=1}^{n} \overline{x_k} y_k$, then*

[70] We will only need to consider complex inner product spaces in this course.

[70] Since we do not require the inequality to be strict, a more natural name for this property might be merely "nonnegativity." However, when we also impose the subsequent property (v), we do get the strict inequality in all but the case $f = 0$.

[70] The "only if" here is redundant thanks to property (ii).

[71] "Sesqui," from the Latin for "one and a half." The inner product is "one and a half" times linear since we must conjugate the scalar on removing it from multiplication in the second slot. A genuine **BILINEAR** form would be a map $\mathcal{B} \colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that $\mathcal{B}[f, \cdot]$ and $\mathcal{B}[\cdot, g]$ are linear operators on $\mathcal{X}$ for each fixed f, $g \in \mathcal{X}$.

we would have $(\alpha\mathbf{x}\cdot\mathbf{y}) = \overline{\alpha}(\mathbf{x}\cdot\mathbf{y})$, which is not what property (ii) in Definition C.4.1 specifies. Of course, this is largely immaterial: pick a convention and stick with it. And this is our choice.

C.4.4 Lemma.

Let $\mathcal{X}$ be an inner product space. The function

$$\|f\| := \sqrt{\langle f, f\rangle} \tag{C.4.2}$$

is a norm on $\mathcal{X}$, called the NORM INDUCED BY THE INNER PRODUCT. *If $\langle\cdot,\cdot\rangle$ is semi-definite, then $\|\cdot\|$ is merely a seminorm.*

Proof. We know $\langle f, f\rangle \geq 0$ for all $f \in \mathcal{X}$, so $\sqrt{\langle f, f\rangle} \geq 0$. And $\|f\| = 0$ if and only if $\|f\|^2 = 0$, which happens if and only if $\langle f, f\rangle = 0$, which in turn holds if and only if $f = 0$.

Next, given $\alpha \in \mathbb{C}$, we calculate

$$\|\alpha f\|^2 = \langle \alpha f, \alpha f\rangle = \alpha\,\langle f, \alpha f\rangle = \alpha\overline{\alpha}\,\langle f, f\rangle = |\alpha|^2\,\|f\|^2,$$

from which we have $\|\alpha f\| = |\alpha|\,\|f\|$.

To prove the triangle inequality, we first expand

$$\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f+g, f\rangle + \langle f+g, g\rangle = \langle f, f\rangle + \langle g, f\rangle + \langle f, g\rangle + \langle g, g\rangle$$

$$= \|f\|^2 + \|g\|^2 + \langle f, g\rangle + \overline{\langle f, g\rangle} = \|f\|^2 + \|g\|^2 + 2\operatorname{Re}[\langle f, g\rangle]. \tag{C.4.3}$$

Suppose for the moment that $\operatorname{Re}[\langle f, g\rangle] \leq \|f\|\,\|g\|$; we derive this below as a consequence of the Cauchy-Schwarz inequality. Then (C.4.3) implies

$$\|f+g\|^2 \leq \|f\|^2 + 2\,\|f\|\,\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2,$$

from which the triangle inequality follows. ∎

C.4.5 Example.

(i) *The 2-norm on $\mathbb{C}^n$ is derived from the dot product and therefore, automatically, is a norm:*

$$\|\mathbf{x}\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} = \left(\sum_{k=1}^{n} x_k\overline{x_k}\right)^{1/2} = \sqrt{\mathbf{x}\cdot\mathbf{x}}.$$

(ii) *The L^2-norm on $\mathcal{C}([a,b])$ is derived from the L^2-inner product and therefore, automatically, is a norm:*

$$\|f\|_{L^2([a,b])} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2} = \left(\int_a^b f(x)\overline{f(x)}\,dx\right)^{1/2} = \sqrt{\langle f, f\rangle_{L^2([a,b])}}.$$

[72]"A mathematical physicist is a mathematician believing that [an inner product] is conjugate-linear in the first variable and linear in the second" [25, p. 80].

C.4.6 Lemma (Cauchy-Schwarz inequality).

Let $\mathcal{X}$ be an (semi-definite) inner product space. Then

$$|\langle f, g\rangle| \leq \|f\| \|g\|.$$

Proof. There are many proofs of this inequality, and all seem to involve some very clever trick. This is perhaps one of the more straightforward versions, and it appears in [29].

If either $f = 0$ or $g = 0$, then both sides are zero. So, we assume both $f \neq 0$ and $g \neq 0$, in which case the desired inequality is equivalent to

$$\left|\left\langle \frac{f}{\|f\|}, \frac{g}{\|g\|}\right\rangle\right| \leq 1.$$

It therefore suffices to show that if u, $v \in \mathcal{X}$ with $\|u\| \leq 1$ and $\|g\| \leq 1$, then $|\langle u, v\rangle| \leq 1$. One can do this by decomposing u as a sum of a vector "parallel" to v and "perpendicular" to v: write

$$u = \langle u, v\rangle v + \big(u - \langle u, v\rangle v\big).$$

For simplicity, set $\alpha = \langle u, v\rangle$, so that our goal is to show $|\alpha| \leq 1$. By (C.4.3), we have

$$1 = \|u\|^2 = \big\|\alpha v + \big(u - \alpha v\big)\big\|^2 = \|\alpha v\|^2 + 2\operatorname{Re}[\langle \alpha v, u - \alpha v\rangle] + \|u - \alpha v\|^2. \tag{C.4.4}$$

We have

$$\|\alpha v\|^2 = |\alpha|^2 \|v\|^2 = |\alpha|^2, \tag{C.4.5}$$

and we calculate

$$\langle \alpha v, u - \alpha v\rangle = \langle \alpha v, u\rangle - \langle \alpha v, \alpha v\rangle = \alpha \langle v, u\rangle - \alpha\overline{\alpha} \langle v, v\rangle = \alpha\overline{\langle u, v\rangle} - \alpha\overline{\alpha} \|v\|^2 = \alpha\overline{\alpha} - \alpha\overline{\alpha} = 0. \tag{C.4.6}$$

Replacing terms on the right side of (C.4.4) with (C.4.5) and (C.4.6), we find

$$1 = |\alpha|^2 + \|u - \alpha v\|^2 \geq |\alpha|^2,$$

and so $|\alpha| \leq 1$, as desired. ■

C.5. Orthonormal bases and generalized Fourier series.

The material in this section draws heavily on [13, 18].

C.5.1. Finite-dimensional theory.

The STANDARD BASIS for $\mathbb{C}^n$ is the set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_k$ is 0 in all of its entries except the kth, where it is 1. The standard basis has a special interaction with the dot product from (C.4.1):

$$\mathbf{e}_k \cdot \mathbf{e}_j = \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases} \tag{C.5.1}$$

Consequently, with the 2-norm on $\mathbb{C}^n$ from (C.3.1), we have

$$\|\mathbf{e}_k\|_2 = \sqrt{\mathbf{e}_k \cdot \mathbf{e}_k} = 1.$$

Any vector $\mathbf{x} \in \mathbb{C}$ has the (necessarily unique) representation

$$\mathbf{x} = \sum_{k=1}^{n} \alpha_k \mathbf{e}_k \tag{C.5.2}$$

for some $\alpha_k \in \mathbb{C}$. More precisely,

$$\mathbf{x} \cdot \mathbf{e}_j = \left(\sum_{k=1}^{n} \alpha_k \mathbf{e}_k \right) \cdot \mathbf{e}_j = \sum_{k=1}^{n} \alpha_k \left(\mathbf{e}_k \cdot \mathbf{e}_j \right) = \alpha_j \tag{C.5.3}$$

to show how the coefficients of $\mathbf{x}$ in the representation are determined by the vectors $\mathbf{e}_k$.

