MATH 2306
ORDINARY DIFFERENTIAL EQUATIONS

TIMOTHY E. FAVER

December 5, 2022

Contents

1. Introduction

[T]he world is changing: I feel it in the water,
I feel it in the earth, and I smell it in the air. —*Treebeard, The Lord of the Rings*

1.1. Main goals and key statements.

The goal of this course is to learn how to predict the future. Knowledge of the future is equivalent to answering the *key question*

$$\text{“What are things like at this moment in time?”,} \tag{KQ}$$

where "this moment in time" can be *any* moment in time. Throughout the course, whatever we are doing, a good way to gain perspective is to ask ourselves (KQ) in the context of the particular problem that we are solving.

One plausible answer to the vague question (KQ) is the following *key statement*:

$$\text{“How things are depends on how things were and how things changed.} \tag{KS1}$$

Hopefully a moment's thought and reflection on your own personal life experiences will indicate the plausibility of (KS1).

This is a mathematics course, so let's pose our key question (KQ) and (first) key statement (KS1) in mathematical language and notation — specifically in the language and notation of calculus. First, by "things" we will mean the values of a function — maybe the number of rabbits in a certain geographic region, the percentage of a population that has a disease, or the position of an object relative to some point of origin. Call this function x and let t be its independent variable. Then "how things are" at time t is the value $x(t)$.

Next, as soon as we hear the word "change" in a mathematical context, we probably should think of the derivative. In this course we will typically denote the derivative with "dot" notation: the derivative of the function x at time t is[1]

$$\dot{x}(t) := x'(t) = \frac{dx}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \lim_{\tau \to t} \frac{x(t) - x(\tau)}{t - \tau}.$$

By the way, the symbol τ is the Greek letter "tau," and we will use it often when we want to write something that looks almost like t.

Third, the key statement (KS1) makes reference to "how things were" — that is, to some prior point in time. Suppose that we know "how things were" at some time t_0 in the past, and now we want to know "how things are" at time t, with $t_0 < t$. That is, we know the value $x(t_0)$, and we want to know the value $x(t)$. We can figure this out if we know "how things changed." This will involve the derivative, but not just the derivative in isolation.

Namely, calculus tells us that the net change in "things" from time t_0 to time t is the integral

$$\int_{t_0}^{t} \dot{x}(\tau)\, d\tau.$$

[1] I will frequently use the ":=" notation when defining a quantity for the first time. We should read the sentence "$\dot{x}(t) := x'(t)$" as "$\dot{x}(t)$ is defined to be equal to $x'(t)$," and of course $x'(t)$ is the derivative of x at t.

And, in particular, the fundamental theorem of calculus tells us

$$\int_{t_0}^{t} \dot{x}(\tau)\, d\tau = x(t) - x(t_0).$$

Rearranging, we have

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\tau)\, d\tau. \tag{FTC}$$

The statement (FTC) is the exact mathematical formulation of our first key statement (KS1).

$$\boxed{\text{how things are} \longrightarrow x(t) = \underset{\substack{\uparrow \\ \text{how things were}}}{x(t_0)} + \overbrace{\int_{t_0}^{t} \dot{x}(\tau)\, d\tau}^{\text{how things changed}}}$$

And so it looks like our key question (KQ) is easy to resolve: figure out one value in the past and the derivative, then integrate and add. All too easy!

How do we figure out that derivative? The derivative is "how things are changing" at a moment in time, and some thought about "how things change" should suggest the plausibility of a *second* key statement:

"How things change depends on when change happens and how things are then." (KS2)

For example, consider the growth of a population of rabbits. (We will be doing this *a lot*.) The growth rate probably depends on the amount of food available to the rabbits, and that amount should vary over the course of the year. And if there are more rabbits, it is likely that there are more mating pairs available, and thus even more rabbits to come. But if there are too many rabbits, maybe they will eat all the food, and the population will decline. So, the growth rate of the rabbit population should depend on both time and the number of rabbits.

Thus we might recast our second key statement (KS2) by saying that the derivative should depend on both time (which we know it already does) *and* the value of the original function at that point in time. We write this symbolically as

$$\dot{x}(t) = f(t, x(t)). \tag{ODE}$$

Here f is a function of the *two* variables t and x, and so we will often consider values of the form $f(t, x)$. Using x as both the *dependent* variable of a function that depends on t and as the *independent* variable of a function that also depends on x can lead to no end of confusion, but it's something we'll have to live with.

We have now met the principal object of study in this course: the ordinary differential equation. An **ORDINARY DIFFERENTIAL EQUATION** is just an equation of the form (ODE), where the function f is given to us, and we want to find a function x that solves

(ODE) for all times t at which x is defined. We will spend most of the course solving equations like (ODE), as well as contemplating what "solving" actually means.

For now, here's why we have a problem. The equation (FTC) expresses the values of x in terms of one past value $x(t_0)$ and the derivative $\dot{x}$. But (ODE) says that $\dot{x}$ depends on x! If we combine (FTC) and (ODE), then we get

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))\ d\tau. \tag{FP}$$

The equation (FP) is a "fixed point" equation for x: the function x equals some expression depending on x (and so doing the right hand side to x doesn't change, but rather "fixes," x).

This seems to be useless. To know the value $x(t)$, we have to be able to calculate that integral over the interval $[t_0, t]$, and so we will need to know the value $f(t, x(t))$ — which means we need to know $x(t)$. But that is exactly what we are trying to find.

A course in ordinary differential equations resolves the tension among the statements (FTC)*,* (ODE)*, and* (FP). Specifically, we will learn how to write a variety of worthwhile problems in the form (ODE), or a form like that. (The thing that everyone learns in physics, "force = mass × acceleration," would involve a second derivative on x, not $\dot{x}$.) Then we will learn techniques for *solving* (ODE), which will depend greatly on what f is.

Setting up a differential equation to model a particular phenomenon — figuring out f in (ODE) — is itself a nontrivial task, and so most of our models and phenomena will be "trivial" constructions; the goal here is to get to the differential equations, not model the whole world. "Solving" an equation will involve three related approaches. In the *analytic* approach, we find explicit formulas for solutions to problems; this is probably what we think "solving" an equation means (and for good reason), but only very special equations have explicit solution formulas. In the the *qualitative* approach, we use certain features of problems to guarantee the existence of solutions and then predict their behavior; often knowing how a solution behaves over long times is more useful than knowing its precise formula. In the *numerical* approach, we convert our "continuous" problems to "discrete" ones that a computer can be taught to solve with results that a human brain can be taught to interpret.

This is not to disparage "formulaic" or "symbolic" techniques, and indeed there are a handful (at least four, but probably no more than seven) that you simply *must* know to be successful in this course and whatever requires differential equations afterward. We will certainly strive to develop a robust understanding of those analytic techniques. Let's not kid ourselves, though: a computer can do most of those techniques — but a computer probably can't *interpret* the results of those techniques, or even set up the problem for you in the first place so that it's amenable to those techniques.

Now, if you have to go to a computer for every little thing, like calculating

$$\frac{d}{dt}[t^2] = 2t \quad \text{or} \quad \int \cos(3t)\ dt = \frac{\sin(3t)}{3} + C,$$

then something has gone very wrong with your education. (Harsh but true.) On the other hand, if you don't have a computer to help you with a problem that actually matters to your life and the larger world, then you probably have much bigger problems!

The following *analyst's creed* summarizes our relationship with formulas in this course:

"Having a formula for something is not the same as understanding that thing." (AC)

This is where we finished on Monday, August 15, 2022.

1.2. Three models of population growth.

The previous section was very, and intentionally, vague. What exactly are the "things" that we might study? For a particular kind of "thing," what function f should we use to govern how that thing changes, as in (ODE)?

Here we will practice developing some simple models of "things" and how they change. That is, we will construct some differential equations. The perennial challenge of any mathematical model is maintaining the balance between a *physically realistic* model that captures "enough" of the "real world" and a *mathematically tractable* model that can be analyzed with a reasonable amount of time and effort.

Once we have the models, we will try to learn what solutions to the underlying differential equations do and interpret that behavior in light of the "real world." In particular, we will try to do this without having explicit formulas for solutions, because very often there aren't any! Rather, we will use our knowledge of calculus to divine the behavior of solutions.

Going forward, many of our models will arise because of *proportional relationships* between quantities. Intuitively, two quantities are proportional if one is always a multiple of another; for example, the circumference of every circle is proportional to its radius (equivalently, its diameter). Here is a formalization of this concept for future use.

1.2.1 Definition.

Two time-dependent quantities A and B are **PROPORTIONAL** *if there exists a nonzero real number r such that*

$$A(t) = rB(t)$$

for all t at which A and B are defined.

We will study lots of *population models* because (1) they are fairly easy to construct, (2) many important kinds of differential equations have incarnations as some kind of population model, and (3) we have a good deal of intuition about how populations might grow, so we can test the worth of our eventual solutions. A calculus caveat: tpically when we count populations, we do so with integers. But calculus is inherently continuous, and taking the derivative of an integer-valued function should make us uneasy. We'll always assume that either the population is so large, or our units of measurement are so skewed, that taking noninteger, fractional counts of this population makes sense — like saying that 8.8 million people live in New York City right now.

1.2.1. Exponential growth.

How fast a population is growing depends on many factors. As we noted earlier, a higher population allows for more interactions among members and thus more mating opportunities and thus more offspring; a lower population does the opposite. One very simple model of population growth, then, is to assume that a population's rate of change is directly proportional to the current population. (There are any number of things wrong about this assumption; we'll address those presently, so just suspend disbelief for now.)

Suppose that at time t there are $x(t)$ members of this population, so the rate of change of this population at time t is $\dot{x}(t)$. Then we are assuming that x and $\dot{x}$ are proportional, so Definition 1.2.1 gives us a nonzero real number r such that

$$\dot{x}(t) = rx(t) \tag{1.2.1}$$

for all times t.

The simple equation (1.2.1) will be a source of many ideas and techniques that we will use repeatedly in this course. We will analyze it in the following steps.

1. Right now, the least important thing about (1.2.1) is its analytic solution, which is $x(t) = ce^{rt}$ for any constant c. You probably remember that from calculus, and you can check that this formula works out, but we'll defer figuring out this formula from scratch for some time.

2. Let's check notation. With $r \neq 0$ fixed, define a function f of the two variables t and x by

$$f(t, x) := rx.$$

Note that f is really independent of t. Then the equation (1.2.1) has the form

$$\dot{x}(t) = f(t, x(t)),$$

just as we saw in (ODE).

3. The number r is a **PARAMETER** of the problem (1.2.1) — a number that is constant in a given incarnation of the problem but whose value could change to allow the problem to model different scenarios. Depending on the type of population that we are trying to model with exponential growth, we will probably need different values of r.

4. For simplicity, we often suppress some of the t-dependence in our notation. For example, we write

$$\dot{x} = rx$$

instead of (1.2.1), or

$$\dot{x} = f(t, x).$$

Context will often make clear whether we are referring to x as the independent variable of f or the dependent variable of t.

5. Populations typically don't arise *ex nihilo*. Say that we are tracking the growth of this population from time $t = 0$. (There will be plenty of circumstances when we want to track growth starting at time $t_0 \neq 0$, but we'll do that later.) Assume that we know the initial population: $x(0) = x_0$ for some number x_0. Then we want to solve a more specific problem than (1.2.1): the *pair* of equations

$$\begin{cases} \dot{x}(t) = rx(t) \\ x(0) = x_0. \end{cases} \tag{1.2.2}$$

This pair of equations is not merely an ordinary differential equation but rather an **INITIAL VALUE PROBLEM (IVP)**, since it asks for the solution x of the ODE $\dot{x} = rx$ with the "initial value" $x(0) = x_0$.

6. Again, a moment of calculus tells us that the solution is $x(t) = x_0 e^{rt}$. Say that we didn't know this. What can we learn about the IVP (1.2.2) *without* jumping to a formula for the solution? There will be plenty of times to come when we simply can't find an explicit formula, but we can be assured that, nonetheless, a solution "exists."

So, let's work backward: assume that (1.2.2) has a solution. What do we know about it? This is a safe situation, since we know what solutions to exponential growth really are, and so we can always check our predictions against those sweet, sweet formulas.

7. The data in (1.2.2) are the parameter r and the initial value x_0. We have to start somewhere, so let's assume $r = 0$. Then the differential equation $\dot{x} = rx$ just becomes

$$\dot{x}(t) = 0$$

for all t at which x is defined. This means that x is not changing, and so x is constant:

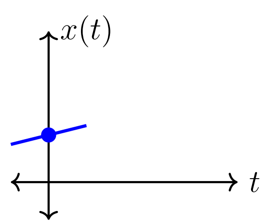
$$x(t) = x(0) = x_0$$

for all times t at which x is defined. And since constant functions are defined at all real numbers, it looks like the solution is just $x(t) = x_0$ for all t.

8. Now suppose $r > 0$. It's reasonable to assume that for any "real" population, the initial number of members is positive: $x_0 > 0$. Then since $r > 0$ and $x_0 > 0$, we have

$$\dot{x}(0) = rx(0) = rx_0 > 0.$$

Thus x is increasing at time $t = 0$; equivalently, the slope of x at $t = 0$ is positive. So, if we look at the graph of x "close" to $t = 0$, it looks like this.



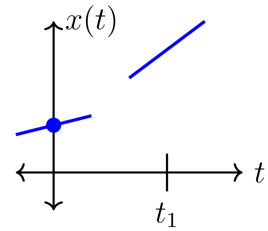
We know more: since x is increasing at time $t = 0$, for a time t_1 "close to" but greater than $t = 0$, we should have $x(0) < x(t_1)$. In particular, $x(t_1) > 0$. Thus

$$\dot{x}(t_1) = rx(t_1) > 0,$$

and so x is again increasing at time $t = t_1$. Moreover, x is increasing *faster* at time $t = t_1$ than at $t = 0$:

$$\dot{x}(t_1) = rx(t_1) > rx(0) = \dot{x}(0).$$

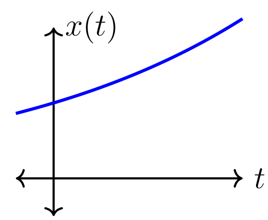
Then if we sketch the slopes of x at both times $t = 0$ and $t = t_1$, we get a picture like this.



We can then iterate this analysis starting at time $t = t_1$ to suggest that x is strictly increasing on its domain. Moreover, we can study the concavity of x by calculating its second derivative, which we denote by $\ddot{x}$. We have

$$\ddot{x}(t) = \frac{d}{dt}[\dot{x}(t)] = \frac{d}{dt}[rx(t)] = r\dot{x}(t) = r(rx(t)) = r^2x(t). \tag{1.2.3}$$

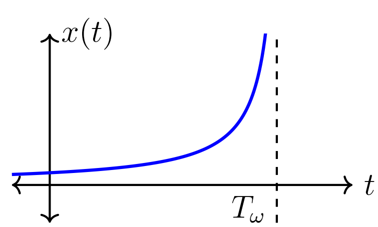
Thus $\ddot{x}(t) > 0$ whenever $x(t) > 0$. Since $x(0) > 0$ and x is strictly increasing, we expect $\ddot{x}(t) > 0$ for all times t at which x is defined. And so x is concave up. Here, then, is a candidate for the graph of x.



In particular, because x is increasing and always getting steeper, we expect the values of x to blow up to ∞ over long times. That is, if x is defined for all times t, we expect

$$\lim_{t \to \infty} x(t) = \infty.$$

There's just one problem: we haven't figured out precisely all those times at which x is defined. Plenty of functions aren't defined for all real numbers; maybe there is some "end" time $t = T_\omega$ at which x fails to be defined. Then the graph of x might have a vertical asymptote at $t = T_\omega$ like the picture below.



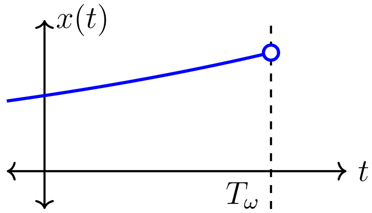
Again, because x is increasing and concave up, we expect that if $T_\omega < \infty$, then we still have the same end behavior as before:

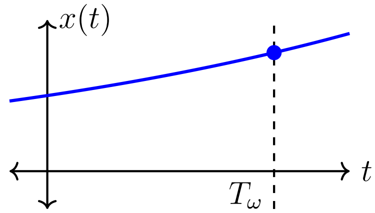
$$L := \lim_{t \to T_\omega^-} x(t) = \infty.$$

We can be reasonably sure that this limit isn't finite, for if $L < \infty$, then

$$\lim_{t \to T_\omega^-} \dot{x}(t) = \lim_{t \to T_\omega^-} rx(t) = rL.$$

Thus x would have finite slope at $t = T_\omega$, and so it looks like we could just continue drawing the graph of x past $t = T_\omega$. But then x could be defined for values of t larger than T_ω, which goes against our assumption above.





9. Since we know that $x(t) = x_0 e^{rt}$ solves (1.2.2), we could have figured all of this out from the formula (and in particular deduced $T_\omega = \infty$), but hopefully this discussion illustrates the power of *qualitative* methods. Without a formula for (1.2.2), we got a pretty good idea of the population's behavior over long times (it explodes). Nonetheless, we did not (1) assure ourselves of the existence of a solution to the problem, (2) figure out if the solution exists for all time or not, and (3) suss out enough information about that solution to speak cogently of its behavior at particular *finite* moments in time. Moreover, even though we can check that the analytic formula $x(t) = x_0 e^{rt}$ solves (1.2.2), we probably don't have the tools at hand to prove that this is the *only* solution. And that's an important step — we don't want two competing formulas giving us two totally different predictions of the future.

10. We did all the analysis above assuming $r \ge 0$ and $x_0 > 0$. If $x_0 = 0$, then

$$\dot{x}(0) = rx(0) = rx_0 = 0,$$

and so the graph of x has a horizontal tangent at $t = 0$. This doesn't tell us if x is increasing, or decreasing, near $t = 0$.

However, if we stare at the differential equation $\dot{x} = rx$ long enough (it's always a good idea to stare long and hard at differential equations), we might see that plugging in $x = 0$ on both sides makes for a true equality. That is, suppose $x(t) = 0$ for all t. Then

$$\dot{x}(t) = \frac{d}{dt}[0] = 0 \quad \text{and} \quad rx(t) = r \cdot 0 = 0.$$

Thus taking $x(t) = 0$ for all t solves $\dot{x} = rx$, and consequently the initial value problem with $x_0 = 0$.

In other words, a nonexistent population that grows exponentially. . .remains nonexistent. In particular,

$$\lim_{t\to\infty} x(t) = 0,$$

a pretty sharp contrast to the case $x_0 > 0$.

This is where we finished on Wednesday, August 17, 2022.

11. Suppose $x_0 < 0$, but keep $r > 0$. From the modeling point of view, a negative population is probably useless, but mathematically it's worth checking out. Let's do the prior analysis more succinctly. We have

$$\dot{x}(0) = rx_0 < 0,$$

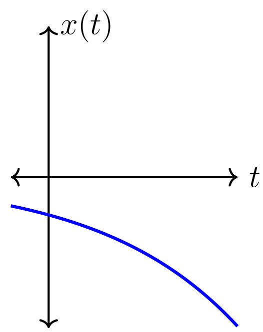
so x is decreasing at time $t = 0$. Then for times t close but greater than 0, we have

$$x(t) < x(0) < 0,$$

so x is decreasing and becoming "more negative." Also, by (1.2.3), we have

$$\ddot{x}(t) = rx(t) < 0,$$

so x is concave down. Then the graph of x might be

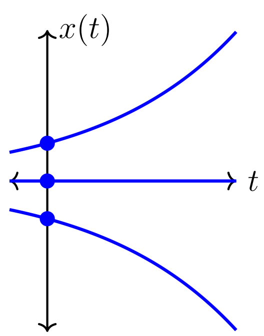


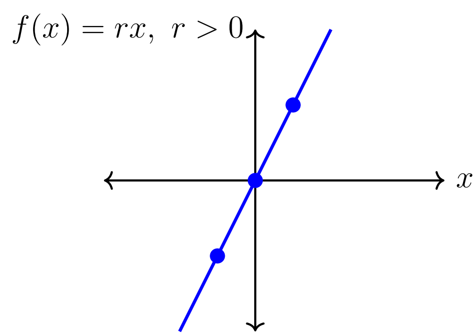
This suggests that, if x is defined for all time,

$$\lim_{t\to\infty} x(t) = -\infty,$$

which is a remarkable change in the end behavior of x from the cases $x_0 > 0$ and $x_0 = 0$!

12. We can summarize all of the work above by graphing three putative solutions to $\dot{x} = rx$ with $r > 0$ and different signs on $x(0) = x_0$. We see that as soon as the initial value $x(0)$ "passes through" the constant solution $x = 0$, the end behavior changes radically.





To the right above we have graphed $f(x) = rx$ with $r > 0$. (Note that in the left graph, the independent variable is t and the dependent variable is x, while on the right x is independent and f is dependent. Notation is a nightmare!) It is no accident that the graphs of increasing solutions x have initial conditions $x(0) = x_0$ where $f(x_0) > 0$, while the graphs of decreasing solutions have initial conditions $x(0) = x_0$ with $f(x_0) < 0$. Also, the constant solution $x = 0$ arose by solving $f(x) = 0$. This suggests a strategy going forward: to gain intuition about the behavior of solutions to a problem $\dot{x} = f(x)$, study f.

1.2.2. Logistic growth.

Exponential growth models, as the name indicates, allow for only two kinds of end behavior for a population: either the population explodes to possess infinitely many members ($x(t) \to \infty$), or it dies off ($x(t) \to 0$). This is wholly unrealistic for many populations, which often exhibit neither kind of extreme behavior. But "real" populations do not just grow in a manner dependent on the current population. Many other factors affect (negatively) the rate of a population's growth: internal conflict/interaction within the population or with another population; limited food, space, or necessary resources; the spread of a disease through the population; harvesting of the population by some outside source; birth control. Pretty much all of these population constraints can be realized as a differential equation, and we'll look at many of them (and the rich underlying theory) throughout the course.

We'll start here by modeling the negative contribution of interaction among members of a population. In doing so, we need a paradigm that will reappear frequently. Suppose that a certain quantity changes via both an "input" source and an "output" source. In terms of population, "input" could be births and "output" could be deaths. Then the rate of change of that quantity satisfies

$$\text{Rate of change} = \text{Rate in} - \text{Rate out}. \tag{RI–RO}$$

To keep things simple, suppose that we have a population that increases via exponential growth. Then the "rate in" is proportional to the existing population; say that the "rate in" term is αx. We are writing α, not r, intentionally. Since we want growth, we take $\alpha > 0$.

Now suppose that the population loses members due to interactions among that population — competition for food, spreading disease, eating each other because they're delicious, any number of horrible things that we do to each other. If there are x members in the population, then any one member of that population can interact with $x - 1$ other members. Then there are $x(x-1)$ possible interaction pairings in that population and thus $x(x-1)/2$ *distinct*[2] possible interaction pairings. If the attrition rate due to interaction is proportional to the number of distinct interaction pairings, then the "rate out" term is $\beta x(x-1)/2$ for

[2] Broadly, this follows from the counting principle that if you choose one option from among m options, and then choose another option from among n options, you can make mn choices. We choose one member from among the x members in the population, and then we choose one member from the remaining $x - 1$ members who are not that first member. This gives us $x(x-1)$ choices. But because an interaction of Member A with Member B is probably (though not necessarily) the same as an interaction of Member B with Member A, we divide by 2 so we don't overcount. Honestly, because of the constant of proportionality β that we introduce below, the factor of $1/2$ is not hugely important. If this confuses/excites you, go study discrete math.

some constant $\beta > 0$. Thus the population's rate of change is

$$\dot{x} = \alpha x - \frac{\beta x(x-1)}{2}. \tag{1.2.4}$$

It will be convenient (trust me!) to factor this as

$$\dot{x} = rx\left(1 - \frac{x}{N}\right), \qquad r := \frac{2\alpha + \beta}{2}, \qquad N := \frac{2\alpha + \beta}{\beta}. \tag{1.2.5}$$

Go ahead and do the algebra to check that (1.2.4) and (1.2.5) are the same. The differential equation in (1.2.5) is the **LOGISTIC EQUATION**; note that $r > 0$ and $N > 0$.

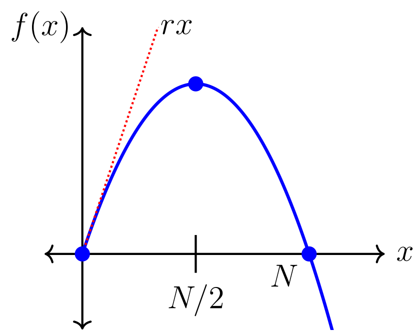
What do solutions to the logistic equation do? What we know from calculus probably won't give us a clue as to *formulas* for solutions, so let's use some of the qualitative ideas that we developed for exponential growth to get some ideas.

We first abbreviate

$$f(x) = rx\left(1 - \frac{x}{N}\right), \tag{1.2.6}$$

so the logistic equation is just $\dot{x} = f(x)$. Good luck solving this as easily as $\dot{x} = rx$!

But motivated by exponential growth, we might expect the end behavior of a solution to $\dot{x} = f(x)$ to relate to the sign of f at the initial condition. That is, is $f(x_0)$ positive, negative, or zero? If we stare at the formula for f for a while (and maybe take some "easy" values of the parameters r and N, like $r = 1$ and $N = 1$), eventually we see that the graph of f always looks like the sketch below.



So, f is positive on the interval $(0, N)$ and negative on $(-\infty, 0)$ and (N, ∞). Moreover, $f(0) = f(N) = 0$. This tells us the values of x (not t!) at which $\dot{x}$ is positive and negative and thus where x is increasing and decreasing.

With this in mind, we analyze the behavior of solutions depending on where their initial condition $x(0) = x_0$ falls.

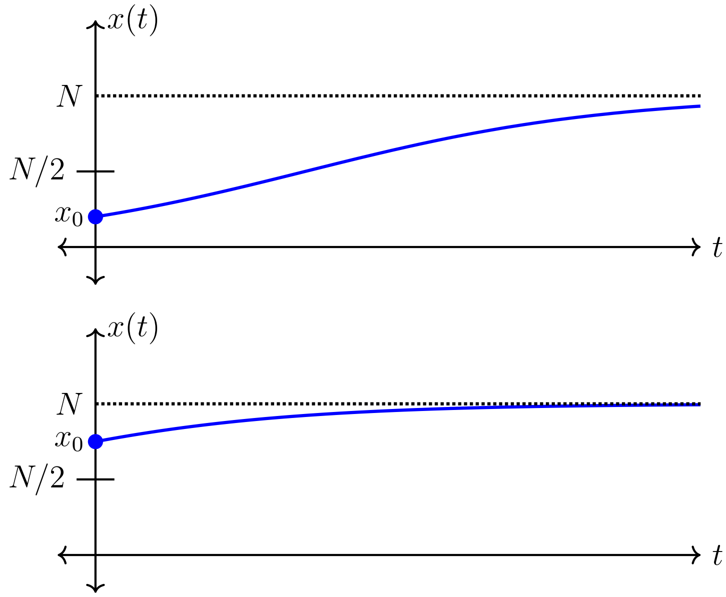
Case 1: $0 < x_0 < N$. If $0 < x_0 < N$, then $\dot{x}(0) > 0$, and so x is increasing at time $t = 0$. As usual, for $t > 0$ but close to 0, we have $x(t) > x(0) = x_0$. As long as $x(t) < N$, we will have $f(x(t)) > 0$, and so $\dot{x}(t) > 0$, and so x is increasing at time t.

Can x increase forever, like exponential growth, and explode to ∞? As the values of x get closer to N, we see from the graph of f that the values of $f(x)$ get closer to 0, and so $\dot{x}$

gets closer to 0. That is, the graph of x gets flatter as it gets closer to N — maybe N is a horizontal asymptote of x and

$$\lim_{t\to\infty} x(t) = N,$$

at least if x is defined for all times t. Consequently, it's reasonable that the graph of x with $0 < x(0) < N$ could look like one of the sketches below. (There is a subtle distinction depending on whether $0 < x_0 < N/2$ or $N/2 < x_0 < N$. Can you figure it out? It involves concavity.)



We'd need to do a lot more work to figure out (1) the values of t for which a solution exists and (2) if x could ever hit N or go beyond.

This is where we finished on Friday, August 19, 2022.

Case 2: $x_0 = N$. Then $\dot{x}(0) = f(x(0)) = f(N) = 0$, so x is neither increasing or decreasing at $t = 0$. However, we saw this before with exponential growth, and so we might expect that the constant function $x(t) = 0$ solves the logistic equation for all times t.

$$\dot{x}(t) = \frac{d}{dt}[N] = 0$$

for all t. On the other hand,

$$f(x(t)) = f(N) = 0$$

for all t. Indeed, $\dot{x}(t) = f(x(t))$ for all t.

Now, this *doesn't* rule out the possibility that there is another solution — call it y — to the logistic equation with $y(0) = 0$ and $y(t) \neq 0$ for some $t \neq 0$. We need deeper theory to prove that. But at least we have *a* solution.

You can imagine that other cases include $x_0 > N$ and, less physically realistic (but still mathematically interesting) $x_0 \leq 0$. Can you try to analyze those cases based on whether x is increasing or decreasing at $t = 0$?

The two cases above suggest that the parameter N is somehow special in this model. If the initial population is positive but less than N, then the population seems to tend toward N in the long run; if the population starts at N, the model allows (perhaps compels!) it to stay at N forever. The value N is called the **carrying capacity** of the logistic model, and it represents a sort of "ideal" population that is in balance with the existing competition among members.

I hope the analysis so far makes sense (all those properties of increasing/decreasing functions from calculus), but I hope you're also starting to get annoyed. We are using weasel words such as "seems like" or "should" or "can imagine," but we haven't done anything too mathematically rigorous. In particular, for the logistic equation we have no firm evidence that solutions even exist in the case of $0 < x(0) < N$! We will slowly and surely remedy these annoyances. Before doing so, we consider one final population model.

1.2.3. A silly model of time-dependent population growth.

Our exponential and logistic growth models have been time-independent in the sense that both (1.2.1) and (1.2.6) have the form

$$\dot{x} = f(x), \tag{1.2.7}$$

where f does not depend on time t. (Of course, the solution x definitely depends on t.) The rate of change of x depends only on the value of x, not at the moment in time at which we are consider x. For example, if x satisfies (1.2.7) and $x(50) = x(100)$, then $\dot{x}(50) = \dot{x}(100)$. The fact that we are considering the rate of change at times 50 and 100 is irrelevant.

However, there are plenty of reasons to consider population growth with time affecting the rate of growth — fertility or mating cycles, weather, and outside harvesting patterns (crop harvesting, hunting season) can all affect growth rate in different ways at different times. One very simple way to incorporate time-dependence into the growth rate is to allow, contrary to Definition 1.2.1, time-dependence in the constant of proportionality. Say that a population's growth rate satisfies

$$\dot{x}(t) = r(t)x(t),$$

where now the "constant" of proportionality r can vary as a function $r(t)$ of time. This could allow a faster growth rate at some times (those times t when $r(t)$ is relatively large) and a slower growth rate, or even a decline in growth, at other times (when $r(t)$ is a small positive number, or maybe a negative number).

Taking

$$f(t,x) := r(t)x,$$

we can write this growth rate as

$$\dot{x} = f(t,x),$$

as usual. However, unlike the exponential and logistic models, the function f here now depends on time t, not just the "state" x.

We will not analyze this model in any detail right now; the point is just to convince you that time-dependent differential equations are worth considering. And consider them we shall — a lot.

1.3. Fundamental terminology and guiding questions.

So far, we have seen three problems that reasonably could be called differential equations:

$$\dot{x} = rx, \qquad \dot{x} = rx\left(1 - \frac{x}{N}\right), \qquad \text{and} \qquad \dot{x} = r(t)x.$$

All of these problems are equations (there's an $=$) involving derivatives (there's a $\cdot$), and more precisely they have the form of the equation (ODE). If we are going to talk sensibly about differential equations, we should have a precise vocabulary to set expectations and eliminate ambiguities.

1.3.1. The true definition of an ODE.

1.3.1 Definition.

An **ORDINARY DIFFERENTIAL EQUATION (ODE)** *is an equation of the form*

$$\dot{x} = f(t, x), \tag{1.3.1}$$

where f is a function defined for t in some interval (a, b) and x in some interval (c, d). The values $a = -\infty$, $c = -\infty$, $b = \infty$, and $d = \infty$ are all allowed.

A **SOLUTION** *to the equation* (1.3.1) *is a differentiable function x defined on an interval I such that*

$$\dot{x}(t) = f(t, x(t)) \tag{1.3.2}$$

for all x in I and that $\dot{x}$ is continuous on I. (For (1.3.2) *to make sense, tacitly we require I to be a subinterval of (a, b) and $x(t)$ to belong to (c, d).)*

Why *this* definition? Definitions are not handed down to us from on high, even though it often looks like that; definitions exist because, over time and after thought, people come to realize that those definitions are the best way to capture a concept. I hope we can all agree that the "pointwise" condition (1.3.2) is essential.

Why does a solution have to be defined on an interval? Remember that we should be thinking of t as time. If our model only predicts the future from, say, 9 am to 12:19 pm, and then again from 1:11 pm to 5 pm, that would be a pretty strange model — it just stops working for one hour during the day. Requiring the solution to an ODE to be defined on an interval helps keep the flow of time unbroken. Do not neglect consideration of the domain of a solution to an ODE; often our solutions will end up defined for all time in $(-\infty, \infty)$, but not always. *Whenever you find a formula for a solution to a differential equation — pause and think about the analyst's creed* (AC) *— you must be able to state its domain.*

Why should the derivative be continuous? Most of the time, things not only change continuously; their rate of change evolves continuously. (Not always: flip a switch.) Requiring $\dot{x}$ to be continuous affords our model extra control over reality.

1.3.2 Example.

Which of the following is an ODE?

(i) $2t\dot{x} + t^2 x = \sin(t)$

(ii) $(\dot{x})^2 = x$

Solution. An equation involving a function and its derivative is an ODE only if we can rewrite it in the form (1.3.1). Algebraically, we must be able to isolate[3] $\dot{x}$.

(i) It is fairly easy to solve for $\dot{x}$, I think:

$$\dot{x} = \frac{\sin(t) - t^2 x}{2t} =: f(t, x).$$

And so this equation is an ODE. Note that f is not defined at $t = 0$.

(ii) If we have $(\dot{x})^2 = x$, then we get either $\dot{x} = \sqrt{x}$ or $\dot{x} = -\sqrt{x}$. We do not have a single "formula" for $\dot{x}$, and so this is not an ODE. If we were given the problem $\dot{x} = \sqrt{x}$, then, yes, that's an ODE, but the $\pm$ ambiguity of $(\dot{x})^2 = x$ ruins things. ▲

There is something very comforting about differential equations: we can pretty much always check our work. In general, when someone asks you to "check" that a certain function solves an ODE, you do not have to come up with the solution from scratch and get what they got. Rather, plug and chug.

1.3.3 Example.

Check that the function $x(t) = e^{\cos(t)}$ *solves the ODE* $\dot{x} = -\sin(t)x$.

Solution. We differentiate

$$\dot{x}(t) = \frac{d}{dt}[e^{\cos(t)}] = e^{\cos(t)}\frac{d}{dt}[\cos(t)] = e^{\cos(t)}[-\sin(t)].$$

Then we rearrange:

$$\dot{x}(t) = [-\sin(t)]e^{\cos(t)} = -\sin(t)x(t).$$

And so x is a solution. ▲

As we saw in our population models, most of the time in a "physical" scenario we do not meet just an ODE by itself, but rather one with an initial condition appended.

[3] I don't want to say "solve for $\dot{x}$," because the goal will be to solve for x, really.

1.3.4 Definition.

An **INITIAL VALUE PROBLEM (IVP)** *is a pair of equations of the form*

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0, \end{cases} \tag{1.3.3}$$

where $\dot{x} = f(t,x)$ is an ODE, and t_0 and x_0 are given real numbers with f defined at $t = t_0$. A **SOLUTION** *to the IVP* (1.3.3) *is a function x that solves the ODE $\dot{x} = f(t,x)$ in the sense of Definition 1.3.1, that is defined at $t = t_0$, and that satisfies $x(t_0) = x_0$.*

1.3.2. Direct integration.

It turns out that you already know how to solve a huge class of ODE. Suppose that in the equation $\dot{x} = f(t,x)$ the function f is independent of x. By virtue of making it through calculus, you are an expert in solving $\dot{x} = f(t)$.

1.3.5 Example.

Find all functions x such that $\dot{x} = t$.

Solution. This is asking us to find all functions x that differentiate to t. In other words, what are all the antiderivatives of t? They are

$$\int t\ dt = \frac{t^2}{2} + C,$$

where C is a constant independent of t. In other words, if $\dot{x} = t$, then there is a constant C such that

$$x(t) = \frac{t^2}{2} + C$$

for all t. By the way, the domain of x is always $(-\infty, \infty)$, no matter what C is. Since changing the value of C changes the value of x, there are infinitely many solutions to this ODE; specifically, changing C shifts the graph of the solution up or down the x-axis. ▲

I have a love/hate relationship with the indefinite integral, and by the end of the course, I hope you do, too. Recall that the symbol

$$\int f(t)\ dt$$

denotes the *set* of all functions whose derivative is the function f, i.e., the set of antiderivatives of f. If F is one antiderivative of f, then calculus tells us that any other antiderivative G has the form $G(t) = F(t) + C$ for some constant C. Even though the symbol $\int f(t)\ dt$ denotes a set, we still say things like

$$x(t) = \int f(t)\ dt \tag{1.3.4}$$

and treat x as a single function. This leads to all sorts of notational unhappiness. For example, we typically agree that changing the variable of integration in an indefinite integral doesn't matter:

$$\int f(t)\ dt = \int f(u)\ du.$$

But then if we try to define a function x by (1.3.4), we get something like

$$x(t) = \int f(t)\ dt = \int f(u)\ du = x(u).$$

So $x(t) = x(u)$?!

We can avoid this sort of confusion via two strategies: (1) *Don't think too hard about it* and (2) *Use a definite integral whenever possible.* Definite integrals play the precise role of antiderivatives thanks to the fundamental theorem of calculus (FTC).

1.3.6 Theorem (FTC).

Let f be continuous on the interval $[a, b]$ and define, for t in $[a, b]$,

$$F(t) := \int_a^t f(\tau)\ d\tau.$$

Then F is differentiable on $[a, b]$ and $\dot{F}(t) = f(t)$ for all t in $[a, b]$.

We can use the FTC to solve, once and for all, every ODE of the form $\dot{x} = f(t)$. More exactly, we will do this for the related IVP.

1.3.7 Theorem (Direct integration).

Let f be continuous on the interval I, let t_0 be a point in I, and let x_0 be a real number. Then the only solution to the IVP

$$\begin{cases} \dot{x} = f(t) \\ x(t_0) = x_0 \end{cases} \tag{1.3.5}$$

is

$$x(t) = x_0 + \int_{t_0}^t f(\tau)\ d\tau.$$

Moreover, x is defined on all of I.

Proof. First let's verify that x as defined is a solution. We use the FTC to calculate

$$\dot{x}(t) = \frac{d}{dt}\left[x_0 + \int_{t_0}^t f(\tau)\ d\tau\right] = 0 + \frac{d}{dt}\left[\int_{t_0}^t f(\tau)\ d\tau\right] = f(t).$$

Then we use properties of integrals to calculate

$$x(t_0) = x_0 + \int_{t_0}^{t_0} f(\tau)\ d\tau = x_0 + 0 = x_0.$$

Finally, suppose that y also solves the IVP (1.3.5). We need to show that $y(t) = x(t)$ for all t. Since $\dot{y} = f(t)$, we can use the FTC to write

$$y(t) - y(t_0) = \int_{t_0}^{t} \dot{y}(\tau)\, d\tau = \int_{t_0}^{t} f(\tau)\, d\tau,$$

and thus, since $y(t_0) = x_0$,

$$y(t) = y(t_0) + \int_{t_0}^{t} f(\tau)\, d\tau = x_0 + \int_{t_0}^{t} f(\tau)\, d\tau = x(t). \qquad \square$$

Theorem 1.3.7 is our first *existence and uniqueness* result, and one of roughly two that we will be able to prove ourselves without breaking too much of a sweat. This theorem tells us that an IVP has a solution (existence) and that there is only one possible solution (uniqueness). Additionally, the theorem tells us explicitly the domain of the solution. Subsequent existence and uniqueness theorems will assure us that solutions to more complicated ODE exist and that if we incorporate an initial condition, then the associated IVP has only one solution. However, we will often have to work harder to find domains.

1.3.8 Example.

Find all solutions to the IVP

$$\begin{cases} \dot{x} = e^{t^2} \\ x(0) = 0. \end{cases}$$

Solution. The direct integration method tells us that the *only* solution is

$$x(t) = 0 + \int_0^t e^{\tau^2}\, d\tau = \int_0^t e^{\tau^2}\, d\tau.$$

We cannot make further progress on simplifying that integral using "elementary" functions, however. ▲

Direct integration is a "complete success" story: we have a formulaic method for solving any direct integration problem, and we know exactly which functions are solutions. (Of course, we may not be able to do the "symbolic" integration easily, or at all!) We will have relatively few "complete successes" in this course, so we should cherish them when we find them.

However, let us recite the analyst's creed (AC) once again: just because we have an integral formula for the solution to a direct integration problem doesn't mean we're done. The integral can help tremendously, but important questions about predicting the future remain.

This is where we finished on Monday, August 22, 2022.

1.3.3. The guiding questions.

Motivated by some of our evocative, but ultimately imprecise, calculus-based analysis of population models, and spurred by our success with direct integration, here are the fundamental questions of our course. They are all deeper, more nuanced versions of our original key question (KQ), now couched in the framework of ODE.

1. Do solutions to an ODE (IVP) exist? If so, how do we know that they exist? Our profession of the analyst's creed (AC) notwithstanding, is there a procedure for finding formulas for those solutions? Most broadly, *does this ODE (IVP) model allow us to predict the future?*

2. Are solutions to an ODE (IVP) unique? (We have infinitely many solutions in Example 1.3.5 but only one solution in Theorem 1.3.7 — but the latter involved an IVP.) If we have found a solution, is it the only one? Most broadly, *does this ODE (IVP) model predict only one*[4] *future?*

3. If solutions exist to an ODE (IVP), where are they defined? What is their domain? Are they defined for all real numbers[5] in $(-\infty, \infty)$ or strictly for a subinterval? Most broadly, *for how long does this ODE (IVP) model allow us to predict the future?*

4. What do solutions to an ODE (IVP) do at the limit of their existence? For example, if a solution x is defined for all time $t \geq 0$, does the limit $\lim_{t\to\infty} x(t)$ exist as a finite real number, or as an extended real number ($\pm\infty$)? If the solution does not have a limit at ∞, is it *asymptotic* to some more familiar function x_∞ in the sense that $\lim_{t\to\infty}[x(t) - x_\infty(t)] = 0$? If a numerical or asymptotic limit exists, can we quantify "how fast" x approaches that limit? And if a solution is only defined up to some finite time T_ω, what happens as t gets close to T_ω? Is there some "breakdown" of the model at T_ω? Most broadly, *what happens in the future?!*

[4] Or will we commit crimes against the Sacred Timeline by getting multiple solutions and predicting multiple futures?

[5] For all time. Always.

2. First-Order Equations

The ODE $\dot{x} = f(t,x)$ presented in Definition 1.3.1 lacked one fundamental word that I intentionally omitted at the time: it's a **first-order** ODE, because there is only a first-order derivative in the problem. We will spend a great deal of time in this course on first-order problems and then mostly focus on second-order problems, which, as you can imagine, involve two derivatives ($\ddot{x}$). We have already seen that first-order equations serve as reasonable population models, and most of the other first-order models that we'll consider will also be population-based. First-order equations have tremendous value as models and are also rich in accessible theory; in particular, many of the results that we develop for first-order equations will motivate and illuminate techniques for second-order problems and beyond.

2.1. Separation of variables.

The three population models of Section 1.2 were

$$\dot{x} = rx, \qquad \dot{x} = rx\left(1 - \frac{x}{N}\right), \qquad \text{and} \qquad \dot{x} = r(t)x.$$

Each of these models can be written in the form

$$\dot{x} = g(t)h(x).$$

Note that for exponential and logistic growth we have $h(x) = 1$, while in the third model $h(x) = x$. Such differential equations have a special name and a special analytic solution technique associated with them.

2.1.1 Definition.

An ODE $\dot{x} = f(t,x)$ is **separable** *if f has the special form $f(t,x) = g(t)h(x)$ for two functions g and h. That is, a separable ODE is an equation of the form*

$$\dot{x} = g(t)h(x).$$

Separable differential equations are "separable" because they "separate" the t- and x-dependencies very precisely. Separable equations arise in many models, including but not limited to the population ones above, and there is a reasonably successful technique for solving them analytically (i.e., with formulas). It turns out that solving the population models above will introduce some distracting technical challenges, so we'll begin with a tamer toy problem.

2.1.1. A "toy" separable problem.

Consider the ODE

$$\dot{x} = \frac{e^t}{x^2}. \tag{2.1.1}$$

We can rewrite the right side as the product

$$\frac{e^t}{x^2} = e^t\left(\frac{1}{x^2}\right),$$

and so we see that (2.1.1) is separable with

$$g(t) = e^t \quad \text{and} \quad h(x) = \frac{1}{x^2}.$$

A bad way of proceeding is to imitate our prior success with direct integration and just integrate both sides:

$$\dot{x} = \frac{e^t}{x^2} \Longrightarrow \int \dot{x}(t)\ dt = \int \frac{e^t}{x(t)^2}\ dt \Longrightarrow x(t) = \int \frac{e^t}{x(t)^2}\ dt + C.$$

Now x appears on both sides of the equation, but we have no way of evaluating that integral…because we don't know what x is! In particular, don't try to factor $x(t)$ out of the integral; you can't, because $x(t)$ depends on t and isn't constant.

A better way of proceeding is to try to *turn the given problem into a direct integration problem.* The following steps may not feel obvious, but they're the right ones. Observe that (2.1.1) is the same as

$$x^2\dot{x} = e^t, \tag{2.1.2}$$

once we multiply both sides by x^2. Now, stare at the left side for a while, and maybe replace $\dot{x}$ with the more familiar Leibniz notation:

$$x^2\dot{x} = x^2\frac{dx}{dt}.$$

When in our prior calculus lives did we see the derivative appear as a factor in a product? I claim that this happens all the time when you do the chain rule. So, how can we make the function

$$x^2\frac{dx}{dt}$$

look like a chain rule derivative of something? We know a lot about polynomials, and x^2 shows up in the derivative of x^3. Just be careful in that x depends on t:

$$\frac{d}{dt}[x(t)^3] = 3x(t)^2\frac{dx}{dt}.$$

This is not quite what we have in our toy problem. But we can move that 3 around:

$$\frac{d}{dt}\left[\frac{x(t)^3}{3}\right] = x(t)^2\frac{dx}{dt}. \tag{2.1.3}$$

This is a major breakthrough: our toy problem (2.1.1) is the same as

$$\frac{d}{dt}\left[\frac{x(t)^3}{3}\right] = e^t.$$

And this equation is saying that the derivative of $x^3/3$ is just e^t. So let's pop an integral on both sides:

$$\int \left(\frac{d}{dt}\left[\frac{x(t)^3}{3}\right]\right)\ dt = \int e^t\ dt.$$

We can do the integral on the right immediately:

$$\int e^t \, dt = e^t + C_1.$$

There are going to be a couple of constants of integration before this is all over, and for now I want to distinguish them, thus C_1, not C.

The integral on the left is also easy *because we are integrating a derivative*:

$$\int \left(\frac{d}{dt} \left[\frac{x(t)^3}{3} \right] \right) \, dt = \frac{x(t)^3}{3} + C_2.$$

And so we get

$$\frac{x(t)^3}{3} + C_2 = e^t + C_1. \tag{2.1.4}$$

Our first move will be to combine these two constants of integration into one: put $K = C_1 - C_2$. Since C_1 and C_2 are arbitrary numbers, so is K. (You did this all the time in calculus when combining a bunch of indefinite integrals, right?) Then

$$\frac{x(t)^3}{3} = e^t + K.$$

Now we will start to solve for x:

$$x(t)^3 = 3e^t + 3K.$$

Again, since K is arbitrary, so is $3K$: put $C = 3K$. Thus

$$x(t)^3 = 3e^t + C.$$

Finally, we can solve for x by taking the cube root:

$$x(t) = [3e^t + C]^{1/3}.$$

And the cube root is defined for any number (not like those nasty square roots).

So, it looks like we have solved for x. Of course, you can, and should, check by differentiating x that we got the right answer. I'm not going to do it here, but please do as I say, not as I do.

I think that all our success came from two ideas: "separating" variables to go from (2.1.1) to (2.1.2), and then recognizing the chain rule correctly in (2.1.3). How could we see that antiderivative if we didn't know it was there in the first place? We know that the product $x^2\dot{x}$ should be the derivative of *something*. . .so let's just introduce that something via an indefinite integral. Specifically, integrate both sides of (2.1.2) to get

$$\int \left(x(t)^2 \dot{x}(t) \right) \, dt = \int e^t \, dt.$$

I claim that the integral on the left is just made for u-substitution: with $u = x(t)$, we get $du = \dot{x}(t) \, dt$, and so

$$\int \left(x(t)^2 \dot{x}(t) \right) \, dt = \int u^2 \, du = \frac{u^3}{3} + C = \frac{x(t)^3}{3} + C.$$

This gets us, effectively, to (2.1.4).

2.1.2. The general method of separation of variables.

Here we distill the results from our toy problem into a more general method for solving separable differential equations. The third column is a reference to the steps from this toy problem.

1. Obtain a separable problem.	$\dot{x} = g(t)h(x)$	$\dot{x} = e^t \left(\frac{1}{x^2}\right)$
2. Separate variables.	$\frac{1}{h(x)}\dot{x} = g(t)$	$x^2\dot{x} = g(t)$
3. Integrate with respect to t.	$\int \frac{1}{h(x(t))}\dot{x}(t)\ dt = \int g(t)\ dt$	$\int x(t)^2\dot{x}(t)\ dt = \int e^t\ dt$
4. Change variables in x.	$\int \frac{1}{h(x(t))}\dot{x}(t)\ dt = \int \frac{dx}{h(x)}$	$\int x(t)^2\dot{x}(t)\ dt = \int x^2\ dx$
5. Try to evaluate integrals.	$\int \frac{dx}{h(x)} =?,\ \int g(t) =?$	$\int x^2\ dx = \frac{x^3}{3} + C,$ $\int e^t\ dt = e^t + C$
6. Try to solve for x.	???	$\frac{x^3}{3} = e^t + C$ $\Longrightarrow x(t) = [3e^t + C]^{1/3}$

2.1.2 Example.

Use separation of variables to produce solutions to

$$\dot{x} = e^t x^2.$$

Solution. This problem is separable, with $g(t) = e^t$ and $h(x) = x^2$. We separate variables:

$$\frac{1}{x^2}\dot{x} = e^t.$$

We integrate with respect to t:

$$\int \frac{1}{x(t)^2}\dot{x}(t)\ dt = \int e^t\ dt.$$

We change variables in the first integral:

$$\int \frac{1}{x(t)^2}\dot{x}(t)\ dt = \int \frac{dx}{x^2}.$$

We evaluate the integrals

$$\int \frac{dx}{x^2} = \int x^{-2}\, dx = -x^{-1} + C \quad \text{and} \quad \int e^t\, dt = e^t + C.$$

We try to solve for x, starting from

$$-x^{-1} = e^t + C.$$

We find

$$x^{-1} = -e^t + C,$$

and thus

$$x(t) = \frac{1}{-e^t + C}.$$

Each value of C gives us a different solution to the ODE. ▲

2.1.3 Example.

Solve the IVP

$$\begin{cases} \dot{x} = e^t x^2 \\ x(0) = 1. \end{cases}$$

Solution. We already know that defining x by

$$x(t) = \frac{1}{-e^t + C}$$

with C constant produces solutions to the ODE. We need to choose C to meet the initial condition. We want

$$1 = x(0) = \frac{1}{-e^0 + C} = \frac{1}{C - 1},$$

and so C has to satisfy

$$C - 1 = 1.$$

Thus $C = 2$, and the solution is

$$x(t) = \frac{1}{-e^t + 2}.$$ ▲

This is where we finished on Wednesday, August 24, 2022.

The analyst's creed (AC) reminds us that we should never just accept a formula and consider ourselves done. Let's answer some of the guiding questions for the problem that we just studied.

2.1.4 Example.

(i) *With $x(t) = 1/(-e^t + C)$ as the solution that we obtained in Example 2.1.2, what is the domain of x?*

(ii) *If x is defined on an interval of the form (t_0, ∞), what can you say about $\lim_{t\to\infty} x(t)$?*

Solution. **(i)** This function x is defined at all t such that $-e^t + C \neq 0$. That is, we need $e^t \neq C$. If $C \leq 0$, then $e^t \neq C$ for all t, in which case the domain of x can be $(-\infty, \infty)$. If $C > 0$, then x is undefined when $e^t = C$, which happens exactly when $t = \ln(C)$. (Note that $\ln(C)$ is not defined for $C \leq 0$.) Thus for $C > 0$, the domain of x can be any interval[6] that does not contain $\ln(C)$. The two "biggest" such intervals are $(-\infty, \ln(C))$ and $(\ln(C), \infty)$.

(ii) The preceding work shows that x is defined on $(-\infty, \infty)$ if $C \leq 0$ and on $(\ln(C), \infty)$ if $C > 0$. In either case, then, it makes sense to ask about $\lim_{t\to\infty} x(t)$. We think about calculus and obtain

$$\lim_{t\to\infty} e^t = \infty \Longrightarrow \lim_{t\to\infty} -e^t = -\infty \Longrightarrow \lim_{t\to\infty} -e^t + C = -\infty \Longrightarrow \lim_{t\to\infty} \frac{1}{-e^t + C} = 0.$$ ▲

2.1.3. Equilibrium solutions.

We are now going to "break" the good results from Example 2.1.2 in an illustrative way.

2.1.5 Example.

Can you use the solutions $x(t) = 1/(-e^t + C)$ to $\dot{x} = e^t x^2$ from Example 2.1.2 to solve the IVP

$$\begin{cases} \dot{x} = e^t x^2 \\ x(0) = 0? \end{cases}$$

Solution. Let's try. We want to find C such that

$$0 = x(0) = \frac{1}{-e^0 + C} = \frac{1}{-1 + C}.$$

If we multiply both sides by $-1 + C$, we get $0 = 1$, which is impossible. Thus there is no way to choose C to meet the initial condition $x(0) = 0$.

However, we might recall what happened with our introductory population models, where we saw that an initial population of 0 could remain 0 for all time and satisfy the model. If we put $x(t) = 0$ for all t, then we have both

$$\dot{x}(t) = \frac{d}{dt}[0] = 0 \quad \text{and} \quad e^t x(t)^2 = e^t \cdot 0^2 = 0,$$

and so $x(t) = 0$ solves the ODE $\dot{x} = e^t x^2$ with $x(0) = 0$. That is, $x(t) = 0$ solves the IVP. ▲

[6] Recall that we demand that the domain of a solution to a differential equation be an interval.

How can we reconcile this "missing" solution $x(t) = 0$ with the ones that we found from separation of variables? The answer lies in a division problem. To separate variables, we assumed that x solved $\dot{x} = e^t x^2$, and then we divided by x^2 to find

$$\frac{1}{x^2}\dot{x} = e^t.$$

From there, we integrated. However, we have no right to divide if $x^2 = 0$! So, when we did divide by x^2, tacitly we were assuming $x^2 \neq 0$ (and thus $x \neq 0$). This caused us to "miss" the solution $x(t) = 0$.

The following is an abstraction of how we can avoid this problem in the future.

2.1.6 Theorem.

Let g and h be functions and suppose that x_* is a number such that $h(x_*) = 0$. Then the function $x(t) := x_*$ solves the separable problem $\dot{x} = g(t)h(x)$. Such a solution is an **EQUILIBRIUM** *solution.*

Proof. We calculate

$$\dot{x}(t) = \frac{d}{dt}[x_*] = 0 \quad \text{and} \quad g(t)h(x(t)) = g(t)h(x_*) = g(t) \cdot 0 = 0.$$

Thus $\dot{x}(t) = g(t)h(x(t))$ for all t. (Note that we did not need to know anything about g.) $\square$

Here is the new rule of law: *from now on, when we separate variables, we must check for equilibrium solutions.*

2.1.7 Example.

Use separation of variables to find solutions to

$$\dot{x} = t\cos^2(x).$$

If a solution is defined on an interval of the form (t_0, ∞), what is its limit at ∞?

Solution. This equation is separable, since it has the form $\dot{x} = g(t)h(x)$ with $g(t) = t$ and $h(x) = \cos^2(x)$. We first check for equilibrium solutions: $h(x) = 0$ if and only if $\cos(x) = 0$. We recall (or look up) that the roots of the cosine are $x = (2k+1)\pi/2$ for any integer k (i.e., $k = 0, \pm 1, \pm 2, \ldots$). Thus the equilibrium solutions are $x = (2k+1)\pi/2$ for k an integer. Of course, equilibrium solutions are constant for all time, so that is their behavior as $t \to \infty$.

Now we assume that $\cos^2(x) \neq 0$ and separate to find

$$\frac{1}{\cos^2(x)}\dot{x} = t.$$

We integrate both sides with respect to t to get

$$\int \frac{1}{\cos^2(x)}\dot{x}\ dt = \int t\ dt.$$

The integral on the right is
$$\int t\ dt = \frac{t^2}{2} + C.$$
We change variables in the integral on the left to find
$$\int \frac{1}{\cos^2(x)}\dot{x}\ dt = \int \frac{dx}{\cos^2(x)} = \int \sec^2(x)\ dx = \tan(x) + C.$$
Thus
$$\tan(x) = \frac{t^2}{2} + C,$$
and so
$$x = \arctan\left(\frac{t^2}{2} + C\right).$$
Note that the arctangent is defined for all real numbers, and so the domain of x here is $(-\infty, \infty)$.

Last, since $\lim_{X\to\infty} \arctan(X) = \pi/2$ and $\lim_{t\to\infty} t^2 = \infty$, we have
$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \arctan\left(\frac{t^2}{2} + C\right) = \infty. \qquad \blacktriangle$$

This is where we finished on Friday, August 26, 2022.

2.1.4. Some unhappier examples.

Separation of variables does not always produce results as "clean" as the ones that we have seen. There are two common difficulties: evaluating the integrals may be hard or impossible, and solving for x may be hard or impossible.

2.1.8 Example.

Get as far as you can with applying separation of variables to
$$\dot{x} = \frac{\cos(t)}{5x^4 + 1}.$$

Solution. This problem is separable as it has the form $\dot{x} = g(t)h(x)$, where $g(t) = \cos(t)$ and $h(x) = 1/(5x^4 + 1)$. Note that $h(x) \neq 0$ for all x, so there are no equilibrium solutions. We separate variables to find
$$(5x^4 + 1)\dot{x} = \cos(t) \Longrightarrow \int (5x^4 + 1)\dot{x}\ dt = \int \cos(t)\ dt \Longrightarrow \int (5x^4 + 1)\ dx = \int \cos(t)\ dt$$
$$\Longrightarrow x^5 + x = \sin(t) + C.$$
However, we cannot solve for x as a function of t. (Seriously, try it.) Thus we are left with our solution defined IMPLICITLY by the equation
$$x^5 + x = \sin(t) + C. \qquad \blacktriangle$$

2.1.9 Example.

Get as far as you can with applying separation of variables to

$$\dot{x} = \frac{e^{t^2}}{x^4}.$$

Solution. This problem is separable with $g(t) = e^{t^2}$ and $h(x) = 1/x^4$ and has no equilibrium solutions, since $h(x) > 0$ for all x. We separate variables to find

$$x^4\dot{x} = e^{t^2} \Longrightarrow \int x^4\dot{x}\ dt = \int e^{t^2}\ dt \Longrightarrow \int x^4\ dx = \int e^{t^2}\ dt.$$

The power rule gives

$$\int x^4\ dx = \frac{x^5}{5} + C.$$

However, we cannot evaluate $\int e^{t^2}\ dt$ "in terms of elementary functions." This is one of those things that you learn in calculus; there is no "familiar" function that is the antiderivative of e^{t^2}. And so we might rearrange

$$\frac{x^5}{5} + C = \int e^{t^2}\ dt$$

into

$$x = \left(5\int e^{t^2}\ dt + C\right)^{1/5}.$$

Note that we have no problem taking fifth roots of any number.

If this feels weird (and defining functions by indefinite integrals always feels weird to me — I mean, t is being overworked as both an independent variable and the variable of integration), let's think about what this is saying. The symbol $\int e^{t^2}\ dt$ here denotes any antiderivative of e^{t^2}. If you want to pick a particular antiderivative, we could fix limits of integration: the function

$$F(t) := \int_{t_0}^{t} e^{\tau^2}\ d\tau$$

satisfies $\dot{F}(t) = e^{t^2}$ for any real number t_0. Then the function

$$x(t) := \left(5\int_{t_0}^{t} e^{\tau^2}\ d\tau + C\right)^{1/5}$$

solves $\dot{x} = e^{t^2}/x^4$ for any choice of the constant C. ▲

In addition to avoiding the inherent squick that comes with trying to decide if a variable is really important or just a variable of integration, the definite integral also lends itself to numerical analysis; on fixing t_0, we could use any number of numerical integration schemes to calculate $\int_{t_0}^{t} e^{\tau^2}\ d\tau$ for different upper limits t. Such numerical results might be more useful than any of the formulas here. Here's the point: *if you can't evaluate the integral, try writing it as a definite integral with the problem's independent variable as the upper limit of integration.*

2.1.10 Example.

Solve the initial value problem

$$\begin{cases} \dot{x} = \dfrac{e^{t^2}}{x^4} \\ x(1) = 2. \end{cases}$$

Solution. We know that the function

$$x(t) := \left(5\int_{t_0}^{t} e^{\tau^2}\, d\tau + C\right)^{1/5}$$

solves the ODE for any choice of t_0 and C. We are interested in the behavior of this solution at $t = 1$, so let's choose $t_0 = 1$, as that will make the integral term utterly trivial when $t = 1$. Then we want to pick C such that if

$$x(t) := \left(5\int_{1}^{t} e^{\tau^2}\, d\tau + C\right)^{1/5},$$

then $x(1) = 2$.

This means that we need

$$2 = \left(5\int_{1}^{1} e^{\tau^2}\, d\tau + C\right)^{1/5} = C^{1/5},$$

and so $C = 2^5 = 32$. Here we used the wonderful fact that $\int_a^a f(\tau)\, d\tau = 0$ for any function f: *the area under a single point is zero.* Thus the solution is

$$x(t) := \left(5\int_{1}^{t} e^{\tau^2}\, d\tau + 32\right)^{1/5}.$$

▲

2.1.5. *Analytic solutions for exponential and logistic growth.*

Now that we have a reasonable command of separation of variables, we will return to the population models and solve them. For the exponential growth model

$$\dot{x} = rx,$$

we are already conditioned to think that solutions have the form $x(t) = Ce^{rt}$ for some constant C. How can we see this using separation of variables?

First, we look for equilibrium solutions. This involves solving $rx = 0$, and thus $x = 0$. This solution has the expected form $x = Ce^{rt}$ with $C = 0$.

For $x \neq 0$, we separate variables:

$$\frac{1}{x}\dot{x} = r \Longrightarrow \int \frac{1}{x}\dot{x}\, dt = \int r\, dt \Longrightarrow \int \frac{dx}{x} = \int r\, dt \Longrightarrow \ln(|x|) = rt + K.$$

Here I am intentionally writing the constant of integration as K, not C. We exponentiate both sides:

$$\ln(|x|) = rt + K \Longrightarrow e^{\ln(|x|)} = e^{rt+K} \Longrightarrow |x| = e^K e^{rt}.$$

How do we get rid of absolute value? For that matter, what is absolute value? We define

$$|x| := \begin{cases} x, \ x \geq 0 \\ -x, \ x < 0. \end{cases}$$

So we think that having $|x(t)| = e^K e^{rt}$ means that, for a given time t, either $x(t) = e^K e^{rt}$ or $x(t) = -e^K e^{rt}$.

This is where we finished on Monday, August 29, 2022.

Can we ever have both? That is, could our solution x satisfy

$$x(t_1) = e^K e^{rt_1} \quad \text{and} \quad x(t_2) = -e^K e^{rt_2} \tag{2.1.5}$$

for some $t_1 \neq t_2$?

I say no. Here's why: x is the solution to a differential equation, so x is differentiable and therefore continuous. If (2.1.5) holds, then $x(t_1) > 0$ and $x(t_2) < 0$. But then the intermediate value theorem tells us that there is t_3 "between"[7] t_1 and t_2 such that $x(t_3) = 0$. And so

$$0 = |x(t_3)| = e^K e^{rt_3}.$$

How can the product $e^K e^{rt_3}$ ever equal 0?!

So, either our solution satisfies $x(t) = e^K e^{rt}$ for all t, or $x(t) = -e^K e^{rt}$ for all t. Since e^K can be any positive real number, $-e^K$ can be any negative real number. In either case, x has the form $x(t) = Ce^{rt}$ where C is some nonzero real number; previously, when considering the equilibrium solutions, we saw that C could be 0. And so separation of variables tells us that $x(t) = Ce^{rt}$ solves $\dot{x} = rx$ for arbitrary C.

The following lemma generalizes the argument above.

2.1.11 Lemma (Absolute value).

Let A and B be continuous functions on some interval I. Suppose that $B(t) > 0$ for all t in I and $|A(t)| = B(t)$ for all t in I. Then either $A(t) = B(t)$ for all t in I or $A(t) = -B(t)$ for all t in I.

A comparatively more complicated separation of variables argument will help us solve the logistic equation. We will actually treat the associated IVP to help us see how initial conditions affect the long-time behavior of the solution.

2.1.12 Example.

Use separation of variables and the absolute value arguments above to solve the logistic IVP

$$\begin{cases} \dot{x} = x(1-x) \\ x(0) = x_0. \end{cases}$$

[7] By this I mean $t_1 < t_3 < t_2$ if it's the case that $t_1 < t_2$, and otherwise $t_2 < t_3 < t_1$.

Solution. Here we have taken $r = N = 1$ in (1.2.6) for simplicity. We first check for equilibrium solutions and solve $x(1-x)$ to find $x = 0$ and $x = 1$.

Next, we assume both $x \neq 0$ and $x \neq 1$ so that we can separate variables:

$$\frac{1}{x(1-x)}\dot{x} = 1 \Longrightarrow \int \frac{1}{x(1-x)}\dot{x}\ dt = \int 1\ dt \Longrightarrow \int \frac{dx}{x(1-x)} = t + C.$$

To evaluate the integral on the left, we need the partial fractions decomposition[8]

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}.$$

Then

$$\int \frac{dx}{(1-x)} = \int \left(\frac{1}{x} + \frac{1}{1-x}\right)\ dx = \int \frac{dx}{x} + \int \frac{dx}{1-x} = \ln(|x|) - \ln(|1-x|) = \ln\left(\frac{|x|}{|1-x|}\right).$$

Make sure that you understand the reasoning behind each $=$ here.

We then have the implicit equation

$$\ln\left(\left|\frac{x}{1-x}\right|\right) = t + K$$

for the solution to the logistic equation. Once again, I'm using K for the constant of integration. Exponentiate to get

$$\left|\frac{x}{1-x}\right| = e^K e^t.$$

Use the absolute value lemma with $A = x/(1-x)$ and $B = e^K e^t$ to see that either

$$\frac{x(t)}{1-x(t)} = e^K e^t \text{ for all } t \quad \text{or} \quad \frac{x(t)}{1-x(t)} = -e^K e^t \text{ for all } t.$$

We conclude that

$$\frac{x(t)}{1-x(t)} = Ce^t \tag{2.1.6}$$

for all t, where C is any nonzero number.

---

[8] This is a very useful, but algebraically tedious, procedure. We want to write $1/x(1-x)$ as the sum

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

for some numbers A and B. This is equivalent to

$$1 = A(1-x) + Bx = (B-A)x + A.$$

This is an equality of polynomials, and it's a fact that two polynomials are equal if and only if the coefficients on their corresponding powers are equal. The coefficient on the power of x^1 on the left is 0 and on the right is $B - A$, and so we need $B - A = 0$ and thus $B = A$. The coefficient on the power of x^0 on the left is 1 and on the right is A, and so we need $A = 1$. Thus $B = 1$, too.

We can then solve the equation

$$\frac{x}{1-x} = Ce^t$$

for x. Multiply both sides by $1-x$ to find

$$x = Ce^t(1-x),$$

and rearrange to isolate x:

$$x = Ce^t(1-x) \Longrightarrow x = Ce^t - Ce^t x \Longrightarrow x + Ce^t x = Ce^t \Longrightarrow x(1+Ce^t) = Ce^t \Longrightarrow x = \frac{Ce^t}{1+Ce^t}.$$

Now, note that if we allow $C = 0$, then we get $x(t) = 0$ for all t, and this was one of our equilibrium solutions. So, separation of variables yields the solutions

$$x(t) = \frac{Ce^t}{1+Ce^t} \quad \text{and} \quad x(t) = 1$$

to the logistic equation.

To incorporate the initial condition $x(0) = x_0$, we first note that if $x_0 = 1$, then the constant solution $x(t) = 1$ works. Otherwise, for $x_0 \neq 1$, we could use (2.1.8) to figure out C in terms of x_0, but a more efficient way is to use (2.1.6) and compute

$$C = Ce^0 = \frac{x_0}{1-x_0}.$$

Note that since $x_0 \neq 1$, there is no danger of division by 0 here. And so the solution to the logistic equation in the case $x_0 \neq 1$ is

$$x(t) = \frac{\dfrac{x_0 e^t}{1-x_0}}{1 + \dfrac{x_0 e^t}{1-x_0}}.$$

What a piece of junk! We have fractions within fractions here. I claim that you can factor $1/(1-x_0)$ out of the numerator and denominator to get

$$\frac{\dfrac{x_0 e^t}{1-x_0}}{1 + \dfrac{x_0 e^t}{1-x_0}} = \frac{x_0 e^t}{(1-x_0) + x_0 e^t}. \tag{2.1.7}$$

That's cleaner. Also, note that this expression is defined for $x_0 = 1$, and, in fact, it equals 1 for all t in that case. And so

$$x(t) = \frac{x_0 e^t}{(1-x_0) + x_0 e^t} \tag{2.1.8}$$

is the solution to the logistic IVP for *any* choice of x_0. ▲

This is where we finished on Wednesday, August 31, 2022.

2.1.13 Example.

What can you say about the domain of solutions to the logistic IVP and their long-time behavior in terms of the initial condition $x(0) = x_0$? Assume $x_0 > 0$, which is the physically meaningful case when the logistic IVP models population growth.

Solution. We use the solutions given in (2.1.8).

Our first task is to determine the domain. A solution is defined at time t unless the denominator is zero, which means

$$(1 - x_0) + x_0 e^t = 0.$$

And this is the same as having

$$x_0 e^t = x_0 - 1.$$

Since $x_0 > 0$, we can divide to find

$$e^t = \frac{x_0 - 1}{x_0} = 1 - \frac{1}{x_0}.$$

This equality can only happen if $0 < 1 - 1/x_0$. In turn, if $x_0 > 0$, then $0 < 1 - 1/x_0$ if and only if $1 < x_0$. And so we have established the following: for $x_0 > 0$, the solution is defined for all t if $0 < x_0 \le 1$, while if $x_0 > 1$, then the solution is undefined at $t = \ln(1 - 1/x_0)$. I claim that $\ln(1 - 1/x_0) < 0$, but I'll leave that for you to check. Consequently, if $x_0 > 0$, then the solution is always defined on at least the interval $[0, \infty)$.

Now let's see how the initial condition $x(0) = x_0$ could influence the behavior of x as $t \to \infty$. If we try to calculate the limit as $t \to \infty$ from the formula (2.1.8), we quickly run into an ∞/∞ situation. We could use L'Hospital's rule, but we could also factor:

$$x(t) = \frac{x_0 e^t}{(1 - x_0) + x_0 e^t} = \frac{x_0}{(1 - x_0)e^{-t} + x_0}. \tag{2.1.9}$$

Then since $\lim_{t\to\infty} e^{-t} = 0$, we find

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x_0}{(1 - x_0)e^{-t} + x_0} = \frac{x_0}{(1 - x_0) \cdot 0 + x_0} = \frac{x_0}{x_0} = 1.$$

Using the terminology of Section 1.2.2, the value 1 is the carrying capacity of the population whose growth is governed by the logistic equation $\dot{x} = x(1 - x)$. The limit above analytically confirms our prediction that over long times, the population would tend to the carrying capacity. ▲

For all of my blather about the analyst's creed (AC), having a formula for something can be really powerful if we know how to use it. It would be difficult for us to obtain such precise and rigorous control over solutions to the logistic equation without either a formula or some powerful, abstract theory on a scale heretofore undreamt of in our class.

2.1.6. Outlook: life beyond separable problems.

The technique of separation of variables is powerful but not without limitations. We have to evaluate two integrals, either of which could be challenging or impossible to express in terms of "elementary" functions, and then we have to solve a typically nonlinear equation for the function under consideration. Also, there are plenty of nonseparable ODE to which we simply cannot apply the method; while $\dot{x} = tx$ is separable, $\dot{x} = t + x$ is not. And so there we will have to do something different.

Going forward, we will step away from separable problems, and from analytic methods entirely, for a while. Instead, we will explore some numerical, theoretical, and qualitative techniques that apply to more general ODE $\dot{x} = f(t, x)$, separable and nonseparable alike.

2.2. Slope fields.

What is the derivative? Analytically, it's the limit of a difference quotient. Geometrically, it's the slope of a curve. Specifically, let x be a function. Then the slope of x at the point $(t, x(t))$ in the tx-plane is $\dot{x}(t)$. So if x solves an ODE $\dot{x} = f(t, x)$, then the slope of x at $(t, x(t))$ is just $f(t, x(t))$.

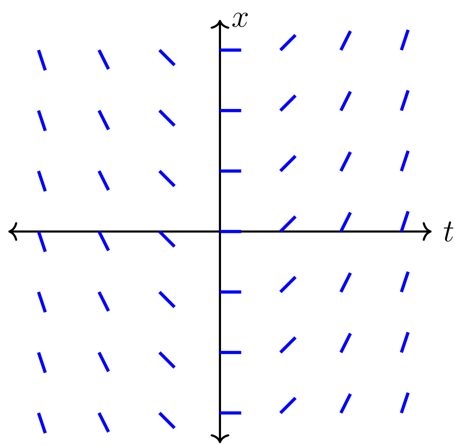
Here, then, is the key geometric insight: if we know that x solves the ODE $\dot{x} = f(t, x)$, and if we know that x passes through the point (t_0, x_0) in the tx-plane, then the slope of x at that point is $f(t_0, x_0)$. *We can calculate the slopes of a solution to $\dot{x} = f(t, x)$ without having a formula for x!* And if we know enough slopes, we might know how a solution is "flowing" through the tx-plane.

We will make this systematic by constructing **SLOPE FIELDS**, also called **DIRECTION FIELDS**. Given an ODE $\dot{x} = f(t, x)$, at the point (t, x) draw a small line segment with slope $f(t, x)$. If we draw enough of these segments and fill the tx-plane with a "field" of them, we will start to see a "flow" of curves in the plane. Those curves are potential solutions to $\dot{x} = f(t, x)$. This is a task ideally left to the computer, but we'll do one example "by hand."

2.2.1 Example.

Draw a slope field for $\dot{x} = t$. (Yes, all solutions are $x(t) = t^2/2 + C$. Do you see this in the slope field?)

Solution. Starting from each point (t, x) with integer coordinates for $-3 \le t \le 3$ and $-3 \le x \le 3$, I will draw a short line segment with slope t. To keep the tx-plane relatively uncluttered, I won't label points on the axes, and I won't mark the starting endpoint of the line segments.



If you stare at these pictures for a little while, hopefully you start to see the parabolas $x = t^2/2 + C$ emerging, however crudely and inchoately. ▲

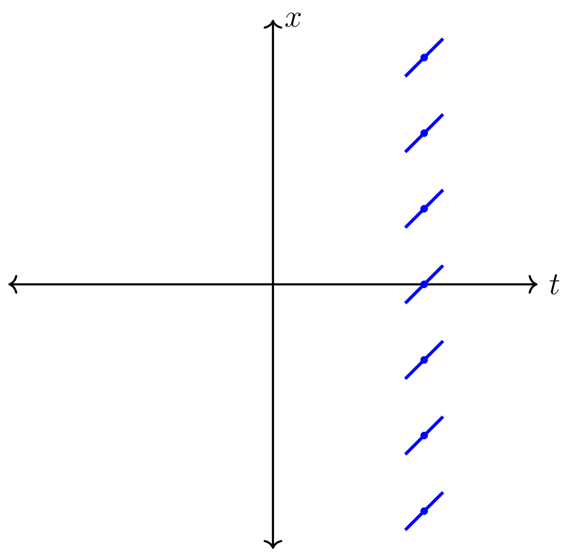
We don't need a slope field to understand the behavior of solutions to $\dot{x} = t$, since we can easily find their formulas. But the process above both taught us how to draw slope fields, and it reminded us of some things that we should know about the "direct integration" problem $\dot{x} = f(t)$. Namely, the following statements all mean the same thing.

- The function x solves the ODE $\dot{x} = f(t)$.

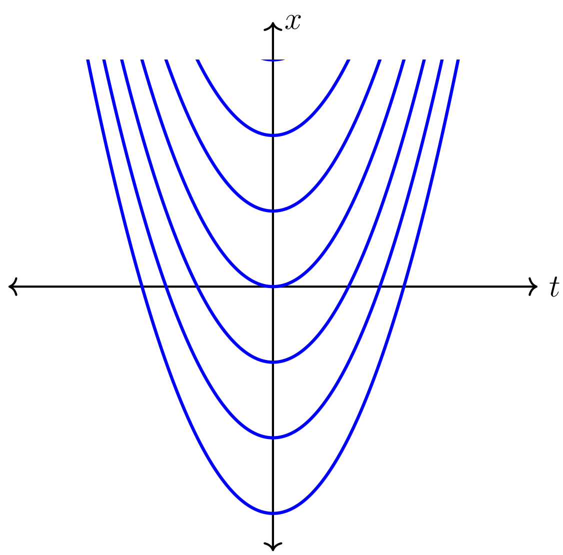
- The slope of the curve at a point depends on the t-coordinate of that point but not the x-coordinate.

- At two given points, the slopes are the same if the t-coordinates of both points are the same.

- The slopes are the same along any vertical line.



- All solutions are just vertical translates of one fixed solution. (All solutions are $x = \int f(t)\ dt + C$.)



Let's consider next something of the opposite situation: $\dot{x} = f(x)$.

2.2.2 Example.

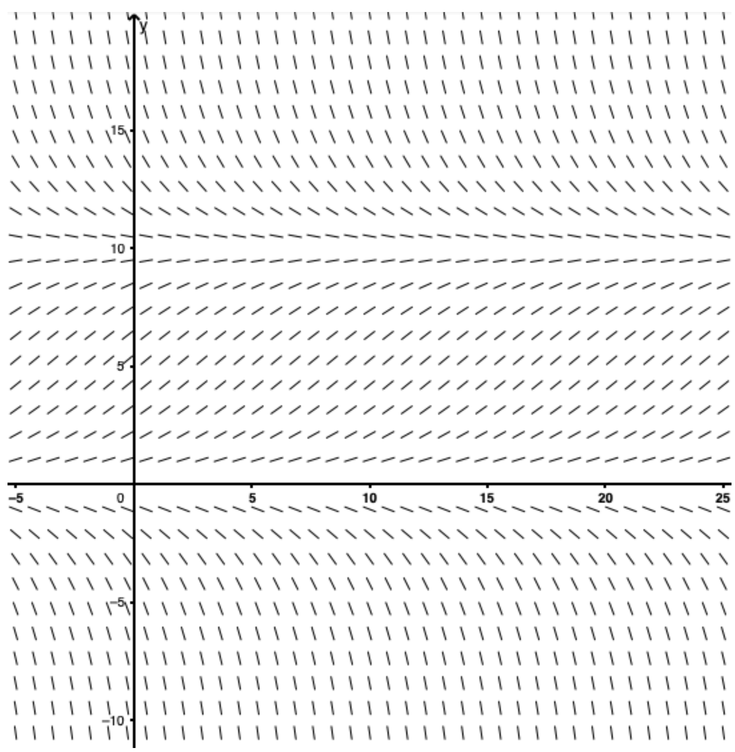
What can you learn from a slope field for $\dot{x} = (x/3)(1 - x/10)$*?*

Solution. To be clear, this is a logistic equation with $r = 3$ and $N = 10$. I chose those values because I think they make for a more evocative plot than $r = N = 1$.