Also, if $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{C}^n$, then

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \left(\sum_{k=1}^{n} (\mathbf{x} \cdot \mathbf{e}_k) \mathbf{e}_k \right) \cdot \left(\sum_{j=1}^{n} (\mathbf{y} \cdot \mathbf{e}_j) \mathbf{e}_j \right) = \sum_{k=1}^{n} (\mathbf{x} \cdot \mathbf{e}_k) \left(\mathbf{e}_k \cdot \sum_{j=1}^{n} \overline{(\mathbf{y} \cdot \mathbf{e}_j)} \mathbf{e}_j \right) \\ &= \sum_{k=1}^{n} (\mathbf{x} \cdot \mathbf{e}_k) \sum_{j=1}^{n} \overline{(\mathbf{y} \cdot \mathbf{e}_j)} (\mathbf{e}_k \cdot \mathbf{e}_j) = \sum_{k=1}^{n} (\mathbf{x} \cdot \mathbf{e}_k) \overline{(\mathbf{y} \cdot \mathbf{e}_k)} \quad \left(= \sum_{k=1}^{n} x_k \overline{y_k} \right). \end{aligned} \tag{C.5.4}$$

Taking $\mathbf{x} = \mathbf{y}$, we have

$$\|\mathbf{x}\|_2^2 = \mathbf{x} \cdot \mathbf{x} = \sum_{k=1}^{n} (\mathbf{x} \cdot \mathbf{e}_k) \overline{(\mathbf{x} \cdot \mathbf{e}_k)} = \sum_{k=1}^{n} |\mathbf{x} \cdot \mathbf{e}_k|^2 \quad \left(= \sum_{k=1}^{n} |x_k|^2 \right). \tag{C.5.5}$$

Of course, there is nothing particularly special about the identities (C.5.3), (C.5.4), and (C.5.5). From the particular structure of $\mathbf{e}_j$, it is obvious that if $\mathbf{x} = (x_1, \ldots, x_n)$, then $\mathbf{x} \cdot \mathbf{e}_j = x_j$, and so (C.5.4) and (C.5.5) also follow from the definition of the 2-norm (C.3.1) and the dot product (C.4.1). What *is* special, however, is that we did not have to use the componentwise expression $\mathbf{x} = (x_1, \ldots, x_n)$ to obtain any of these identities; we just used algebraic properties of the dot product and the special relation (C.5.1) on the vectors $\mathbf{e}_k$.

If $\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_n \in \mathbb{C}^n$ with

$$\widetilde{\mathbf{e}}_k \cdot \widetilde{\mathbf{e}}_j = \begin{cases} 1, & j = k \\ 0, & j \neq k, \end{cases} \tag{C.5.6}$$

then the three relations (C.5.3), (C.5.4), and (C.5.5) still hold, with each $\mathbf{e}_k$ replaced by $\widetilde{\mathbf{e}}_k$. Thus the property (C.5.1), equivalently (C.5.6), is what really matters. With this *orthonormality* property, many fundamental computations become far more transparent in terms of the orthonormal vectors. We will see in the next section the power of an orthonormal set in an abstract inner product space and derive analogues of the identities (C.5.3), (C.5.4), and (C.5.5) outside Euclidean space.

C.5.2. Infinite-dimensional theory.

C.5.1 Definition.

Let $\mathcal{X}$ be a (semi-definite) inner product space.

(i) *Two vectors f, $g \in \mathcal{X}$ are* **ORTHOGONAL** *if $\langle f, g \rangle = 0$.*

(ii) *A set $\mathcal{U} \subseteq \mathcal{X}$ is* **ORTHONORMAL** *if the vectors in $\mathcal{U}$ are mutually orthogonal and each vector has norm 1. That is, $\mathcal{U}$ is orthonormal if and only if*

$$\langle f, g\rangle = \begin{cases} 1, & f = g \\ 0, & f \neq g \end{cases}$$

for all $f,\ g \in \mathcal{U}$.

C.5.2 Lemma.

Any orthonormal set is linearly independent.

Proof. Suppose $\mathcal{U} \subseteq \mathcal{X}$ is orthonormal and $\phi_1, \ldots, \phi_n \in \mathcal{U}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ such that $\sum_{k=1}^{n} \alpha_k \phi_k = 0$. Fix $j \in \{1, \ldots, n\}$ and calculate

$$0 = \left\langle \sum_{k=1}^{n} \alpha_k \phi_k, \phi_j \right\rangle = \sum_{k=1}^{n} \alpha_k \langle \phi_k, \phi_j \rangle = \alpha_j.$$

Hence $\alpha_j = 0$ for all j. ■

If $\mathcal{X}$ is a (semi-definite) inner product space, then $\mathcal{X}$ has a **BASIS**: there is a set $\mathcal{B}$ of linearly independent vectors in $\mathcal{X}$ such that any $f \in \mathcal{X}$ can be written as a finite linear combination of vectors in $\mathcal{B}$, i.e., the **SPAN** of $\mathcal{B}$ equals $\mathcal{X}$. In symbols,

$$\mathcal{X} = \operatorname{span}(\mathcal{B}) := \left\{ \sum_{k=0}^{n} \alpha_k f_k \;\middle|\; n \in \mathbb{N}, f_1, \ldots, f_n \in \mathcal{B}, \alpha_1, \ldots, \alpha_n \in \mathbb{C} \right\}.$$

For clarity, we will call such a linearly independent spanning set a **HAMEL BASIS**. There is nothing special about the existence of a Hamel basis for an inner product space; every vector space has one. But the vectors in $\mathcal{B}$ need not "interact well" with the inner product, although sometimes they do, as we saw in Appendix C.5.1. Under certain technical hypotheses on a general inner product, it is possible for an inner product space to have an "orthonormal basis" that permits the (necessarily unique) representation of any vector in the space as an *infinite* linear combination of vectors in this basis set. To make sense of an infinite linear combination of vectors, we need to define what it means for a series to converge in an inner product space, or, more generally, a (semi)normed space.

C.5.3 Definition.