I'll use the Geogebra slope field plotter by Dr. Adrian Jannetta, available at

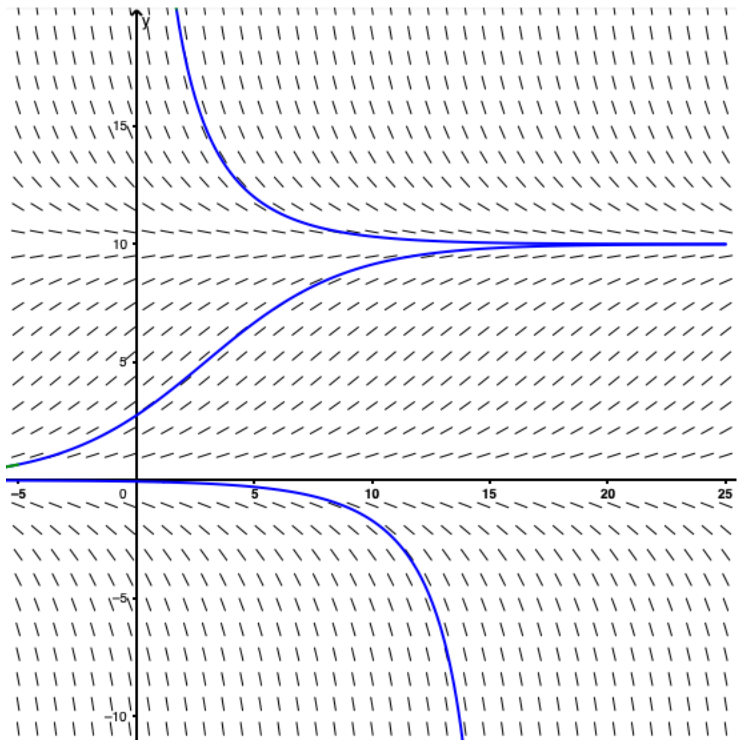
`https://www.geogebra.org/m/W7dAdgqc`.

If you try this yourself, and I strongly suggest you do, play around with the density and length parameters. I think that helps make the emergence of the curves more obvious. I had to wiggle the parameters quite a bit to get the screenshot below.



Here's what I see. The slopes are horizontal at $x = 10$ (and also at $x = 0$, but you probably don't see that, because of the overlap with the t-axis); if we put $x(t) = 10$ for all t, then we get an equilibrium solution to the ODE, right? The slopes between $x = 0$ and $x = 10$ are positive, but they get less steep as they get closer to $x = 10$. The slopes above $x = 10$ are negative, and they get less steep as they get closer to $x = 10$. The slopes below $x = 0$ are negative, and they get steeper as x becomes more negative.

I think I can see some of the logistic curves that we sketched back in Section 1.2.2. Let me turn on the numerical solver feature of the Geogebra app and plot a few solutions. (We will discuss how a numerical solver works soon and very soon.)

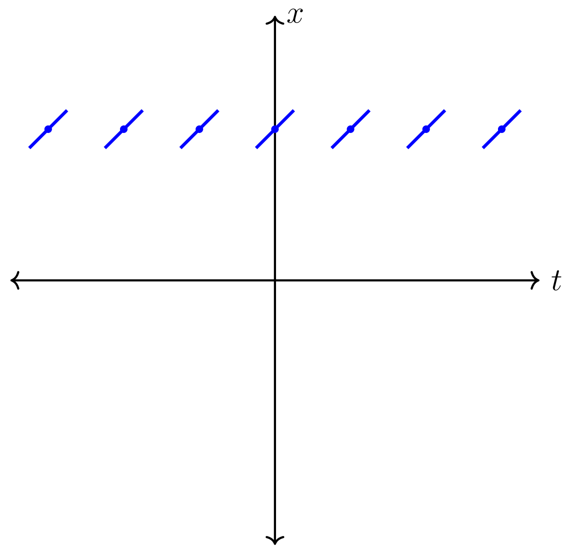


I got Geogebra to plot three solutions along the slope field, and those three do pretty much what I thought the slope field predicted. The "lowest" solution in the plot is strictly decreasing, and seems to tend rapidly toward $-\infty$. The "middle" solution strictly increases up to $x = 10$ and seems to have a horizontal asymptote there. And the "highest" solution strictly decreases down to $x = 10$ and, again, seems to have a horizontal asymptote there. We could use separation of variables on $\dot{x} = (x/3)(1 - x/10)$, and I claim that our analytic solutions would demonstrate the same behavior depending on how we take the initial condition.

One other thing (and you should try this in Geogebra): I claim that if we "slide" any of these solutions horizontally to the left or the right, the curve still stays on the slope field. That is, horizontally translating a given solution produces another soluion. ▲

The preceding example might teach us that the following statements all mean the same thing.

- The function x solves the ODE $\dot{x} = f(x)$.
- The slope of the curve at a point depends on the x-coordinate of that point but not the t-coordinate.
- At two given points, the slopes are the same if the x-coordinates of both points are the same.
- The slopes are the same along any horizontal line.



- If we translate the graph of one solution horizontally, then we get another solution.

The last insight here is something new, I think. Recall that if x is a function, then the graph of $y(t) := x(t + t_*)$ is just the graph of x shifted to the left (if $t_* > 0$) or to the right (if $t_* < 0$). Let's formalize that insight.

2.2.3 Theorem (Time-shifted solutions to autonomous ODE).

Suppose that x solves $\dot{x} = f(x)$. Fix a number t_* and define $y(t) := x(t+t_*)$. Then y also solves $\dot{y} = f(y)$.

Proof. We need to show that $\dot{y}(t) = f(y(t))$ for all t. First,

$$\dot{y}(t) = \frac{d}{dt}[x(t + t_*)] = \dot{x}(t + t_*)\frac{d}{dt}[t + t_*] = \dot{x}(t + t_*)$$

by the chain rule. Since $\dot{x}(\tau) = f(x(\tau))$ for all τ, we get

$$\dot{y}(t) = \dot{x}(t + t_*) = f(x(t + t_*)) = f(y(t)). \qquad \square$$

We have studied differential equations of the form $\dot{x} = f(x)$ several times already; both exponential and logistic growth have this form, and differential equations of this form are separable. (Reread Definition 2.1.1 and take $g(t) = 1$ and $h(x) = f(x)$.) Equations of the form $\dot{x} = f(x)$ are called **AUTONOMOUS** because they depend only on the "state" x and not time t. Now we know that if we have one solution to an autonomous problem, we get many more just by shifting horizontally.

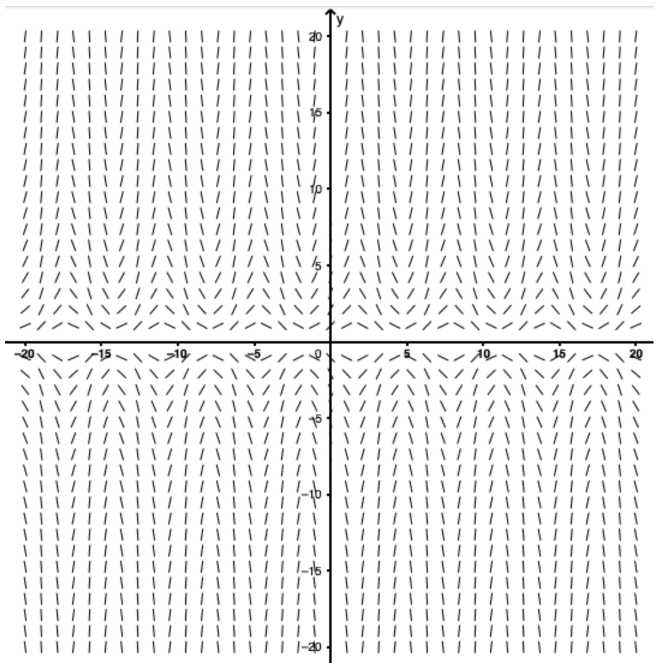
Here is an example of a nonautonomous problem that, I think, illustrates some of the limitations of slope fields by themselves.

2.2.4 Example.

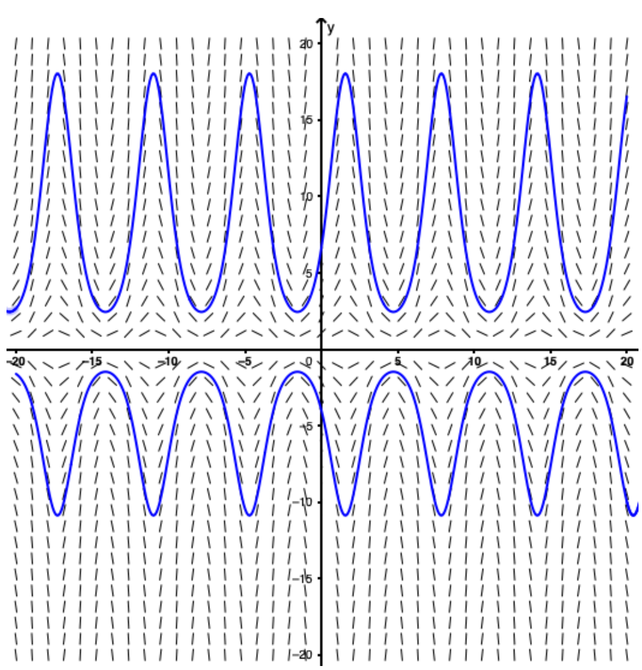
What can you learn, or not learn, from a slope field for $\dot{x} = \cos(t)x$?

Solution. We saw an ODE like this in Section 1.2.3.

There is a periodic function in the problem, so we might expect some periodicity in the solution. In fact, the problem is separable, so we can try finding an analytic solution if we want, but I don't want to right now. Here is the slope field from Geogebra.



I definitely see a periodic pattern in the slopes, but, honestly, it's tough for me to "connect" them into any continuous curve. Around the horizontal axis, I see sort of a periodic ripple, but as we go up the vertical axis, I worry that some of the slopes could be getting so steep that they're indicating vertical asymptotes. Let's try Geogebra's numerical solver.



Now those steeper slopes make more sense. The solutions *are* pretty steep there, but not steep enough to be a vertical asymptote. ▲

Drawing a slope field for $\dot{x} = f(t, x)$ in principle requires no actual "math" on your part; you go to a computer, plug in your formula $f(t, x)$, and fool around until you think you see a picture or a pattern. You do a little work, and you get a little insight. If you are studying an ODE and have no other clue about what to do, get your hands on the slope field, and maybe that will nudge you in a useful direction. And if you don't see any patterns in the slope field, don't worry; sometimes you have to know what you're looking for to see it. But at least the slope field didn't require you to do any calculus.

In practice, you will never work with a slope field in isolation; you will always combine it with another tool — an analytic method to find formulas, a numerical method to discretize the problem and approximate particular values of the solution. We'll try the latter approach next.

This is where "we" finished on Friday, September 2, 2022.

2.3. Euler's method.

Slope fields help us make qualitative predictions about the behavior of solutions to ODE — crude, tentative predictions, but predictions nonetheless, and predictions that do not require us to do any calculus. However, slope fields do not tell us the exact values of solutions at particular moments in time. Separation of variables can give us a formula from which we could compute exact values, but we have seen that separation of variables doesn't always work — and not all problems need to have the special separable form, anyway.

We now need a third tool beyond the analytic and qualitative methods: the numeric. There is a broad array of numerical methods that can approximate solutions to ODE; a full study of these methods reveals (1) where they come from, (2) how they work, and (3) how good they are — that is, what kind of errors might a numerical method unfortunately introduce. We will study just one numerical method in this course, and we will really focus on just the issues (1) and (2).

This method is called (as the title of this section foretells) Euler's method. I've seen two different derivations of the method, both of which are worth knowing, as they teach us different things about ODE and calculus. We'll derive the method and implement it in pseudocode. Then we'll go to a computer to do the arithmetic. You should be comfortable explaining how the method arises and how to work it out, but I won't ask you to do any painful calculations by hand.

2.3.1. Origins of Euler's method: the integral approach.

As usual, we start by working backward. Suppose that we have a solution to the ODE $\dot{x} = f(t, x)$. We are not assuming that this problem is separable. For definiteness, let's add an initial condition: $x(t_0) = x_0$. So, we are really going to solve the IVP

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0. \end{cases} \tag{2.3.1}$$

If we want to learn about x, and we know absolutely nothing specific about f, one good way to make x appear is to integrate. The fundamental theorem of calculus gives, as always,

$$x(t) - x(t_0) = \int_{t_0}^{t} \dot{x}(\tau)\ d\tau = \int_{t_0}^{t} f(\tau, x(\tau))\ d\tau,$$

and so

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))\ d\tau = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\ d\tau. \tag{2.3.2}$$

This is exactly what we obtained in (FTC) at the start of the course, and we complained then, because this equation defines x in terms of x, which is not very helpful.

However, we can turn this into a good *approximation* for the value $x(t)$ by recalling the "left-hand rule" for approximating integrals:

$$\int_{a}^{b} g(\tau)\ d\tau \approx (b-a)g(a), \tag{LHR}$$

at least if a and b are "close" (whatever that means). Thus

$$\int_{t_0}^{t} f(\tau, x(\tau))\ d\tau \approx (t - t_0)f(t_0, x(t_0)) = (t - t_0)f(t_0, x_0) \tag{2.3.3}$$

when t is "close" to t_0. The "=" on the right really is genuine; it's the initial condition $x(t_0) = x_0$. And so if we combine (2.3.2) and (2.3.3), we get

$$x(t) \approx x_0 + (t - t_0)f(t_0, x_0) \quad \text{for} \quad t \approx t_0. \tag{2.3.4}$$

Let's make "close" a little more precise (but not a lot). Fix a small positive number h, maybe with $0 < h < 1$. Define $t_1 := t_0 + h$. Then (2.3.4) just says

$$x(t_1) \approx x_0 + hf(t_0, x_0).$$

Let's abbreviate

$$x_1 := x_0 + hf(t_0, x_0).$$

Then $x(t_1) \approx x_1$. Note that we calculated x_1 just using the given information of f and the initial data t_0 and x_0. We did not do any calculus.

Now let's jump a bit forward into the future. Put $t_2 := t_1 + h = t_0 + 2h$, so t_2 is not too far away from t_1. Then

$$\begin{aligned}
x(t_2) &= x(t_1) + \int_{t_1}^{t_2} \dot{x}(\tau)\ d\tau \text{ by the fundamental theorem of calculus} \\
&= x(t_1) + \int_{t_1}^{t_2} f(\tau, x(\tau))\ d\tau \text{ since } x \text{ solves } \dot{x} = f(t, x) \\
&\approx x_1 + \int_{t_1}^{t_2} f(\tau, x(\tau))\ d\tau \text{ since } x(t_1) \approx x_1 \\
&\approx x_1 + (t_2 - t_1)f(t_1, x(t_1)) \text{ by the left-hand approximation for integrals}
\end{aligned}$$

$$= x_1 + hf(t_1, x_1) \text{ since } t_2 = t_1 + h.$$

Let's abbreviate

$$x_2 := x_1 + hf(t_1, x_1).$$

Then $x(t_2) \approx x_2$, and we calculated x_2 just by using the given information of f and the previously calculated data t_1 and x_1. Again, we did not do any calculus. Note that there were two uses of $\approx$ above: when we replaced $x(t_1)$ with x_1 and when we approximated the integral.

These two steps suggest a scheme for numerically approximating the solution to the IVP (2.3.1). First, fix a small **TIME STEP** $h > 0$. For integers $k \ge 0$, define

$$t_k := \begin{cases} t_0, & k = 0 \\ t_{k-1} + h, & k \ge 1, \end{cases}$$

or, if you prefer,

$$t_k := t_0 + kh, \ k \ge 0.$$

And define

$$x_k := \begin{cases} x_0, & k = 0 \\ x_{k-1} + hf(t_{k-1}, x_{k-1}), & k \ge 1. \end{cases}$$

Then we expect that the true solution x to the IVP (2.3.1) enjoys the approximation

$$x(t_k) \approx x_k.$$

If we run this iteration some $n \ge 1$ times, then we generate $n+1$ approximations to the value of x on the interval $[t_0, t_0 + nh]$. These are

$$x(t_0) = x_0, \qquad x(t_1) \approx x_1, \qquad \ldots, \qquad x(t_n) \approx x_n.$$

This is where we finished on Wednesday, September 7, 2022.

2.3.2. Pseudocode and sample implementations.

Here is a summary of Euler's method.

Define the function f.
Define the starting time t_0.
Define the initial value x_0.

Choose a time step $h > 0$.
Choose a number of iterations $n \ge 1$.

For $k = 1, \ldots, n$, iterate

$$\begin{cases} t_k := t_0 + kh \\ x_k := x_{k-1} + hf(t_{k-1}, x_{k-1}). \end{cases}$$

2.3.1 Example.

Use Euler's method to approximate the solution to

$$\begin{cases} \dot{x} = t \\ x(0) = 0 \end{cases}$$

with five and the time step $h = 0.2$.

Solution. You and I both know that the solution is $x(t) = t^2/2$. I'm just doing this example because I think it's good for us to see the arithmetic of Euler's method spelled out explicitly once, and only once, before we start stuffing it into a computer. And, all things considered, I think the arithmetic here is pretty easy. (But still feel free to go to a computer to do it.)

In the notation of our pseudocode above, we are taking $f(t, x) = t$, $t_0 = 0$, and $x_0 = 1$. We fill in the following table.

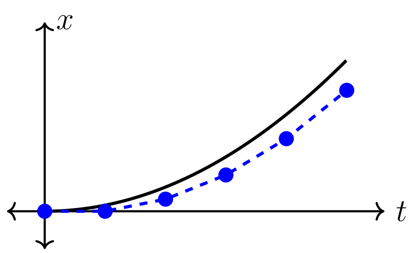
k	t_k	x_k	$f(t_k, x_k)$	$x_{k+1} = x_k + hf(t_k, x_k) = x_k + ht_k$
0	0	0	0	$0 + (0.2 \cdot 0) = 0$
1	0.2	0	0.2	$0 + (0.2 \cdot 0.2) = 0.04$
2	0.4	0.04	0.4	$0.04 + (0.2 \cdot 0.4) = 0.12$
3	0.6	0.12	0.6	$0.12 + (0.2 \cdot 0.6) = 0.24$
4	0.8	0.24	0.8	$0.24 + (0.2 \cdot 0.8) = 0.4$
5	1	0.4	1	$0.4 + (0.2 \cdot 1) = 0.6$

Let's compare the approximations to the exact value of the known solution $x(t) = t^2/2$ at the values t_k.

k	t_k	x_k	$x(t_k) = t_k^2/2$
0	0	0	0
1	0.2	0	0.02
2	0.4	0.04	0.08
3	0.6	0.12	0.18
4	0.8	0.24	0.32
5	1	0.4	0.5

It looks like our Euler's method results consistently under-approximate the true solution, but the values are definitely strictly increasing.

We can see this with plots. I'll graph the true solution $x(t) = t^2/2$ in solid black; I'll plot the points (t_k, x_k) and connect them by dotted lines, all in blue. Plots of your numerical results can be very useful, and, like the numerical results themselves, they're best produced by a computer.



▲

2.3.2 Example.

Use as many tools as you can to study the solutions of

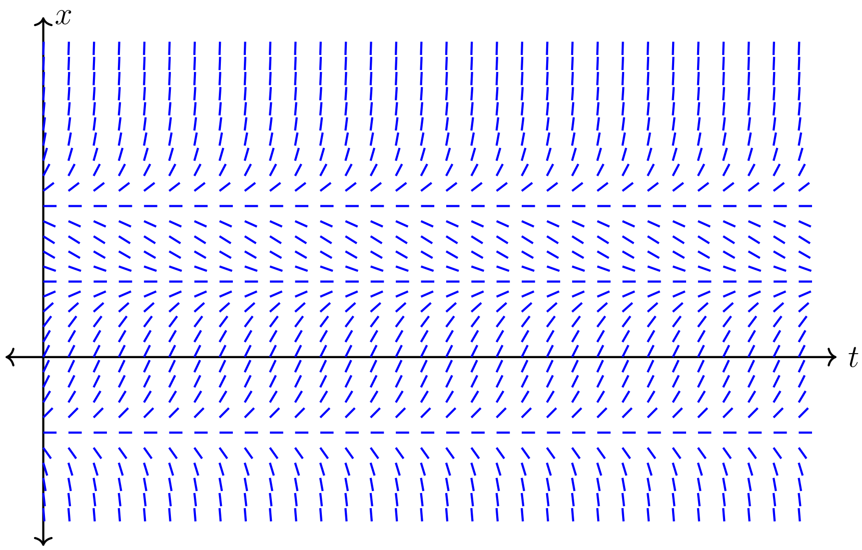
$$\dot{x} = (x+1)(x-1)(x-2).$$

Solution. This is a separable equation (take $g(t) = 1$ and $h(x) = (x+1)(x-1)(x-2)$) and in fact autonomous. We set $(x+1)(x-1)(x-2) = 0$ to find the equilibrium solutions; they are $x = \pm 1$, 2. We could try to separate variables, and initially we find

$$\int \frac{dx}{(x+1)(x-1)(x-2)} = t + C.$$

The integral on the left demands partial fractions; the denominator consists of a product of distinct linear factors, which is probably the least painful form of partial fractions. Nonetheless, it's partial fractions, and the algebra to solve for x at the end could be even more intense than what we did with the logistic equation — or maybe impossible!

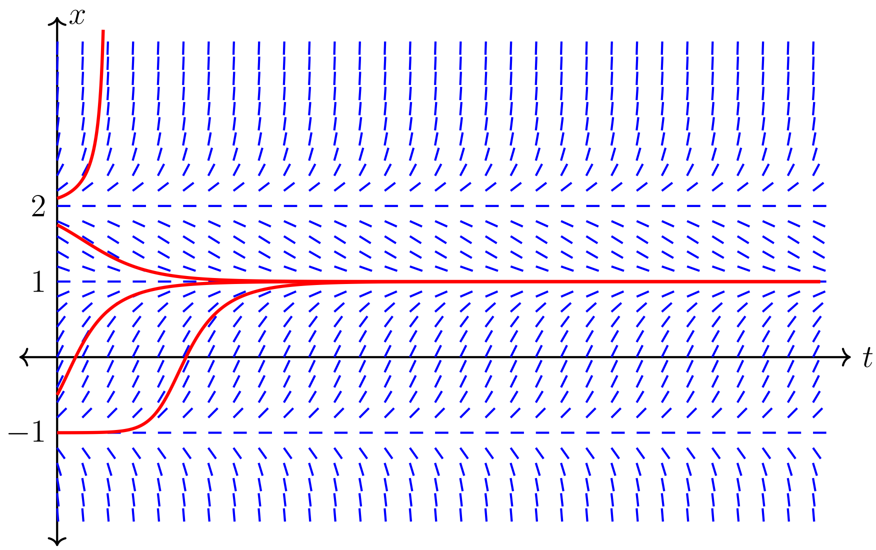
Drawing a slope field is a good idea.



Depending on the quality of our slope field (and our comfort with the computer program that is generating it), hopefully we see the three equilibrium solutions at $x = \pm 1$, 2. These correspond to the three (roughly) horizontal sets of slopes.

If we stare at the slope field for a while, we might observe four "bands" in the graph: the region below $x = -1$, the region between $x = -1$ and $x = 1$, the region between $x = 1$ and $x = 2$, and the region above $x = 2$. The flow of the slopes changes from region to region. For example, slopes seem to be decreasing below $x = -1$ and increasing between $x = -1$ and $x = 1$. Thus we might suspect that a solution with $x(0) < -1$ will be decreasing and a solution with $-1 < x(0) < 1$ will be increasing.

We can lend further credence to these suspicions by implementing Euler's method with the initial condition x_0 taken at different points along the x-axis. Here are some of my results, superimposed in red over the slope field.



I took the time step to be pretty small ($h = 0.01$). It looks to me like the equilibrium solutions are closely related to the behavior as $t \to \infty$ of some (though not all) of the solutions. Also, we can see the time-shifted solutions promised by Theorem 2.2.3. ▲

In "real life," you will never use any one of the tools from this class in isolation. If you are studying a differential equation that really matters to you, you will (probably) have a reference text on hand to look up analytic techniques (formulas); a computer algebra system to do the dirty work of constructing those formulas or at least simplifying/checking your "by hand" calculus and algebra; a graphing program to produce slope fields; and a programming language in which to implement numerical methods. While it is important to be able to use and appreciate each of these tools individually, never feel that you have to understand an ODE completely just from one of them.

This is where we finished on Friday, September 9, 2022.

2.4. Existence and uniqueness theory.

We now have several powerful tools for studying differential equations. We have the analytic method of separation of variables to find formulas for solutions to a broad (though not all-encompassing) class of problems; the qualitative approach of slope fields, which suggests, with minimal work, patterns of behavior of solutions; and Euler's method, which can approximate the value of a solution at a point (and give a better sense of a solution's graph than a slope field). The validity of these tools, however, is predicated on a somewhat hidden assumption: *when using these tools, we have always been assuming that a solution exists.*

Think about separating variables for the problem $\dot{x} = g(t)h(x)$. What happens? We check for equilibrium solutions by solving $h(x) = 0$ and then, probably, dive right into separating variables to get

$$\frac{1}{h(x)}\dot{x} = g(t), \tag{2.4.1}$$

and then we integrate away. In doing so, we tacitly assumed that a solution[9] x existed. *If* there is a function x such that $\dot{x} = g(t)h(x)$, and if $h(x) \neq 0$, *then* we must have (2.4.1), and from there we can integrate to find a formula for x. But why are we allowed to assume that a solution exists in the first place?

Now, working backward by assuming that a solution to our problem exists is nothing new; you've done that every time you've solved for x since your first algebra class. However, if we uncritically assume that solutions to our problems always exist, we could get into all sorts of trouble. Perhaps the problem at hand doesn't have a solution, and so we would be wasting our time trying to find an analytic formula, or we would be deceiving ourselves by staring at slope fields or trying to run Euler's method.

If we are going to predict the future using differential equations, as we set out to do at the start of the course, then we need to be sure that the problems that we're studying really do have solutions. Moreover, we want to be sure that we're predicting only *one* future. Our experience with calculus suggests that most differential equations will not have unique solutions; even something as simple as the direct integration problem $\dot{x} = t$ has infinitely many solutions of the form

$$x(t) = \frac{t^2}{2} + C$$

for constants C. Consequently, we probably need to add more data to our problems to ensure uniqueness of solutions.

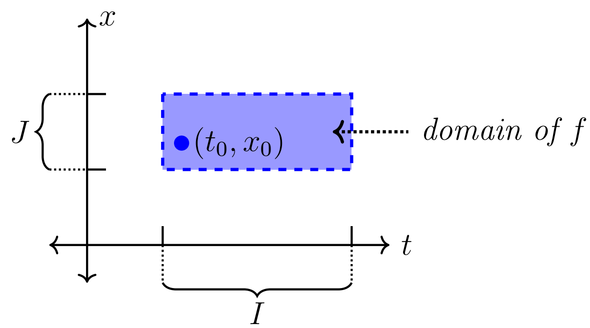
We will use the following theorem without proof. It is a gentler version of the "standard" existence and uniqueness results in most ODE classes, which require a passing familiarity with multivariable calculus.

2.4.1 Theorem (Existence and uniqueness).

Suppose that $g_1, \ldots, g_n$ are continuous functions defined on the open interval I, and $h_1, \ldots, h_n$ are differentiable functions defined on the open interval J. (The intervals I and J may be bounded or unbounded.) Assume that h_k' is continuous on J for each $k = 1, \ldots, n$. Define

$$f(t,x) := \sum_{k=1}^{n} g_k(t)h_k(x). \tag{2.4.2}$$

[9] Now is a very good time to reread Definition 1.3.1.



(i) [Existence] *Let t_0 be a point in I and x_0 be a point in J. There exists $\epsilon > 0$ and a function x defined on $(t_0 - \epsilon, t_0 + \epsilon)$ such that x solves the IVP*

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0. \end{cases}$$

(ii) [Uniqueness] *Suppose that y is another function on $(t_0 - \epsilon, t_0 + \epsilon)$ that solves the IVP*

$$\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = x_0. \end{cases}$$

Then $x(t) = y(t)$ for all t.

This a deep and technically worded result, so let's run through some commentary.

2.4.2 Remark.

(i) *The form of the function f in (2.4.2) is certainly strange. Nonetheless, all the "governing" functions f that we will encounter in this class will have this form. In particular, taking $n = 1$ gives the separable problem $\dot{x} = g_1(t)h_1(x)$. A more general f could be allowed if we presumed knowledge of multivariable calculus and altered the hypotheses of the theorem a bit.*

(ii) *The existence result is just that: existence. It does not give us a procedure for finding the solution x, nor does it tell us anything about ϵ. It does not tell us anything about the behavior of x. If we want to understand a specific problem better, we have lots more work to do.*

(iii) *The uniqueness result should be comforting: not only can we predict the future, if we impose initial data, then we can only predict one future. (All hail the Sacred Timeline!) Note that in the IVP in part (ii), it is essential that the initial data be the same as in part (i). In other words, uniqueness only holds for a given IVP.*

We will work, as is our wont, a few toy problems here to show how one can use, or fail to use, the existence and uniqueness theorem.

2.4.3 Example.

Explain why the IVP

$$\begin{cases} \dot{x} = \dfrac{e^{t^2}}{5x^4+1} \\ x(0) = 0 \end{cases}$$

has a unique solution.

Solution. Good luck solving this monster with separation of variables; you'll fail to evaluate the t-integral, and you won't be able to solve for x. However, since this problem is separable, we rewrite it as

$$\begin{cases} \dot{x} = g_1(t)h_1(x) \\ x(0) = 0, \end{cases} \qquad \text{where} \quad g_1(t) = e^{t^2} \quad \text{and} \quad h_1(x) = \frac{1}{5x^4+1}.$$

After staring at these functions for a moment[10], we see that g_1 is continuous on $(-\infty, \infty)$, and h_1 is continuously differentiable on $(-\infty, \infty)$. Thus the existence and uniqueness theorem applies to produce a unique solution to the IVP. If we want to know more about this solution, we would have to do more work — slope fields and numerics would be a good start. ▲

2.4.4 Example.

Let

$$g(t) := \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

Can you apply the existence and uniqueness theorem to the IVP

$$\begin{cases} \dot{x} = g(t) \\ x(0) = 0? \end{cases}$$

What, if anything, can you say about solutions?

Solution. The ODE is in the form $\dot{x} = g(t)h(x)$, where g is defined above and $h(x) = 1$. However, g is definitely not continuous at $t = 0$, and so the existence and uniqueness theorem simply doesn't apply here.

Nonetheless, just because we can't *use* the existence and uniqueness theorem doesn't mean we can conclude that there is, or isn't, a solution. I'm going to bet that you haven't seen many differentiable functions with discontinuous derivatives. Let's build on this gut instinct and assume that there *is* a solution x and see if anything goes wrong.

Working backward, if $\dot{x} = g(t)$ with $x(0) = 0$, then the fundamental theorem of calculus tells us

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)\ d\tau = 0 + \int_0^t g(\tau)\ d\tau = \int_0^t g(\tau)\ d\tau.$$

[10] A legitimate mathematical technique called "proof by inspection."

For $t > 0$, we have
$$x(t) = \int_0^t 1 \, d\tau = t,$$
while for $t < 0$, we have
$$x(t) = \int_0^t 0 \, d\tau = 0.$$
Thus
$$x(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0. \end{cases}$$
I claim that this function is not differentiable at $t = 0$, something that a picture will suggest to you immediately. I suggest you use the definition of the derivative to confirm this.

So what's the problem? We assumed that the IVP had a solution and we found a very explicit formula for it by direct integration. But this formula showed that the solution was not differentiable at $t = 0$. Contrast this with Definitions 1.3.1 and 1.3.4 — a solution to an IVP with the initial condition at time $t = t_0$ better be differentiable at time $t = t_0$! ▲

This is where we finished on Monday, September 12, 2022.

2.4.5 Example.

Does the IVP
$$\begin{cases} \dot{x} = |x| \\ x(0) = 0 \end{cases}$$
have a solution? How does this fit with the existence and uniqueness theorem?

Solution. Sure it does: take $x(t) = 0$ for all t. The ODE here has the form $\dot{x} = g_1(t)h_1(x)$, where $g_1(t) = 1$ and $h_1(x) = |x|$. Then h_1 is not continuously differentiable on any interval containing the point $x = 0$, so the existence and uniqueness theorem doesn't apply. But the failure of the theorem to apply doesn't preclude the (non)existence of a solution to the IVP; it just means that we have to use a different tool (our brains). ▲

2.4.6 Example.

Let x_0 be a real number. Previously, in Example 2.1.12, we saw from separation of variables that a solution to the logistic IVP
$$\begin{cases} \dot{x} = x(1 - x) \\ x(0) = x_0 \end{cases}$$
was
$$x(t) = \frac{x_0 e^t}{(1 - x_0) + x_0 e^t}.$$
Why is this the only solution?

Solution. Put $g(t) = 1$ and $h(x) = x(1-x)$. Then g and h satisfy the hypotheses of the existence and uniqueness theorem, so the logistic IVP has exactly one solution. And we can check via calculus and algebra that the formula above already gives a solution. So, it is the only solution. ▲

The preceding example should comfort the skeptical separator. Suppose that we use separation of variables to eke out a formula for the solution to $\dot{x} = g(t)h(x)$, where $f(t,x) = g(t)h(x)$ satisfies the conditions of the existence and uniqueness theorem. Once we check[11] that the formula solves this separable problem, the existence and uniqueness theorem guarantees that it is the unique solution satisfying $x(t_0) = x_0$ for our favorite choice of t_0 in the domain and x_0 in the range. And so maybe separation of variables isn't as suspicious as I always think it is.

2.4.7 Example.

In Example 2.3.2, both the slope field and the results from Euler's method predicted that if a solution x to

$$\dot{x} = (x+1)(x-1)(x-2)$$

satisfied $x(0) > 2$, then $x(t) > 2$ for all t in the domain of x. Use the existence and uniqueness theorem to confirm this prediction.

Solution. What's special about 2 here? The function $y(t) = 2$ is an equilibrium solution to the ODE. Also, the existence and uniqueness theorem definitely applies to ODE/IVP of the form $\dot{x} = h(x)$, where $h(x) = (x+1)(x-1)(x-2)$. After all, h is a polynomial, which is infinitely differentiable.

We want to show that if x solves $\dot{x} = h(x)$ with $x(0) > 2$, then $x(t) > 2$ for all t in the domain of x. Short of having a formula for x, which would involve an icky, and maybe impenetrable, separation of variables argument, I don't see a way of concluding this directly. Instead, let's ask what goes wrong if the inequality $x(t) > 2$ is not always true. What if $x(t_0) \le 2$ for some t_0? Necessarily $t_0 \neq 0$ since $x(0) > 2$.

First, suppose $x(t_0) = 2$. Then x solves the IVP

$$\begin{cases} \dot{x} = h(x) \\ x(t_0) = 2. \end{cases}$$

But we already have a solution to this IVP: the equilibrium solution $y(t) = 2$. That is,

$$\begin{cases} \dot{y} = h(y) \\ y(t_0) = 2, \end{cases}$$

too. In this case, the uniqueness result forces $x(t) = y(t)$ for all t that belong to both the domain of x and the domain of y. The domain of y is $(-\infty, \infty)$, so then $x(t) = 2$ for all t in the domain of x. In particular, $x(0) = 12$, which contradicts our assumption of $x(0) > 2$.

[11] Recall from Example 1.3.3 the difference between checking that a given function solves a differential equation and finding the solution to that differential equation.

Now suppose $x(t_0) < 2$. Since $x(0) > 2$, the intermediate value theorem provides t_2 such that $x(t_1) = 2$. But then we are back in the case above and get another contradiction. ▲

This is where we finished on Wednesday, September 14, 2022.

Going forward, we will use the existence and uniqueness theorem whenever we need it to assure ourselves that our problems have unique solutions. However, we will continue to develop new techniques so that we can describe properties of solutions to ODE and IVP in much more detail. In particular, we will explore in detail the question of the domain of a solution to an ODE/IVP: what can we say about that number ϵ in Theorem 2.4.1?

2.5. Autonomous equations and the phase line.

We will now take up the study of a special kind of ODE, one that does not depend on time but only on the current "state" of the quantity under consideration. We have met these ODE many times in the past, most recently in Theorem 2.2.3.

2.5.1 Definition.

An ODE of the form $\dot{x} = f(x)$, where f is a function of the single real variable x, is called **AUTONOMOUS**.

Any autonomous ODE is separable, and so if we want a formula for the solution to $\dot{x} = f(x)$, we should first find the equilibrium solutions (if any exist) by solving $f(x) = 0$, and then we separate variables and integrate to obtain

$$\int \frac{dx}{f(x)} = t + C.$$

After that, it's anyone's guess as to how to antidifferentiate and solve for x as a function of t. This, as we know, could go very badly, and so now is a good time to recite the Analyst's Creed AC and think beyond formulas. It turns out that a combination of our recently elucidated existence/uniqueness theory and familiar calculus techniques can tell us quite a lot about the solutions to such equations. For motivation, and to consider why our prior techniques need augmentation for this new project, we develop a new population model.

2.5.1. Motivation: the modified logistic equation.

. I claim that both exponential and logistic growth models are too optimistic. Recall that the exponential growth model has the form

$$\dot{x} = rx,$$

and logistic growth is

$$\dot{x} = rx\left(1 - \frac{x}{N}\right), \tag{2.5.1}$$

where in both cases we'll take $r > 0$, and in the second $N > 0$. Our explicit solutions for these two models, obtained via separation of variables, do not allow a population that is initially nonzero ever to become extinct. Specifically, in exponential growth, if $x(0) > 0$, we have $\lim_{t\to\infty} x(t) = \infty$, while in logistic growth we have $\lim_{t\to\infty} x(t) = N$. (We called N the carrying capacity.) So, whether the population explodes or settles down, it never dies out.

This is not how life always goes, right? Any number of horrible things can happen to a species — predators, hunters, asteroids. We will consider such deleterious "external" influences later. Here we will add the realistic constraint that a population should decrease, and maybe die off, if it has too few members. After all, if the population is too small, then mating pairs might not find each other, or they might need to reject each other to maintain genetic diversity.

Let's state our qualitative features in plain English and then translate them to mathspeak. Here's what we want our model to reflect.

1. If the population is too large, it should decrease to a manageable level. (This is a good feature of the original logistic model.)

2. If the population is too small, it should go extinct.

3. If the population is ever zero, then it stays zero forever. (There is no spontaneous generation.)

If we put the first two conditions together, then we arrive at a sort of "Goldilocks" requirement: the population should only be increasing if it is not too small and not too large.

Now for the math. As usual, $x(t)$ will be the population count at time t. Here's how the three features above can be rewritten.

1. There is $N > 0$ such that if $x(t) > N$, then $\dot{x}(t) < 0$. The number N should be large enough to capture our notion of "too large" for the given population.

2. There is $M > 0$ such that if $x(t) < M$, then $\dot{x}(t) < 0$. The number M should be small enough to capture our notion of "too small" for the given population. Also, since we are associating N and "large," let's assume $M < N$.

3. If $x(t) = 0$ for some time t, then $\dot{x}(t) = 0$ for all t.

The "Goldilocks" conclusion might be phrased as follows: if $M < x(t) < N$, then $\dot{x}(t) > 0$.

Our prior experience with exponential and logistic growth suggests that we create a function f so that the ODE $\dot{x} = f(x)$ has the three features above. Recall that, as usual, the letter x is playing a dual role: sometimes it's the function $x = x(t)$, and other times it's the independent variable of f. We can rewrite the three features above as conditions on f.

1. If $x > N$, then $f(x) < 0$.

2. If $x < M$, then $f(x) < 0$.

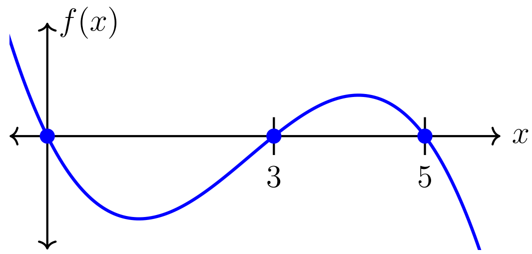
3. $f(0) = 0$.

And, once more, the Goldilocks condition could be $f(x) > 0$ for $M < x < N$.

There are many, many functions f that could do what we want. One of the simplest is a slight variation on the logistic function from (2.5.1):

$$f(x) = rx\left(1 - \frac{x}{N}\right)\left(\frac{x}{M} - 1\right). \tag{2.5.2}$$

It's a good exercise in inequalities to check that f does indeed satisfy the final set of three conditions above. Less rigorously, we could graph f for different positive values of r, M, and N. Here's a graph for $r = 2$, $M = 3$, and $N = 5$.



We call the ODE

$$\dot{x} = rx\left(1 - \frac{x}{N}\right)\left(\frac{x}{M} - 1\right) \tag{2.5.3}$$

the **MODIFIED LOGISTIC EQUATION**. So, do solutions to this ODE do what they should do? (The existence of solutions should be easy to deduce by now, right?) Namely, what is $\lim_{t\to\infty} x(t)$ for a solution x to this problem? Slope fields and Euler's method give us lots of detail at a "local" level, over finite time intervals, but we can't take a slope field to ∞ nor keep doing Euler's method forever.

We certainly could try to separate variables. I claim that the equilibrium solutions are $x = 0$, N, and M — do you see that? But for nonequilibrium solutions, we would need to solve

$$\int \frac{dx}{rx\left(1 - \frac{x}{N}\right)\left(\frac{x}{M} - 1\right)}\, dx = \int 1\, dt + C.$$

We can all do the integral on the right, but how about the one on the left? That would require even more partial fractions than we did with the logistic equation, and then there's the whole task of solving for x. And after that we would *still* have to calculate $\lim_{t\to\infty} x(t)$, if x is even defined for all time.

There is a much better way of proceeding. We will develop fairly simple techniques for predicting the end behavior of solutions to autonomous ODE like (2.5.3) that do not rely on formulas, or even a whole lot of calculus.

2.5.2. Maximal existence.

We can first assure ourselves that solutions to autonomous ODE exist. The following is a direct consequence of Theorem 2.4.1.

2.5.2 Theorem (Existence and uniqueness for autonomous ODE).

Let f be differentiable on the interval (a, b), and suppose that f' is continuous on (a, b). (The values $a = -\infty$ and/or $b = \infty$ are allowed.) Let x_0 be a point in (a, b). Then there is $\epsilon > 0$ such that the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

has a unique solution x defined on $(-\epsilon, \epsilon)$.

The only difference with the IVP here compared to those in Theorem 2.4.1 is that we have taken $t_0 = 0$, purely for convenience. That is, when studying autonomous ODE, we will always place our initial time at $t_0 = 0$. This really makes no difference other than simplifying notation (and according with our gut instinct that time begins at 0). After all, the ODE $\dot{x} = f(x)$ is independent of time!

While this result is comforting, its use by itself is limited. The theorem tells us nothing about the long-time properties of x, and nothing, in principle, about ϵ. For just how long in time does x exist?

This is where we finished on Monday, September 19, 2022.

2.5.3 Example.

Here are five different autonomous (and hence separable) IVP. All have the form

$$\begin{cases} \dot{x} = f(x) \\ x(0) = 1, \end{cases}$$

and finding the formulas below is a good exercise for you. In each case, a domain ("Dom.") for the solution ("Soln.") is given. This is not necessarily the largest interval on which the solution is defined but rather the largest open interval on which the solution is defined and on which it solves the IVP. (Of course, any domain to an IVP with the initial condition at $t = 0$ must contain the point $t = 0$.) The limit ("Lim.") as t approaches the right endpoint of the domain from the left is given. What do you observe?

	(1)	**(2)**	**(3)**	**(4)**	**(5)**
ODE	$\dot{x} = x$	$\dot{x} = x(1-x)$	$\dot{x} = x^2$	$\dot{x} = -\dfrac{1}{x}$	$\dot{x} = \sqrt{5-x}$
IC	$x(0) = 1$	$x(0) = 1$	$x(0) = 1$	$x(0) = 1$	$x(0) = 1$
Soln.	$x(t) = e^t$	$x(t) = 1$	$x(t) = \dfrac{1}{1-t}$	$x(t) = \sqrt{1-2t}$	$x(t) = 5 - \dfrac{(t-4)^2}{4}$
Dom.	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, 1)$	$\left(-\infty, \dfrac{1}{2}\right)$	$(-\infty, 4)$
Lim.	∞	1	∞	0	5

Solution. The first two IVP are not very exciting anymore; they are exponential and logistic growth, and so we know their solutions are defined for all time. In particular, the solution for logistic growth here is the equilibrium solution $x(t) = 1$.

The other three IVP are more interesting. Each has the form $\dot{x} = f(x)$ for a relatively "tame" function f — you probably wouldn't have minded meeting any of these three f on an exam in calculus. But the given domains all have a finite right endpoint.

In (3), we have the "catastrophic" situation

$$\lim_{t \to 1^-} x(t) = \lim_{t \to 1^-} \frac{1}{1-t} = \infty$$

I hope it's easy to see why we can't push this solution past time $t = 1$: the solution blows up to $+\infty$! Where else can it go beyond $t = 1$ when it's already at ∞?

In (4), however, the solution $x(t) = \sqrt{1-2t}$ is defined and continuous at time $t = 1/2$, and yet we exclude $t = 1/2$ from the domain in (4). Why? Here we need to think carefully about the square root: *the square root is defined and continuous but not differentiable at 0.* Indeed, if $g(s) = \sqrt{s}$, then $g'(s) = -s^{-1/2}$, which is not defined at $s = 0$; graphically, the slope of the square root becomes infinitely steep as $s \to 0^+$. (Go check it out in your favorite graphing program and zoom way in.) Recall from Definitions 1.3.1 and 1.3.4 that a solution to an ODE/IVP must not only be defined and continuous but also differentiable at all points of its domain. Thus $t = 1/2$ cannot belong to the domain of $x(t) = \sqrt{1-2t}$ if we are considering x not merely as a function but as a function that solves $\dot{x} = -1/x$.

It is also interesting to note in (4) that if we put $f(x) = -1/x$, then f is not defined at $x = 0$. But the solution $x(t) = \sqrt{1-2t}$ tends to 0 as t approaches $1/2$ from the left. Not only is the solution not differentiable at $t = 1/2$, it approaches a value outside the domain of f. How can we possibly plug this solution into $\dot{x} = -1/x$ at time $t = 1/2$ and get a numerical result that makes sense?

Finally, in (5), as a function of t, ignoring the ODE/IVP context, the function $x(t) = 5 - (t-4)^2/4$ is defined for all t. It's just a quadratic polynomial, after all. However, you can check that defining x in this way gives

$$\dot{x} = 2 - \frac{t}{2} \quad \text{and} \quad \sqrt{5-x} = \left|2 - \frac{t}{2}\right|.$$

(Here we need the rule $\sqrt{A^2} = |A|$ for any real number A.) And so to have $\dot{x(t)} = \sqrt{5 - x(t)}$, we need

$$2 - \frac{t}{2} = \left|2 - \frac{t}{2}\right|,$$

and thus $2 - t/2 \geq 0$, hence $t \leq 4$. My choice of the strict inequality $t < 4$ is not one of mathematical necessity here but to keep things in line with (3) and (4), where we were forced to exclude the right endpoint. Last, taking $f(x) = \sqrt{5-x}$ in (5), we note that f is defined but not differentiable at $x = 5$, which is the limit of our solution as t approaches 4 from the left (and, for that matter, from the right). ▲

Here is what I hope you are seeing: when a solution fails to be defined for the entire interval $[0, \infty)$, something "interesting" happens at the finite time when the solution fails to be defined. Either the solution explodes at the endpoint of its domain, or it behaves more tamely, but in a "bad" way relative to the underlying ODE. The moral is that solutions to ODE do not simply stop after a finite time or "vanish" into thin air at a particular moment — *something has to happen.*

The following theorem is a precise statement of that moral. This statement is technical and worth parsing slowly and carefully.

2.5.4 Theorem (Maximal existence).

Let f be continuously differentiable on the interval (a, b), and let x_0 be a point in (a, b). There exist numbers T_α and T_ω, with $T_\alpha < 0 < T_\omega$, and a unique solution x to the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

such that x is defined on the interval (T_α, T_ω) and (T_α, T_ω) is "maximal" in the sense x cannot be defined outside this interval and remain a solution to the IVP. More precisely, one, and only one, of the following three alternatives holds for T_ω.

(Ω1) $T_\omega = \infty$.

(Ω2) $T_\omega < \infty$ *and either* $\lim_{t \to T_\omega^-} x(t) = \infty$ *or* $\lim_{t \to T_\omega^-} x(t) = -\infty$.

(Ω3) $T_\omega < \infty$ *and either* $\lim_{t \to T_\omega^-} x(t) = a$ *or* $\lim_{t \to T_\omega^-} x(t) = b$.

Identical statements hold for T_α if we replace $T_\omega = \infty$ with $T_\alpha = -\infty$ in part (Ω1) and the left limit $\lim_{t \to T_\omega^-}$ with the right limit $\lim_{t \to T_\alpha^+}$ in parts (Ω2) and (Ω3).

This is a demanding theorem, so we'll paraphrase its conclusions more informally.

2.5.5 Remark.

(i) *The possibility* $T_\omega = \infty$ *in part* (Ω*1*) *of Theorem 2.5.4 is a sort of "ideal" result. It says that we can predict the future forever in our given model. However, it does not help us predict the behavior of our solution at* ∞*; we have no statement about* $\lim_{t\to\infty} x(t)$ *in part* (Ω*1*)*.*

(ii) *The possibilities* $\lim_{t\to T_\omega^-} x(t) = \pm\infty$ *in part* (Ω*2*) *of Theorem 2.5.4 are a sort of "catastrophic" result. Our solution simply explodes! People often call this phenomenon* BLOW-UP IN FINITE TIME. *This might represent a natural and expected result — say, the unbounded growth of a species given certain ideal environmental conditions — or maybe a flaw in our model.*

(iii) *The possibilities* $\lim_{t\to T_\omega^-} x(t) = a$ *or* $\lim_{t\to T_\omega^-} x(t) = b$ *in part* (Ω*3*) *of Theorem 2.5.4 are, perhaps, the most subtle. Recall that the ODE under consideration is* $\dot{x} = f(x)$*, and* f *is guaranteed to be continuously differentiable only on the interval* (a, b)*. Recall also that such nice behavior of* f *is a hypothesis of the existence and uniqueness theorem (Theorem 2.5.2). Saying that* $x(t)$ *tends to* a *or* b *as* t *approaches* T_ω *from the left means that* $x(t)$ *is leaving the domain of* f*. The domain of* f *is the value of "states" for which the model is valid. Once the solution leaves this realm of validity, we can no longer make predictions about its behavior from our original model.*

This is where we finished on Wednesday, September 21, 2022.

Theorem 2.5.4 is powerful, because it finally answers our question of "What happens in the future?" However, it does not give a definite answer: there are three possibilities, and there is no "test" presented to determine which one happens for a given IVP. It is possible to present such tests in fairly refined and excruciating detail via some demanding mathematical rigor; instead, we will develop a (somewhat less excruciating) tool (the phase line) to determine maximal domains and end behavior (with somewhat less rigor).

2.5.3. A toy version of the modified logistic equation.

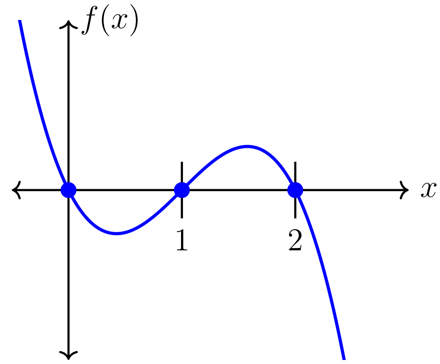
The modified logistic equation (2.5.3) has three parameters and fractions, so it's complicated. I claim that the ODE

$$\dot{x} = x(x-1)(2-x)$$

has all the same technical features as (2.5.3) with much simpler arithmetic and algebra. Indeed, if we put

$$f(x) := x(x-1)(2-x), \tag{2.5.4}$$

we can see that the graph of f has the same general shape as the graph of the more complicated function defined in (2.5.2).



For the remainder of this section, we will define f via (2.5.4). Note that f is just a cubic polynomial, so f is continuously differentiable on $(-\infty, \infty)$. Also, $f(x) = 0$ if and only if $x = 0$, $x = 1$, or $x = 2$. Hence the equilibrium solutions of $\dot{x} = f(x)$ are $x = 0$, $x = 1$, and $x = 2$.

Since f is continuously differentiable on $(-\infty, \infty)$, the maximal existence theorem (Theorem 2.5.4) tells us that the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

has a unique solution x defined on a maximal interval (T_α, T_ω) for any initial condition x_0. Here $T_\alpha < 0 < T_\omega$. In the language of Theorem 2.5.4, we have $a = -\infty$ and $b = \infty$ here, and so alternatives (Ω2) and (Ω3) of that theorem are really the same. Thus if $T_\omega < \infty$, then either $\lim_{t \to T_\omega^-} x(t) = \infty$ or $\lim_{t \to T_\omega^+} x(t) = -\infty$. That is, if the maximal endpoint T_ω is finite, we must have a blow-up in finite time.

We will study how the choice of x_0 influences the end behavior of x. Since we are really interested in population models, we will make the simplifying assumption $x_0 \geq 0$. Also, for simplicity, we will only study T_ω; that is, we will only predict behavior in the future, not in the past.

We break our work into the following steps.

1. Suppose that x solves the IVP with $0 < x_0 < 1$. Recommended Problem #3 from Week 5 says that $0 < x(t) < 1$ for all t in (T_α, T_ω). In particular, x is bounded below (by 0).

2. We try to determine how x is behaving at time $t = 0$ by calculating

$$\dot{x}(0) = f(x(0)) = f(x_0) < 0,$$

since $0 < x_0 < 1$. Thus x is decreasing at time $t = 0$.

3. I claim that x is always decreasing. Otherwise, at some time $t_1 > 0$, x would be increasing. Then $\dot{x}(t_1) > 0$. But since $\dot{x}$ is continuous[12], and since $\dot{x}(0) < 0 < \dot{x}(t_1)$, the intermediate

[12] This is why Definitions 1.3.1 and 1.3.4 are important.

value theorem provides t_2 in the interval $(0, t_1)$ such that $\dot{x}(t_2) = 0$. Then

$$0 = \dot{x}(t_2) = f(x(t_2)).$$

The only zeros of f are 0, 1, and 2, so $x(t_2)$ must equal one of these three numbers. But we know that $0 < x(t_2) < 1$, so $x(t_2)$ cannot equal any of the roots of f. Thus it is not possible for x to be increasing, and so x must be decreasing.

4. The function x is therefore decreasing and bounded below on $(0, T_\omega)$. A theorem from calculus then asserts that the limit $L := \lim_{t \to T_\omega^-} x(t)$ exists. This is a pretty deep theorem, but we can think about it graphically: the values of $x(t)$ keep getting smaller as time goes on. But they are always trapped below by 0. So what else can they do but "bunch up" around some limit L?

5. The existence of L as a finite real number rules out alternatives (Ω2) and (Ω3) from Theorem 2.5.4. In particular, T_ω cannot be finite, and so $T_\omega = \infty$. We have thus ensured that the solution continues for all time, and so we can predict the future forever!

6. We can say more about L than just existence. Since $0 < x(t) < 1$, properties of limits force $0 \le L \le 1$. (Limits do not have to maintain strict inequalities.) But x is decreasing, so more properties of limits imply $L < x(0)$. In particular, $0 \le L < 1$.

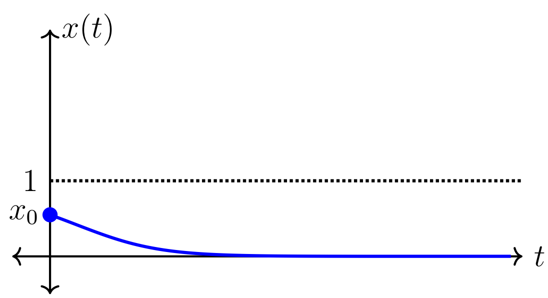
7. So now we know $\lim_{t\to\infty} x(t) = L$. That is, x has the horizontal asymptote L as $t \to \infty$. Horizontal asymptotes should call to mind "flat" graphs, and we should expect that the slope of x gets close to 0 as $t \to \infty$. In other words, we expect $\lim_{t\to\infty} \dot{x}(t) = 0$. Then we can use the fact that x solves $\dot{x} = f(x)$ to calculate

$$0 = \lim_{t\to\infty} \dot{x}(t) = \lim_{t\to\infty} f(x(t)) = f\left(\lim_{t\to\infty} x(t)\right) = f(L).$$

The third equality follows from the continuity of f. And so L is a root of f, which means L is one of the three numbers 0, 1, or 2. But we also know $0 \le L < 1$. The only possibility is $L = 0$.

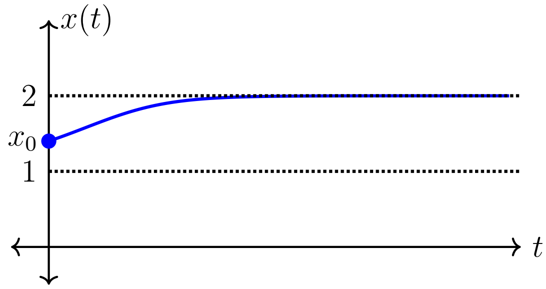
8. We conclude that if x solves the IVP with $0 < x(0) < 1$, then x is defined for all time in $[0, \infty)$, x is strictly decreasing, $0 < x(t) < 1$ for all t, and $\lim_{t\to\infty} x(t) = 0$. This is a staggering amount of information about x. (In particular, it's effectively our wish list from our very early explorations in Section 1.2.) We got all this information without needing any formulas!

Let's celebrate with a graph of x.



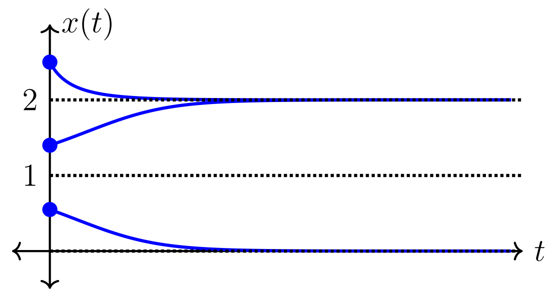
9. Now let's suppose that x solves the IVP with $1 < x(0) < 2$. Exactly the same reasoning as in Step 1 tells us that $1 < x(t) < 2$ for all t; as in Steps 2 and 3 that x is increasing (not decreasing) for all t in $(0, T_\omega)$; as in Step 4 and 6 that $L := \lim_{t \to T_\omega^-} x(t)$ exists and that $1 < L \leq 2$ (here we need the fact that if x is increasing and bounded above on $(0, T_\omega)$, then the limit L exists); as in Step 5 that $T_\omega = \infty$; and as in Step 7 that $f(L) = 0$. Since $1 < L \leq 2$, we obtain $L = 2$. And so x is defined and strictly increasing on $(0, \infty)$ with $1 < x(t) < 2$ for all t and $\lim_{t\to\infty} x(t) = 2$.

Here's a graph.



10. Finally, if x solves the IVP with $2 < x(0)$, I claim that all the work above can be adapted again to show that $2 < x(t)$ for all t, $T_\omega = \infty$, and x is strictly decreasing on $(0, \infty)$ with $\lim_{t\to\infty} x(t) = 2$.

Here's a graph of all possible solutions to this problem (assuming $x_0 \geq 0$). The equilibrium solutions are dotted and the nonequilibrium (one example per type of initial value) solutions are solid.

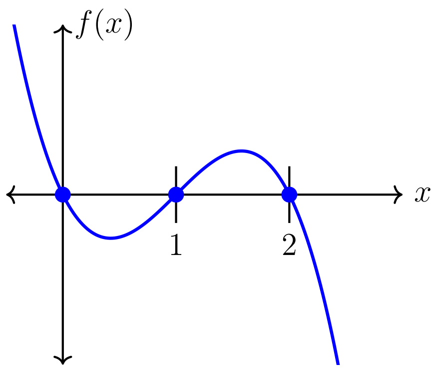


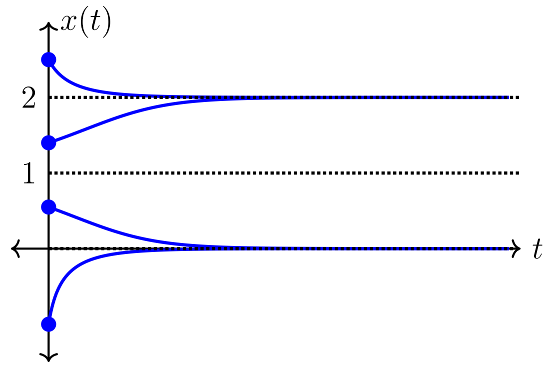
We can generalize the demanding ideas behind this analysis to some fairly simple "tests" for the behavior of solutions to autonomous IVP depending on where the initial condition falls relative to the equilibrium solutions. These tests will effectively allow us to predict the future *completely*, at least if we are expecting the future to be governed by an autonomous IVP. (Unfortunately, always in motion, the future is.)

This is where we finished on Friday, September 23, 2022.

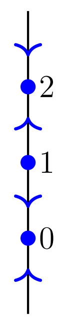
2.5.4. Constructing and interpreting phase lines.

All of our work on the toy modified logistic problem $\dot{x} = f(x)$, where $f(x) = x(x-1)(2-x)$, boils down to two pictures. On the left, we have the graph of f as a function of the "state" variable x. On the right, we have the graphs of different solutions x to $\dot{x} = f(x)$ as functions of time t. (This time I have also drawn a solution with $x(0) < 0$.)





If all we care about is the end behavior of solutions to $\dot{x} = f(x)$, then there is an easy way to encode this information. Draw a vertical line and mark dots on it corresponding to the equilibrium solutions $x = 0$, 1, 2. Between the dots, draw an upwards-pointing arrow if a solution with initial value in that interval is increasing, and a downwards-pointing arrow if a solution with initial value in that interval is decreasing.



The drawing above is called the **phase line** for the ODE $\dot{x} = x(x-1)(2-x)$. It tells us what the equilibrium solutions are and how solutions that start at nonequilibrium values behave over long times — whether those solutions are increasing or decreasing and what their long-time limits[13] are. Here is how you draw a phase line in general for the ODE $\dot{x} = f(x)$.

1. Find the equilibrium solutions by solving $f(x) = 0$. Draw a vertical line and mark the equilibrium on the line with dots; label them, too.

2. Consider a solution to $\dot{x} = f(x)$. If $f(x(0)) = 0$, then $x(0)$ is an equilibrium solution to $\dot{x} = f(x)$, and $x(t) = x(0)$ for all t.

3. If $f(x(0)) > 0$, then x is strictly increasing on its domain. If there is an equilibrium

[13] Here I am being cagey about the domain of the solution. By "long-time limits" I mean the limit as time approaches the right endpoint of the maximal interval of existence from the left, i.e., $\lim_{t \to T_\omega^-} x(t)$. I am not saying whether $T_\omega = \infty$ or $T_\omega < \infty$.

solution greater than $x(0)$, then the domain includes[14] $[0, \infty)$ and $\lim_{t\to\infty} x(t)$ equals the smallest equilibrium solution greater than $x(0)$. Otherwise, if there is no equilibrium solution greater than $x(0)$, then $x(t) \to \infty$ as t approaches the right endpoint of the domain of x.

4. If $f(x(0)) < 0$, then x is strictly decreasing on its domain. If there is an equilibrium solution less than $x(0)$, then the domain includes $[0, \infty)$, and $\lim_{t\to\infty} x(t)$ equals the largest equilibrium solution less than $x(0)$. Otherwise, if there is no equilibrium solution less than $x(0)$, then $x(t) \to -\infty$ as t approaches the right endpoint of the domain of x.

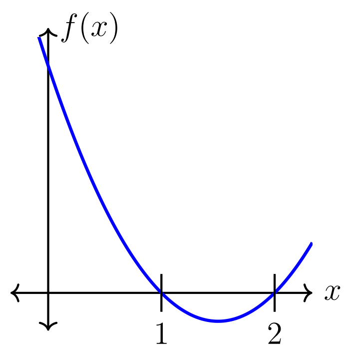
2.5.6 Example.

Draw the phase line for

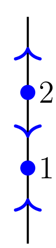
$$\dot{x} = (x-1)(x-2)$$

and use the phase line to sketch all solutions to this ODE.

Solution. The equilibrium solutions are $x = 1$, 2. First we graph $f(x) = (x-1)(x-2)$ against x to help us see where f is positive and negative.



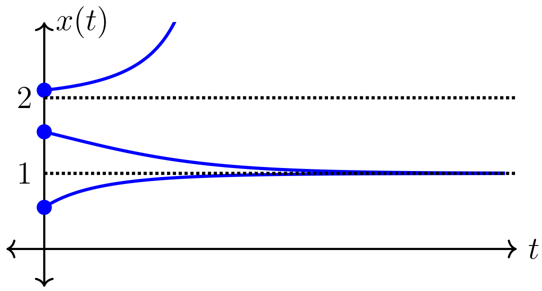
Our graph reveals that if $x < 1$, then $f(x) > 0$, if $1 < x < 2$, then $f(x) < 0$, and if $x > 2$, then $f(x) > 0$. From this we draw the phase line.



So, solutions that start below 1 are increasing and tend to 1 over long times; solutions that start between 1 and 2 are decreasing and (also) tend to 1 over long times; and solutions

[14] This follows from the maximal existence theorem. We already know that x is increasing on its domain, which is (T_α, T_ω); if there is an equilibrium solution x_* greater than $x(0)$, then $x(t) < x_*$ for all t in $[0, T_\omega)$. Hence x is increasing and bounded above, so $L := \lim_{t\to T_\omega^-} x(t)$ exists as a finite real number, in particular $L \le x_*$. The only way we can have $T_\omega < \infty$ is to also have $L = \pm\infty$, and that doesn't happen here.

that start above 2 are increasing and tend to ∞ over long times. Here is a sketch of all the solutions; we draw the equilibrium solutions as dotted black lines and (representatives of) the nonequilibrium solutions as solid blue curves.



▲

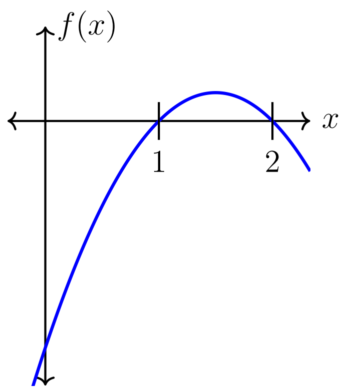
2.5.7 Example.

Draw the phase line for

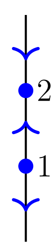
$$\dot{x} = (1-x)(x-2)$$

and use the phase line to sketch all solutions for this ODE.

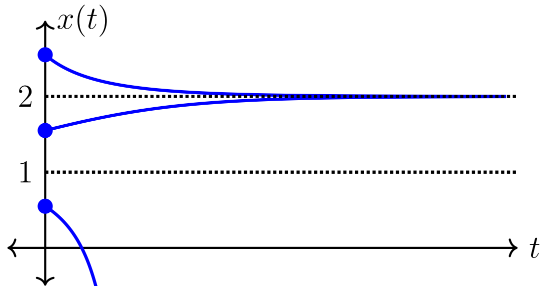
Solution. The equilibrium solutions are (again) $x = 1, 2$. Here is the graph of $f(x) = (1-x)(x-2)$; note that $(1-x)(x-2) = -(x-1)(x-2)$ from Example 2.5.6.



Our graph shows that if $x < 1$, then $f(x) < 0$; if $1 < x < 2$, then $f(x) > 0$; and if $x > 2$, then $f(x) < 0$. In other words, this is exactly the opposite of Example 2.5.6. From this we draw the phase line.



Solutions that start below 1 are decreasing to $-\infty$; solutions that start between 1 and 2 are increasing to 2; and solutions that start above 2 are decreasing to 2.

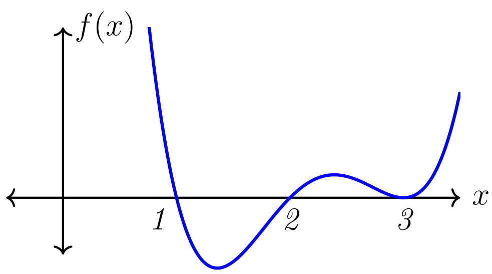


Again, this is the opposite of what happened in Example 2.5.6; flipping the sign of the derivative changes a function from increasing to decreasing, or from decreasing to increasing. ▲

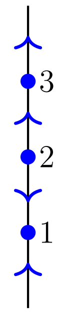
To draw a phase line for $\dot{x} = f(x)$, we don't necessarily need (or want) a formula for f. Rather, we need to know the roots (zeros) of f and f is positive or negative between those roots; in fact, if we assume that f is continuous (as we almost always do), then calculus tells us that the sign of f doesn't change between those roots. (This is the intermediate value theorem.) Given a good graph of f, we can get all of this information.

2.5.8 Example.

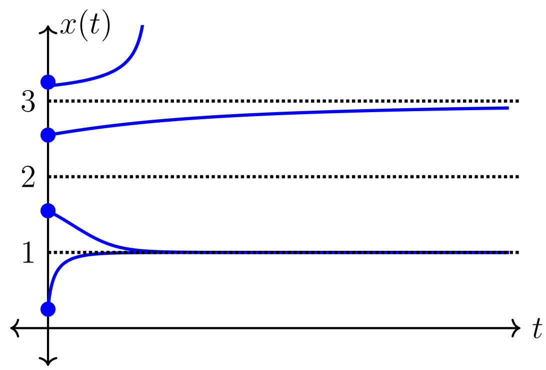
Use the graph of the function f drawn below to sketch the phase line for $\dot{x} = f(x)$ and then solutions for this ODE.



Solution. Assuming that f maintains its apparent behavior outside the given snippet, the only roots of f are $x = 1$, 2, 3, and so these are the equilibrium solutions. We see that if $x < 1$, then $f(x) > 0$; if $1 < x < 2$, then $f(x) < 0$; if $2 < x < 3$, then $f(x) > 0$; and if $x > 3$, then $f(x) > 0$, still. Here is the resulting phase line.



Here is the resulting behavior of solutions. A solution that starts below 1 increases to 1; a solution that starts between 1 and 2 decreases to 1; a solution that starts between 2 and 3 increases to 3; and a solution that starts above 3 increases to ∞.



▲

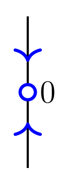
In the preceding examples of phase lines for the autonomous problem $\dot{x} = f(x)$, the function f was defined and continuously differentiable on $(-\infty, \infty)$. If f fails to be defined at some point, we cannot consider that point as an equilibrium solution to the ODE, but that point is nonetheless likely to influence the long-time behavior of solutions. We will mark points at which f fails to be defined by circles, not dots, on the phase line.

2.5.9 Example.

Draw the phase line for $\dot{x} = -1/x$. What does this tell you about solutions to the ODE?

Solution. Here is the graph of $f(x) = -1/x$.

This function is not defined at $x = 0$, so we will mark 0 with a circle, not a dot, on the phase line. If $x < 0$, then $f(x) > 0$, while if $x > 0$, then $f(x) < 0$. So, we draw the phase line as usual.



This tells us that solutions that start below 0 increase to 0, while solutions that start above 0 decrease to 0. Some care, however, is needed if we want to sketch the graphs of solutions.

Recall from Example 2.5.3 that

$$\begin{cases} \dot{x} = -1/x \\ x(0) = 1. \end{cases}$$

has the analytic solution $x(t) = \sqrt{1-2t}$ on $(-\infty, 1/2)$. This solution is defined and continuous, but not differentiable (and therefore not a solution), at $t = 1/2$. We have

$$\lim_{t \to 1/2^-} \sqrt{1-2t} = 0,$$

and the ODE is not defined at $x = 0$. Specifically, the ODE is $\dot{x} = f(x)$, where $f(x) = -1/x$. As the solution approaches $t = 1/2$ from the left, the values of the solution leave the domain of the ODE; we might say, euphemistically, that the values of the solution "fall into a hole."

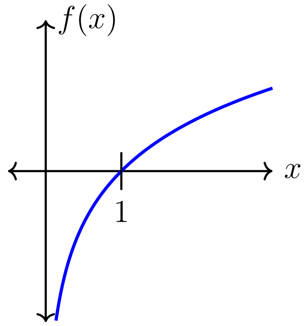
More generally, solutions to $\dot{x} = -1/x$ will not be defined on the whole interval $[0, \infty)$, and so they don't have horizontal asymptotes at 0. Nonetheless, they still limit to 0 at the right endpoint of their domain. ▲

This is where we finished on Monday, September 26, 2022.

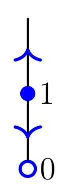
2.5.10 Example.

Draw the phase line and sketch some solutions for $\dot{x} = \ln(x)$.

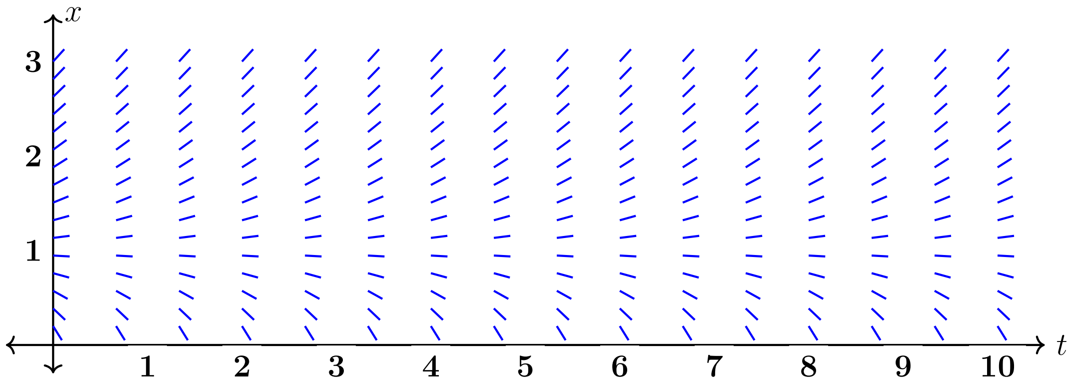
Solution. First, recall that $f(x) := \ln(x)$ is not defined when $x \le 0$. Otherwise, we have $\ln(x) < 0$ for $0 < x < 1$, with a vertical asymptote at $x = 0$; $\ln(1) = 0$, and so $x = 1$ is an equilibrium solution; and $\ln(x) > 0$ for $x > 0$ (with, by the way, $\lim_{x\to\infty} \ln(x) = \infty$ very slowly). Here's a graph.



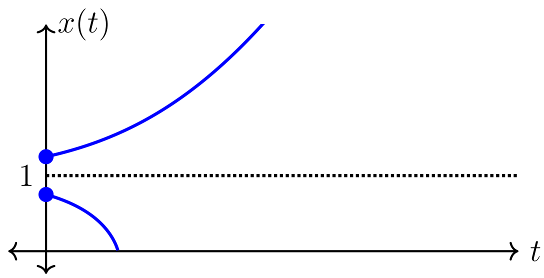
To draw the phase line, we need to exclude $x < 0$ from consideration, as the problem $\dot{x} = \ln(x)$ does not have a solution if we demand $x(0) \le 0$. We do draw a circle, not a dot, at $x = 0$, as this is the "first" point at which the natural log is not defined.



Solutions cannot start at or below 0; solutions that start between 0 and 1 decrease to 0; and solutions that start above 1 increase to ∞. Without further information, however, it is difficult to attempt a sketch — we do not know the concavity of these solutions, and we also don't know if they are defined for all time. For example, could a solution starting between 0 and 1 reach 0 in finite time? Here is a slope field to help.



I think that solutions that start between 0 and 1 will, in fact, reach 0 in finite time (i.e., the solutions x are defined on $[0, T_\omega)$ with $\lim_{t \to T_\omega^-} x(t) = 0$), and so here is a cautious sketch.



▲

Phase lines convey useful and concise information about predicting the future, chiefly the range of solutions (are they bounded between equilibrium points or unbounded?) and their long-time limits. However, phase lines by themselves also lack lots of information. Here are some complaints.

1. The domain of a solution, in particular if a solution is defined on $[0, \infty)$, may not be apparent from a phase line.

2. The concavity of a solution may not be apparent from a phase line.

3. If a solution converges to a finite limit over long time, the *rate* of convergence may not be apparent from a phase line. For example,

$$\lim_{t \to \infty} \frac{1}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} e^{-t} = 0,$$

but the latter function converges much more quickly to 0 than the former — look at the graphs to see which one "flattens out" first. A phase line can't tell you that.

4. The exact value of a solution at a particular moment in time is definitely not something that a phase line can tell you, unless that solution is an equilibrium solution.

I don't make these remarks to disparage phase lines. Rather, I want to remind you, again, that no one ODE tool will tell you all that you want, or need, to know about a problem. If your life depends on understanding the solutions to an ODE[15], then you should use all the tools available to you — formulas, slope fields, numerical methods, phase lines, and more.

2.6. Linear equations.

Two classes of first-order ODE (with some overlap between them) broadly appear in models and theory. We have already met separable problems; now we will study a class called **LINEAR**[16] ODE. Our solutions to linear ODE will be among our greatest successes; in particular, we will derive a relatively painless procedure for finding a formula for the solution *any* linear ODE, and this process will be consistently more successful than separation of variables. To motivate the value of linear equations, we study first a new population problem.

2.6.1. Harvesting.

Life was good for our population models in prior examples. With the exception of a certain bad regime in the modified logistic equation, either our populations always exploded to ∞, or they happily leveled out around a carrying capacity. Either way, they survived, and probably prospered.

The good times are over! Famine, pestilence, and peril are on the horizon! Suppose that we have a population that, in the absence of external malice, grows exponentially. With $x(t)$ as the population at time t, we expect $\dot{x} = rx$. Now, however, because this population is useful and/or delicious, we decide to harvest (hunt, remove) some members of the population.

Say that at time t, we harvest $H(t)$ units of the population per unit time. Since we are removing members of the population and not adding them, we expect $H(t) \geq 0$. Recalling that the general rate of change of a quantity subject to both influx and removal is "Rate in minus Rate out," per (RI−RO), we have

$$\dot{x} = rx - H(t). \tag{2.6.1}$$

If H is not constant, then this problem is not separable; in particular, it is not autonomous. Of course, we could use slope fields and Euler's method to analyze it, given a formula for H.

How might we choose H? There are lots of valid harvesting schemes, but the one I like here is *periodic harvesting*. After all, this is probably how we harvest crops and hunt game on a seasonal/annual basis. So, we want H to be a nonnegative periodic[17] function.

There are many such functions, but maybe the most familiar nonconstant periodic function is the sine. However, $-1 \leq \sin(t) \leq 1$, so let's add 1 to get $0 \leq 1 + \sin(t) \leq 2$. If we take $H(t) = 1 + \sin(t)$, then, yes, we get a nonconstant, nonnegative periodic function.

[15] Note that I intentionally did not say "If your life depends on solving an ODE."

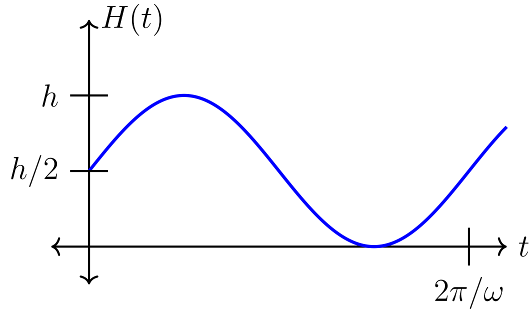
[16] In general in math, when you hear the word "linear," you should expect that good things will happen when you add things and/or multiply by constants. This will be apparent shortly.

[17] A function H is **PERIODIC** if there is a number $P \neq 0$ such that $H(t + P) = H(t)$ for all t.

But this would only allow us to harvest with a rate ranging between 0 and 2. If we multiply by a number $h > 0$, then taking $H(t) = h(1 + \sin(t))$ allows us to remove anywhere between 0 and $2h$ members of the population per unit time. Let me refine this further to $H(t) = h(1+\sin(t))/2$ so that we are harvesting between 0 and h members of the population per unit time. Finally, the sine is 2π-periodic, and we probably don't want to harvest with 2π-periodicity. To give us control over the frequency of harvesting, let me incorporate one more parameter: put

$$H(t) = \frac{h\big(1 + \sin(\omega t)\big)}{2}. \tag{2.6.2}$$

Here's a graph of H, which, by the way, is $2\pi/\omega$-periodic (why?).



The ODE that governs our exponentially growing population subject to harvesting is now

$$\dot{x} = rx - \frac{h\big(1 + \sin(\omega t)\big)}{2}. \tag{2.6.3}$$

There are three parameters in this ODE: the positive numbers ω, h, and r. We might wonder what effect tweaking the values of these three parameters has on the solutions. Of course, we could go to slope fields and/or numerics and make observations. But suppose we wanted to answer definitively the following question: *are there values of the parameters ω, h, and r that cause the population to go extinct?* That is, can we harvest in such a way that we kill off the population, and, if so, how "sensitive" is the population's behavior to the values of ω, h, and r? While we have long since accepted that having formulas is not the same as understanding (inhale, recite (AC), exhale), but I think this situation is exactly why we want precise formulas for the solution to (2.6.3). Qualitative and numerical methods alone simply won't cut it.

2.6.2. Definitions.

The harvesting ODE (2.6.3) has the following form.