Let $\mathcal{X}$ be a (semi)normed space If (f_k) is a sequence in $\mathcal{X}$ and $f \in \mathcal{X}$, then we write $f \cong \sum_{k=0}^{\infty} f_k$ if

$$\lim_{n\to\infty} \left\| f - \sum_{k=0}^{n} f_k \right\|. \tag{C.5.7}$$

If $\mathcal{X}$ is a normed space and (C.5.7) *holds, then the vector f is necessarily unique*[73]*, in which case we write $f = \sum_{k=0}^{\infty} f_k$. As in Definition A.2.1, we use the symbol $\sum_{k=0}^{\infty} f_k$ to denote both the sequence of partial sums $\left(\sum_{k=0}^{n} f_k\right)$ and the limit(s) of this sequence, if it (they) exist.*

From now on we will assume that our (semi-definite) inner product space $\mathcal{X}$ is infinite-dimensional, i.e., it has a Hamel basis that is not finite. Otherwise, all the theory would reduce to linear algebra on $\mathbb{C}^n$.

C.5.4 Definition.

Let $\mathcal{X}$ be a (semi-definite) inner product space. An orthonormal set $\mathcal{U} \subseteq \mathcal{X}$ is an **ORTHONORMAL BASIS** *for $\mathcal{X}$ if for any $f \in \mathcal{X}$, there are sets $\{\phi_k\}_{k=0}^{\infty} \subseteq \mathcal{U}$ and $\{\alpha_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$ such that*

$$f \cong \sum_{k=0}^{\infty} \alpha_k \phi_k, \tag{C.5.8}$$

i.e., we have

$$\lim_{n\to\infty} \left\| f - \sum_{k=0}^{n} \alpha_k \phi_k \right\|,$$

where $\|\cdot\|$ is the (semi)norm (C.4.2) *induced by the (semi-definite) inner product on $\mathcal{X}$. The representation of $f \in \mathcal{X}$ as the series* (C.5.8) *is its* **FOURIER SERIES WITH RESPECT TO THE ORTHONORMAL BASIS** *$\mathcal{U}$, and the coefficients α_k are the* **FOURIER COEFFICIENTS OF** *f* **WITH RESPECT TO** *$\mathcal{U}$.*

It is not necessarily the case that a given (semi-definite) inner product space has an orthonormal basis; indeed, taking $\mathcal{X} = \mathcal{C}([a, b])$ with the L^2-inner product is such a space. We will not attempt to characterize those spaces that do have orthonormal bases (but see [14, 25]). Instead, we will merely assume that a space *does* have an orthonormal basis and then see what more we can learn.

Although an orthonormal basis involves an infinite sum, it shares some familiar properties with a Hamel basis, beyond the linear independence endowed by Lemma C.5.2.

- First, the Fourier coefficients of $f \in \mathcal{X}$ from the representation (C.5.8) are uniquely determined by the simple relation $\alpha_k = \langle f, \phi_k \rangle$.

- Second, the vectors ϕ_k that appear in (C.5.8) are unique, up to adding terms from the orthonormal basis paired with coefficients equal to zero. In other words, we cannot change the basis vectors that represent f, not counting the trivial case of adding zero.

- Third, the sum (C.5.8) converges regardless of the ordering of its terms, just like a finite sum from a Hamel basis.

We state these properties formally below.

[73] If there is another such vector, say, g, with $\lim_{n\to\infty} \|g - \sum_{k=0}^{n} f_k\| = 0$, then we have

$$\|f - g\| = \left\| \left(f - \sum_{k=0}^{n} f_k\right) + \left(g - \sum_{k=0}^{n} f_k\right) \right\| \leq \left\| f - \sum_{k=0}^{n} f_k \right\| + \left\| g - \sum_{k=0}^{n} f_k \right\| \to 0 + 0 \text{ as } n \to \infty,$$

thus $\|f - g\| = 0$. Since $\|\cdot\|$ is a norm, not a seminorm, in this case, then $f - g = 0$, so $f = g$.

C.5.5 Lemma.

Suppose that $\mathcal{U}$ is an orthonormal basis for the (semi-definite) inner product space $\mathcal{X}$ and take $f \in \mathcal{X}$.

(i) *If $f \cong \sum_{k=0}^{\infty} \alpha_k \phi_k$ for some $\{\phi_k\}_{k=0}^{\infty} \subseteq \mathcal{U}$ and $\{\alpha_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$, then $\alpha_k = \langle f, \phi_k \rangle$.*

(ii) *If $f \cong \sum_{k=0}^{\infty} \beta_k \psi_k$ for some other sets $\{\psi_k\}_{k=0}^{\infty} \subseteq \mathcal{U}$ and $\{\beta_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$, then*

$$\{\alpha_k\}_{k=0}^{\infty} \setminus \{0\} = \{\beta_k\}_{k=0}^{\infty} \setminus \{0\} \quad \text{and} \quad \{\phi_k \mid \alpha_k \neq 0\} = \{\psi_k \mid \beta_k \neq 0\}$$

(iii) *The series $\sum_{k=0}^{\infty} \langle f, \phi_k \rangle \phi_k$ converges unconditionally to f in the sense that any rearrangement of it also converges to f. That is, if $\sigma \colon \mathbb{N} \to \mathbb{N}$ is one-to-one and onto, then*

$$f \cong \sum_{k=0}^{\infty} \left\langle f, \phi_{\sigma(k)} \right\rangle \phi_{\sigma(k)}.$$

Proof. We prove only part (i), the others requiring more technical tools from analysis; see [13, 18]. First, fix k and write, for $n \geq j$,

$$\langle f, \phi_k \rangle = \left\langle \left(f - \sum_{j=1}^{n} \alpha_j \phi_j \right) + \sum_{k=0}^{n} \alpha_j \phi_j, \phi_k \right\rangle = \underbrace{\left\langle f - \sum_{j=1}^{n} \alpha_j \phi_j, \phi_k \right\rangle}_{I_n} + \underbrace{\left\langle \sum_{k=0}^{n} \alpha_j \phi_j, \phi_k \right\rangle}_{II_n}.$$

The Cauchy-Schwarz inequality implies

$$\|I_n\| \leq \left\| f - \sum_{j=1}^{n} \alpha_j \phi_j \right\| \|\phi_k\| \to 0 \text{ as } n \to \infty$$

since $f \cong \sum_{j=1}^{n} \alpha_j \phi_j$. Next, since $\mathcal{U}$ is orthonormal and $n \geq j$, we have

$$II_n = \sum_{k=0}^{n} \alpha_j \langle \phi_j, \phi_k \rangle = \alpha_k.$$

Thus

$$\langle f, \phi_k \rangle = I_n + \alpha_k,$$

where $I_n \to 0$ as $n \to \infty$. Taking this limit, we conclude $\langle f, \phi_k \rangle = \alpha_k$. ■

The careful reader will note that we did not mention in this lemma an analogue of the familiar property that if two vectors have the same coefficients with respect to a Hamel basis, then those two vectors are unique. This is, unfortunately, not necessarily true if we work in a semi-definite inner product space. For example, if $\mathcal{X}$ is a semi-definite inner product space, and $f \in \mathcal{X}$ with $\|f\| = 0$ and $f \neq 0$, then for any $g \in \mathcal{X}$, the Cauchy-Schwarz inequality gives

$$|\langle f, g \rangle| \leq \|f\| \|g\| = 0 \cdot \|g\| = 0.$$

Hence $\langle f, g \rangle = 0$. In particular, the coefficients of f with respect to any orthonormal basis of $\mathcal{X}$ must all be 0, even though f is not the zero vector. We construct a concrete example of this situation in Example 2.4.17.