2.6.1 Definition.

An ODE $\dot{x} = f(t, x)$ is **LINEAR** *if f has the special form $f(t, x) = a(t)x+b(t)$ for functions a and b. That is, a linear ODE is an equation of the form*

$$\dot{x} = a(t)x + b(t). \tag{2.6.4}$$

The function a is the **COEFFICIENT** *and the function b is the* **FORCING** *or* **DRIVING** *term.*

Specifically, in (2.6.3), we had $a(t) = r$ and $b(t) = -H(t)$, with H defined in (2.6.2). We call a the coefficient because a multiplies x, and coefficients are supposed to multiply things; calling b the forcing/driving term is a convention that stems from terminology for *second*-order ODE, which naturally involve things being forced or driven by external[18] influences.

Various special choices of a and b make a linear ODE fairly easy to solve. For example, if $a(t) = 0$ for all t, then the ODE is just $\dot{x} = b(t)$, which can be solved with direct integration. Or, if the functions a and b are constant, then the ODE is $\dot{x} = ax + b$, which is autonomous, and which therefore can be solved with separation of variables. We will develop a general method for solving linear ODE, and it will be illustrative to compare the results of that method to the results of direct integration or separation of variables when one of those techniques applies, too.

2.6.3. The structure of solutions to homogeneous linear ODE.

There is one special case of the linear ODE that is both very easy to solve with separation of variables and very worth knowing for use in the near future.

2.6.2 Definition.

The linear ODE (2.6.4) *is* **HOMOGENEOUS** *if* $b(t) = 0$ *for all* t*. That is, a homogeneous ODE is an equation of the form*

$$\dot{x} = a(t)x. \tag{2.6.5}$$

The ODE (2.6.4) *is* **NONHOMOGENEOUS** *(sometimes* **INHOMOGENEOUS***) if it is not homogeneous, i.e., if* $b(t) \neq 0$ *for at least one* t*.*

We can solve (2.6.5) via separation of variables. First, the only equilibrium solution is $x = 0$. Assuming $x \neq 0$, divide and integrate to find

$$\int \frac{dx}{x} = \int a(t)\ dt + C. \tag{2.6.6}$$

If we assume that a is continuous on some interval, then a always has an antiderivative on that interval; call the antiderivative A. That is, $\dot{A} = a$. Then we integrate on the left in (2.6.6) to find

$$\ln(|x|) = A(t) + C.$$

Exponentiating, we have

$$|x| = e^C e^{A(t)},$$

and so

$$x = \pm e^C e^{A(t)},$$

which we collapse, as usual, into

$$x = Ce^{A(t)}.$$

We summarize this result more formally.

[18] See Footnote 28.

2.6.3 Theorem (Linear structure I: homogeneous ODE).

Let a be continuous on the interval I. If x solves $\dot{x} = a(t)x$ on I, then we can write x in the form

$$x(t) = Ce^{A(t)},$$

where C is a constant and A is an antiderivative of a on I.

Proof. Separation of variables essentially gives us this result, but it is also a nice opportunity to check our work using existence and uniqueness. First, fix some point t_0 in I and let $x_0 := x(t_0)$. The fundamental theorem of calculus (Theorem 1.3.6) allows us to define an antiderivative A of a on I by

$$A(t) := \int_{t_0}^{t} a(\tau)\, d\tau;$$

note that $A(t_0) = 0$.

Now put

$$y(t) := x_0 e^{A(t)}.$$

We calculate

$$y(t_0) = x_0 e^{A(t_0)} = x_0 e^0 = x_0 \cdot 1 = x_0 \quad \text{and} \quad \dot{y}(t) = x_0 e^{A(t)} \dot{A}(t) = x_0 e^{A(t)} a(t) = a(t)y(t),$$

and so y solves the IVP

$$\begin{cases} \dot{y} = a(t)y \\ y(t_0) = x_0. \end{cases}$$

But x clearly solves this IVP, too, since we are assuming that $\dot{x} = a(t)x$, and since we defined $x_0 = x(t_0)$.

Last, observe that $f(t,x) := a(t)x$ satisfies the hypotheses of Theorem 2.4.1. Thus $x(t) = y(t)$ for all t, and so $x(t) = x_0 e^{A(t)}$. $\square$

The proof above is, I think, a nice illustration of how we can always check our work in ODE. No matter how dubious the calculations that wrest from the very clay of the earth a formula for a putative solution to an ODE, we can always check that formula by plugging it into our equation. And if we take our time, we can probably use the existence and uniqueness theorem to explain why that formula is the *only* formula that works.

This is where we finished on Wednesday, September 28, 2022.

2.6.4 Remark.

Contrary to everything that we learned in calculus, we do not need to include a constant of integration when calculating the antiderivative A in Theorem 2.6.3. Indeed, if we do include a constant of integration, which we call K here, the solution x would have the form

$$x(t) = Ce^{A(t)+K} = (Ce^K)e^{A(t)}.$$

And so we could just lump the constant of integration in with the free constant C.

2.6.5 Example.

Determine all solutions to $\dot{x} = \sin(t)x$ *and, in particular, find the solution satisfying* $x(0) = 5$.

Solution. We could separate variables, but we could also use the abstract results above. By Theorem 2.6.3, all solutions to $\dot{x} = \sin(t)x$ have the form

$$x = Ce^{\int \sin(t)\ dt} = Ce^{-\cos(t)}.$$

To find x with $x(0) = 5$, we need $5 = Ce^{-\cos(0)} = Ce^{-1}$. We solve for C: $C = 5(e^{-1})^{-1} = 5e$. Thus the solution with $x(0) = 5$ is $x(t) = (5e)e^{-\cos(t)} = 5e^{1-\cos(t)}$. ▲

2.6.4. The structure of solutions to nonhomogeneous linear ODE.

We now have complete control over linear homogeneous ODE, so we turn to the nonhomogeneous case. What can we learn about the linear ODE $\dot{x} = a(t)x + b(t)$ without actually solving it? The right idea, which may not be the obvious idea, is to suppose that we have two solutions x and y and study their difference. Specifically, suppose

$$\dot{x} = a(t)x + b(t) \quad \text{and} \quad \dot{y} = a(t)y + b(t)$$

and set

$$z := x - y.$$

Then z measures how different x and y are.

You know what they say: "if it moves, differentiate it." We calculate

$$\begin{aligned}
\dot{z} &= \frac{d}{dt}[x - y] \\
&= \dot{x} - \dot{y} \\
&= \big(a(t)x + b(t)\big) - \big(a(t)y + b(t)\big) \\
&= a(t)x + b(t) - a(t)y - b(t) \\
&= a(t)x - a(t)y \\
&= a(t)\big(x - y\big) \\
&= a(t)z.
\end{aligned}$$

Thus z satisfies the linear homogeneous ODE

$$\dot{z} = a(t)z,$$

and so there is a constant C such that

$$z(t) = Ce^{A(t)},$$

where A is an antiderivative of a.

Thus

$$x - y = z = Ce^{A(t)},$$

and so

$$x = y + Ce^{A(t)}.$$

This cuts down our workload dramatically. It tells us that if we have found one solution y to our linear ODE, then every other solution x has the form above; all we have to do is choose the constant C correctly. Let's state this formally and then informally.

2.6.6 Theorem (Linear structure II: nonhomogeneous ODE).

Let a be continuous on the interval I, let b be a function on I, and suppose that y solves

$$\dot{y} = a(t)y + b(t).$$

Let x be another solution to this ODE:

$$\dot{x} = a(t)x + b(t).$$

Then there is a constant C such that

$$x(t) = y(t) + Ce^{A(t)} \tag{2.6.7}$$

for all t, where A is an antiderivative of a.

We can paraphrase this theorem in what I hope is an evocative manner, but first we need a definition.

2.6.7 Definition.

Let a and b be functions. The ODE $\dot{z} = a(t)z$ is the **ASSOCIATED HOMOGENEOUS ODE** *for the nonhomogeneous problem $\dot{x} = a(t)x + b(t)$.*

Note that in (2.6.7), the function $z(t) := e^{A(t)}$ solves the associated homogeneous ODE $\dot{z} = a(t)z$.

2.6.8 Remark.

Here is the paraphrase in words of the second structure theorem: every solution to a nonhomogeneous ODE is the sum of a particular solution to the nonhomogeneous ODE and a constant multiple of a solution to the associated homogeneous ODE. (The first structure theorem, by the way, just says that every solution to a homogeneous ODE is a constant multiple of one particular solution to that homogeneous ODE.)

Here is how we can use the second structure theorem.

2.6.9 Example.

One can (and you should) verify that the function $y(t) := 3te^{-t^2}$ solves $\dot{y} = -2ty + 3e^{-t^2}$. Use this fact to find all functions x that solve $\dot{x} = -2tx + 3e^{-t^2}$.

Solution. The associated homogeneous ODE is $\dot{z} = -2tz$, and all solutions to this ODE have the form

$$z(t) = Ce^{\int -2t \, dt} = Ce^{-t^2}.$$

The second structure theorem then guarantees that every solution x to $\dot{x} = -2tx + 3e^{-t^2}$ has the form

$$x(t) = 3te^{-t^2} + Ce^{-t^2}$$

for some constant C. It will be worth noting for later use that both terms in this formula for x, the particular solution $y(t) = 3te^{-t^2}$ and the homogeneous solution $z(t) = Ce^{-t^2}$, contain the factor e^{-t^2}, and $z_{\mathsf{h}}(t) := e^{-t^2}$ solves the homogeneous problem $\dot{z}_{\mathsf{h}} = -2tz_{\mathsf{h}}$. ▲

2.6.5. The product rule and an illustrative toy problem.

The second structure theorem tells us that to find all solutions to a linear nonhomogeneous problem, all we have to do is get our hands on *one* solution to the problem. Everything else is antidifferentiating and adding! So how do we find *one* such solution? We will develop a remarkable procedure that *always* works. It hinges on seeing the product rule in surprising places, so we'll do an example of that. Recall that if x and v are functions, then

$$\frac{d}{dt}[xv] = \dot{x}v + x\dot{v}. \tag{2.6.8}$$

2.6.10 Example.

(i) *Use the product rule to calculate*

$$\frac{d}{dt}[t\sin(t^2)].$$

(ii) *Can you recognize the expression*

$$2e^{2t}\cos(3t) - 3te^{2t}\sin(3t)$$

as the product rule-style derivative of a function?

Solution. (i) The product rule says

$$\frac{d}{dt}[t\sin(t^2)] = \frac{d}{dt}[t]\sin(t^2) + t\frac{d}{dt}[\sin(t^2)] = 1\cdot\sin(t^2) + t\cos(t^2)(2t) = \sin(t^2) + 2t^2\cos(t^2).$$

(ii) No fair just integrating this expression. (Do you really want to? You'd have to integrate by parts on things like $\int e^{2t}\cos(3t)\, dt$, and that's a chore.)

Here is what I see: we have two terms, each with a factor of e^{2t}, and each with a cosine or sine evaluated at $3t$. We know that cosine and sine are related by differentiation, and we expect a 2 to pop out of differentiating e^{2t} and a 3 to pop out of differentiating a composition with $3t$. So, after some fumbling, I say

$$2e^{2t}\cos(3t) - 3te^{2t}\sin(3t) = \big[2e^{2t}\big]\cos(3t) + e^{2t}\big[-3\sin(3t)\big]$$

$$= \frac{d}{dt}[e^{2t}]\cos(3t) + e^{2t}\frac{d}{dt}[\cos(3t)]$$
$$= \frac{d}{dt}[e^{2t}\cos(3t)]. \qquad \blacktriangle$$

This is where we finished on Friday, September 30, 2022.

We will now, at last, develop a general method for solving linear ODE. We'll extract the general details from the specific toy problem of

$$\dot{x} = -2tx + 3e^{-t^2}, \tag{2.6.9}$$

which we previously studied in Example 2.6.9. Some of the following steps may not seem obvious — hopefully they feel logically correct, but the motivation or insight behind them may be obscure. Don't worry; these ideas have unfolded over centuries of study, and we only have a semester together.

The first thing to do is to rewrite (2.6.9) as

$$\dot{x} + x(2t) = 3e^{-t^2}. \tag{2.6.10}$$

The left side looks very vaguely like the product rule as we wrote it in (2.6.8): it's a sum of two terms, one term has a factor of $\dot{x}$, and the other term has a factor of x. If the left side were a genuine product rule-style derivative, then we would have

$$\dot{x} + 2tx = \dot{x}v + x\dot{v} = \frac{d}{dt}[xv]$$

for some function v. That would certainly simplify things.

What's wrong? The left side of (2.6.10) doesn't have any t-dependent factor on $\dot{x}$. (Okay, $\dot{x} = \dot{x} \cdot 1$, but that's not helpful.) How do we introduce a missing factor in a math problem? Go forth and multiply!

Specifically, let us enter the land of wishful thinking: suppose that μ is a function[19], and multiply both sides of (2.6.9) by $\mu(t)$ to find that x must solve

$$\dot{x}(t)\mu(t) + x(t)[2t\mu(t)] = 3e^{-t^2}\mu(t). \tag{2.6.11}$$

I am now writing the dependence of x and $\dot{x}$ on t explicitly. So, if x solves our original problem (2.6.9), then x also solves (2.6.11). Conversely, if we can divide out $\mu(t)$ from both sides of (2.6.11), then we get back (2.6.9). So, let us impose our first wish:

$$\mu(t) \neq 0 \text{ for all } t. \tag{μ1}$$

If this wish is met, then we know that any function x that solves (2.6.11) will solve (2.6.9), and so we can just focus on (2.6.11). Now, (2.6.11) looks a lot more like a product rule, since each term contains two factors, one involving x, the other not. We really want

$$\dot{x}(t)\mu(t) + x(t)[2t\mu(t)] = \dot{x}(t)v(t) + x(t)\dot{v}(t)$$

[19] The symbol μ is the Greek letter "mu," and it is traditional to use μ here because we are *mu*ltiplying, or, should I say, μltiplying.

for some function v, and we can achieve this by putting $v = \mu$. But then we need a second wish:

$$\dot{\mu}(t) = 2t\mu(t) \tag{$\mu 2$}$$

for all t.

Suppose that our wishes have been granted. Then (2.6.11) becomes

$$\dot{x}(t)\mu(t) + x(t)\dot{\mu}(t) = 3e^{-t^2}\mu(t). \tag{2.6.12}$$

We recognize the left side as the product rule:

$$\dot{x}(t)\mu(t) + x(t)\dot{\mu}(t) = \frac{d}{dt}[x\mu].$$

Thus (2.6.12) is the same as

$$\frac{d}{dt}[x\mu] = 3e^{-t^2}\mu(t). \tag{2.6.13}$$

In fact, if μ satisfies our wishes ($\mu 1$) and ($\mu 2$), then (2.6.9), (2.6.10), (2.6.11), (2.6.12), and (2.6.13) are all the same equation, just dressed up differently.

The goal of life is always to solve for x; in (2.6.13), x is stuck inside a derivative. We remove derivatives by integrating. That is, if we have (2.6.13), then we must have

$$\int \frac{d}{dt}[x\mu]\ dt = \int 3e^{-t^2}\mu(t)\ dt. \tag{2.6.14}$$

Since

$$\int \frac{d}{dt}[x\mu]\ dt = x\mu + C,$$

where C is a constant of integration, (2.6.14) becomes

$$x(t)\mu(t) = 3\int e^{-t^2}\mu(t)\ dt + C. \tag{2.6.15}$$

Here we are performing the usual calculus sleight-of-hand of putting the constant of integration on the right and still calling it C. Last, since $\mu(t) \neq 0$ by our wish ($\mu 1$), we may divide both sides of (2.6.15) by $\mu(t)$ and thereby solve for x:

$$x(t) = \frac{3}{\mu(t)}\int e^{-t^2}\mu(t)\ dt + \frac{C}{\mu(t)}. \tag{2.6.16}$$

Up to determining what μ is, we have solved for x; unlike separation of variables, there was no question of solving an "implicit" equation for x — we just got x explicitly! By the way, in (2.6.16), we are committing the notational awkwardness of letting the independent variable be t and also letting the variable of integration be t. The point is that $\int e^{-t^2}\mu(t)\ dt$ represents an antiderivative of the function $q(t) := e^{-t^2}\mu(t)$.

Now, how can we make our wishes ($\mu 1$) and ($\mu 2$) come true? The key is ($\mu 2$): this is a homogeneous ODE for μ. Indeed, we know how to solve

$$\dot{\mu} = 2t\mu \tag{2.6.17}$$

by now: all solutions, by Theorem 2.6.3 have the form

$$\mu(t) = Ke^{\int 2t\ dt} = Ke^{t^2}$$

for some constant K. To guarantee our wish ($\mu 1$), we want to avoid $K = 0$. Moreover, the wish ($\mu 2$) doesn't call for any specific solution to the ODE (2.6.17), so let's keep life simple and take $K = 1$. That is, we define

$$\mu(t) := e^{t^2}.$$

We will call μ the **INTEGRATING FACTOR** for our ODE (2.6.9), because multiplying both sides of (2.6.9) effectively reduces solving the ODE to performing a direct integration.

Plugging this formula for μ into (2.6.16) gives

$$x(t) = \frac{3}{e^{t^2}} \int e^{-t^2} e^{t^2}\ dt + \frac{C}{e^{t^2}} = 3e^{-t^2} \int\ dt + Ce^{-t^2} = 3te^{-t^2} + Ce^{-t^2}.$$

And this, you might recall, is exactly what we got in Example 2.6.9.

2.6.6. Integrating factors.

Here we distill the results from the preceding toy problem into a more general method for solving linear ODE.

1. Obtain a linear ODE.	$\dot{x} = -2tx + 3e^{-t^2}$	$\dot{x} = a(t)x + b(t)$
2. Subtract the x-term from both sides.	$\dot{x} + 2tx = 3e^{-t^2}$ $\dot{x}(t) + x(t)[2t] = 3e^{-t^2}$	$\dot{x} - a(t)x = b(t)$ $g(t) := -a(t)$ $\dot{x}(t) + x(t)g(t) = b(t)$
3. Obtain the integrating factor.	Solve $\dot{\mu} = 2t\mu$. Obtain $\mu(t) = e^{\int 2t} = e^{t^2}$.	Solve $\dot{\mu} = g(t)\mu$. Obtain $\mu(t) = e^{\int g(t)} = e^{G(t)}$.
4. Multiply through by the integrating factor.	$\dot{x}(t)e^{t^2} + x(t)[2te^{t^2}] = 3e^{-t^2}e^{t^2}$	$\dot{x}(t)e^{G(t)} + x(t)[g(t)e^{G(t)}]. = b(t)e^{G(t)}$
5. Find the product rule.	$\dot{x}(t)e^{t^2} + x(t)[2te^{t^2}] = \dot{x}(t)e^{t^2} + x(t)\frac{d}{dt}[e^{t^2}] = \frac{d}{dt}[x(t)e^{t^2}]$	$\dot{x}(t)e^{G(t)} + x(t)[g(t)e^{G(t)}] = \dot{x}(t)e^{G(t)} + x(t)\frac{d}{dt}[e^{G(t)}] = \frac{d}{dt}[x(t)e^{G(t)}]$
6. Rewrite and simplify.	$\frac{d}{dt}[x(t)e^{t^2}] = 3$	$\frac{d}{dt}[x(t)e^{G(t)}] = b(t)e^{G(t)}$
7. Integrate.	$\int \frac{d}{dt}[x(t)e^{t^2}]\ dt = \int 3\ dt$ $x(t)e^{t^2} = 3t + C$	$\int \frac{d}{dt}[x(t)e^{G(t)}]\ dt = \int b(t)e^{G(t)}\ dt$ $x(t)e^{G(t)} = \int b(t)e^{G(t)}\ dt + C$
8. Solve for x.	$x(t) = 3te^{-t^2} + Ce^{-t^2}$	$x(t) = e^{-G(t)} \int b(t)e^{G(t)}\ dt + Ce^{-G(t)}$

This is where we finished on Monday, October 3, 2022.

The function μ that we used several times above has a special name.

2.6.11 Definition.

An INTEGRATING FACTOR *for the ODE* $\dot{x} = a(t)x + b(t)$ *is a function* μ *of the form* $\mu(t) = e^{-A(t)}$*, where* A *is any antiderivative of* a*.*

The "integrating factor method" outlined in the table above can be summarized as *subtract the homogeneous term* $a(t)x$ *from both sides of the equation, multiply both sides of the result by the integrating factor, look out for the product rule, and hope for the best.*

2.6.12 Example.

Find all solutions to

$$\dot{x} = \cos(t)x + \cos(t).$$

Solution. This is a linear nonhomogeneous ODE with $a(t) = b(t) = \cos(t)$. We rearrange the ODE as

$$\dot{x} - \cos(t)x = \cos(t).$$

To reflect further the anticipated structure of the product rule, we regroup the terms on the left:

$$\dot{x} + x[-\cos(t)] = \cos(t).$$

The integrating factor is

$$\mu(t) = e^{\int -\cos(t)\ dt} = e^{-\int \cos(t)\ dt} = e^{-\sin(t)}.$$

We only need *one* antiderivative of a in the integrating factor, so we do not include a constant of integration.

Now we multiply both sides of our rearranged ODE by the integrating factor:

$$\dot{x}(t)e^{-\sin(t)} + x(t)[-\cos(t)]e^{-\sin(t)} = \cos(t)e^{-\sin(t)}.$$

We check our work by looking for the product rule structure on the left. We have

$$\frac{d}{dt}[e^{-\sin(t)}] = e^{-\sin(t)}\frac{d}{dt}[-\sin(t)] = e^{-\sin(t)}[-\cos(t)],$$

and so

$$\dot{x}(t)e^{-\sin(t)} + x(t)[-\cos(t)]e^{-\sin(t)} = \dot{x}(t)e^{-\sin(t)} + x(t)\frac{d}{dt}[e^{-\sin(t)}] = \frac{d}{dt}[x(t)e^{-\sin(t)}]$$

by the product rule.

Thus our solution x must satisfy

$$\frac{d}{dt}[x(t)e^{-\sin(t)}] = \cos(t)e^{-\sin(t)}.$$

We integrate both sides to find

$$\int \frac{d}{dt}[x(t)e^{-\sin(t)}]\ dt = \int \cos(t)e^{-\sin(t)}\ dt. \tag{2.6.18}$$

The integral on the left is

$$\int \frac{d}{dt}[x(t)e^{-\sin(t)}]\ dt = x(t)e^{-\sin(t)} + C$$

for some constant C.

For the integral on the right, we substitute $u = -\sin(t)$ to find $du = -\cos(t)\ dt$ and $\cos(t)\ dt = -du$, and so

$$\int \cos(t)e^{-\sin(t)}\ dt = \int -e^u\ du = -e^u + K = -e^{-\sin(t)} + K,$$

where K is (also) a constant of integration. We conclude

$$x(t)e^{-\sin(t)} + C = -e^{-\sin(t)} + K.$$

We perform the usual (and suspicious) algebra of combining both constants of integration into one constant, which we naturally call C. Thus

$$x(t)e^{-\sin(t)} = -e^{-\sin(t)} + C,$$

and so we solve for x as

$$x(t) = e^{\sin(t)}[-e^{-\sin(t)} + C] = -e^{\sin(t)-\sin(t)} + Ce^{-\sin(t)} = -1 + Ce^{-\sin(t)}.$$ ▲

2.6.13 Example.

Find all solutions to, again,

$$\dot{x} = \cos(t)x + \cos(t)$$

by interpreting the equation as a separable problem, finding an equilibrium solution, and then using one of the structure theorems.

Solution. We factor

$$\cos(t)x + \cos(t) = \cos(t)(x+1)$$

to see that the ODE has the separable form

$$\dot{x} = \cos(t)(x+1).$$

We could therefore find all solutions, presumably, by separating variables; we won't go that far, but we do note that solving $x+1 = 0$ for $x = -1$ will give an equilibrium solution. That is, the constant function $x(t) = -1$ solves the ODE; this is easy to check, since $\dot{x}(t) = 0$ when x is constant, and

$$\cos(t)x(t) + \cos(t) = \cos(t)(-1) + \cos(t) = 0$$

for all t, as well.

So, we have found one particular solution to the nonhomogeneous problem. And all solutions to the associated homogeneous problem $\dot{z} = \cos(t)z$ have the form $z(t) = Ce^{\int \cos(t)\ dt} = Ce^{\sin(t)}$ for some constant C. (Here we are, as usual, interpreting the indefinite integral in the exponent as one particular antiderivative of $a(t) = \cos(t)$.) The second structure theorem then tells us that all solutions to the nonhomogeneous problem $\dot{x} = \cos(t)x + \cos(t)$ have the form

$$x(t) = -1 + Ce^{\sin(t)}$$

for some constant C, exactly as we found in the previous example. ▲

2.6.14 Remark.

It is natural to find the competing constants of integration C and K in Example 2.6.12 confusing. Rewrite (2.6.18) *as*

$$\int \frac{d}{dt}[x(t)e^{-\sin(t)}]\ dt - \int \cos(t)e^{-\sin(t)}\ dt = 0. \tag{2.6.19}$$

You have evaluated the sum of indefinite integrals in calculus plenty of times; for example,

$$\int t^2\ dt + \int e^t\ dt = \frac{t^3}{3} + e^t + C.$$

We usually do not hesitate to write only one constant of integration when dealing with the sum of two or more indefinite integrals. In a similar way, we could evaluate (relying on the work in Example 2.6.12)

$$\begin{aligned}\int \frac{d}{dt}[x(t)e^{-\sin(t)}]\ dt - \int \cos(t)e^{-\sin(t)}\ dt &= x(t)e^{-\sin(t)} - [-e^{-\sin(t)}] + C \\ &= x(t)e^{-\sin(t)} + e^{-\sin(t)} + C.\end{aligned}$$

Then (2.6.19) *reads*

$$x(t)e^{-\sin(t)} + e^{-\sin(t)} + C = 0.$$

Divide through by $e^{-\sin(t)}$ to get

$$x(t) + 1 + Ce^{\sin(t)} = 0$$

and solve for x:

$$x(t) = -1 - Ce^{\sin(t)}. \tag{2.6.20}$$

Since C, as a constant of integration, is allowed to be any real number, $-C$ can also be any real number. We follow the usual conventions from calculus and just write

$$x(t) = -1 + Ce^{\sin(t)}$$

in (2.6.20) *instead.*

Many of the ambiguities that constants of integration introduce could be eliminated if we worked with definite integrals instead. Weasel words like the sentence "Since C, as a *constant of integration, is allowed to be any real number, $-C$ can also be any real number" are designed to handle the fact that an indefinite integral represents a set of functions (the set of all antiderivatives of a given function), and so doing "algebra" with this set can be slippery. The definite integral is just that: definite. The challenge of the definite integral, which practice makes easier, is choosing the right limits of integration.*

This is where we finished on Wednesday, October 5, 2022.

The solution to the following IVP illustrates how using a definite integral can cut down on some work and ambiguities related to the constant of integration.

2.6.15 Example.

Solve the IVP

$$\begin{cases} \dot{x} = \dfrac{x}{t} + t \\ x(1) = 2. \end{cases}$$

Solution. This ODE has the linear structure $\dot{x} = a(t)x + b(t)$ with $a(t) = 1/t$ and $b(t) = t$. First we rewrite the ODE as

$$\dot{x} - \frac{x}{t} = t,$$

and then we rewrite that as

$$\dot{x}(t) + x(t)\left(-\frac{1}{t}\right) = t,$$

in an effort to make things look as much like the product rule as possible.

We introduce the integrating factor

$$\mu(t) = e^{\int(-1/t)\ dt} = e^{-\ln(|t|)} = e^{\ln(|t|^{-1})} = |t|^{-1}.$$

Here we have used some properties of the natural logarithm. Now, both the coefficient a and the integrating factor μ are undefined at $t = 0$. However, the initial value calls for $x(1) = 1$, and so we will just solve this ODE for $t > 0$; that will certainly include $t = 1$. Thus we can simplify

$$\mu(t) = |t|^{-1} = t^{-1} = \frac{1}{t}$$

for $t > 0$.

We multiply both sides of our rearranged ODE by μ to find

$$\dot{x}(t)\left(\frac{1}{t}\right) + x(t)\left(-\frac{1}{t}\right)\left(\frac{1}{t}\right) = t\left(\frac{1}{t}\right),$$

and thus

$$\dot{x}(t)\left(\frac{1}{t}\right) + x(t)\left(-\frac{1}{t^2}\right) = 1.$$

We check that we have a product rule-style structure on the left:

$$\frac{d}{dt}\left[\frac{1}{t}\right] = \frac{d}{dt}[t^{-1}] = -t^{-2} = -\frac{1}{t^2},$$

and so

$$\dot{x}(t)\left(\frac{1}{t}\right) + x(t)\left(-\frac{1}{t^2}\right) = \frac{d}{dt}\left[x(t)\left(\frac{1}{t}\right)\right].$$

Thus x solves

$$\frac{d}{dt}\left[x(t)\left(\frac{1}{t}\right)\right] = 1.$$

Naturally, we want to integrate both sides to solve for x. However, since we are dealing with an IVP here, let's integrate with a definite integral:

$$\int_1^t \frac{d}{d\tau}\left[x(\tau)\left(\frac{1}{\tau}\right)\right]\ d\tau = \int_1^t 1\ d\tau. \tag{2.6.21}$$

We always want x to be a function of t, so I am using τ as the variable of integration and t as the upper limit of integration. The lower limit of integration is 1 because that is the time at which the initial value is taken.

We find

$$x(t)\left(\frac{1}{t}\right) - x(1)\left(\frac{1}{1}\right) = t - 1.$$

This simplifies slightly to

$$\frac{x(t)}{t} - x(1) = t - 1,$$

and we want $x(1) = 2$. Thus

$$\frac{x(t)}{t} - 2 = t - 1,$$

and so we solve for x:

$$x(t) = t(t+1).$$ ▲

2.7. Three more perspectives on linear ODE.

The integrating factor method is possibly our one complete success in this course. While it is not short, we can always solve for x. We didn't have this guarantee of explicit success with separation of variables. Although every linear problem can be solved with integrating factors, there are several additional perspectives on linear ODE that can enrich and embiggen our lives.

2.7.1. Duhamel's formula.

The integrating factor method gives us a much better result than a well-intentioned (but naive) application of the existence and uniqueness theorem will; solutions to a linear ODE are defined at least as long as the coefficient a and the driving term b are both continuous. Here is a precise statement of that result.

2.7.1 Theorem.

Let a and b be continuous on the interval I, let t_0 be a point in I, and let x_0 be a real number. Put $A(t) := \int_{t_0}^{t} a(\tau)\ d\tau$. Then the unique solution to the IVP

$$\begin{cases} \dot{x} = a(t)x + b(t) \\ x(t_0) = x_0 \end{cases}$$

is

$$x(t) = x_0 e^{A(t)} + e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} b(\tau)\ d\tau. \qquad (2.7.1)$$

This solution x is defined on the entire interval I. The formula (2.7.1) *is sometimes called* **Duhamel's formula**.

Proof. First, observe that A is defined for all t in I since a is continuous on I. The fundamental theorem of calculus then says that A is differentiable $\dot{A}(t) = a(t)$ for all t in I. Likewise, the integral $\int_{t_0}^{t} e^{-A(\tau)} b(\tau)\ d\tau$ is defined for all t in I since the integrand is continuous on I. (The forcing term b does not have to be differentiable on I.) We can then check directly using the product rule and the fundamental theorem of calculus that x as defined in (2.7.1) solves the IVP. The existence and uniqueness theorem applies (why does it apply to this IVP?) to show that x as defined here is the *only* solution.

Here is another, more constructive, proof of existence. Suppose that x solves this IVP: we have $\dot{x}(t) = a(t)x(t) + b(t)$ for all t in I and $x(t_0) = x_0$. Use the integrating factor method as outlined at the start of this section to produce the formula (2.7.1). Take

$$\mu(t) = e^{-\int_{t_0}^{t} a(\tau)\ d\tau}$$

as the integrating factor, i.e., $G(t) = -\int_{t_0}^{t} a(\tau)\ d\tau$; when integrating in Step 7, use a definite integral and integrate from t_0 to t. (Call the variable of integration τ, not t, as we did in (2.6.21) for clarity.) Don't succumb to the temptation to multiply $e^{A(t)}$ and $e^{-A(\tau)}$ together and get 1; one factor has t, the other τ. $\square$

There is just one downside to the integrating factor method: we have to evaluate two integrals, one to construct the integrating factor, and one to antidifferentiate the product of the integrating factor and the forcing function. In Theorem 2.7.1, these are the integrals $A(t) = \int_{t_0}^{t} a(\tau)\ d\tau$ and $\int_{t_0}^{t} e^{-A(\tau)} b(\tau)\ d\tau$. It may not be possible to evaluate these integrals explicitly. (Okay, *two* downsides.) Very much the same happened in separation of variables, although here we can always solve for x. By the way, you could try to memorize (2.7.1), but it's detailed and quite easy to get wrong. Best to be comfortable with the integrating factor method and above all with the role of the product rule.

2.7.2 Example.

Find all solutions to

$$\dot{x} + 3t^2 x = e^{t^2 - t^3}.$$

Solution. Our equation is already in the form $\dot{x} + g(t)x = b(t)$ — this is sometimes called **STANDARD FORM**, by the way — and so we can get the integrating factor as

$$\mu(t) = e^{\int 3t^2 \ dt} = e^{t^3}.$$

Multiplying both sides of the ODE by μ, we find

$$\dot{x}(t)e^{t^3} + x(t)[3t^2 e^{t^3}] = e^{t^2}.$$

We recognize the left as the product rule

$$\frac{d}{dt}[x(t)e^{t^3}],$$

and we recognize the right as something that's hard (impossible) to antidifferentiate in terms of "elementary" functions.

This is a good time to introduce a definite integral. Both the coefficient on x and the forcing function in this ODE are defined at all real numbers, so let's integrate both sides of

$$\frac{d}{dt}[x(t)e^{t^3}] = e^{t^2}$$

from 0 to t; here the lower limit of 0 is just convenient. We find

$$\int_0^t \frac{d}{d\tau}[x(\tau)e^{\tau^3}] \ d\tau = \int_0^t e^{\tau^2} \ d\tau.$$

We can't get any further on the right, but on the left we can use the fundamental theorem of calculus:

$$\int_0^t \frac{d}{d\tau}[x(\tau)e^{\tau^3}] \ d\tau = x(t)e^{t^3} - x(0).$$

Thus

$$x(t)e^{t^3} - x(0) = \int_0^t e^{\tau^2} \ d\tau,$$

and so we solve for x as

$$x(t) = x(0)e^{-t^3} + e^{-t^3}\int_0^t e^{\tau^2} \ d\tau.$$

We can view the unknown value $x(0)$ as the "free parameter" in this solution, which we would ordinarily write as C. ▲

We will now explore two alternatives to the integrating factor method that, at least partially, avoid some of the integrals.

2.7.2. Variation of parameters.

Look back at Duhamel's formula in (2.7.1). It says that the solution x to $\dot{x} = a(t)x + b(t)$ has the form

$$x(t) = \left(x(t_0) + \int_{t_0}^{t} e^{-A(\tau)}b(\tau)\ d\tau\right) e^{A(t)},$$

where A is an antiderivative of a. Here we are assuming that a and b are continuous on the interval I and t_0 is a point in I. Call the big factor in parentheses u:

$$u(t) = x(t_0) + \int_{t_0}^{t} e^{-A(\tau)}b(\tau)\ d\tau. \tag{2.7.2}$$

Then we can paraphrase Duhamel's formula as follows: every solution x to $\dot{x} = a(t)x + b(t)$ has the form $x(t) = u(t)e^{A(t)}$, where A is an antiderivative of a. (Note that the factor $e^{A(t)}$ is not the integrating factor, which is $\mu(t) = e^{-A(t)}$. Thus Duhamel's formula *really* says that every solution is the product of the reciprocal of the integrating factor and some other function.)

This motivates an alternative approach to the integrating factor method: guess that the solution x to $\dot{x} = a(t)x + b(t)$ has the form $x(t) = u(t)e^{A(t)}$ for some $u$known function u, with A an antiderivative of a. Plug this guess into the ODE and solve for u. We guess all the time in math, but that doesn't mean we have to feel good about it. Maybe a fancy German word will help. Making an "educated guess" that a solution to an ODE has a particular form is called making an **ANSATZ** for that ODE; here the ansatz is

$$x(t) = u(t)e^{A(t)}, \tag{2.7.3}$$

and our job is to figure out what u is. Of course, we'll end up with something like (2.7.2), but maybe the guess will be faster.

This is where we finished on Friday, October 7, 2022.

This method of guessing $x(t) = u(t)e^{A(t)}$ and solving for u is sometimes called "variation of parameters," and also "variation of constants." (Caution: some people also call Duhamel's formula the "variation of parameters" formula.) I think this is because the solution to the homogeneous problem $\dot{y} = a(t)y$ is $y(t) = Ce^{A(t)}$ for a constant C, but now we are solving the nonhomogeneous problem $\dot{x} = a(t)x + b(t)$ with $x(t) = u(t)e^{A(t)}$: we have "varied" the parameter/constant C into the function $u(t)$.

2.7.3 Example.

Every solution x to

$$\dot{x} = \cos(t)x + 4\sin(t)e^{1+\sin(t)}$$

has the form $x(t) = u(t)e^{A(t)}$, where A is an antiderivative of $a(t) = \cos(t)$. Guess that x has such a form. What ODE must u solve? What is u?

Solution. We guess that $x(t) = u(t)e^{\sin(t)}$ solves $\dot{x} = \cos(t)x + 4\sin(t)e^{1+\sin(t)}$. With this guess, we have

$$\dot{x}(t) = \frac{d}{dt}[u(t)e^{\sin(t)}] = \dot{u}(t)e^{\sin(t)} + u(t)e^{\sin(t)}\cos(t),$$

and also

$$\cos(t)x + 4\sin(t)e^{1-\sin(t)} = \cos(t)[u(t)e^{\sin(t)}] + 4\sin(t)e^{1+\sin(t)}.$$

So, we need u to solve

$$\dot{u}(t)e^{\sin(t)} + u(t)e^{\sin(t)}\cos(t) = \cos(t)[u(t)e^{\sin(t)}] + 4\sin(t)e^{1+\sin(t)}.$$

We can subtract the common term $u(t)e^{\sin(t)}\cos(t)$ from both sides to find

$$\dot{u}(t)e^{\sin(t)} = 4\sin(t)e^{1+\sin(t)}.$$

Since $e^{1+\sin(t)} = e^{\sin(t)}e$, we can divide both sides by the common factor $e^{\sin(t)}$ to see that u solves

$$\dot{u} = 4e\sin(t).$$

This is a direct integration problem for u, and so

$$u = \int 4e\sin(t)\ dt = -4e\cos(t) + C$$

for some constant C. Thus every solution x to $\dot{x} = \cos(t)x + 4\sin(t)e^{1-\sin(t)}$ has the form

$$x(t) = \big(-4e\cos(t) + C\big)e^{\sin(t)}.$$ ▲

We didn't really get around having to evaluate two integrals in the work above, but if we were absolutely stuck and couldn't remember the integrating factor method, or the nitty-gritty details of Duhamel's formula, we might still have hope of remembering the ansatz $x(t) = u(t)e^{A(t)}$ and try solving for u. I claim that every integrating factor problem can be done with the variation of parameters ansatz (2.7.3). However, we wouldn't have arrived at this ansatz if we hadn't done the integrating factor method to get Duhamel's formula.

2.7.3. Constant-coefficient problems.

We started this whole project of linear ODE to handle the harvesting problem (2.6.3). Let me remind you what that problem said:

$$\begin{cases} \dot{x} = rx - \dfrac{h\big(1+\sin(\omega t)\big)}{2} \\ x(0) = x_0. \end{cases} \tag{2.7.4}$$

Here x is, of course, the harvested population; $r > 0$ is the growth rate of that population; and h and ω are positive numbers, with h controlling the maximum rate of harvesting and ω the frequency of harvesting. Finally, I have specified that the initial population (at time $t = 0$, when we first start harvesting) is x_0.

The integrating factor for this problem is easy to calculate; it's $\mu(t) = e^{-rt}$. However, if we were to work through the integrating factor method, or go to Duhamel's formula, we would have to compute the integral

$$\int_0^t e^{-r\tau} \left(\frac{h\big(1+\sin(\omega\tau)\big)}{2} \right) \, d\tau = \frac{h}{2} \int_0^t e^{-r\tau} \, d\tau + \frac{h}{2} \int_0^t e^{-r\tau} \sin(\omega\tau) \, d\tau.$$

The first integral on the right is easy, but the second requires some clever integration by parts. (We've been clever in this course so far, but we haven't been clever enough to integrate by parts.)

The result is so messy that I'm not even going to tell it to you here[20]. Instead, this sort of problem begs the question *Do we really have to integrate?* The answer, happily, is no, at least for some special problems that arise with reasonable frequency. These special problems have a name.

2.7.4 Definition.

A linear ODE $\dot{x} = a(t)x + b(t)$ *is* **CONSTANT-COEFFICIENT** *if the coefficient* a *is constant. That is, a constant-coefficient linear ODE has the form*

$$\dot{x} = ax + b(t),$$

where a *is a real number and* b *is a function (which does not have to be constant).*

Note that (2.7.4) is constant-coefficient, since here $a(t) = r$, and r is a fixed real number. In (2.7.4) we have $b(t) = -h(1 + \sin(\omega t))/2$, and this b is definitely *not* constant.

It turns out that a clever observation, and an optimistic guess, convert the problem $\dot{x} = ax + b(t)$ into a question of *algebra*, not calculus, when b has some fairly special (but also, in applications, reasonably common) forms.

2.7.5 Example.

What sort of function x *can solve*

$$\dot{x} = -3x + 2e^{4t}?$$

Guess that x *has this form, and see what happens.*

Solution. This ODE says that the derivative of x is a multiple of itself and an exponential. What sort of functions have exponentials in their derivatives? Experience, I hope, says *func-*

[20] Maple tells me

$$x(t) = \frac{\big(2r^3x_0 - hr^2 + (2\omega^2 x_0 - h\omega)r - h\omega^2\big)e^{rt} + h\big(\sin(\omega t)r^2 + \cos(\omega t)\omega r + r^2 + \omega^2\big)}{r(\omega^2 + r^2)},$$

which is ghastly. Question for you: what does x do over very long time? It all depends on whether $2r^3x_0 - hr^2 + (2\omega^2x_0 - h\omega)r - h\omega^2$ is positive, negative, or zero. You'll have the pleasure of answering this question later, when we'll fix $r = \omega = 1$ for simplicity.

tions that involve exponentials. And functions that involve things other than exponentials, like sines, cosines, or logarithms, typically don't pop out exponentials in their derivatives.

Since the problem has the term $2e^{4t}$, it's reasonable to guess that the solution will involve an e^{4t} somewhere. The simplest nontrivial function of this form is $x(t) = \alpha e^{4t}$, where α is a number that we'll determine. Let's guess (make the ansatz) that x has this form.

We evaluate each side of the ODE at the ansatz $x(t) = \alpha e^{4t}$ to find

$$\dot{x}(t) = 4\alpha e^{4t} \tag{2.7.5}$$

and

$$-3x(t) + 2e^{4t} = -3\alpha e^{4t} + 2e^{4t} = (2 - 3\alpha)e^{4t}. \tag{2.7.6}$$

We equate (2.7.5) and (2.7.6) to find that we need

$$4\alpha e^{4t} = (2 - 3\alpha)e^{4t}. \tag{2.7.7}$$

We can cancel the common factor of e^{4t}, since it is always positive, to find

$$4\alpha = 2 - 3\alpha. \tag{2.7.8}$$

This turns a calculus problem into algebra: solve for α. And we do so to find $\alpha = 2/7$.

It is then fairly easy to check that

$$x(t) = \frac{2}{7}e^{4t}$$

solves the ODE $\dot{x} = -3x + 2e^{4t}$. We then add a constant multiple of the solution to the homogeneous problem $\dot{z} = -3z$, which is $z(t) = Ce^{-3t}$. Thus all solutions to $\dot{x} = -3x + 2e^{4t}$ have the form

$$x(t) = \frac{2e^{4t}}{7} + Ce^{-3t}.$$

By the way, we could also view the passage from (2.7.7) to (2.7.8) as follows. We want (2.7.7) to hold for all t. So, let's choose t so that (2.7.7) becomes very simple. One way to do that is to pick $t = 0$. This yields (2.7.8).

This is where we finished on Monday, October 10, 2022.

Now, what happens? We haven't asked about long-time behavior of solutions recently, in part because Duhamel's formula is so involved that we need pretty specialized knowledge of the "data" of our ODE (the coefficient a and the forcing function b) to make sense of the limit as $t \to \infty$. Here, however, the formula for x is very transparent. The first term explodes as $t \to \infty$ because of the factor of e^{4t}, while the second term vanishes, because of the factor of e^{-3t}. In particular, C has *no effect* on the long-time behavior of the solution: it is always the case that

$$\lim_{t\to\infty} x(t) = \frac{2}{7}\lim_{t\to\infty} e^{4t} + C\lim_{t\to\infty} e^{-3t} = \infty. \qquad \blacktriangle$$

The preceding example gave us a constant-coefficient problem of the form

$$\dot{x} = ax + be^{rt},$$

with a, b, and r constant, and we guessed

$$x(t) = \alpha e^{rt}$$

with α as our "undetermined coefficient." In other words, an exponential forcing function manifested itself in the solution as...an exponential!

2.7.6 Example.

What sort of function can solve $\dot{x} = -x + \cos(2t)$? Guess that x has this form, and see what happens.

Solution. The derivative of x has to involve a cosine, so it's a good bet that x does, too. We might guess $x(t) = \alpha\cos(2t)$, but that just differentiates into a sine, and sine doesn't appear anywhere in x or in the term $\cos(4t)$. Instead, we use undetermined coefficient*s*: guess

$$x(t) = \alpha\cos(2t) + \beta\sin(2t)$$

for some to-be-determined numbers α and β.

We plug this guess into our ODE. First, we have

$$\dot{x}(t) = -2\alpha\sin(2t) + 2\beta\cos(2t). \tag{2.7.9}$$

Next, the right side is

$$-x(t) + \cos(2t) = -\alpha\cos(2t) - \beta\sin(2t) + \cos(2t) \tag{2.7.10}$$

We equate (2.7.9) and (2.7.10) to find that we need

$$-2\alpha\sin(2t) + 2\beta\cos(2t) = -\alpha\cos(2t) - \beta\sin(2t) + \cos(2t). \tag{2.7.11}$$

There are no common factors here that we can divide out. Instead, it might pay off to combine "like terms," and so we collect everything in (2.7.11) together on one side and simplify to get

$$(-2\alpha + \beta)\sin(2t) + (2\beta + \alpha - 1)\cos(2t) = 0. \tag{2.7.12}$$

This equality has to be true for every value of t. Let's pick some friendly values. If we take $t = 0$, we can remove the sine term (since $\sin(0) = 0$ but $\cos(0) = 1$) and find that we need

$$2\beta + \alpha - 1 = 0. \tag{2.7.13}$$

To remove the cosine term, we should take t such that $\cos(2t) = 0$ but $\sin(2t) \neq 0$; one way to do this is $t = \pi/4$, since $\cos(\pi/2) = 0$ but $\sin(\pi/2) = 1$. Then we find

$$-2\alpha + \beta = 0,$$

This means $\beta = 2\alpha$, so we substitute that in (2.7.14) to find $5\alpha - 1 = 0$. Thus $\alpha = 1/5$ and so $\beta = 2/5$.

We conclude that a particular solution to $\dot{x} = -x + \cos(2t)$ is

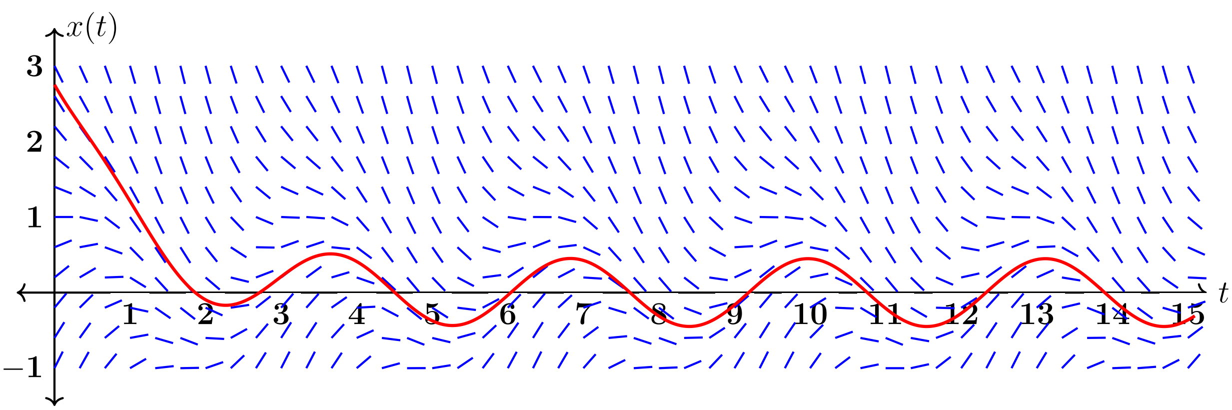
$$x(t) = \frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5}.$$

The associated homogeneous ODE is $\dot{z} = -z$, and all solutions to this ODE are $z(t) = Ce^{-t}$. Then all solutions to $\dot{x} = -x + \cos(2t)$ have the form

$$x(t) = Ce^{-t} + \frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5}. \tag{2.7.14}$$

So what happens? The exponential term Ce^{-t} dies out for any C as time goes on, but the sine and the cosine terms keep oscillating. The solution x does not have a limit as $t \to \infty$. However, the solution x exhibits *asymptotically sinusoidal* behavior over long times.

This is best shown in pictures. Here's a slope field with Euler's method superimposed on top.



We can phrase this analytically, too: (2.7.14) says that any solution x satisfies

$$x(t) - \left[\frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5}\right] = Ce^{-t},$$

so when t is large, Ce^{-t} is small, and therefore $x(t)$ is close to $\cos(2t)/5 - 2\sin(2t)/5$.

Of course, we could have solved this whole problem with the integrating factor method, but I claim that would require us to evaluate $\int e^t \cos(2t)\,dt$ along the way. We're probably happier not doing that. ▲

The preceding example gave us a constant-coefficient problem of the form

$$\dot{x} = ax + b_1\cos(\omega t) + b_2\sin(\omega t)$$

with a, b_1, b_2, and ω constant, and we guessed

$$x(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$$

with α and β as our "undetermined coefficients." In other words, a sinusoidal forcing function manifested itself in the solution as. . .a sinusoidal term.

A first guess is not always the best guess. If we altered Example 2.7.5 to $\dot{x} = -3x + 2e^{-3t}$ and guessed $x(t) = \alpha e^{-3t}$, we would need α to satisfy

$$-3\alpha e^{-3t} = -3\alpha e^{-3t} + 2e^{-3t},$$

which gives the incorrect identity $e^{-3t} = 0$ for all t. What went wrong? Our guess solved the associated homogeneous problem $\dot{y} = -3y$.

This is far from the end of the world, since we know how to solve $\dot{x} = -3x + 2e^{-3t}$ with the integrating factor method, or Duhamel's formula, or variation of parameters. Any of those methods (try them) will give us a solution in the form $x(t) = 2te^{-3t}$. This is almost our guess, except for the factor of t. And so here is a good idea for the future: if the first guess fails, whatever that guess was, multiply it by t. More precisely, when faced with a constant-coefficient problem of the form

$$\dot{x} = ax + be^{at},$$

where a and b are real numbers, guess

$$x(t) = \alpha te^{at}.$$

2.8. Laplace transform methods for first-order linear ODE.

Different problems in ODE (and beyond) call for different tools. We have learned many such tools: qualitative methods, like the phase line and slope fields, which also straddle the domain of numerical methods, like Euler's method, and analytic methods for finding formulas. Our two primary analytic methods of separation of variables and the integrating factor method are closely linked to fundamental techniques of calculus: the chain and product rules, respectively.

We will now explore a very different technique for solving constant-coefficient linear ODE: the method of *Laplace transforms*. We will present this method for first-order problems, but (unlike separation of variables and integrating factors) it generalizes easily to ODE with more derivatives. We don't really need yet another method for constant-coefficient problems on top of integrating factors and undetermined coefficients; rather, the true value of the Laplace transform is how it handles *forcing terms with discontinuities*.

The following semi-apocryphal historical account will motivate studying ODE with discontinuous forcing terms and suggest that our present methods, while adequate, are far from *adroit* at approaching such problems.

2.8.1. Harvesting with discontinuous rates.

Consider the following historical situation (adapted from pp. 56–57 of *Unintended Consequences of Human Actions* by Elena Ermolaeva and Jessica Ross). In 1859, a Thomas Austin released 24 rabbits near his home in Australia; previously rabbits had never grown to any sizable number on the continent. By 1881, farmers had started to abandon their farms due to the rabbits' extreme consumption of vegetation. By 1926, there were over *ten billion* rabbits in Australia.

Here is my question for us. What if Australia had engaged in a comprehensive rabbit harvesting (hunting) program at some point between 1859 and 1926? Could the overpopulation of rabbits have been curbed?

We will phrase and solve this question using ODE. Let $x(t)$ be the population of rabbits at time t, with t measured in years and $t = 0$ corresponding to the year 1859. So, $x(0) = 24$. The population in 1926 is then $x(67) = 10^{10}$. Let's assume that, in the absence of predators, the rabbit population grows exponentially, so $x(t) = 24e^{rt}$ for some constant r. In order that $10^{10} = x(67) = 24e^{67r}$, it suffices to take[21] $r \approx 1/4$. So, we will assume that the rabbit population satisfies $x(t) = 24e^{t/4}$.

It is actually more convenient to view the population not as a formula but as the solution to the IVP

$$\begin{cases} \dot{x} = x/4 \\ x(0) = 24. \end{cases} \tag{2.8.1}$$

Then if we harvest (hunt) $h(t)$ rabbits per year, our work in Section 2.6.1 tells us that the rabbit population will satisfy

$$\begin{cases} \dot{x} = x/4 - h(t) \\ x(0) = 24. \end{cases} \tag{2.8.2}$$

If h is continuous, we can solve this problem with the integrating factor method.

What if, however, h is not continuous? Suppose that the Australian farmers noticed that rabbits were becoming a problem in 1871, i.e., at time $t = 12$, about ten years before farms were abandoned in the "real" history. Then, in this imaginary history, the farmers decided to hunt a constant $d > 0$ rabbits per year (d is for "dead"). The harvesting function h would then be *piecewise-continuous* and have the form

$$h(t) = \begin{cases} 0, & t < 12 \\ d, & 12 \le t. \end{cases} \tag{2.8.3}$$

Is there a way to choose d (i.e., a national yearly quota of dead bunnies) to make the rabbit population go extinct? That is, with h defined in (2.8.3), will the solution x to (2.8.2) ever be 0?

We use this definition of h to see that for times $0 \le t < 12$, the IVP (2.8.2) is just (2.8.1), and the solution to that is just $x(t) = 24e^{t/4}$. Note that $x(12) = 24e^3$. Then the IVP for

[21] Here is how we get this value. We want $10^{10} = 24e^{67r}$, so $e^{67r} = 10^{10}/24$. Take the natural log of both sides:

$$67r = \ln\left(\frac{10^{10}}{24}\right) = \ln(10^{10}) - \ln(24) = 10\ln(10) - \ln(24).$$

Thus

$$r = \frac{10\ln(10) - \ln(24)}{67} \approx \frac{1}{4}.$$

By using log properties, we can avoid trying to compute $\ln(10^{10})$; since 10^{10} is so large, this might introduce numerical errors.

times $t \geq 12$ is

$$\begin{cases} \dot{x} = x/4 - d \\ x(12) = 24e^3. \end{cases}$$

We can also solve this with an integrating factor (or even undetermined coefficients, since the forcing term is the constant exponential $-d = -de^{0 \cdot t}$) to find

$$x(t) = (24 - 4de^{-3})e^{t/4} + 4d, \ t \geq 12.$$

We put all this together to conclude

$$x(t) = \begin{cases} 24e^{t/4}, \ 0 \leq t < 12 \\ (24 - 4de^{-3})e^{t/4} + 4d, \ t \geq 12. \end{cases} \tag{2.8.4}$$

This is where we finished on Wednesday, October 12, 2022.

The goal is now to choose d so that $x(t) = 0$ for some t. I hope it is obvious that if $24 - 4de^{-3} > 0$, then $x(t) > 0$ for all t. So, we (or, more precisely, the Australians of yore) should choose d to satisfy $24 - 4de^{-3} \leq 0$. Note that if we choose $24 - 4de^{-3} = 0$, then $x(t) = 4d$ for $t \geq 12$, and the rabbit population is just constant, but nonzero. So, we want $24 - 4de^{-3} < 0$, and that can be arranged by taking $d > 121$. (Why? Look at a graph of $y(d) := 24 - 4de^{-3}$.) In other words, if, starting in the year 1871, the Australians harvested just 121 rabbits per year, the explosion in rabbit population never would have occurred!

We were able to use the integrating factor method to solve this problem. Suppose now that the farmers wanted to change the rabbit harvesting rate after a year or so, maybe to take more, maybe less. Then the harvesting function might have the form

$$h(t) = \begin{cases} 0, \ t < 12 \\ d_1, \ 12 \leq t < 13 \\ d_2, \ 13 \leq t. \end{cases}$$

Then we would have to consider three IVP: one for $0 \leq t < 12$, which gives us the initial condition for $x(12)$, then a second IVP for $12 \leq t < 13$, which gives us the initial condition for $x(13)$, and last the IVP for $t \geq 13$. This could get out of hand pretty quickly and motivates the following question. *Is there a better way to solve constant-coefficient linear ODE when the forcing function is piecewise-continuous?*

The answer is yes (although "better," perhaps, is subjective), and it requires the development of a technique unlike anything that we've seen before. One other thing: look closely at the function x defined in (2.8.4). I claim that x is continuous on $(-\infty, \infty)$ and differentiable on $(-\infty, 12)$ and $(12, \infty)$ but not differentiable at $t = 12$, unless $d = 0$. Can you figure out why? This should not be too surprising or disappointing; we forced the ODE $\dot{x} = x/4 - h(t)$ by a discontinuous function h. Duhamel's formula requires the forcing term to be continuous. This will be a pattern in our solutions to come: solutions are usually not differentiable at points where the forcing function is discontinuous. We might have to relax our heretofore rigid adherence to Definitions 1.3.1 and 1.3.4!

2.8.2. Definition and elementary properties of the Laplace transform.

The tool that will allow us to crack linear constant-coefficient ODE with discontinuous forcing terms is an improper integral. Suppose that x is a **LOCALLY INTEGRABLE** function on $[0,\infty)$: for each $R > 0$, the function x is integrable on $[0,R]$. Then we define the **IMPROPER INTEGRAL** of x on $[0,\infty)$ as

$$\int_0^\infty x(t)\ dt := \lim_{R\to\infty}\int_0^R x(t)\ dt$$

if this limit exists. If the limit above exists, we say that $\int_0^\infty x(t)\ dt$ **CONVERGES**; otherwise, $\int_0^\infty x(t)\ dt$ **DIVERGES**.

The Laplace transform[22] of a locally integrable function x on $[0,\infty)$ is defined by taking an improper integral involving x.

2.8.1 Definition.

Let x be locally integrable on $[0,\infty)$ and let s be a real number. The **LAPLACE TRANSFORM OF x AT** *s is the number*

$$\mathscr{L}[x](s) := \int_0^\infty x(t)e^{-st}\ dt \tag{2.8.5}$$

if this improper integral converges. If the integral (2.8.5) diverges, then we say that the Laplace transform at s **DIVERGES** *as well. We may also write $\mathscr{L}[x](s) = \widetilde{x}(s)$.*

The independent variable of our function x above is t, as always, for time, but the variable of the Laplace transform is s. Some books denote the Laplace transform of x at s by the uppercase $X(s)$, but this veers too close to the usual uppercase notation for the antiderivative of x for my comfort.

The Laplace transform is, at first glance, a very strange beast. What possible relevance could it have to ODE? Trust me for now that the Laplace transform is the right "lens" through which to view certain ODE.

It will probably help (or, at least, not hurt) to do a computational example.

---

[22] Here is some cultural background that you may freely skip. In general, **TRANSFORM** is mathematical parlance for an operation that turns one function into another by means of an integral. An "*integral* transform" of a function $x = x(t)$ is a new function that I'll denote by $\mathcal{I}[x]$ — my idea is that the brackets around x emphasize that this new function $\mathcal{I}[x]$ inherently depends on the function x. Since $\mathcal{I}[x]$ itself is going to be a function, let's refer to its independent variable as s and write $\mathcal{I}[x](s)$ to denote the evaluation of the function $\mathcal{I}[x]$ at the point s. The general formula for $\mathcal{I}[x](s)$ will look like

$$\mathcal{I}[x](s) = \int_a^b \mathcal{K}(s,t)x(t)\ dt.$$

Here $\mathcal{K}$ is a function of the two variables s and t, called the **KERNEL** of the transform, and the limits of integration may be finite or infinite. There are a lot of choices here, and so there are lots of integral transforms. You'll see later with the Laplace transform that $a = 0$, $b = \infty$, and $\mathcal{K}(s,t) = e^{-st}$. A close cousin of the Laplace transform is the Fourier transform, which takes $a = -\infty$, $b = \infty$, and $\mathcal{K}(s,t) = e^{-ist}$, where $i^2 = -1$. (We will define exponentials with factors of i at a later time.)

2.8.2 Example.

Let $x(t) = e^{3t}$. Determine all values of s for which the Laplace transform $\mathscr{L}[x](s)$ is defined and calculate a formula for $\mathscr{L}[x](s)$ at those values of s.

Solution. By definition,

$$\mathscr{L}[x](s) = \int_0^\infty e^{3t}e^{-st}\ dt = \lim_{R\to\infty}\int_0^R e^{(3-s)t}\ dt.$$

First we evaluate the definite integral for R fixed:

$$\int_0^R e^{(3-s)t}\ dt = \frac{e^{(3-s)t}}{3-s}\Bigg|_{t=0}^{t=R} = \frac{e^{(3-s)R}}{3-s} - \frac{1}{3-s}.$$

This, by the way, is only valid for $3-s \neq 0$; we will handle $3-s=0$ momentarily. Hence (if $3-s\neq 0$) we have

$$\mathscr{L}[x](s) = \lim_{R\to\infty}\left(\frac{e^{(3-s)R}}{3-s} - \frac{1}{3-s}\right).$$

Recall that for a given real number $q \neq 0$, we have

$$\lim_{\tau\to\infty} e^{q\tau} = \begin{cases} 0, & q<0 \\ \infty, & q>0. \end{cases}$$

Thus if $3-s<0$, we have

$$\mathscr{L}[x](s) = \lim_{R\to\infty}\left(\frac{e^{(3-s)R}}{3-s} - \frac{1}{3-s}\right) = -\frac{1}{3-s} = \frac{1}{s-3}.$$

But if $3-s>0$, the limit does not exist; the improper integral does not converge; and the Laplace transform is undefined. That is,

$$\mathscr{L}[x](s) = \begin{cases} \dfrac{1}{s-3}, & s>3 \\ \\ \text{undefined}, & s<3. \end{cases}$$

Finally, to handle the case $s-3=0$, or $s=3$, we appeal to the definition once again:

$$\mathscr{L}[x](3) = \int_0^\infty e^{3t}e^{-3t}\ dt = \int_0^\infty 1\ dt = \lim_{R\to\infty}\int_0^R 1\ dt = \lim_{R\to\infty} R = \infty.$$

Thus $\mathscr{L}[x](3)$ is undefined, and we conclude

$$\mathscr{L}[x](s) = \begin{cases} \dfrac{1}{s-3}, & s>3 \\ \\ \text{undefined}, & s\leq 3. \end{cases}$$

▲

It is a bit sloppy, but very evocative, to replace x in $\mathscr{L}[x](s)$ with the formula for x as a function of t; thus we would say

$$\mathscr{L}[e^{3t}](s) = \frac{1}{s-3}, \ s > 3.$$

More generally, by replacing every instance of 3 in Example 2.8.2 with a fixed number r, we have the following result.

2.8.3 Lemma.

Let a be a real number. Then

$$\mathscr{L}[e^{at}](s) = \begin{cases} \dfrac{1}{s-a}, & s > a \\ \\ \textit{undefined}, & s \le a. \end{cases}$$

Taking $a = 0$, we have

$$\mathscr{L}[1](s) = \begin{cases} \dfrac{1}{s}, & s > 0 \\ \\ \textit{undefined}, s \le 0. \end{cases}$$

At the best of times, the Laplace transform of a function really is another function, possibly on a different domain. As we just saw above, while $x(t) := e^{at}$ is defined for all real numbers t, its Laplace transform is only defined on the interval (a, ∞). The Laplace transform $\mathscr{L}$ is a map or operator on a set of functions: it turns a function x defined on $[0, \infty)$ into a function $\mathscr{L}[x]$ defined...somewhere.

The Laplace transform interacts nicely with addition and multiplication by a constant. For this reason, it will interact very nicely with constant-coefficient linear ODE). Here is a theorem that I encourage you to prove using the linearity of the integral.

2.8.4 Theorem (Linearity of the Laplace transform).

Suppose that x and y are locally integrable functions defined on $[0, \infty)$ and that the Laplace transforms $\mathscr{L}[x](s)$ and $\mathscr{L}[y](s)$ exist for some number s. Then for any constants α and β, the Laplace transform $\mathscr{L}[\alpha x + \beta y](s)$ exists and satisfies

$$\mathscr{L}[\alpha x + \beta y](s) = \alpha \mathscr{L}[x](s) + \beta \mathscr{L}[y](s).$$

This is where we finished on Friday, October 14, 2022.

This is a course on differential equations, so we probably should ask how the Laplace transform interacts with the derivative. Suppose that x is locally integrable and differentiable on $[0, \infty)$. What, if anything, can we say about $\mathscr{L}[\dot{x}](s)$ at a given number s?

The Laplace transform $\mathscr{L}[\dot{x}](s)$ exists if and only if the limit

$$\lim_{R\to\infty} \int_0^R \dot{x}(t)e^{-st}\ dt$$

exists. At the very least, then, $\dot{x}$ should be integrable on $[0, R]$ for any $R > 0$.

Let's examine this integral carefully. We have the product of the derivative $\dot{x}$ and a pretty tame exponential; this is exactly why we have integration by parts. Put

$$u = e^{-st} \qquad dv = \dot{x}(t)\ dt$$
$$du = -se^{-st}\ dt \qquad v = x(t)$$

to find

$$\int_0^R \dot{x}(t)e^{-st}\ dt = e^{-st}x(t)\Big|_{t=0}^{t=R} - \int_0^R x(t)[-se^{-st}]\ dt = e^{-sR}x(R) - x(0) + s\int_0^R x(t)e^{-st}\ dt.$$

Now we want to take the limit as $R \to \infty$. We know that the integral term on the right will just turn into the Laplace transform of x: $\lim_{R\to\infty} \int_0^R e^{-st}x(t)\ dt = \mathscr{L}[x](s)$. If we also know

$$\lim_{R\to\infty} e^{-sR}x(R) = 0, \tag{2.8.6}$$

then we will have

$$\int_0^R \dot{x}(t)e^{-st}\ dt = \lim_{R\to\infty}\left(e^{-sR}x(R) - x(0) + s\int_0^R x(t)e^{-st}\ dt\right) = -x(0) + s\int_0^\infty x(t)e^{-st}\ dt$$
$$= s\mathscr{L}[x](s) - x(0).$$

That is,

$$\lim_{R\to\infty} e^{-sR}x(R) = 0 \Longrightarrow \mathscr{L}[\dot{x}](s) = s\mathscr{L}[x](s) - x(0). \tag{2.8.7}$$

This seems like a pretty nice relationship between the Laplace transform of x and the Laplace transform of its derivative! So, how do we guarantee that a limit like (2.8.6) holds for a function x and a given number s? Actually, it would be nice to have a condition on x that guarantees (2.8.6) for a whole range of s, not just a single s.

One way to guess at that condition is to think about what the limit (2.8.6) means: when R is large, $e^{-sR}x(R)$ is small. In particular, there is $R_0 > 0$ such that if $R > R_0$, then $e^{-sR}x(R) \le 1$. That is, for $R > R_0$, we have $x(R) \le e^{sR}$. And so x can be no bigger than an exponential that grows with rate s. Unfortunately, if x is exactly that exponential, then we can have problems; specifically, if $x(t) = e^{st}$, then (2.8.6) doesn't hold. However, if x grows exponentially at a rate slightly smaller than s, then we can still get (2.8.6). Let's quantify this.

2.8.5 Definition.

Let q be a real number. A function x on $[0, \infty)$ **HAS EXPONENTIAL ORDER** *q if there is a time $M > 0$ and a constant C such that $|x(t)| \le Ce^{qt}$ for all $t \ge M$.*

Here is the utility of this condition: it implies (2.8.6) and, as an added benefit, it also implies the convergence of the original Laplace transform of x!

2.8.6 Theorem.

Suppose that x is locally integrable on $[0, \infty)$ and has exponential order q.

(i) *$\mathscr{L}[x](s)$ converges for all $s > q$. That is, $\mathscr{L}[x]$ is defined on (q, ∞).*

(ii) *If x is also differentiable on $[0, \infty)$, and if $\dot{x}$ is locally integrable, then $\mathscr{L}[\dot{x}](s)$ converges for all $s > q$. In particular,*

$$\mathscr{L}[\dot{x}](s) = s\mathscr{L}[x](s) - x(0). \tag{2.8.8}$$

I am going to guide you through the proof of this theorem as an exercise later on. For now, let's apply it.

2.8.7 Example.

Let $x(t) = t$.

(i) *Find a real number q such that x has exponential order q.*

(ii) *Use (2.8.8) to compute $\mathscr{L}[x](s)$ without evaluating any integrals. Check from the formula that you find for $\mathscr{L}[x](s)$ that the domain of $\mathscr{L}[x]$ is what Theorem 2.8.6 predicts.*

Solution. (i) First, if $t \geq 0$, then $|x(t)| = |t| = t$. So, we can drop the absolute values here.

We need to find real numbers q for which there are constants $C > 0$ and $M > 0$ such that $|t| \leq Ce^{qt}$ for all $t \geq M$. (This is a long sentence. Reread and parse it slowly as many times as you need until it feels comfortable.)

If we have no other ideas, one good step is to start graphing. I'll graph $x(t) = t$ in blue and some exponentials in red. What do you observe?

I say that when $q > 0$, eventually $|t| < e^{qt}$ for t large enough. This is the inequality that we want with $C = 1$. When $q \leq 0$, however, we have $e^{qt} < t$ for t large. This should be absolutely unsurprising: exponentials either explode or vanish depending on the sign of the exponent. Specifically,

$$\lim_{t \to \infty} e^{qt} = \begin{cases} \infty, & q > 0 \\ 1, & q = 0 \\ 0, & q < 0. \end{cases}$$

I claim that for any $q > 0$, there is a number $M_q > 0$ such that if $t \geq M_q$, then $t \leq e^{qt}$. Thus x will have exponential order q for any positive q (and so exponential order is not unique). We have $t \leq e^{qt}$ if and only if $te^{-qt} \leq 1$. I claim that since $q > 0$, we have

$$\lim_{t \to \infty} te^{-qt} = 0, \tag{2.8.9}$$

and so, by definition of that limit, there is $M_q > 0$ such that if $t \geq M_q$, then $te^{-qt} \leq 1$. That is what we want. How can you justify my claim in (2.8.9)? Use L'Hospital's rule.

(ii) We now know that $x(t) = t$ has exponential order q for any $q > 0$. I claim this means that $\mathscr{L}[x](s) = \mathscr{L}[t](s)$ is defined for any $s > 0$. Here's why. Fix your favorite $s > 0$ and take $q = s/2$. Then (1) $q > 0$, (2) x has exponential order q, and (3) $s > q$, so we know that the Laplace transform $\mathscr{L}[x](s)$ is defined.

To compute $\mathscr{L}[t](s)$ from the definition, we would need to evaluate $\int_0^\infty te^{-st}\,dt$ for $s > 0$. This requires integration by parts and then a limit to handle the improper integral. Instead, here is a great trick. The identity (2.8.8) relates $\mathscr{L}[x](s)$ and $\mathscr{L}[\dot{x}](s)$. Here $\dot{x}(t) = 1$ and $x(0) = 0$, so (2.8.8) reads

$$s\mathscr{L}[t](s) = s\mathscr{L}[x](s) = s\mathscr{L}[x](s) - x(0) = \mathscr{L}[\dot{x}](s) = \mathscr{L}[1](s).$$

Since $s > 0$, we can solve for $\mathscr{L}[t](s)$ as

$$\mathscr{L}[t](s) = \frac{\mathscr{L}[1](s)}{s}.$$

Now, none of this is worth doing if $\mathscr{L}[1]$ isn't easy to compute. But $1 = e^{0\cdot t}$, so Lemma 2.8.3 with $a = 0$ tells us

$$\mathscr{L}[1](s) = \mathscr{L}[e^{0\cdot t}](s) = \frac{1}{s-0} = \frac{1}{s}.$$

Thus

$$\mathscr{L}[t](s) = \frac{1/s}{s} = \frac{1}{s^2}.$$

▲

This is where we finished on Monday, October 17, 2022.

2.8.3. The Laplace transform and linear constant-coefficient ODE.

The last thing we may want right now is yet another method for solving constant-coefficient linear ODE, i.e., problems of the form $\dot{x} = ax + b(t)$, where a is a real number and b is a function. But, it's what we need, because it will teach us how to solve harder problems.

2.8.8 Example.

Pretend that we do not know how to solve the IVP

$$\begin{cases} \dot{x} = -x + e^t \\ x(0) = 0. \end{cases}$$

Instead, make the two large assumptions that (1) this problem does have a solution x and (2) x is of some exponential order, so that x and $\dot{x}$ have Laplace transforms for s large enough. What can we learn about the Laplace transform of x, and what does this teach us about x itself?

Solution. We work backward and assume (1) and (2): there is a solution x to the IVP defined on $[0,\infty)$ and x has exponential order, so that x and $\dot{x}$ have Laplace transforms defined on some interval (q,∞). Hereafter, let's assume $s > q$. Since $\dot{x}(t) = -x(t) + e^t$ for all t, we must have

$$\mathscr{L}[\dot{x}(t)](s) = \mathscr{L}[-x(t) + e^t](s). \tag{2.8.10}$$

Here I am following the somewhat awkward custom of putting the t-dependence inside the square brackets of $\mathscr{L}$.

On the left in (2.8.11) we have

$$\mathscr{L}[\dot{x}(t)](s) = s\widetilde{x}(s) - x(0) = s\widetilde{x}(s). \tag{2.8.11}$$

Here the initial condition $x(0) = 0$ helps, and I am now writing $\widetilde{x}(s)$ instead of $\mathscr{L}[x](s)$ or $\mathscr{L}[x(t)](s)$.

On the right in (2.8.11), we first get

$$\mathscr{L}[-x(t) + e^t](s) = \mathscr{L}[-x(t)](s) + \mathscr{L}[e^t](s) = -\mathscr{L}[x(t)](s) + \mathscr{L}[e^t](s) \tag{2.8.12}$$

by linearity. Lemma 2.8.3 with $a = 1$ tells us

$$\mathscr{L}[e^t](s) = \frac{1}{s-1},$$

at least if we assume $s > 1$, which we will do from now on. Thus (2.8.12) reads

$$\mathscr{L}[-x(t) + e^t](s) = -\widetilde{x}(s) + \frac{1}{s-1}. \tag{2.8.13}$$

Let's put it all back together: combine (2.8.11) and (2.8.13) to get

$$s\widetilde{x}(s) = -\widetilde{x}(s) + \frac{1}{s-1}. \tag{2.8.14}$$

This is, of course,

$$s\widetilde{x}(s) + \widetilde{x}(s) = \frac{1}{s-1}, \tag{2.8.15}$$

and so we factor

$$s\widetilde{x}(s) + \widetilde{x}(s) = (s+1)\widetilde{x}(s)$$

to solve (2.8.15) for $\widetilde{x}$:

$$\widetilde{x}(s) = \frac{1}{(s+1)(s-1)} \tag{2.8.16}$$

Here is a tremendously important observation: *viewing the IVP through the lens of the Laplace transform turned the IVP into an algebraic equation for the Laplace transform* $\widetilde{x}$. And now we have solved that algebraic equation as (2.8.16). So, whatever x is, we know what its Laplace transform is. Now, we only know two transforms so far, $\mathscr{L}[e^{rt}](s)$ and $\mathscr{L}[t](s)$, and the right side $1/(s-1)(s+1)$ above does not look like either of those transforms. The correct, although perhaps not obvious, idea is to appeal to the dreaded method of partial fractions and rewrite

$$\frac{1}{(s-1)(s+1)} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)}. \tag{2.8.17}$$

We recognize that

$$\frac{1}{2(s-1)} = \frac{1}{2}\mathscr{L}[e^t](s) \quad \text{and} \quad \frac{1}{2(s+1)} = \frac{1}{2(s-(-1))} = \mathscr{L}[e^{-t}](s).$$

Thus x satisfies

$$\mathscr{L}[x](s) = \frac{\mathscr{L}[e^t](s)}{2} - \frac{\mathscr{L}[e^{-t}](s)}{2} = \mathscr{L}\left[\frac{e^t - e^{-t}}{2}\right](s),$$

where the second equality uses, once again, the linearity of the Laplace transform.

This gives us the strong suspicion that the solution to our problem is

$$x(t) = \frac{e^t - e^{-t}}{2},$$

and a moment of calculus and algebra shows that this is indeed the case. You can (and should) check that this solution has exponential order 1, as well. Of course, we could have found this with the integrating factor method or undetermined coefficients. ▲

This is not an exciting result. Of the many things our world needs right now, a new method for solving *constant-coefficient linear first-order equations* (not even variable-coefficient equations!) probably isn't one of them. But it does give us a good idea: if we have no clue about how to solve a differential equation, take the Laplace transform of everything and see what we can learn about the Laplace transform of the solution. Of course, if we want to use Laplace transforms to solve ODE with forcing terms other than exponentials, we will have to learn how to compute more Laplace transforms. It turns out that many commonly occurring functions, such as polynomials and trig functions, have straightforward Laplace transforms; however, repeating the work of Example 2.8.8 with such forcing functions may lead to more uncomfortable partial fractions manipulations than those in (2.8.17). We will not cover such problems but instead do something slightly more exciting.

2.8.4. Inverse Laplace transforms.

In Example 2.8.8, we assumed that a solution x to $\dot{x} = -x + e^t$ with $x(0) = 0$ existed (not a big assumption, because we know how to solve linear ODE now) and that x had a Laplace transform (a slightly bigger assumption). We then found that the transform of x should satisfy

$$\mathscr{L}[x](s) = \mathscr{L}\left[\frac{e^t - e^{-t}}{2}\right](s), \tag{2.8.18}$$

which suggested to us that

$$x(t) = \frac{e^t - e^{-t}}{2}.$$

Given this formula for x, it is only a matter of calculus and algebra to show that x does solve the IVP.

This example suggests that the following more general scenario could occur. Suppose that we use the Laplace transform to show that a putative solution x to an IVP satisfies

$$\mathscr{L}[x](s) = \mathscr{L}[y](s) \tag{2.8.19}$$

for some known function y. Should we automatically conclude $x = y$?

Maybe. First, we have to restrain ourselves and realize that we are working backward. We took an IVP, assumed that it had a solution x amenable to the Laplace transform, and we found a formula for $\mathscr{L}[x]$ as (2.8.19). This by no means ensures that y solves the IVP; we'd need to plug y in and check. Fortunately, this is pretty easy, since we (more or less) know how to do calculus and algebra.

However, we are accustomed to thinking that IVP have unique solutions. Could this Laplace transform method miss a solution? We think that the only solution is y, but is there another? The following theorem says no.

2.8.9 Theorem.

Suppose that x and y are continuous functions on $[0, \infty)$ and there is a real number q such that $\mathscr{L}[x](s) = \mathscr{L}[y](s)$ for all $s > q$. Then $x(t) = y(t)$ for all $t \geq 0$.

Look back at (2.8.18). This equation says that if x solves the IVP, then $\mathscr{L}[x](s) = \mathscr{L}[y](s)$, where $y(t) = (e^t - e^{-t})/2$. Clearly y is continuous. Since we are assuming that x solves an IVP, x must be continuous. Theorem 2.8.9 then applies to let us conclude $x = y$. So, *if* there exists a solution x to the IVP (small if) and *if* x has a Laplace transform (big if), *then* x must equal y.

Embiggened with this uniqueness result, we make a formal definition.