If we strengthen our setting to a genuine inner product space, then, yes, a vector *is* uniquely determined by its Fourier coefficients.

C.5.6 Lemma.

Let $\mathcal{X}$ be an inner product space with orthonormal basis $\mathcal{U}$. If for some $g \in \mathcal{X}$ we have $\langle f, \phi \rangle = \langle g, \phi \rangle$ for all $\phi \in \mathcal{U}$, then $f = g$. That is, the Fourier coefficients of f determine f uniquely.

The orthonormal bases that typically arise in practice are "countable" in the sense that we may "enumerate" $\mathcal{U}$ in the form $\mathcal{U} = \{\phi_k\}_{k=0}^{\infty}$. Indeed, an orthonormal basis for an infinite-dimensional space must be at least "countably infinite," for if such a basis $\mathcal{U}$ were finite, then all the Fourier series representations (C.5.8) would be finite sums, and $\mathcal{U}$ would be a Hamel basis consisting of finitely many vectors. And then the vector space would be finite-dimensional.

However, a Hamel basis $\mathcal{B}$ for an infinite-dimensional vector space may be "uncountable" in the sense that we cannot write it in the form $\mathcal{B} = \{\psi_k\}_{k=0}^{\infty}$. This is the case for most "interesting" infinite-dimensional spaces. In particular, then, a *countable* orthonormal basis $\mathcal{U} = \{\phi_k\}_{k=0}^{\infty}$ almost never forms a Hamel basis for an inner product space.

Working with the infinite sum in (C.5.8) is a small price to pay for the other conveniences of an orthonormal basis. Namely, we can read off some important properties from an orthonormal basis representation that a Hamel basis will not inherently provide. In fact, these properties even hold if the definiteness property is eliminated from the inner product.

C.5.7 Theorem.

Let $\mathcal{X}$ be a semi-definite inner product space. Suppose that $\{\phi_k\}_{k=0}^{\infty}$ is an orthonormal basis for $\mathcal{X}$ and let $f, g \in \mathcal{X}$.

(i) [Parseval's identity] *Suppose $f, g \in \mathcal{X}$ with $f \cong \sum_{k=0}^{\infty} \alpha_k \phi_k$ and $g \cong \sum_{k=0}^{\infty} \beta_k \phi_k$. Then*

$$\langle f, g \rangle = \sum_{k=0}^{\infty} \alpha_k \overline{\beta_k} = \sum_{k=0}^{\infty} \langle f, \phi_k \rangle \langle \phi_k, g \rangle .$$

(ii) [Plancherel's[74] identity] *If $f \in \mathcal{X}$ with $f \cong \sum_{k=0}^{\infty} \alpha_k \phi_k$, then*

$$\|f\|^2 = \sum_{k=0}^{\infty} |\alpha_k|^2 = \sum_{k=0}^{\infty} | \langle f, \phi_k \rangle |^2.$$

Proof. Plancherel's identity follows directly from Parseval's by taking $f = g$, so we prove only the former. One can calculate that, for any $n \geq 1$, we have[75]

$$\langle f, g \rangle = \underbrace{\left\langle f - \sum_{k=0}^{n} \alpha_k \phi_k, g \right\rangle}_{I_n} + \underbrace{\left\langle \sum_{k=0}^{n} \alpha_k \phi_k, g - \sum_{j=1}^{n} \beta_j \phi_j \right\rangle}_{II_n} + \underbrace{\left\langle \sum_{k=0}^{n} \alpha_k \phi_k, \sum_{j=1}^{n} \beta_j \phi_j \right\rangle}_{III_n} .$$

[74] The inequality $\sum_{k=0}^{\infty} | \langle f, \phi_k \rangle |^2 \leq \|f\|^2$ is called Bessel's inequality.

[75] When one is working with two different sums in an inner product, it is often beneficial to index them with different letters.

The Cauchy-Schwarz inequality and the definition of series convergence (C.5.7) imply

$$|I_n| \leq \left\| f - \sum_{k=0}^{n} \alpha_k \phi_k \right\| \|g\| \to 0 \text{ as } n \to \infty.$$

Take n so large that $\|f - \sum_{k=0}^{n} \alpha_k \phi_k\| \leq 1$. Then the triangle inequality implies

$$\left\| \sum_{k=0}^{n} \alpha_k \phi_k \right\| \leq \left\| f - \sum_{k=0}^{n} \alpha_k \phi_k \right\| + \|f\| \leq 1 + \|f\|,$$

and so we have

$$|II_n| \leq \left\| \sum_{k=0}^{n} \alpha_k \phi_k \right\| \left\| g - \sum_{j=1}^{n} \beta_j \right\| \leq (1 + \|f\|) \left\| g - \sum_{j=1}^{n} \beta_j \right\| \to 0 \text{ as } n \to \infty.$$

Last, since $\{\phi_k\}_{k=0}^{\infty}$ is orthonormal, the sesquilinearity of the inner product yields

$$III_n = \sum_{k=0}^{n} \sum_{j=1}^{n} \alpha_k \overline{\beta}_j \langle \phi_k, \phi_j \rangle = \sum_{k=0}^{n} \alpha_k \overline{\beta}_k.$$

Thus

$$\left| \langle f, g \rangle - \sum_{k=0}^{n} \alpha_k \overline{\beta}_k \right| \leq |I_n| + |II_n| \to 0 \text{ as } n \to \infty.$$

■

The Fourier coefficients $\langle f, \phi_k \rangle$ of f with respect to a countable orthonormal basis $\mathcal{U} = \{\phi_k\}_{k=0}^{\infty}$ of the inner product space $\mathcal{X}$ not only give the unique "coordinates" of $f \in \mathcal{X}$ with respect to $\mathcal{U}$, they also are the "best" coefficients to use to represent f with respect to a truncation of this basis. That is, given $f \in \mathcal{X}$, it may not be computationally expedient to consider the whole series $\sum_{k=0}^{\infty} \langle f, \phi_k \rangle \phi_k$. Perhaps we want to approximate $f \approx \sum_{k=0}^{n} \alpha_k \phi_k$ for some finite $n \in \mathbb{N}$. Can we do better than taking $\alpha_k = \langle f, \phi_k \rangle$ for $k = 0, \ldots, n$? No.

C.5.8 Theorem (Least-squares[76] approximation).

Let $\{\phi_k\}_{k=0}^{\infty}$ be an orthonormal subset of the semi-definite inner product space $\mathcal{X}$ (not necessarily an orthonormal basis). Then

$$\min_{\alpha_1, \ldots, \alpha_n \in \mathbb{C}} \left\| f - \sum_{k=0}^{n} \alpha_k \phi_k \right\| = \left\| f - \sum_{k=0}^{n} \langle f, \phi_k \rangle \phi_k \right\| \tag{C.5.9}$$

for any $n \geq 1$.