2.8.10 Definition.

Let z be a function defined on an interval (q, ∞) and suppose there is a continuous function x defined on $[0, \infty)$ such that $\mathscr{L}[x](s) = z(s)$ for all $s > q$. (There is at most one such function x by Theorem 2.8.9.) Then we call x the **INVERSE LAPLACE TRANSFORM** *of z and we write*

$$x(t) = \mathscr{L}^{-1}[z](t) = \mathscr{L}^{-1}[z(s)](t).$$

We can therefore amend our strategy for solving IVP with the Laplace transform to read: *Assume there is a solution x and find a formula for its Laplace transform as $\mathscr{L}[x](s) = z(s)$ for some function z. Then a solution candidate is $x(t) = \mathscr{L}^{-1}[z(s)](t)$.*

2.8.11 Example.

What is

$$\mathscr{L}^{-1}\left[\frac{2}{s+5}\right](t)?$$

Solution. We recognize from Lemma 2.8.3 that

$$\frac{1}{s+5} = \frac{1}{s-(-5)} = \mathscr{L}[e^{-5t}](s),$$

and so

$$\frac{5}{s-2} = 5\mathscr{L}[e^{2t}](s) = \mathscr{L}[5e^{2t}](s).$$

Moreover, $x(t) = 5e^{2t}$ is continuous. Thus

$$\mathscr{L}^{-1}\left[\frac{1}{s-2}\right](t) = e^{2t}. \qquad \blacktriangle$$

Unfortunately, given a function z, there is no "easy" or "transparent" formula for its inverse Laplace transform, if the inverse transform exist. A semester or so of complex analysis will produce a nice theoretical result, but for us it has little computational relevance. Instead, we often calculate inverse Laplace transforms using the same catch-as-catch-can strategy that we do antiderivatives: rewrite the function so that more elementary inverse transforms appear and hope for the best. Often this rewriting involves a number of partial fractions decompositions, and while it is worthwhile to know "conceptually" how to perform such decompositions, any intense calculations are better left for a computer.

2.8.12 Example.

Suppose that x and y are locally integrable functions on $[0, \infty)$ whose Laplace transforms are defined on (q, ∞) for some $q > 0$. If $\mathscr{L}[x](s) = \mathscr{L}[y](s)$ for all $s > q$, must we have $x(t) = y(t)$ for all $t \geq 0$?

Solution. Alas, no. Consider the following situation. Take

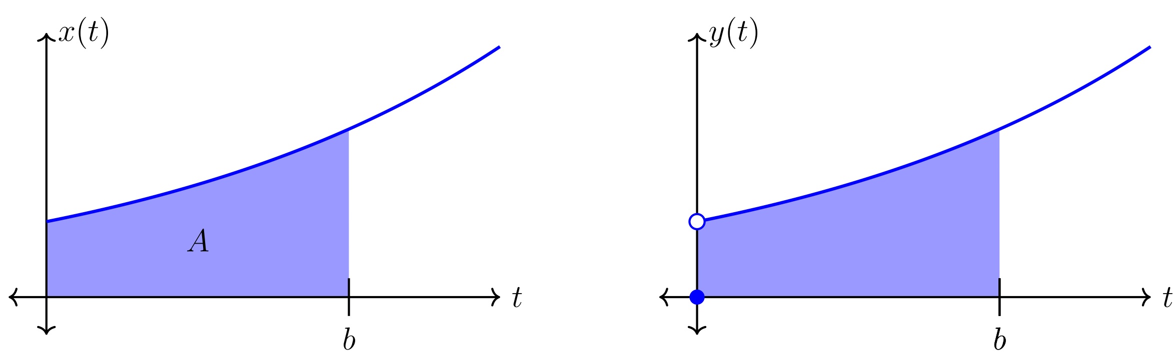
$$x(t) = 0 \quad \text{and} \quad y(t) = \begin{cases} 1, & x = 0 \\ 0, & x > 0. \end{cases}$$

Clearly $x(0) \neq y(0)$, so x and y are not the same function. However, for any $R > 0$ and any real number s, it is the case that

$$\int_0^R x(t)e^{-st}\, dt = \int_0^R y(t)e^{-st}\, dt = 0.$$

Thus $\mathscr{L}[x](s) = \mathscr{L}[y](s) = 0$ for all s. ▲

The previous example exploited the following property of integrals: *changing the value of the integrand at one point does not change the value of the integral*, or more geometrically, *the area under a point is zero*. In the following pictures, two functions x and y are graphed; they agree except at $t = 0$. Thus the area under x and y from 0 to any $b > 0$ is the same.



This is where we finished on Monday, October 24, 2022.

2.8.5. ODE with discontinuous forcing terms.

We used the Laplace transform to solve a constant-coefficient linear IVP in Example 2.8.8. This example was a paradigm of how we could use the Laplace transform to solve *any* constant-coefficient problem. Here's the breakdown, which we will use over and over.

1. Start with a constant-coefficient linear IVP.	
$\begin{cases} \dot{x} = ax + b(t) \\ x(0) = x_0 \end{cases}$	$\begin{cases} \dot{x} = -x + e^t \\ x(0) = 0 \end{cases}$
2. Take the Laplace transform of both sides of the ODE.	
$\mathscr{L}[\dot{x}](s) = \mathscr{L}[ax + b(t)](s)$	$\mathscr{L}[\dot{x}](s) = \mathscr{L}[-x + e^t](s)$
3. Simplify using properties of the Laplace transform ($\mathscr{L} = \widetilde{\ }$).	
$s\widetilde{x}(s) - x(0) = a\widetilde{x}(s) + \widetilde{b}(s)$ $s\widetilde{x}(s) - x_0 = a\widetilde{x}(s) + \widetilde{b}(s)$	$s\widetilde{x}(s) - x(0) = -\widetilde{x}(s) + \mathscr{L}[e^t](s)$ $s\widetilde{x}(s) = -\widetilde{x}(s) + \dfrac{1}{s-1}$
4. Solve (algebraically) for the Laplace transform.	
$s\widetilde{x}(s) - a\widetilde{x}(s) = x_0 + \widetilde{b}(s)$ $(s-a)\widetilde{x}(s) = x_0 + \widetilde{b}(s)$ $\widetilde{x}(s) = \dfrac{x_0}{s-a} + \dfrac{\widetilde{b}(s)}{s-a}$	$s\widetilde{x}(s) + \widetilde{x}(s) = \dfrac{1}{s-1}$ $(s+1)\widetilde{x}(s) = \dfrac{1}{s-1}$ $\widetilde{x}(s) = \dfrac{1}{(s+1)(s-1)}$
5. Invert the Laplace transform.	
$x(t) = \mathscr{L}^{-1}\left[\dfrac{x_0}{s-a} + \dfrac{\widetilde{b}(s)}{s-a}\right](t)$ $= x_0\mathscr{L}^{-1}\left[\dfrac{1}{s-a}\right](t) + \mathscr{L}^{-1}\left[\dfrac{\widetilde{b}(s)}{s-a}\right](t)$ $= x_0 e^{at} + \mathscr{L}^{-1}\left[\dfrac{\widetilde{b}(s)}{s-a}\right](t)$	$x(t) = \mathscr{L}^{-1}\left[\dfrac{1}{(s+1)(s-1)}\right](t)$ $= \mathscr{L}^{-1}\left[\dfrac{1}{2(s-1)} - \dfrac{1}{2(s+1)}\right](t)$ $= \dfrac{e^t}{2} - \dfrac{e^{-t}}{2}$

The real utility of the Laplace transform for us will be its interaction with discontinuous functions, like the harvesting term that appeared in our Australian rabbit IVP. Specifically, we wanted to solve an ODE like

$$\dot{x} = \frac{x}{4} - h(t), \qquad h(t) = \begin{cases} 0, & t < 12 \\ d, & t \geq 12, \end{cases}$$

where d was a fixed number. While we could solve this brute-force[23] with integrating factors, the Laplace transform will offer an efficient alternative, especially if we replace h with a piecewise function that has more pieces. The following tool will be central to our forthcoming work.

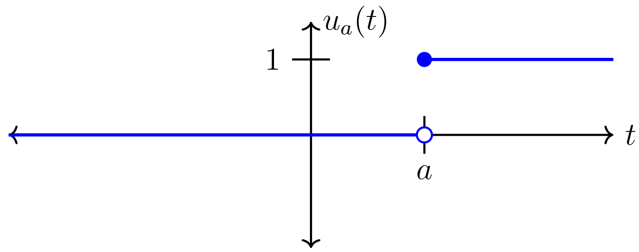
[23] And brute force is often the best force.

2.8.13 Definition.

A **STEP FUNCTION** *or* **HEAVISIDE FUNCTION** *is a function of the form*

$$u_a(t) := \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} \tag{2.8.20}$$

for a given real number a.



I should make a few remarks before we return to Laplace transforms and differential equations. Sometimes the function

$$u_0(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{2.8.21}$$

is called *the* Heaviside function (not "*a* Heaviside function") or the **UNIT STEP FUNCTION**. You should use the formulas (2.8.20) and (2.8.21) to check that

$$u_a(t) = u_0(t-a)$$

for all t and a. By the way, there is nothing really special about having the Heaviside function be 0 for t strictly less than a; we could swap $<$ and $\geq$ in (2.8.20) and not suffer at all. Last, I hope you see that the original discontinuous harvesting function (2.8.3) for the rabbit problem has the form

$$h(t) = \begin{cases} 0, & t < 12 \\ d, & t \geq 12 \end{cases} = du_{12}(t),$$

and so the rabbit IVP is really

$$\begin{cases} \dot{x} = \dfrac{x}{4} + du_{12}(t) \\ x(0) = 24. \end{cases} \tag{2.8.22}$$

Naturally, we want to know how the Laplace transform acts on Heaviside functions. The following is more or less a direct calculation from the definition of a Heaviside function and the definition of the Laplace transform — so direct, in fact, that I'll let you do it.

2.8.14 Lemma.

Let a be a real number. Then

$$\mathscr{L}[u_a](s) = \begin{cases} \dfrac{e^{-sa}}{s}, & s > 0 \\ \\ \textit{undefined}, & s \le 0. \end{cases}$$

This is an interesting result. The Heaviside function u_a is defined on $(-\infty, \infty)$ but discontinuous at $t = 0$. The Laplace transform $\mathscr{L}[u_a]$ is defined only on $(0, \infty)$, but it is continuous (in fact, infinitely differentiable) on $(0, \infty)$. Thus the function u_a has a larger domain than $\mathscr{L}[u_a]$, but $\mathscr{L}[u_a]$ is more nicely behaved on its (smaller) domain!

The following example studies an IVP whose forcing term is a Heaviside function, much like the rabbit IVP (2.8.22), just with slightly easier (maybe?) numbers.

2.8.15 Example.

Suppose that x solves the IVP

$$\begin{cases} \dot{x} = x + u_3(t) \\ x(0) = 1. \end{cases}$$

Determine a formula for $\mathscr{L}[x](s)$.

Solution. Taking the Laplace transform of both sides gives

$$\mathscr{L}[\dot{x}](s) = \mathscr{L}[x + 2u_3(t)](s). \tag{2.8.23}$$

On the left we have

$$\mathscr{L}[\dot{x}](s) = s\mathscr{L}[x](s) - x(0) = s\widetilde{x}(s) - 1. \tag{2.8.24}$$

For slightly greater notational simplicity, I am once again writing $\widetilde{x}(s) = \mathscr{L}[x](s)$. On the right we have

$$\mathscr{L}[x + 2u_3(t)](s) = \mathscr{L}[x](s) + 2\mathscr{L}[u_3](s) = \widetilde{x}(s) + \frac{2e^{-3s}}{s}. \tag{2.8.25}$$

Now we rewrite (2.8.23) using (2.8.24) and (2.8.25). We find

$$s\widetilde{x}(s) - 1 = \widetilde{x}(s) + \frac{2e^{-3s}}{s}. \tag{2.8.26}$$

We want to solve for $\widetilde{x}(s)$, so we rearrange (2.8.26) into

$$s\widetilde{x}(s) - \widetilde{x}(s) = 1 + \frac{2e^{-3s}}{s}, \tag{2.8.27}$$

and thus

$$\widetilde{x}(s) = \mathscr{L}[x](s) = \frac{1}{s-1} + \frac{2e^{-3s}}{s(s-1)}. \tag{2.8.28}$$

▲

The previous example leads us to the following ruminations. We have a formula (2.8.28) for the Laplace transform $\mathscr{L}[x]$ of the solution x to an IVP. We want to say that the solution is

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{2e^{-3s}}{s(s-1)}\right](t).$$

How do we calculate this inverse Laplace transform? We should expect

$$\mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{2e^{-3s}}{s(s-1)}\right](t) = \mathscr{L}^{-1}\left[\frac{1}{s-1}\right](t) + 2\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s(s-1)}\right](t) = e^t + 2\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s(s-1)}\right](t).$$

How do we handle the remaining inverse transform?

One idea that has worked before is partial fractions. We can rewrite

$$\frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s},$$

and so

$$\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s(s-1)}\right](t) = \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s-1} - \frac{e^{-3s}}{s}\right](t) = \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s-1}\right](t) - \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s}\right](t).$$

Look at the first term on the right. We want to calculate

$$\mathscr{L}^{-1}\left[\left(\frac{1}{s-1}\right)e^{-3s}\right](t). \tag{2.8.29}$$

We know that

$$\mathscr{L}^{-1}\left[\frac{1}{s-1}\right](t) = e^t.$$

Can we relate that to the current situation of multiplying by e^{-3s} in (2.8.29)?

More generally, the situation above begs the following question. Suppose that x is a function whose Laplace transform is defined at the number s, and let a also be a real number. Is there a function y such that

$$e^{-as}\mathscr{L}[x](s) = \mathscr{L}[y](s)?$$

In other words, what is

$$\mathscr{L}^{-1}\big[e^{-as}\mathscr{L}[x](s)\big](t)?$$

This is where we finished on Wednesday, October 26, 2022.

Remember that we do not really have an explicit formula for $\mathscr{L}^{-1}$ the way we do for $\mathscr{L}$; the sentence $y(t) = \mathscr{L}^{-1}[z](t)$ for two functions y and z just means $\mathscr{L}[y](s) = z(s)$. So, we may as well try to manipulate the expression $e^{-as}\mathscr{L}[x](s)$ and see if anything pops out. Since x has a Laplace transform at s, the improper integral below is defined:

$$e^{-as}\mathscr{L}[x](s) = e^{-as}\int_0^\infty x(t)e^{-st}\,dt = \int_0^\infty x(t)e^{-(a+t)s}\,dt. \tag{2.8.30}$$

The integral on the right looks vaguely like a Laplace transform, but the exponent is too busy. Let's substitute[24] $\tau = a + t$ to find $d\tau = dt$ and $t = \tau - a$, and thus

$$\int_0^\infty x(t)e^{-(a+t)s}\, dt = \int_a^\infty x(\tau - a)e^{-\tau s}\, d\tau. \tag{2.8.31}$$

This sort of substitution in improper integrals can be justified by using the limit definition of the improper integral. I'll leave this as an exercise for you to cheerfully ignore.

The integral on the right almost looks like a Laplace transform for the "shifted" function $\tau \mapsto x(\tau - a)$. The problem is that the integral on the right is "only" over $[a, \infty)$, not $[0, \infty)$. However, we should have Heaviside functions on our mind, so perhaps it's not a stretch to see that

$$\int_a^\infty x(\tau - a)e^{-\tau s}\, d\tau = \int_a^\infty 1 \cdot x(\tau - a)e^{-\tau s}\, d\tau = \int_a^\infty u_a(\tau)x(\tau - a)e^{-\tau s}\, d\tau, \tag{2.8.32}$$

since $u_a(\tau) = 1$ for $\tau \ge a$. And since $u_a(\tau) = 0$ for $\tau < a$, we have

$$\begin{aligned}\int_a^\infty u_a(\tau)x(\tau - a)e^{-\tau s}\, d\tau = 0 + \int_a^\infty u_a(\tau)x(\tau - a)e^{-\tau s}\, d\tau \\ &= \int_0^a u_a(\tau)x(\tau - a)e^{-\tau s}\, d\tau + \int_a^\infty u_a(\tau)x(\tau - a)e^{-\tau s}\, d\tau \\ &= \int_0^\infty u_a(\tau)x(\tau - a)e^{-\tau s}\, d\tau \\ &= \mathscr{L}[u_a(\tau)x(\tau - a)](s).\end{aligned} \tag{2.8.33}$$

It has taken some work, but we can chase the equalities from (2.8.30) to (2.8.31) to (2.8.32) to (2.8.33) to conclude

$$e^{-as}\mathscr{L}[x(t)](s) = \mathscr{L}[u_a(t)x(t - a)](s).$$

Here, then, is the answer to our question.

2.8.16 Lemma.

Suppose that x is a locally integrable function on $[0, \infty)$ and that there is a real number q such that the Laplace transform $\mathscr{L}[x](s)$ is defined for $s > q$. Then

$$\mathscr{L}^{-1}\big[e^{-as}\mathscr{L}[x](s)\big](t) = u_a(t)x(t - a). \tag{2.8.34}$$

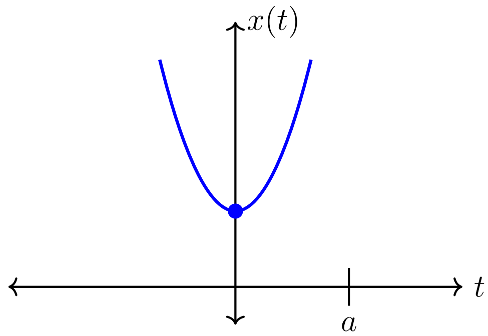
Equivalently,

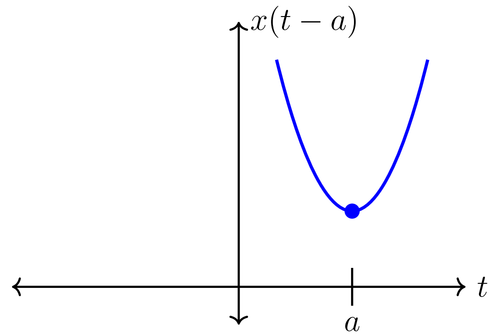
$$\mathscr{L}[u_a(t)x(t - a)](s) = e^{-as}\mathscr{L}[x](s).$$

There is one ticklish point in the formula (2.8.34). Maybe $a > 0$, in which case $t - a < 0$ for $0 \le t < a$. But if we only assume that x is defined on $[0, \infty)$, then x is not defined at these negative $t - a$. However, the factor $u_a(t)$ is 0 for $t < a$, i.e., when $t - a < 0$, and so we just interpret $u_a(t)x(t - a)$ as 0 even if $x(t - a)$ isn't defined.

[24] We are using τ in this substitution, not u, since u is playing a role above with u_a.

The graphical interpretation of the function $t \mapsto u_a(t)x(t-a)$ may not be immediately obvious from the formula, so it is worth pausing for a moment to draw. For simplicity, take $a > 0$. Then the graph of $t \mapsto x(t-a)$ is the graph of x shifted a units to the right[25] on the t-axis.

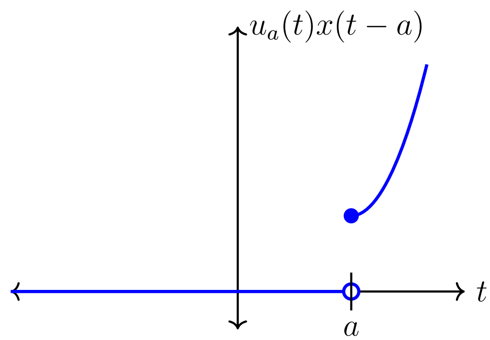




Since

$$u_a(t)x(t-a) = \begin{cases} 0, & t < a \\ x(t-a), & t \geq a, \end{cases}$$

the graph of $t \mapsto u_a(t)x(t-a)$ is just 0 for $t < a$ and then, starting at $t = a$, the graph of x shifted to the right by t units.



2.8.17 Example.

In Example 2.8.15, we saw that a solution x to

$$\begin{cases} \dot{x} = x + 2u_3(t) \\ x(0) = 1 \end{cases}$$

should satisfy

$$x(t) = e^t + \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s-1}\right](t) - \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s}\right](t).$$

Compute these inverse Laplace transforms. What do you learn about x?

[25] Note that if $a < 0$, then we are considering $t \mapsto x(t+a)$, and this is a shift left. Shifting to the right by a negative number means shifting to the left.

Solution. We use the formula $\mathscr{L}^{-1}\big[e^{-as}\mathscr{L}[y](s)\big](t) = u_a(t)y(t-a)$ from (2.8.34) to write

$$\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s-1}\right](t) = \mathscr{L}^{-1}\big[e^{-3s}\mathscr{L}[e^t](s)\big](t) = u_3(t)e^{t-3}$$

and

$$\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s}\right](t) = \mathscr{L}^{-1}\big[e^{-3s}\mathscr{L}[e^{0\cdot t}](s)\big](t) = u_3(t)e^{0\cdot(t-3)} = u_3(t).$$

Thus the solution to the IVP is

$$x(t) = e^t + u_3(t)e^{t-3} - u_3(t) = e^t + u_3(t)\big(e^{t-3} - 1\big).$$

This is a nice formula. Now is a good time to recite the Analyst's Creed (AC) and ask what this formula is doing? We know that $u_3(t) = 0$ for $t < 3$, and so before time $t = 3$, the solution is just $x(t) = e^t$. At time $t = 3$, the other term in the solution "turns on," and thereafter $x(t) = e^t + e^{t-3} - 1$. That is,

$$x(t) = \begin{cases} e^t, \ t < 3 \\ e^t + e^{t-3} - 1, \ t \geq 3. \end{cases}$$

As we expect (since x solves an IVP), x is continuous everywhere; at $t = 3$ we have

$$\lim_{t\to 3^-} x(t) = \lim_{t\to 3^-} e^t = e^3 \quad \text{and} \quad \lim_{t\to 3^+} x(t) = \lim_{t\to 3^+} \big(e^t + e^{t-3} - 1\big) = e^3 + e^{3-3} - 1 = e^3.$$

However, long experience with piecewise functions might make us suspicious about the differentiability of x at $t = 3$. If we go to a graph and zoom *very carefully*, we might see a corner at $t = 3$. (Or not, the graph is going to be pretty steep there.) More precisely, we could use the definition of the derivative at $t = 3$, and I will suggest that you do this as an exercise, to show that x is not differentiable at $t = 3$.

This should be surprising. We have long prided ourselves in this course on the fact that a solution to a differential equation should be differentiable! Here is the situation: you can only give as good as you get. We have solved ("solved," I say, somewhat skeptically now) an IVP whose forcing term was discontinuous at $t = 3$; this discontinuity manifested itself in the solution ("solution") as a corner at $t = 3$. While the solution is continuous at $t = 3$, it is not differentiable there. We simply have to live with this result; we can check that x is differentiable at every $t \neq 3$, and if $t \neq 3$, then, indeed $\dot{x}(t) = x(t) + 2u_3(t)$, and so x satisfies the ODE except at $t = 3$. ▲

What if the forcing function has more than two pieces? How might we solve

$$\begin{cases} \dot{x} = x + h(t) \\ x(0) = 1 \end{cases} \qquad \text{with} \qquad h(t) := \begin{cases} 1, \ 0 \leq t < 10 \\ 2, \ 10 \leq t < 20 \\ 3, \ 30 \leq t \end{cases} \tag{2.8.35}$$

without resorting to *three* invocations of integrating factors? Such a forcing function could arise in the rabbit problem if we changed our harvesting pattern every few years.

To study this problem, we need just one more tool, which will show up if I tell you a good idea. If you stare at h for a while, I think you'll agree with the following calculations:

$$h(t) = \begin{cases} 1, & 0 \le t < 10 \\ 2, & 10 \le t < 20 \\ 3, & 20 \le t \end{cases} = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t < 10 \\ 0, & 10 \le t \end{cases} + \begin{cases} 0, & t < 10 \\ 2, & 10 \le t < 20 \\ 0, & 20 \le t \end{cases} + \begin{cases} 0, & t < 20 \\ 3, & 20 \le t. \end{cases} \tag{2.8.36}$$

The second equality breaks h up as a sum of piecewise functions that are "mostly 0," except on a bounded interval; this h has three "pieces," and so there are three terms in this sum. The third term is

$$\begin{cases} 0, & t < 20 \\ 3, & 20 \le t \end{cases} = 3u_{20}(t),$$

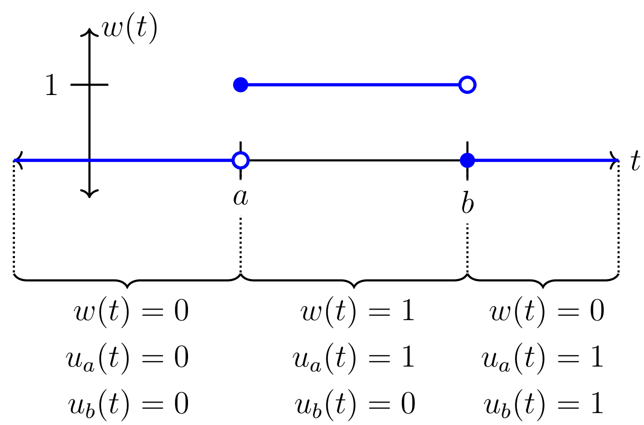
but what about the first two?

This is where we finished on Friday, October 28, 2022.

Both the first and the second terms in (2.8.36) are multiples of a function of the form

$$w(t) := \begin{cases} 0, & t < a \\ 1, & a \le t < b \\ 0, & b \le t. \end{cases}$$

Let's graph w. I went ahead and calculated values of the Heaviside functions u_a and u_b on some intervals here.



The picture therefore tells us that

$$w(t) = u_a(t) - u_b(t)$$

for all t. That is,

$$u_a(t) - u_b(t) = \begin{cases} 0, & t < a \\ 1, & a \le t < b \\ 0, & b \le t. \end{cases}$$

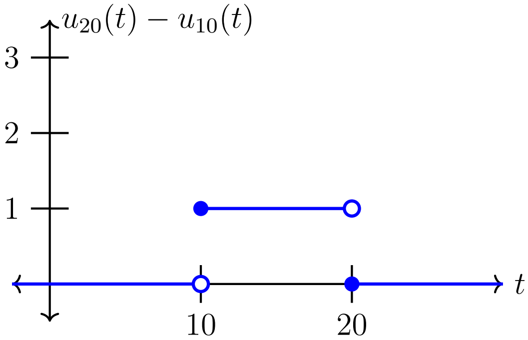
We can now rewrite (2.8.36) as

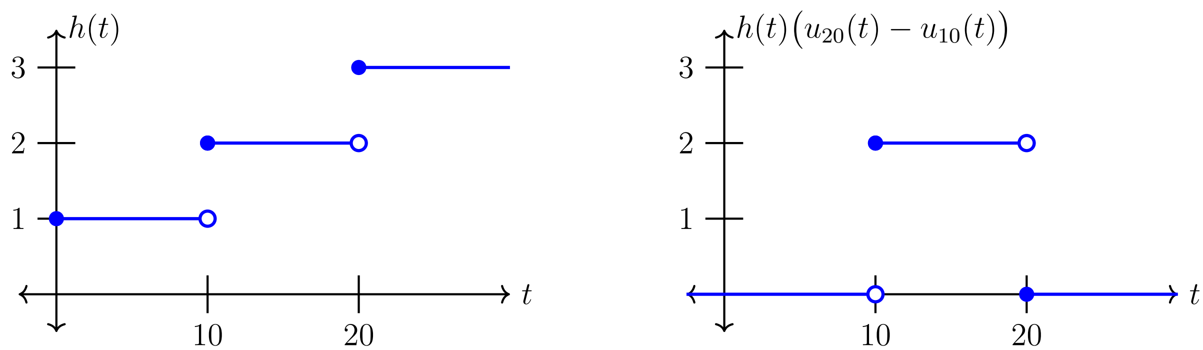
$$h(t) = \big(u_0(t) - u_{10}(t)\big) + 2\big(u_{10}(t) - u_{20}(t)\big) + 3u_{20}(t), \tag{2.8.37}$$

and this simplifies to

$$h(t) = u_0(t) + u_{10}(t) + u_{20}(t). \tag{2.8.38}$$

Here are pictures of how, for example, the second term in (2.8.37) arises.





If we want to solve (2.8.35), which was forced by h, via Laplace transforms, it is now very easy to calculate $\mathscr{L}[h]$ thanks to the new formula for h in (2.8.38). We have

$$\begin{aligned}\mathscr{L}[h](s) = \mathscr{L}[u_0+u_{10}+u_{20}](s) = \mathscr{L}[u_0](s)+\mathscr{L}[u_{10}](s)+\mathscr{L}[u_{20}](s) &= \frac{e^{-0\cdot s}}{s} + \frac{e^{-10s}}{s} + \frac{e^{-20s}}{s} \\ &= \frac{1}{s} + \frac{e^{-10s}}{s} + \frac{e^{-20s}}{s}.\end{aligned}$$

More generally, we can rewrite any piecewise function as a *sum of multiples of differences of Heaviside functions*. That is what (2.8.37) says, as well as its simpler form (2.8.38). Lemma 2.8.16 then tells us how to compute the Laplace transforms of the terms of this sum. So, if we have an ODE forced by some discontinuous function, we can, after a lot of work, arrive at a formula for the transform of the solution and then maybe hope to invert that transform. It will be epic, and awful, but with time and patience *it can be done*. Let us speak no more of these things.

3. Second-Order Linear Equations

We have spent considerable (you might say, "excessive") time studying first-order ODE. This is in part due to my proclivities (you might say, "iniquities") as a mathematician, but in larger part for the following reasons.

1. First-order ODE model many interesting and important natural phenomena.

2. The analytic, qualitative, and numerical techniques used to study first-order ODE teach us many useful things about studying higher-order ODE, i.e., ODE with more than one derivative in play. Not all of these techniques generalize exactly, or easily, to higher-order problems, but it is my firm belief that good ideas about higher-order ODE are often most transparently incarnated for first-order problems.

3. Higher-order problems are, on the whole, more difficult than first-order problems, and, overall, there is less that we can say in sweeping generality about higher-order ODE as compared to first-order ODE. For example, we have a very exact method for finding formulas for solutions to

$$\dot{x} = a(t)x + b(t)$$

for more or less arbitrary continuous functions a and b. Now denote the second derivative of x by $\ddot{x}$. There is *no* one all-encompassing method for finding formulas for solutions to

$$\ddot{x} = p(t)\dot{x} + q(t)x + g(t)$$

for arbitrary functions p, q, and g; even dealing with p and q constant and $g = 0$ is a lot of work.

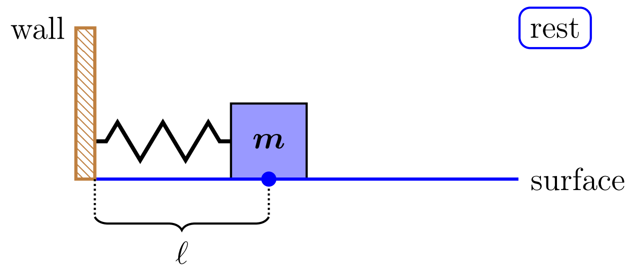
We will therefore focus on one very particular kind of second-order ODE, which arises naturally from the model that we now discuss.

3.1. The harmonic oscillator.

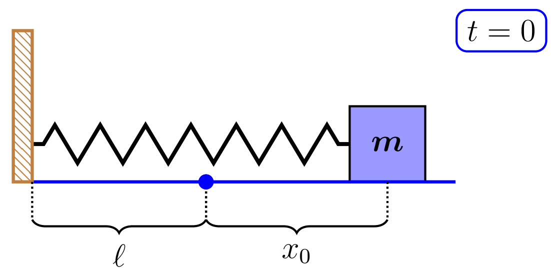
Our models in this course so far have been almost, if not entirely, all population models. Population models naturally hinge on the first derivative: the rate of change ($\dot{x}$) of a population (x) is determined by some function of time and current population ($\dot{x} = f(t, x)$). Models involving second derivatives often come instead from Newton's law: force = mass × acceleration. After all,

$$\text{acceleration} = \frac{d^2}{dt^2}[\text{position}].$$

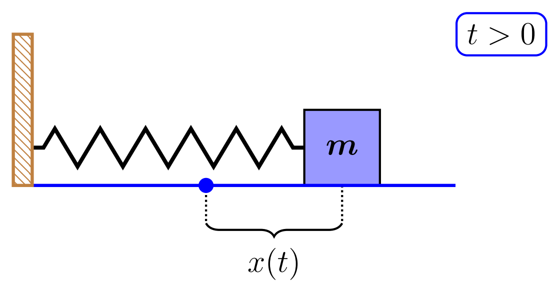
Consider the following physical situation. Place an object of uniform mass $m > 0$ along a horizontal surface. Connect the object to a wall on the left by a spring. At rest the object is ℓ units from the wall. This is a HARMONIC OSCILLATOR.



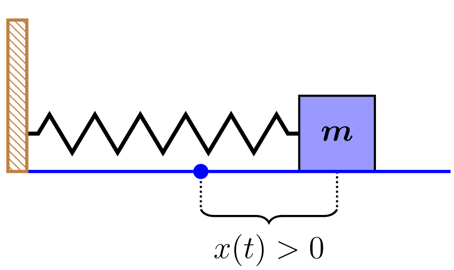
Pull the oscillator some x_0 units to the right (or push to the left; we will interpret "right" as $x_0 > 0$ and "left" as $x_0 < 0$) and let it go, maybe with a little extra oomph, maybe not. What happens? How does the oscillator move?

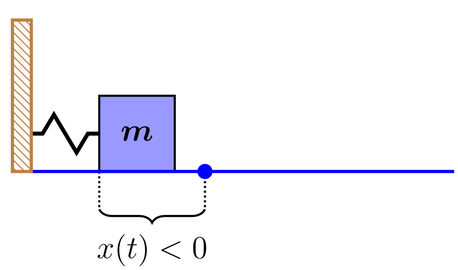


We'll make the fundamental assumption that the oscillator can only move to the left or the right along the surface, i.e., its motion is effectively one-dimensional. This allows us to introduce a coordinate system: denote the oscillator's displacement from its equilibrium position at time t by $x(t)$.



For example, since at the very start we pulled the oscillator a distance x_0 from equilibrium, we have $x(0) = x_0$. Assume that the displacement is positive if the oscillator is to the right of its equilibrium position and negative if the oscillator is to the left of equilibrium. (This is unlike our population models: now $x(t) < 0$ makes physical sense.)





Newton's law will give us an ODE governing the behavior of the oscillator. Here mass is m and acceleration at time t is $\ddot{x}(t)$. Suppose that we can measure all the forces acting on the oscillator at time t by $\mathsf{F}(t)$, where F is some function. Then

$$m\ddot{x}(t) = \mathsf{F}(t).$$

The precise choice of F will determine the precise ODE governing the oscillator (and also our course of study for the foreseeable future). One force that will always be present arises from the spring. Experience[26] teaches that the further you pull a spring, the more force you have to exert. If you stretch a spring a distance x, the spring pulls back with the force $\mathsf{F}_{\mathsf{spr}}(x)$. Experience probably also teaches us that we want $\mathsf{F}_{\mathsf{spr}}(x)$ to (1) be proportional to x and (2) act in the opposite direction to x. So, let's define

$$\mathsf{F}_{\mathsf{spr}}(x) := -\kappa x \tag{3.1.1}$$

for some $\kappa > 0$. The definition (3.1.1) is **Hooke's law**.

3.1.1. *The undamped harmonic oscillator.*

Assume, for the moment, and wholly unrealistically, that the *only* force experienced by the oscillator is the spring force — no friction against the surface, no air resistance, no cats coming by to play. Such an oscillator is called **undamped**. Then $\mathsf{F} = \mathsf{F}_{\mathsf{spr}}$, and so Newton's law says

$$m\ddot{x}(t) = \mathsf{F}_{\mathsf{spr}}(x(t)) = -\kappa x(t),$$

which we more typically write[27] as

$$m\ddot{x} + \kappa x = 0. \tag{3.1.2}$$

This is, of course, a **second-order** ODE, as $\ddot{x}$ appears in the equation, but no higher derivatives of x are there.

So, what happens? Since there is no friction, we might expect the oscillator, once put in motion, to move forever. Nothing is there to slow it down, or speed it up. In particular, it definitely should not settle down to stay motionless at some fixed distance from equilibrium, and so we expect that $\lim_{t\to\infty} x(t)$ does not exist; this presumes that (3.1.2) even has a solution x (of course it does), but we'll come to that presently.

We might make a more precise conjecture in one particular physical situation. Suppose that we do not move the oscillator at all from equilibrium: $x(0) = x_0 = 0$. Suppose that we don't even touch the oscillator; at time $t = 0$, it is motionless: $\dot{x}(0) = 0$. Then the oscillator should never move and thus stay at equilibrium for all time, i.e., we expect $x(t) = 0$ for all t. More formally, we expect that

$$\begin{cases} m\ddot{x} + \kappa x = 0 \\ x(0) = 0 \\ \dot{x}(0) = 0 \end{cases} \Longrightarrow x = 0.$$

[26] In the lab, with Slinkies…

[27] While we usually wrote first-order problems in the form $\dot{x} = f(t,x)$, we usually do *not* isolate the highest derivative in second-order problems or beyond.

This is where we finished on Monday, October 31, 2022.

3.1.2. The damped harmonic oscillator.

Suppose now that the oscillator experiences friction or air resistance in addition to the spring force — but otherwise there are no other forces (no cats coming by to play, not yet...). We now say that the oscillator is **DAMPED**.

Experience suggests that friction is proportional to velocity; let's say that if the oscillator is moving with velocity $\dot{x}$, then the frictional force that it experiences at time t is $\mathsf{F}_{\mathsf{fr}}(t) = -b\dot{x}(t)$ for some $b > 0$. Then the total force that the oscillator experiences is the sum of the spring force and the friction force: $\mathsf{F} = \mathsf{F}_{\mathsf{spr}} + \mathsf{F}_{\mathsf{fr}}$, and so Newton's law now reads

$$m\ddot{x} = -\kappa x - b\dot{x},$$

or, as we will more often write it,

$$m\ddot{x} + b\dot{x} + \kappa x = 0. \tag{3.1.3}$$

In the absence of other forces, then, we expect that friction will slow down the oscillator over long times and cause it to return to its rest position. Thus we expect the long time behavior

$$m\ddot{x} + b\dot{x} + \kappa x \text{ with } b > 0 \Longrightarrow \lim_{t \to \infty} x(t) = 0.$$

3.1.3. The driven harmonic oscillator.

The oscillators considered so far, whether undamped or damped, have experience no "external" forces. That is, to set up the oscillator, there is always a spring connecting the oscillator to a wall and a surface over which the oscillator moves. The spring always contributes a spring force (what else would we call it?), and the surface sometimes contributes a damping force (and sometimes the surface is magical and doesn't). These two kinds of forces are "internal" to the oscillator. But maybe a force "external" to the oscillator influences its motion — an earthquake, shaking the wall to which the oscillator is attached; a microlocalized black hole pulling the oscillator in one direction; a cat walking by and whacking the oscillator with her beefy paw.

If there is an external force, then the total force experienced by the oscillator at time t has the form $\mathsf{F}(t) = \mathsf{F}_{\mathsf{spr}}(t) + \mathsf{F}_{\mathsf{fr}}(t) + f(t)$, where $\mathsf{F}_{\mathsf{spr}}$ is the spring force, F_{fr} is the friction force (we now allow $\mathsf{F}_{\mathsf{fr}} = 0$, so $b = 0$, to incorporate the undamped oscillator), and f is a catch-all term for "all the other forces." The displacement of the oscillator then is

$$m\ddot{x} + b\ddot{x} + \kappa x = f(t).$$

An oscillator experiencing an external force is called **DRIVEN**[28] or (unsurprisingly) **FORCED**; an oscillator without an external force is **FREE**.

[28] Look back at Definition 2.6.1 right now. This is why we used the terms "forcing" and "driving" in that definition.

3.1.4. Guiding questions.

Here is a summary of our work on the ODE governing the displacement of a harmonic oscillator and its attendant terminology.

$m\ddot{x} + b\dot{x} + \kappa x = f(t)$	
Damped: $b > 0$	Undamped: $b = 0$
Free: $f(t) = 0$	Driven: $f(t) \neq 0$

Here are some questions that will guide our work. The first two are reiterations of ones that we previously asked.

1. If friction is present and $b > 0$, do we have $\lim_{t \to \infty} x(t) = 0$?

2. If friction is not present and $b = 0$, how can we quantify the idea that "the oscillator keeps moving forever and doesn't slow down"?

Here are some new questions, now that we can put all aspects of the oscillator together into the one ODE above.

3. How might changing the parameters m, b, and κ affect the solution? We might expect that a "heavier" mass moves "more slowly" than a "lighter" one. We might expect that if we "turn up" the friction, the oscillator returns to equilibrium "more quickly." We might expect that a "stiffer" spring pulls back "more quickly" than a "looser" spring. Overall, *how can we quantify these questions in terms of the parameters m, b, and κ, and how can we see their effects in the solution?*

4. How might a particular driving term f (say, a regular, periodic whacking of the oscillator by our jerk of a cat) manifest itself in the solution x? How can we see explicitly the dependence of a solution on f?

5. What role (if any) do the initial displacement $x(0)$ and velocity $\dot{x}(0)$ of the oscillator play in its future motion and behavior? These are two aspects of its motion that we can quite reasonably control at the very start.

Finally, here are some harder questions, which we will not pursue in this course, but which can nonetheless be addressed with enough work.