Proof. We expand

$$\left\| f - \sum_{k=0}^{n} \alpha_k \phi_k \right\|^2 = \left\langle f - \sum_{k=0}^{n} \alpha_k \phi_k, f - \sum_{j=1}^{n} \alpha_j \phi_j \right\rangle = \|f\|^2 + \sum_{k=0}^{n} \left(|\alpha_k|^2 - \overline{\alpha_k} \langle f, \phi_k \rangle - \alpha_k \overline{\langle f, \phi_k \rangle} \right). \tag{C.5.10}$$

[76] So named because when we work with the 2-norm on $\mathbb{F}^n$ or the L^2-norm on a space of square-integrable functions, computing the minimum in (C.5.9) is equivalent to minimizing a sum of squares or an integral of squares.

Now we use the identity

$$|z|^2 - \overline{z}w - z\overline{w} = |z-w|^2 - |w|^2,$$

valid for any z, $w \in \mathbb{C}$, to convert (C.5.10) into

$$\left\| f - \sum_{k=0}^{n} \alpha_k \phi_k \right\|^2 = \|f\|^2 + \sum_{k=0}^{n} \left(|\alpha_k - \langle f, \phi_k \rangle|^2 - |\langle f, \phi_k \rangle|^2 \right) \geq \|f\|^2 - \sum_{k=0}^{n} |\langle f, \phi_k \rangle|^2.$$

Remarkably, we can compute

$$\|f\|^2 - \sum_{k=0}^{n} |\langle f, \phi_k \rangle|^2 = \left\| f - \sum_{k=0}^{n} \langle f, \phi_k \rangle \phi_k \right\|^2 + \left\langle \sum_{k=0}^{n} \langle f, \phi_k \rangle \phi_k, f - \sum_{j=1}^{n} \langle f, \phi_j \rangle \phi_j \right\rangle$$

and

$$\left\langle \sum_{k=0}^{n} \langle f, \phi_k \rangle \phi_k, f - \sum_{j=1}^{n} \langle f, \phi_j \rangle \phi_j \right\rangle = 0.$$

Thus

$$\left\| f - \sum_{k=0}^{n} \alpha_k \phi_k \right\|^2 \geq \left\| f - \sum_{k=0}^{n} \langle f, \phi_k \rangle \phi_k \right\|^2,$$

and this is the least-squares inequality. ■

C.6. Eigenvalues.

Let T be a linear operator in the vector space $\mathcal{X}$ over $\mathbb{C}$. Denote the domain of T by $\mathfrak{D}(T) \subseteq \mathcal{X}$. A point $\lambda \in \mathbb{C}$ is an **EIGENVALUE** of T if there exists $f \in \mathfrak{D}(T) \setminus \{0\}$ such that $Tf = \lambda f$. The ordered pair $(\lambda, f) \in \mathbb{C} \times \mathcal{X}$ is an **EIGENPAIR** of T. The set of all eigenvalues of T is the **POINT SPECTRUM**[77] of T, and we denote it by $\sigma_{\text{pt}}(T)$.

C.6.1 Example.

The eigenvalues of the matrix operator $A \in \mathbb{C}^{n \times n}$ are the roots of the nth degree **CHARACTERISTIC POLYNOMIAL** $\lambda \mapsto \det(\lambda \mathbb{1}_n - A)$*, where $\mathbb{1}_n$ is the $n \times n$ identity matrix. Indeed, given $\lambda \in \mathbb{C}$, there is a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ if and only if the equation $(\lambda \mathbb{1}_n - A)\mathbf{x} = 0$ has a nontrivial solution, which happens if and only if* $\det(\lambda \mathbb{1}_n - A) = 0$*. One can calculate that* $\det(\lambda \mathbb{1}_n - A)$ *is a polynomial of degree n in λ.*

C.6.2 Example.

Let $\mathcal{X} = \mathcal{C}([0,1])$ and define the operator T with domain $\mathfrak{D}(T) = \mathcal{X}$ by $(Tf)(x) := xf(x)$. Show that T has no eigenvalues.

Solution. Suppose there does exist $\lambda \in \sigma_{\text{pt}}(T)$. Then there is a function $f \in \mathcal{C}([0,1], \mathbb{C})$ such that $xf(x) = \lambda f(x)$ for all $x \in [0,1]$ and, moreover, $f(x) \neq 0$ for at least one

[77] In more abstract operator theory, there are other notions of "spectral" values for an operator, and it is important to distinguish eigenvalues from other "spectral" values with different behaviors.

$x \in [0,1]$. We find $(x-\lambda)f(x) = 0$ for all $x \in [0,1]$. If $\lambda \notin [0,1]$, then $x - \lambda \neq 0$ for all $x \in [0,1]$, and so we may divide to find $f(x) = 0$ for all $x \in [0,1]$, a contradiction. If $\lambda \in [0,1]$, then $x - \lambda \neq 0$ for all $x \in [0,1] \setminus \{\lambda\}$, and so again we may divide, this time to find $f(x) = 0$ for all $x \in [0,1] \setminus \{\lambda\}$. But f is continuous on $[0,1]$, so

$$f(\lambda) = \lim_{x \to \lambda} f(x) = 0,$$

a contradiction once more. ▲

C.6.3 Remark.

Here is the value of defining an operator "in" a vector space, not just "on." If $T \colon \mathcal{X} \to \mathcal{Y}$ is a linear operator from the space $\mathcal{X}$ to the space $\mathcal{Y}$, then for $(\lambda, f) \in \mathbb{C} \times \mathcal{X}$ to be an eigenpair of T, we need $Tf = \lambda f \in \mathcal{X}$. Thus T must map at least some elements of $\mathcal{X}$ back to $\mathcal{X}$, and this would be difficult to guarantee unless $\mathcal{Y} \subseteq \mathcal{X}$. Hence our preference for taking the domain of T to be a subspace of the "target" space $\mathcal{X}$.

On the other hand, we need not have $Tf \in \mathfrak{D}(T)$ for all f; consider $Tf = f'$ with $\mathfrak{D}(T) = \mathcal{C}^1([-1,1])$ and $\mathcal{X} = \mathcal{C}([-1,1])$. If we put $f(x) = \int_0^x |\xi| \, d\xi$, then Tf is not differentiable at 0.

The **eigenspace corresponding to** $\lambda \in \sigma_{\mathrm{pt}}(T)$ is the vector space

$$\mathcal{E}(T,\lambda) := \{ f \in \mathfrak{D}(T) \mid Tf = \lambda f \}.$$

The **geometric multiplicity** of λ is the dimension of $\mathcal{E}(T,\lambda)$. The eigenvalue λ is **simple** if $\dim[\mathcal{E}(T,\lambda)] = 1$. (It is possible to define a notion of algebraic multiplicity)

Let T be a linear operator in $\mathcal{X}$ and $\lambda \in \sigma_{\mathrm{pt}}(T)$. An m-tuple $(f_1, \ldots, f_m) \in \mathfrak{D}(T)^m$ is a **Jordan chain of generalized eigenvectors for** λ if $Tf_{k+1} = f_k$ for $k = 1, \ldots, m-1$ and $Tf_1 = \lambda f_1$.

C.6.4 Example.

(i) *Let*

$$T := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$. Clearly 0 is an eigenvalue of T Then $T\mathbf{e}_3 = \mathbf{e}_2$, $T\mathbf{e}_2 = \mathbf{e}_1$, and $T\mathbf{e}_1 = 0$, so $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a Jordan chain for 0.

(ii) *Let $Tf = f'' - 2f' + f$ with domain $\mathfrak{D}(T) = \mathcal{C}^2(\mathbb{R})$ in $\mathcal{X} = \mathcal{C}(\mathbb{R})$. Set $f_0(x) = e^x$ and $f_1(x) = xe^x$. Then $Tf_1 = f_0$ and $Tf_0 = 0$, so 0 is an eigenvalue of T and (f_0, f_1) forms a partial Jordan chain for 0. We say partial because variation of parameters and/or undetermined coefficients always allows us to find a function f_{k+1} such that*

$$Tf_{k+1} = f_k.$$

C.7. Application: the matrix resolvent equation.

Fix $A \in \mathbb{C}^{n \times n}$ and let $\mathbb{1}_n$ denote the $n \times n$ identity matrix. The **resolvent equation** for A is the equation

$$(A - \lambda \mathbb{1}_n)\mathbf{x} = \mathbf{y}. \tag{C.7.1}$$

Here $\mathbf{y} \in \mathbb{C}^n$ is given, and we want to know the scalars $\lambda \in \mathbb{C}$ for which we can solve this equation (uniquely or not). Of course, the equation has a unique solution if and only if $\det(A - \lambda \mathbb{1}_n) \neq 0$, which is equivalent to $\lambda \notin \sigma_{\text{pt}}(A)$. In this case, the unique solution is $\mathbf{x} = (A - \lambda \mathbb{1}_n)^{-1}\mathbf{y}$. If $\lambda \in \sigma_{\text{pt}}(A)$, then a solution need not exist for all $\mathbf{y}$, and if a solution does exist, then it will not be unique.

There is a situation in which it is possible to construct the unique solution in a more elegant and explicit manner than just using $\mathbf{x} = (A - \lambda \mathbb{1}_n)^{-1}\mathbf{y}$. After all, computing a matrix inverse can be difficult and expensive. Here is the situation: suppose that A has n linearly independent eigenvectors, so that these eigenvectors span $\mathbb{C}^n$. More precisely, let $\{(\lambda_k, \mathbf{u}_k)\}_{k=1}^n$ be a set of eigenpairs of A, i.e., $A\mathbf{u}_k = \lambda_k \mathbf{u}_k$ and furthermore assume[78] that $\{\mathbf{u}_k\}_{k=1}^n$ is orthonormal, i.e., $\mathbf{u}_k \cdot \mathbf{u}_j = 0$ if $j \neq k$ and $\|\mathbf{u}_k\|_2 = 1$. (See part (i) of Example C.3.2 for the definition of this hopefully familiar norm.) We do not assume that the eigenvalues of A are distinct.

Now write $\mathbf{y} = \sum_{k=1}^n \theta_k \mathbf{u}_k$, where $\theta_k = \mathbf{y} \cdot \mathbf{u}_k$, and write $\mathbf{x} = \sum_{k=1}^n \mu_k \mathbf{u}_k$, where the n numbers μ_k are, for now, unknown. Then (C.7.1) becomes

$$(A - \lambda \mathbb{1}_n) \sum_{k=1}^n \mu_k \mathbf{u}_k = \sum_{k=1}^n \theta_k \mathbf{u}_k. \tag{C.7.2}$$

On the left, we have

$$\begin{aligned}(A - \lambda \mathbb{1}_n) \sum_{k=1}^n \mu_k \mathbf{u}_k = \sum_{k=1}^n \mu_k (A - \lambda \mathbb{1}_n)\mathbf{u}_k = \sum_{k=1}^n \mu_k (A\mathbf{u}_k - \lambda \mathbf{u}_k) &= \sum_{k=1}^n \mu_k (\lambda_k \mathbf{u}_k - \lambda \mathbf{u}_k) \\ &= \sum_{k=1}^n \mu_k (\lambda_k - \lambda)\mathbf{u}_k.\end{aligned}$$

We substitute this calculation into (C.7.2) to find

$$\sum_{k=1}^n \mu_k (\lambda_k - \lambda)\mathbf{u}_k = \sum_{k=1}^n \theta_k \mathbf{u}_k.$$

Subtracting and factoring, we have

$$\sum_{k=1}^n \big(\mu_k (\lambda_k - \lambda) - \theta_k\big)\mathbf{u}_k = 0. \tag{C.7.3}$$

Since the set $\{\mathbf{u}_k\}_{k=1}^n$ is orthonormal, it is linearly independent, so the sum above forces

$$\mu_k (\lambda_k - \lambda) - \theta_k = 0 \tag{C.7.4}$$

for each k. Another way to see this is to take the dot product of both sides of (C.7.3) with $\mathbf{u}_j$ for a fixed j:

$$\sum_{k=1}^n \big(\mu_k (\lambda_k - \lambda) - \theta_k\big)\mathbf{u}_k = 0 \Longrightarrow \left(\sum_{k=1}^n \big(\mu_k (\lambda_k - \lambda) - \theta_k\big)\mathbf{u}_k\right) \cdot \mathbf{u}_j = \mathbf{u}_j \cdot 0 \Longrightarrow \mu_j (\lambda_j - \lambda) - \theta_j = 0. \tag{C.7.5}$$

[78]Given a set $\mathcal{V}$ of n linearly independent vectors in an inner product space $\mathcal{X}$, the **Gram-Schmidt orthonormalization procedure** constructs a set $\mathcal{U}$ of n orthonormal vectors in $\mathcal{X}$ such that $\text{span}(\mathcal{V}) = \text{span}(\mathcal{U})$. In particular, if $\mathcal{V}$ is a basis for $\mathcal{X}$, then so is $\mathcal{U}$.

Either way, if (C.7.4) holds and if $\lambda \neq \lambda_k$ for any k, then we can solve for μ_k:

$$\mu_k = \frac{\theta_k}{\lambda_k - \lambda}. \tag{C.7.6}$$

And so

$$\mathbf{x} = \sum_{k=1}^{n} \left(\frac{\theta_k}{\lambda_k - \lambda}\right) \mathbf{u}_k. \tag{C.7.7}$$

This is a rather more explicit formula for $\mathbf{x}$ than just $(A - \lambda \mathbb{1}_n)^{-1}\mathbf{y}$.

Now suppose $\lambda = \lambda_j$ for some j. To make our work easier, assume that the eigenvalues are distinct, so $\lambda \neq \lambda_k$ for $k \neq j$. Then (C.7.4) forces $\theta_j = 0$. And so if we are to have a solution to (C.7.1), the vector $\mathbf{y}$ must satisfy $\mathbf{y} \cdot \mathbf{u}_j = 0$. Thus we *cannot* solve (C.7.1) for all $\mathbf{y}$ in the case that λ is an eigenvalue of A. But suppose also that $\mathbf{y}$ meets this solvability condition $\mathbf{y} \cdot \mathbf{u}_j = 0$. What hope do we have of solving (C.7.1) then?

Since the eigenvalues of A are distinct, we have $\lambda \neq \lambda_k$ for $k \neq j$, and so we can still solve for the other coefficients μ_k of $\mathbf{x}$ as in (C.7.6). But (C.7.4) now tells us nothing about μ_j, and so μ_j is a "free parameter" — we can take it to be any value that we want. Specifically, a direct calculation shows that if we take $c \in \mathbb{C}$ and set

$$\mathbf{x}_c := \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\theta_k}{\lambda_k - \lambda}\right) \mathbf{u}_k + c\mathbf{u}_j, \tag{C.7.8}$$

then $(A - \lambda_j \mathbb{1}_n)\mathbf{x}_c = \mathbf{y}$, assuming $\mathbf{y} \cdot \mathbf{u}_j = 0$. That is, our problem (C.7.2) has infinitely many solutions. This is not surprising: the kernel of $A - \lambda_j \mathbb{1}_n$ is spanned by $\mathbf{u}_j$, and the expression in (C.7.8) has the form "particular solution + scalar multiple of kernel element."

We summarize our results.

C.7.1 Theorem.

Suppose that $A \in \mathbb{C}^{n \times n}$ *has the* n *linearly independent, orthonormal eigenvectors* $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{C}^n$. *Write* $\sigma_{\text{pt}}(A) = \{\lambda_k\}_{k=1}^n$, *so that* $A\mathbf{u}_k = \lambda_k \mathbf{u}_k$, *and fix* $\mathbf{y} \in \mathbb{C}^n$.

(i) *If* $\lambda \notin \sigma_{\text{pt}}(A)$, *then the unique vector* $\mathbf{x} \in \mathbb{C}^n$ *such that* $\lambda\mathbf{x} - A\mathbf{x} = \mathbf{y}$ *is*

$$\mathbf{x} = \sum_{k=1}^{n} \left(\frac{\mathbf{y} \cdot \mathbf{u}_k}{\lambda_k - \lambda}\right) \mathbf{u}_k.$$

(ii) *Given* $\lambda_j \in \sigma_{\text{pt}}(A)$, *there exists* $\mathbf{x} \in \mathbb{C}^n$ *such that* $\lambda_j\mathbf{x} - A\mathbf{x} = \mathbf{y}$ *if and only if* $\mathbf{y} \cdot \mathbf{u}_j = 0$. *In this case, there exist infinitely many such* $\mathbf{x}$, *all of which have the form*

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\theta_k}{\lambda_k - \lambda}\right) \mathbf{u}_k + c\mathbf{u}_j$$

for some $c \in \mathbb{C}$.

C.8. Self-adjoint operators.

Let $\mathcal{X}$ be an inner product space. The operators $T\colon \mathfrak{D}(T) \subseteq \mathcal{X} \to \mathcal{X}$ and $S\colon \mathfrak{D}(S) \subseteq \mathcal{X} \to \mathcal{X}$ are **ADJOINT** to each other[79] if

$$\langle Tf, g\rangle = \langle f, Sg\rangle$$

for all $f \in \mathfrak{D}(T)$ and $g \in \mathfrak{D}(S)$. The operator T is **SELF-ADJOINT**, **SYMMETRIC**, or **HERMITIAN** if T is adjoint to itself, i.e., if

$$\langle Tf, g\rangle = \langle f, Tg\rangle$$

for all f, $g \in \mathfrak{D}(T)$.

C.8.1 Example.

Find an adjoint for each operator with respect to the given inner product. Is the operator self-adjoint?

(i) *Let $A \in \mathbb{C}^{n\times n}$ and let $\mathcal{M}_A\colon \mathbb{C}^n \to \mathbb{C}^n$ be matrix-vector multiplication: $\mathcal{M}_A\mathbf{x} := A\mathbf{x}$.*

(ii) *For a function $f\colon \mathbb{R} \to \mathbb{C}$, let $(Tf)(x) = ixf(x)$. Consider T as an operator in $L^2(\mathbb{R})$, where*

$$\mathfrak{D}(T) = \left\{ f \in L^2(R) \;\middle|\; \int_{-\infty}^{\infty} |xf(x)|^2 \, dx \text{ converges} \right\}.$$

(iii) *Define T in $\mathcal{C}([0,1])$ by $Tf = f'$ with domain*

$$\mathfrak{D}(T) = \left\{ f \in \mathcal{C}^1([0,1]) \;\middle|\; f(0) = f(1) = 0 \right\}.$$

Endow $\mathcal{C}([0,1])$ with the L^2-inner product.

(iv) *Define T as in part (iii), but now take its domain to be*

$$\mathfrak{D}(T) = \left\{ f \in \mathcal{C}^1([0,1]) \;\middle|\; f(0) = 0 \right\}.$$

(v) *Define T in $\mathcal{C}([0,1])$ by $Tf := f''$ with domain*

$$\mathfrak{D}(T) = \left\{ f \in \mathcal{C}^2([0,1]) \;\middle|\; f(0) = f(1) = f'(0) = f'(1) = 0 \right\}.$$

Solution. **(i)** Let A^* be the conjugate transpose of A. Then an adjoint to $\mathcal{M}_A$ is $\mathcal{M}_{A^*}$. Indeed, we have

$$(\mathcal{M}_A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A^*\mathbf{y}) = \mathbf{x} \cdot (\mathcal{M}_{A^*}\mathbf{y}).$$

The operator $\mathcal{M}_A$ is self-adjoint if and only if $A = A^*$, i.e., if and only if A is a Hermitian matrix.

[79] We are tempted to write something like $S = T^*$, as we do with matrices, see part (i) of Example C.8.1. However, the uniqueness (let alone existence) of adjoints for operators defined in infinite-dimensional spaces is a rather delicate issue, as parts (iii) and (iv) of Example C.8.1 will illustrate.

(ii) We claim that an adjoint is $(Sg)(x) = -ixg(x)$, where we put $\mathfrak{D}(S) = \mathfrak{D}(T)$. To see this, we integrate

$$\langle Tf, g\rangle_{L^2} = \int_{-\infty}^{\infty} (Tf)(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} ixf(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} f(x)\overline{(-ixg(x))}\, dx.$$

Clearly $Tf \neq Sf$, but, rather, $Tf = -Sf$ for all f. So, T is not self-adjoint, but T is **SKEW-ADJOINT**.

(iii) We start with an integration by parts:

$$\langle Tf, g\rangle_{L^2} = \int_0^1 (Tf)(x)\overline{g(x)}\, dx = \int_0^1 f'(x)\overline{g(x)}\, dx = f(1)\overline{g(1)} - f(0)\overline{g(0)} - \int_0^1 f(x)\overline{g'(x)}\, dx. \tag{C.8.1}$$

Since $f(0) = f(1) = 0$, this reduces to

$$\langle Tf, g\rangle_{L^2} = \int_0^1 f(x)\overline{(-g'(x))}\, dx,$$

so we take the adjoint of T to be $(Sg)(x) = -g'(x)$. Note that we could take the domain of S to be $\mathfrak{D}(T)$ or all of $\mathcal{C}^1([0,1])$.

(iv) Integrating by parts as in (C.8.1), we use the condition $f(0) = 0$ to find

$$\langle Tf, g\rangle_{L^2} = f(1)\overline{g(1)} - \int_0^1 f(x)\overline{g'(x)}\, dx.$$

If we again set $(Sg)(x) = -g'(x)$ and take the domain of S to be

$$\mathfrak{D}(S) = \left\{ g \in \mathcal{C}^1([0,1]) \;\middle|\; g(1) = 0 \right\},$$

then S will be adjoint to T. Thus changing the domain of T can change the domain of the adjoint — possibly endowing T and its adjoint with different domains. By the definition of "adjoint" above, this is absolutely permissible.

(v) We integrate by parts twice:

$$\begin{aligned}\langle Tf, g\rangle_{L^2} &= \int_0^1 f''(x)\overline{g(x)}\, dx = f'(x)\overline{g(x)}\Big|_{x=0}^{x=1} - \int_0^1 f'(x)\overline{g'(x)}\, dx = -\int_0^1 f'(x)\overline{g'(x)}\, dx \\ &= -f(x)\overline{g'(x)}\Big|_{x=0}^{x=1} + \int_0^1 f(x)\overline{g''(x)}\, dx = \int_0^1 f(x)\overline{g''(x)}\, dx = \langle f, Tg\rangle_{L^2},\end{aligned}$$

and so T is self-adjoint. ▲

As we saw in Appendix C.6, the *existence* of eigenvalues for a linear operator can be posed purely at the level of vector spaces. All that a point $\lambda \in \mathbb{C}$ has to do to be an eigenvalue of the operator $T\colon \mathfrak{D}(T) \subseteq \mathcal{X} \to \mathcal{X}$ is to satisfy $Tf = \lambda f$ for some $f \in \mathfrak{D}(T) \setminus \{0\}$. However, if we impose more structure on the vector space, it is possible to learn more about the eigenvalues.

C.8.2 Theorem.

Let $\mathcal{X}$ be an inner product space and T be a self-adjoint operator in $\mathcal{X}$.

(i) *Every eigenvalue*[80] *of T is real.*

(ii) *Eigenvectors of T corresponding to distinct eigenvalues are orthogonal, i.e., if λ, $\mu \in \sigma_{\mathrm{pt}}(T)$ with $\lambda \neq \mu$ and $Tf = \lambda f$, $Tg = \mu g$, then $\langle f, g\rangle = 0$.*

Proof. This is a rather classical proof; we include it for completeness.

(i) Let $\lambda \in \sigma_{\mathrm{pt}}(T)$ with eigenvector $f \in \mathcal{X} \setminus \{0\}$, i.e., $Tf = \lambda f$. We will show $\lambda = \overline{\lambda}$. To relate λ to its conjugate, we use the linearity of the inner product:

$$\langle Tf, f\rangle = \langle \lambda f, f\rangle = \lambda \langle f, f\rangle = \lambda \|f\|^2$$

and

$$\langle Tf, f\rangle = \langle f, Tf\rangle = \langle f, \lambda f\rangle = \overline{\langle \lambda f, f\rangle} = \overline{\lambda}\overline{\langle f, f\rangle} = \overline{\lambda} \|f\|^2$$

Equating these two expressions and then dividing by $\|f\|^2$, which we may do, since $f \neq 0$, we find $\lambda = \overline{\lambda}$.

(ii) Since we want to show $\langle f, g\rangle = 0$, and all that we know about f and g involves multiplication by λ or μ, let us introduce the eigenvalues into this inner product. We calculate

$$\langle \lambda f, g\rangle = \langle Tf, g\rangle = \langle f, Tg\rangle = \langle f, \mu g\rangle = \overline{\mu} \langle f, g\rangle = \mu \langle f, g\rangle . \tag{C.8.2}$$

For the last equality, we used the result from part (i) that μ is real (and so is λ). Since $\langle \lambda f, g\rangle = \lambda fg$, we may subtract in (C.8.2) to find

$$(\lambda - \mu) \langle f, g\rangle = 0.$$

And since $\lambda - \mu \neq 0$, we may divide to find $\langle f, g\rangle = 0$. ∎

[80] We make no claims about whether or not T *has* any eigenvalues!

D. Set-Theoretic Terminology

Let A and B be sets.

1. The expression $x \in A$ means "x is an element of A." For example, if $A = \{1, 2, 3\}$, then $1 \in A$.

2. The expression $A \subseteq B$ means "A is a subset of B," which in turn means "every element of A is an element of B." In symbols, $A \subseteq B$ if and only if given $x \in A$ we have $x \in B$. For example, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then $A \subseteq B$.

3. If A consists of all elements x such that a property P is true for x, then we often write

$$A = \{x \mid P \text{ is true for } x\}.$$

For example, if P is the property "is a positive real number," then the interval $(0, \infty)$ is the set

$$(0, \infty) = \{x \mid x \text{ is a positive real number}\}.$$

Usually we will consider x to be an element of a larger "universal" set, and we will indicate that universal set in the notation:

$$(0, \infty) = \{x \in \mathbb{R} \mid x > 0\}.$$

4. The expression $A \cup B$ denotes the **union** of A and B. The set $A \cup B$ is the set of all elements in either A or B. For example,

$$\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}.$$

5. The expression $A \cap B$ denotes the **intersection** of A and B. The set $A \cap B$ is the set of all elements in both A and B. For example,

$$\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}.$$

6. If Λ is a set of indices and, for each $\lambda \in \Lambda$, A_λ is a set, then we denote by

$$\bigcup_{\lambda \in \Lambda} A_\lambda$$

the set of all elements x such that $x \in A_\lambda$ for at least one $\lambda \in \Lambda$. Likewise,

$$\bigcap_{\lambda \in \Lambda} A_\lambda$$

denotes the set of all elements x such that $x \in A_\lambda$ for all $\lambda \in \Lambda$.

7. The expression $A \setminus B$ denotes the **set-theoretic difference** of A **from** B. The set $A \setminus B$ consists of all elements of B that are not in A. In symbols, $A \setminus B = \{x \in A \mid x \notin B\}$. For example,

$$\{1, 2, 3, 4\} \setminus \{3, 4, 5, 6\} = \{1, 2\}.$$

8. The symbol $\varnothing$ denotes the empty set, which has the property that $x \notin \varnothing$ for any possible element x.

9. The expression $A \times B$ denotes the **Cartesian product** of A and B. The set $A \times B$ is the set of all ordered pairs whose first coordinate is an element of A and whose second coordinate is an element of B. For example,

$$\{1,2\} \times \{3,4\} = \big\{(1,3),(1,4),(2,3),(2,4)\big\}.$$

10. If $A = B$, we sometimes write $A^2 = A \times A$, $A^3 = (A \times A) \times A$, and so on. For example,

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$$

and

$$\mathbb{C}^3 = \mathbb{C} \times C \times C = \{(z_1,z_2,z_3) \mid z_1,z_2,z_3 \in \mathbb{C}\}\,.$$

List of Notation

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