6. What if the oscillator and its environment "change" over time? Maybe the mass leaks, the surface (over which the oscillator moves) gets rougher, or the spring stiffens. In these cases, we would want the "material" data m, b, and κ to depend on time, and so we would need to solve problems of the form

$$m(t)\ddot{x} + b(t)\dot{x} + \kappa(t)x = f(t).$$

This turns out to be quite hard!

7. What if the spring is stretched over "long" distances? Hooke's law (3.1.1) is really only valid when the spring is stretched a "short" length from equilibrium. Otherwise, we might need to incorporate a nonlinear term; instead of saying that the force exerted by stretching the spring a length x is the linear function $\mathsf{F}_{\mathsf{spr}}(x) = -\kappa x$, we might now use a *nonlinear* force $\mathsf{F}_{\mathsf{spr}}(x) = -\kappa x - \beta x^2$. Here the spring force has a quadratic term; it could be even more complicated. Then the equation of motion for the oscillator is

$$m\ddot{x} + b\dot{x} + \kappa x + \beta x^2 = f(t).$$

This too is quite hard!

While we know by now that there is more to life than formulas — here's your daily reminder to pause and recite the Analyst's Creed (AC) — a good initial attempt at answering these questions might be finding some formulas and playing with them. Specifically, we want to solve the IVP[29]

$$\begin{cases} m\ddot{x} + b\dot{x} + \kappa x = f(t) \\ x(0) = x_0 \\ \dot{x}(0) = y_0 \end{cases} \tag{3.1.4}$$

for given numbers m, b, κ, x_0, and y_0 and a given function f.

We now proceed to do just that.

3.2. The homogeneous problem.

The governing equation for a harmonic oscillator is

$$m\ddot{x} + b\dot{x} + \kappa x = f(t).$$

Since $m > 0$, we may divide both sides by m and put $p := b/m$, $q := \kappa/m$, and $g(t) := f(t)/m$ to find that x must satisfy

$$\ddot{x} + p\dot{x} + qx = g(t). \tag{3.2.1}$$

This is slightly simpler than our original problem and follows our long convention with first-order ODE that the highest derivative (previously $\dot{x}$, now $\ddot{x}$) has only a coefficient of 1.

More generally, we could consider the problem

$$a\ddot{x} + b\dot{x} + cx = f(t), \tag{3.2.2}$$

where a, b, and c are real numbers and f is a function. This is a **CONSTANT-COEFFICIENT LINEAR SECOND-ORDER** ODE, which is a lot of adjectives. This ODE is **HOMOGENEOUS** if $f(t) = 0$ for all t and **NONHOMOGENEOUS** if $f(t) \neq 0$ for at least one t. We require $a \neq 0$, as otherwise the problem reduces to a first-order equation. However, the following mathematical analysis will work for a, b, and c negative, which we did not allow in the harmonic oscillator's equation.

Nonetheless, for notational simplicity, we will stick with (3.2.1), and, if you really have to, you can always convert (3.2.2) into (3.2.1) by dividing by a. We will start with the homogeneous case of (3.2.1), i.e., $g(t) = 0$. That is slightly simpler; moreover, the first-order homogeneous linear problem turned out to be critical to solving the first-order nonhomogeneous problem, and we might expect the same here.

So, what do we do?

[29] We could pose this IVP at the initial time $t = t_0$ for t_0 arbitrary, but we won't; life will be hard, and interesting, enough just starting from time $t = 0$.

3.2.1. An inspired exponential ansatz.

Our immediate goal is to solve (and understand) the homogeneous problem

$$\ddot{x} + p\dot{x} + qx = 0. \tag{3.2.3}$$

Having no better ideas, and nothing better to do, our thinking might follow one of two tracks.

First, we could delete the $\ddot{x}$ term and think about what happens to the ODE $p\dot{x}+qx = 0$. Maybe that will help? Assuming $p \neq 0$, we get

$$\dot{x} = -\frac{q}{p}x.$$

This is a homogeneous linear first-order ODE, and so its solution is

$$x(t) = Ce^{-(q/p)t}$$

for any constant C. Here is what we learn: maybe the second-order problem (3.2.3) has a solution that involves an exponential?

Second, we could think about what is happening in (3.2.3) arithmetically. We take two derivatives of x and add multiples of x, $\dot{x}$, and $\ddot{x}$ together, and we get 0. The three functions x, $\dot{x}$, and $\ddot{x}$ have to "talk" together in a sufficiently nice way so that the combination $\ddot{x}+p\dot{x}+qx$ is always 0. What kind of function talks to its derivatives in a particularly transparent way? One good answer is the exponential.

Here is the right idea, and despite the efforts of the previous two paragraphs, you may not be convinced that it's the right idea until you see that it works. (I'm still not entirely convinced myself.) Let's guess that a solution x to (3.2.3) has the form

$$x(t) = e^{\lambda t}$$

for some real number λ. We calculate

$$\dot{x}(t) = \lambda e^{\lambda t} \quad \text{and} \quad \ddot{x}(t) = \lambda^2 e^{\lambda t},$$

and so we have

$$\ddot{x}(t) + p\dot{x}(t) + qx(t) = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t}.$$

Then, since $e^{\lambda t} > 0$ for all real λ and t, we have

$$\ddot{x}(t) + p\dot{x}(t) + qx(t) = 0 \iff (\lambda^2 + p\lambda + q)e^{\lambda t} \iff \lambda^2 + p\lambda + q = 0.$$

Thus, to get a solution x to $\ddot{x} + p\dot{x} + qx = 0$ of the form $x(t) = e^{\lambda t}$, we just need λ to solve the quadratic equation

$$\lambda^2 + p\lambda + q = 0. \tag{3.2.4}$$

We call (3.2.4) the **characteristic equation** (sometimes, the **auxiliary** equation) of the ODE $\ddot{x} + p\dot{x} + qx = 0$. This is great news! We spent a lot of time in high school, and afterward, thinking about quadratic equations, and we know how to solve them in a flash. The **QUADRATIC FORMULA** for solutions λ to (3.2.4) reads

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

You might remember that the quantity $p^2 - 4q$ is called the **DISCRIMINANT** of the quadratic equation (3.2.4). The discriminant is a real number, so there are three possibilities:

$$p^2 - 4q > 0, \qquad p^2 - 4q = 0, \qquad \text{or} \qquad p^2 - 4q < 0.$$

When the discriminant is positive, the quadratic equation has two distinct (i.e., unequal) real roots; when the discriminant is 0, the quadratic equation has only one (real) root; and when the discriminant is negative, the quadratic equation has two complex, nonreal roots that appear as complex conjugate pairs. Our first task is to interpret what these three root behaviors tell us about solutions to the ODE $\ddot{x} + p\dot{x} + qx = 0$. Then we will interpret what these behaviors say about the harmonic oscillator.

3.2.2. Distinct real roots.

Suppose that $p^2 - 4q > 0$, so the quadratic equation $\lambda^2 + p\lambda + q = 0$ has the two roots

$$\lambda_1 := -\frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2} \qquad \text{and} \qquad \lambda_2 := -\frac{p}{q} + \frac{\sqrt{p^2 - 4q}}{2}.$$

Note that $\lambda_1 < \lambda_2$, so $\lambda_1 \neq \lambda_2$; these roots are "distinct." Then the functions

$$x_1(t) := e^{\lambda_1 t} \qquad \text{and} \qquad x_2(t) := e^{\lambda_2 t}$$

both solve the ODE $\ddot{x} + p\dot{x} + qx = 0$.

Let's do an example.

3.2.1 Example.

Find solutions to $\ddot{x} - x = 0$.

Solution. Here $p = 0$ and $q = -1$, so the characteristic equation is $\lambda^2 - 1 = 0$. This is a difference of perfect squares, which we factor instantly as $(\lambda+1)(\lambda-1) = 0$, and so its roots are $\lambda = 1$ and $\lambda = -1$. Solutions to the ODE are therefore $x_1(t) := e^t$ and $x_2(t) := e^{-t}$. ▲

Let's not be content with our success. How might we solve an IVP like

$$\begin{cases} \ddot{x} + p\dot{x} + qx = 0 \\ x(0) = x_0 \\ \dot{x}(0) = y_0? \end{cases}$$

In the past, our success with first-order ODE came from having a "free constant" in the solution (however we found it); we were able to use algebra to choose the constant correctly to meet the initial condition.

However, in our exponential solutions, there are no (apparent) free constants! Moreover, we are going to have two initial conditions (to encode initial displacement and initial velocity in the harmonic oscillator model), and so we actually we need two free constants. How do we get them? The right idea is to introduce something artificial (but also natural).

Think about the first-order linear homogeneous ODE $\dot{x} = ax$. Any constant multiple of a solution is a solution, and sums of solutions are solutions, too. This is linearity, and it also holds for the second-order linear homogeneous problem (otherwise we wouldn't call it linear). More precisely, we have the following theorem, which you can, and should, prove on your own; note that its statement does not assume that the corresponding characteristic equation has distinct real roots, so it is valid for *all* second-order linear homogeneous problems.

3.2.2 Theorem (Linearity).

Suppose that x_1 and x_2 solve the ODE $\ddot{x} + p\dot{x} + qx = 0$. Let c_1 and c_2 be real numbers. Then $c_1x_1 + c_2x_2$ also solves $\ddot{x} + p\dot{x} + qx = 0$.

Let's put this theorem into action.

3.2.3 Example.

We know from Example 3.2.1 that $x_1(t) = e^t$ and $x_2(t) = e^{-t}$ both solve $\ddot{x} - x = 0$. Choose constants c_1 and c_2 so that $c_1x_1 + c_2x_2$ solves the IVP

$$\begin{cases} \ddot{x} - x = 0 \\ x(0) = 1 \\ \dot{x}(0) = 2. \end{cases}$$

Solution. Let $x(t) = c_1e^t + c_2e^{-t}$. We first need

$$1 = x(0) = c_1e^0 + c_2e^{-0} = c_1 + c_2.$$

Next, we calculate

$$\dot{x}(t) = c_1e^t - c_2e^{-t},$$

and so we need

$$2 = \dot{x}(0) = c_1e^0 - c_2e^{-0} = c_1 - c_2.$$

So, c_1 and c_2 must solve the linear system

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 2. \end{cases} \tag{3.2.5}$$

We have turned a calculus problem into an algebra problem! There are many, many ways of solving a system like (3.2.5). Here is just one of them. Rewrite the second equation as $c_2 = c_1 - 2$, and plug that into the first equation to get

$$1 = c_1 + c_2 = c_1 + (c_1 - 2) = 2c_1 - 2.$$

Add 2 to both sides to get $2c_1 = 3$, and divide by 2 to get $c_1 = 3/2$. Then, from before,

$$c_2 = c_1 - 2 = \frac{3}{2} - 2 = \frac{1}{2}.$$

Thus a solution to the IVP is

$$x(t) = \frac{3e^t}{2} + \frac{e^{-t}}{2}.$$

▲

This is where we finished on Friday, November 4, 2022.

Note that we said "a" solution, not "the" solution. The work in Example 3.2.3 shows that the only solution to the IVP under consideration of the form $x(t) = c_1e^t + c_2e^{-t}$ occurs when $c_1 = 3/2$ and $c_2 = 1/2$; in other words, there is only one way to choose the constants c_1 and c_2. But could there be another solution that is not a linear combination of the exponentials $x_1(t) = e^t$ and $x_2(t) = e^{-t}$?

Happily, the answer is no. Our experience with first-order problems should make us think that second-order IVP also have unique solutions. We were able to prove this from scratch with the integrating factor method for the first-order linear IVP, but a rigorous treatment of the second-order linear IVP is rather more challenging. We will just state the result here.

3.2.4 Theorem (Existence and uniqueness).

Let g be a continuous function on $(-\infty, \infty)$. Then the IVP

$$\begin{cases} \ddot{x} + p\dot{x} + qx = g(t) \\ x(0) = x_0 \\ \dot{x}(0) = y_0 \end{cases}$$

has a unique solution.

Again, this theorem is valid for all second-order linear problems, not just ones whose characteristic equations have distinct real roots, and also not just homogeneous ones. The utility of this theorem will be that if we have found one solution to a second-order IVP, then we will have found all of them.

3.2.5 Example.

Solve the IVP

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 \\ x(0) = -1 \\ \dot{x}(0) = 1. \end{cases}$$

Solution. The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$. Either by factoring (if we see a factorization quickly) or the quadratic formula (if we don't), we find that the roots are $\lambda = 1$ and $\lambda = 2$. We will look for a solution to the IVP of the form $x(t) = c_1e^t + c_2e^{2t}$. We need

$$-1 = x(0) = c_1 + c_2,$$

and, since

$$\dot{x}(t) = c_1e^t + 2c_2e^{2t},$$

we need

$$1 = \dot{x}(0) = c_1 + 2c_2.$$

So, we must solve the system

$$\begin{cases} c_1 + c_2 = -1 \\ c_1 + 2c_2 = 1. \end{cases}$$

The first equation gives $c_1 = -1 - c_2$, so the second equation becomes

$$1 = c_1 + 2c_2 = (-1 - c_2) + 2c_2 = -1 + c_2,$$

thus $c_2 = 2$, and so $c_1 = -1 - 2 = -3$. The solution to the IVP therefore is

$$x(t) = -3e^t + 2e^{2t}.$$ ▲

3.2.3. Repeated real roots.

Suppose now that the characteristic equation $\lambda^2 + p\lambda + q = 0$ for the ODE $\ddot{x} + p\dot{x} + qx = 0$ has just one repeated real root. That is, the discriminant satisfies $p^2 - 4q = 0$, and so, from the quadratic formula the only root is $\lambda = -p/2$. Then, certainly, the function

$$x_1(t) := e^{-(p/2)t}$$

will solve the ODE, and so will any constant multiple of x_1. But we expect to need two solutions, and two free constants, to solve IVP.

The workaround is quite simple, and my favorite second-order problem will illustrate it. Consider the ODE

$$\ddot{x} = 0,$$

so $p = q = 0$. Its characteristic equation is $\lambda^2 = 0$, and certainly the only root here is $\lambda = 0$. Thus a solution is

$$x_1(t) = e^{\lambda t} = e^{0 \cdot t} = 1.$$

But we certainly don't need any high-powered ODE ideas to solve $\ddot{x} = 0$; just antidifferentiate twice to get

$$x(t) = c_1t + c_2$$

for some constants c_1 and c_2. Now here's the clever observation that joins this solution with the repeated root $\lambda = 0$ of the characteristic equation:

$$x(t) = c_1t + c_2 = c_1te^{0 \cdot t} + c_2e^{0 \cdot t} = c_1te^{\lambda t} + c_2e^{\lambda t}, \qquad \lambda = 0.$$

So, here is the lesson of this very simple ODE. If $\lambda = -p/2$ is the repeated real root of the characteristic equation $\lambda^2 + p\lambda + q = 0$ for the ODE $\ddot{x} + p\dot{x} + qx = 0$, then maybe the function

$$x_2(t) := te^{\lambda t} = te^{-(p/2)t}$$

also solves $\ddot{x} + p\dot{x} + qx = 0$. And indeed it does; I will leave that for you to check. (The key idea is that a repeated real root only happens when $p^2 - 4q = 0$, so we have the additional relation $q = p^2/4$ in play.)

3.2.6 Example.

Solve the IVP

$$\begin{cases} \ddot{x} - 2\dot{x} + x = 0 \\ x(0) = 1 \\ \dot{x}(0) = 5. \end{cases}$$

Solution. The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$, which factors into $(\lambda - 1)^2 = 0$. The only root is therefore the repeated root $\lambda = 1$. Following the arguments above, we look for a solution to the IVP of the form

$$x(t) = c_1 e^t + c_2 t e^t.$$

This is where we finished on Monday, November 7, 2022.

We need

$$1 = x(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1,$$

so we figured out c_1 pretty fast. We need to do a bit more work than before to calculate $\dot{x}$, which is

$$\dot{x}(t) = c_1 e^t + (c_2 e^t + c_2 t e^t) = e^t + c_2 e^t + 2te^t.$$

Here we had to use the product rule, but we could also use $c_1 = 1$ from before. So, we want

$$5 = \dot{x}(0) = e^0 + c_2 e^0 + 2 \cdot 0 \cdot e^0 = 1 + c_2,$$

and so $c_2 = 4$. Thus

$$x(t) = e^t + 4te^t. \qquad \blacktriangle$$

3.2.4. Complex conjugate roots.

Suppose now that the characteristic equation $\lambda^2 + p\lambda + q = 0$ for the ODE $\ddot{x} + p\dot{x} + qx = 0$ has a pair of complex conjugate roots. That is, the discriminant satisfies $p^2 - 4q < 0$. We then want to say that the solutions to the characteristic equation are

$$\lambda_1 = -\frac{p}{2} - i\left(\frac{\sqrt{4q - p^2}}{2}\right) \quad \text{and} \quad \lambda_2 = -\frac{p}{2} + i\left(\frac{\sqrt{4q - p^2}}{2}\right).$$

Since $p^2 - 4q < 0$, we have $4q - p^2 = -(p^2 - 4q) > 0$, and so there is no problem taking the square root $\sqrt{4q - p^2}$.

The pesky thing is the factor i, which we interpret as satisfying $i^2 = -1$. We want to say that the functions $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ solve the ODE $\ddot{x} + p\dot{x} + qx = 0$, but what do these exponentials mean?

Let's take a moment to pull back and think about complex numbers. A **COMPLEX NUMBER** is an expression of the form $\alpha + i\beta$, or sometimes $\alpha + \beta i$, where α and β are real numbers and i satisfies $i^2 = -1$; we add and multiply complex numbers just as we do real numbers, and we make absolutely no attempt here to give a rigorous interpretation of the symbolic juxtaposition $i\beta$. (What in the world does "multiplying" a real number β with the symbol i satisfying $i^2 = 1$ mean? Take a course in complex analysis.) Every real number α, by the way, is a complex number; write $\alpha = \alpha + (i \cdot 0)$.

Here is a problem for which complex roots arise in practice:

$$\ddot{x} - 2\dot{x} + 5 = 0.$$

The characteristic equation is

$$\lambda^2 - 2\lambda + 5 = 0.$$

If after about ten seconds of thought, you don't see a quick and easy factorization, go to the quadratic formula:

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

We want to say that solutions are

$$x_1(t) = e^{(1+2i)t} \quad \text{and} \quad x_2(t) = e^{(1-2i)t}. \tag{3.2.6}$$

What on earth does this mean?

At the very least, we expect

$$x_1(t) = e^{t+2it} = e^t e^{2it} \quad \text{and} \quad x_2(t) = e^{t-2it} = e^t e^{-2it}. \tag{3.2.7}$$

So, a better question is how to define $e^{i\beta}$ when β is a real number. This begs the question of what e^{α} is when α is a real number; my favorite answer is power series:

$$e^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}.$$

So, a good *definition* for the **complex exponential** $e^{i\beta}$ is

$$e^{i\beta} := \sum_{k=0}^{\infty} \frac{(i\beta)^k}{k!}. \tag{3.2.8}$$

For example, the numerator of the $k = 3$ term in this series is

$$(iy)^3 = i^3 y^3 = i^2 i y^3 = -iy^3.$$

Now what? What does the series (3.2.8) do? If you work at it a bit, and use the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on, repeating ever thereafter, and if you remember the Taylor series for the sine and cosine, you can arrive at the following identity:

$$e^{i\beta} = \cos(\beta) + i\sin(\beta). \tag{3.2.9}$$

This is called **Euler's formula** for the complex exponential.

We therefore combine the expected arithmetic of (3.2.7) with Euler's formula (3.2.9) to make the following *definition*.

3.2.7 Definition.

Let α and β be real numbers. The symbol $e^{\alpha+i\beta}$ means

$$e^{\alpha+i\beta} := e^{\alpha}\big[\cos(\beta) + i\sin(\beta)\big].$$

Note that since α and β are real numbers, the quantities e^{α}, $\cos(\beta)$, and $\sin(\beta)$ are all defined as usual (say, by their Taylor series). Back to the ODE $\ddot{x} - 2\dot{x} + 5 = 0$ and its putative solutions x_1 and x_2 defined in (3.2.6). We therefore expect these solutions to be

$$x_1(t) = e^{(1+2i)t} = e^{t+i(2t)} = e^{t}\big[\cos(2t) + i\sin(2t)\big] = e^{t}\cos(2t) + ie^{t}\sin(2t)$$

and

$$x_2(t) = e^{(1-2i)t} = e^{t+i(-2t)} = e^{t}\big[\cos(-2t)+i\sin(-2t)\big] = e^{t}\big[\cos(2t)-i\sin(2t)\big] = e^{t}\cos(2t)-ie^{t}\sin(2t).$$

As always, what does this mean? We've (presumably) never done calculus for functions defined at real numbers t but whose outputs are complex (and not real) numbers. Moreover, our original problem $\ddot{x} - 2\dot{x} + 5 = 0$ was stated only with real numbers (and the physical description of the harmonic oscillator model definitely involved only real numbers), and it's at best rude to hand someone a complex-valued answer to a real-coefficient problem.

What are we to do? We could build up the theory of the calculus of complex-valued functions of real variables from scratch (spoiler alert: nothing changes from the calculus that you already know), but a faster way is to hope that since x_1 and x_2 have the forms $u \pm iv$, where $u(t) = e^{t}\cos(2t)$ and $v(t) = e^{t}\sin(2t)$, maybe u and v have some healthy relation to the original ODE. And they do: you can check that

$$\ddot{u} - 2\dot{u} + 5u = \ddot{v} - 2\dot{v} + 5v = 0.$$

The following is more generally true, and, as usual, I invite you to check it.

3.2.8 Lemma.

Suppose that p and q are real numbers with $p^2 - 4q < 0$. Let

$$\alpha := -\frac{p}{2} \quad \text{and} \quad \beta := \frac{\sqrt{4q-p^2}}{2},$$

so that the roots of the quadratic equation $\lambda^2 + p\lambda + q = 0$ are

$$\lambda = \alpha + i\beta \quad \text{and} \quad \lambda = \alpha - i\beta.$$

Then the functions

$$x_1(t) = e^{\alpha t}\cos(\beta t) \quad \text{and} \quad x_2(t) = e^{\alpha t}\sin(\beta t)$$

both solve the ODE $\ddot{x} + p\dot{x} + qx = 0$.

This is where we finished on Wednesday, November 9, 2022.

3.2.9 Example.

Use all of the preceding work to solve the IVP

$$\begin{cases} \ddot{x} - 2\dot{x} + 5x = 0 \\ x(0) = -1 \\ \dot{x}(0) = 1. \end{cases}$$

Solution. The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$, and we know that it has the pair of complex conjugate roots $\lambda = 1 \pm 2i$. Let's write $\alpha = 1$ and $\beta = 2$ here; if you want to use $\beta = -2$, that will still work, but writing -2 takes longer than writing 2. We want to look for a solution to the IVP of the form

$$x(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t).$$

This means we need

$$-1 = x(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = c_1.$$

Next, we differentiate x very carefully to find

$$\dot{x}(t) = c_1\left[e^t\cos(2t) - 2e^t\sin(2t)\right] + c_2\left[e^t\sin(2t) + 2e^t\cos(2t)\right],$$

and so we need

$$1 = \dot{x}(0) = c_1[e^0\cos(0) - 2e^0\sin(0)] + c_2[e^0\sin(0) + 2e^0\cos(0)] = c_1 + 2c_2.$$

But we already know $c_1 = -1$, so c_2 must satisfy

$$1 = -1 + 2c_2,$$

thus $2c_2 = 2$, and so $c_2 = 1$. Hence the solution is

$$x(t) = -e^t\cos(2t) + e^t\sin(2t). \qquad \blacktriangle$$

3.3. The free harmonic oscillator.

We now know how to solve ODE of the special form

$$\ddot{x} + p\dot{x} + x = 0. \tag{3.3.1}$$

Just look at the characteristic equation $\lambda^2 + p\lambda + q = 0$ and interpret its roots. More precisely, we have the following three cases.

1. *Two distinct real roots λ_1 and λ_2 ($p^2 - 4q > 0$)*: all solutions to (3.3.1) have the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

2. *One repeated real root λ ($p^2 - 4q = 0$)*: all solutions to (3.3.1) have the form

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

3. *Two distinct complex conjugate roots $\alpha \pm i\beta$, with $\beta \neq 0$ ($p^2 - 4q < 0$)*: all solutions to (3.3.1) have the form

$$x(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t).$$

To be fair, we never actually proved this, but you can (and should). Rather, we worked through a number of IVP and always found solutions of the forms above; the existence and uniqueness theorem then guaranteed that those were the only solutions. Here's how you do this in general. Start with an arbitrary solution to (3.3.1). Then introduce "artificial" initial data by declaring $x_0 := x(0)$ and $y_0 := \dot{y}(0)$. Then solve the IVP

$$\begin{cases} \ddot{x} + p\dot{x} + qx = 0 \\ x(0) = x_0 \\ \dot{x}(0) = y_0 \end{cases}$$

by considering the three cases above on the discriminant. Show that you can always solve for the coefficients c_1 and c_2 in terms of x_0 and y_0 (and p and q). Sound reasonable?

I won't do this explicitly, because I want to stop talking about formulas and start living out the Analyst's Creed (AC). Specifically, we took the harmonic oscillator as our canonical model for second-order linear ODE. Recall that in the absence of external forces, an oscillator of mass $m > 0$, spring constant $\kappa > 0$, and damping coefficient $b \geq 0$ satisfies

$$m\ddot{x} + b\dot{x} + \kappa x = 0.$$

Dividing through by m and relabeling $p := b/m$ and $q := \kappa/m$, we get (3.3.1). We believe that when $p = 0$, the oscillator should keep moving forever in some obvious fashion, since friction is then absent. But when $p > 0$, we expect $\lim_{t\to\infty} x(t) = 0$, because the oscillator should slow down and return to its equilibrium position.

We will show that the mathematics bears out these physical expectations in the special case $q = 1$, so we'll study

$$\ddot{x} + p\dot{x} + x = 0. \tag{3.3.2}$$

I am setting $q = 1$ purely for simplicity; you can think of it as the special case when the mass m and the spring constant κ are identical. (Yes, I know that masses and spring constants should have different units, but I just can't bring myself to care.) We will focus on the role of p, the damping coefficient. There are really four cases to consider.

1. $p = 0$. Then (3.3.2) is just $\ddot{x} + x = 0$. The characteristic equation here is $\lambda^2 + 1 = 0$, and we see that its roots are $\pm i$. That is, $\alpha = 0$ and $\beta = 1$, so all solutions are $x(t) = c_1 \cos(t) + c_2 \sin(t)$. These solutions clearly oscillate forever and do not tend to a limit as $t \to \infty$ unless $c_1 = c_2 = 0$ (in which case the oscillator isn't moving at all).

2. $p > 0$ and $p^2 - 4 > 0$ (thus $p > 2$ since $p > 0$). Since $p > 0$, damping is "turned on," and so the characteristic equation is now

$$\lambda^2 + p\lambda + 1 = 0. \tag{3.3.3}$$

Its roots are

$$\lambda_1 := -\frac{p}{2} - \frac{\sqrt{p^2-4}}{2} \quad \text{and} \quad \lambda_2 := -\frac{p}{2} + \frac{\sqrt{p^2-4}}{2},$$

and all solutions are $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. Since $p > 0$ and $\sqrt{p^2-4} > 0$, I hope it's clear that $\lambda_1 < 0$, and therefore $\lim_{t\to\infty} c_1 e^{\lambda_1 t} = 0$. To control the other term in x, we want to be sure that $\lambda_2 < 0$. This needs just a little more work: since $0 < p^2 - 4 < p^2$, we have

$$\sqrt{p^2-4} < \sqrt{p^2} = |p| = p.$$

Here we used the fact that the square root is increasing and $|p| = p$ since $p > 0$. Then

$$-p < -\sqrt{p^2-4},$$

and so

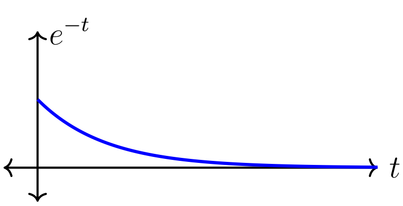
$$-p + \sqrt{p^2-4} < 0.$$

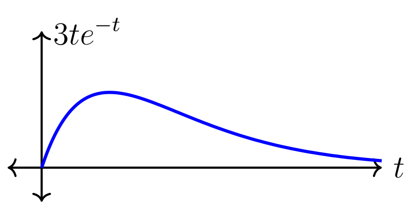
Multiplying through by $1/2$ shows $\lambda_2 < 0$, and so $\lim_{t\to\infty} c_2 e^{\lambda_2 t} = 0$. Thus we get $\lim_{t\to\infty} x(t) = 0$, which is what we wanted.

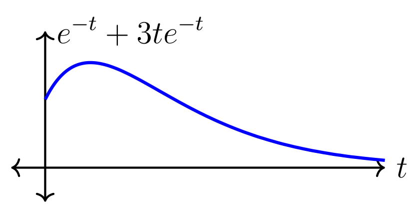
3. $p > 0$ and $p^2 - 4 = 0$ (also known as $p = 2$). Here the characteristic equation (3.3.3) has the repeated real root $-p/2 (= -1)$, so all solutions are $x(t) = c_1 e^{-(p/2)t} + c_2 t e^{-(p/2)t}$. The first term vanishes at infinity by properties of exponentials: $\lim_{t\to\infty} c_2 e^{-(p/2)t} = 0$ since $-p/2 < 0$. So does the second term, but we need to do some more work to prevent it from becoming an $\infty \cdot 0$ indeterminate form. Use L'Hospital's rule:

$$\lim_{t\to\infty} c_2 t e^{-(p/2)t} = \lim_{t\to\infty} \frac{c_2 t}{e^{(p/2)t}} = \lim_{t\to\infty} \frac{\frac{d}{dt}[c_2 t]}{\frac{d}{dt}[e^{(p/2)t}]} = \lim_{t\to\infty} \frac{c_2}{p e^{(p/2)t}/2} = 0.$$

It is interesting (relatively speaking) to play a bit with the parameter p and look at some potential solution graphs when $p = 2$ and $p > 2$. Depending on the choices of c_1 and c_2, which arise from the initial conditions, a variety of graphs can appear. Some are monotonically decreasing, while others might have a little local extremum near time $t = 0$. Here are some graphs that illustrate different possibilities.







You can (and should) ask endless sorts of questions about how the choice of p and the initial data precisely affect the graph, and what that means about the behavior of the oscillator.

This is where we finished on Friday, November 11, 2022.

4. $p > 0$ and $p^2 - 4 < 0$ (thus $0 < p < 2$). Here the characteristic equation (3.3.3) has the complex conjugate pair of roots $\alpha \pm i\beta$, where $\alpha = -p/2$ and $\beta = \sqrt{4-p^2}/2$. It is very important that $\alpha < 0$. Then all solutions are, as usual, $x(t) = c_1e^{\alpha t}\cos(\beta t) + c_2e^{\alpha t}\sin(\beta t)$. We can show that these solutions vanish at infinity via the squeeze theorem: for simplicity, say $c_2 = 0$, and estimate

$$0 \le |c_1e^{\alpha t}\cos(\beta t)| = |c_1||e^{\alpha t}||\cos(\beta t)| \le |c_1|e^{\alpha t}. \tag{3.3.4}$$

Here we used the bound $|\cos(\tau)| \le 1$, valid for all numbers τ. Since $\alpha < 0$, we have $\lim_{t\to\infty} e^{\alpha t} = 0$, and so the estimate (3.3.4) implies $\lim_{t\to\infty} c_1e^{\alpha t}\cos(\beta t) = 0$.

However, the oscillator here is not "just" slowing down and returning to its rest position. This may not be obvious from the formulas, but if we graph a solution of the form $x(t) = e^{\alpha t}\sin(\beta t)$, setting $c_1 = 0$ and $c_2 = 1$ for convenience, we'll see some behaviors that weren't present before.

Here's what I see: the graph of x crosses the t-axis infinitely many times. That means x has infinitely many roots, or zeros. If $x(t) = 0$, then the oscillator is at its equilibrium position at time t. So, in this case, the oscillator passes through its equilibrium point *infinitely* many times. It's not too hard to show when that happens:

$$e^{\alpha t}\sin(\beta t) = 0 \iff \sin(\beta t) = 0 \iff \beta t = n\pi,\ n \text{ an integer} \quad \iff t = \frac{n\pi}{\beta}.$$

Here we used the facts that $e^{\alpha t} > 0$ and that $\beta > 0$ to facilitate division, and we used the root structure of the sine: $\sin(\tau) = 0$ precisely when τ is an integer multiple of π, i.e., $\tau = n\pi$ for some integer n. The numbers $n\pi/\beta$ are the times at which the oscillator passes through equilibrium. I think it's interesting that these times are *evenly spaced apart*; each root of $x(t) = e^{\alpha t}\sin(\beta t)$ is separated by a distance of π/β from the next root. This suggests a surprising uniformity in the behavior of the oscillator; even though it is moving over smaller distances as time goes on and getting closer to equilibrium (this is what $\lim_{t\to\infty} x(t) = 0$ means), it continues moving to the left and right of equilibrium over exactly the same time intervals *forever*.

Math is weird.

As with all of our models, the value of this harmonic oscillator analysis is less (for me) that we have achieved control over some profound physical system that's going to help us rule the world; rather, we've seen how to distill some physical assumptions into mathspeak, analyze the math (using tools as diverse as inequality manipulation, L'Hospital's rule, and the squeeze theorem), and interpret the results physically.

3.4. Nonhomogeneous problems.

We cooked up the ODE

$$m\ddot{x} + b\dot{x} + \kappa x = f(t)$$

to govern the displacement of a harmonic oscillator of mass $m > 0$ and spring constant $\kappa > 0$ with damping constant $b \ge 0$. Here f incorporates any external force influencing the oscillator. Since $m > 0$ and writing three letters is more work than writing two, we divide by m and study instead

$$\ddot{x} + p\dot{x} + qx = g(t), \tag{3.4.1}$$

with $p = b/m$, $q = \kappa/m$, and $g(t) = f(t)/m$. The ODE (3.4.1) is **NONHOMOGENEOUS** when $g(t) \neq 0$ for at least one t.

Now that we have mastered the homogeneous problem, we will turn on g and study the nonhomogeneous problem. While it is possible to develop a solution method for (3.4.1) that handles all continuous g, the actual calculations quickly devolve into endless integration exercises, and we will not consider them here. (It is worth knowing where to look them up in a textbook or on the internet if you ever do need them.) Rather, we will first look at some general properties of (3.4.1), valid for any g, and then develop solution techniques, and qualitative interpretations, for special, and commonly occurring, choices of g.

3.4.1. Consequences of linearity.

To learn about solutions to the nonhomogeneous problem (3.4.1), one natural thing to do is to compare two of them. Say that x_1 and x_2 both solve (3.4.1). How different are they? One way to measure their difference is to subtract. So, what does $x_1 - x_2$ do? You know what they say: if it moves, differentiate it.

I claim that because x_1 and x_2 both solve (3.4.1), and because derivatives interact well (*linearly*) with constant multiples and addition, $x_1 - x_2$ solves the homogeneous problem

$$\ddot{x} + p\dot{x} + qx = 0. \tag{3.4.2}$$

We just find the roots of the characteristic equation $\lambda^2 + p\lambda + q = 0$ and interpret them. Let's abbreviate, then, $x_{\mathsf{h}} := x_1 - x_2$; the h is for, of course, $\mathsf{h}$omogeneous. Then we get

$$x_1 = x_2 + x_{\mathsf{h}}.$$

Huh? The functions x_1 and x_2 both solve the nonhomogeneous problem (3.4.1), but this equality says that x_1 and x_2 are essentially the same, up to adding a solution x_{h} of the homogeneous problem. We saw this before with the first-order linear equation, right?

Here's the formal statement of this result.

3.4.1 Theorem (Linearity).

Suppose that x_{p} solves

$$\ddot{x} + p\dot{x} + qx = g(t)$$

and let x be any other solution of this ODE. Then there is a solution x_{h} of the homogeneous problem

$$\ddot{x} + p\dot{x} + qx = 0$$

such that

$$x = x_{\mathsf{h}} + x_{\mathsf{p}}.$$

In other words, if we know one "$\mathsf{particular}$" solution[30] x_{p} to $\ddot{x} + p\dot{x} + qx = g(t)$, then

[30] I hope it's okay that I'm using the subscript p for the particular solution and also the parameter p in the ODE. $p \neq \mathsf{p}$.

we can find all other solutions to this ODE, since we know how to solve the homogeneous problem $\ddot{x}+p\dot{x}+qx = 0$. So, we will concentrate on finding one particular solution x_{p} and then combining it with homogeneous solutions to get full control over the nonhomogeneous ODE.

This structural result has profound consequences for the harmonic oscillator; recall that if $p \ge 0$ and $q > 0$, then any solution x_{h} to the homogeneous problem $\ddot{x}+p\dot{x}+qx = 0$ satisfies $\lim_{t\to\infty} x_{\mathsf{h}}(t) = 0$. Now suppose that we have one particular solution x_{p} to the nonhomogeneous problem $\ddot{x}+p\dot{x}+qx = g(t)$ and write any other solution x to this nonhomogeneous problem as $x = x_{\mathsf{h}} + x_{\mathsf{p}}$ for some homogeneous solution x_{h}. Subtract to get $x - x_{\mathsf{p}} = x_{\mathsf{h}}$. Then

$$\lim_{t\to\infty}\big(x(t) - x_{\mathsf{p}}(t)\big) = \lim_{t\to\infty} x_{\mathsf{h}}(t) = 0.$$

That is, over very long times, x and x_{p} are essentially the same function; the end behavior of x is always the same as the end behavior of x_{p}. In particular, initial conditions don't matter! No matter how the oscillator starts moving, its long-time behavior will always be governed by the same particular solution x_{p}. Of course, though, this is only guaranteed for $p \ge 0$ and $q > 0$.

This is where we finished on Monday, November 14, 2022.

Now, how do we find those particular solutions x_{p} to the nonhomogeneous problem $\ddot{x}+p\dot{x}+qx = g(t)$? Good news: there is a procedure (commonly called **VARIATION OF PARAMETERS**, very much like our work in Section 2.7.2) that always works for any continuous g. Bad news: it requires a lot of antidifferentiation and a ton of ancillary calculations, and the actual formula for x_{p} is quite bulky and delicate. If your life depends on it, you can always look it up and grind it out.

A better use of our time, I think, is to study two types of forcing terms g that are common in applications and physically relevant. Our strategy will be the educated guessing (or ansatz-making) of the method of "undetermined coefficients," much as we did in Section 2.7.3.

3.4.2. Exponential forcing.

Let p, q, A, and r be numbers. We will study here the problem

$$\ddot{x}+p\dot{x}+qx = Ae^{rt}. \tag{3.4.3}$$

We can think of the forcing term $g(t) = Ae^{rt}$ as a force that either increases dramatically over time (the case $r > 0$) or that weakens substantially over time ($r < 0$). I like to think of black holes sucking a harmonic oscillator into oblivion (but maybe I've watched too much *Star Trek*).

For a problem like (3.4.3), a good idea is to guess that x is an exponential of the same "form" as the forcing function:

$$x(t) = \alpha e^{rt} \tag{3.4.4}$$

for some to-be-determined constant α. This is a good idea for two reasons. First, we might ask ourselves what kind of function x should we have so that $\ddot{x}+p\dot{x}+qx$ adds up to an exponential. Probably an exponential! I mean, if x has sines or logs in it, that won't turn

$\ddot{x} + p\dot{x} + qx$ into an exponential. Second, we might ask ourselves *how the forcing function manifests itself in the solution.* Since e^{rt} appears in the forcing function, we might expect that e^{rt} appears in the solution, and arguably the simplest way for that to happen is if the solution has the form (3.4.4).

Working with arbitrary p, q, A, and r could get confusing, so let's do a concrete example.

3.4.2 Example.

Guess that a function of the form $x(t) = \alpha e^{2t}$ solves

$$\ddot{x} - x = e^{2t}.$$

Find the right value of α for this to work, and then find all other solutions to this nonhomogeneous ODE. And for good measure, solve the IVP

$$\begin{cases} \ddot{x} - x = e^{2t} \\ x(0) = 0 \\ \dot{x}(0) = 1. \end{cases}$$

Solution. With $x(t) = \alpha e^{2t}$, we have $\dot{x}(t) = 2\alpha e^{2t}$ and $\ddot{x}(t) = 4\alpha e^{2t}$, and so we want

$$e^{2t} = \ddot{x}(t) - x(t) = 4\alpha e^{2t} - \alpha e^{2t} = 3\alpha e^{2t}.$$

Dividing both sides of $3\alpha e^{2t} = e^{2t}$ by e^{2t}, we find that α must satisfy $3\alpha = 1$, so $\alpha = 1/3$. You can (and should) then check that

$$x_{\mathsf{p}}(t) := \frac{e^{2t}}{3}$$

solves $\ddot{x} - x = e^{2t}$.

To find all other solutions, we just have to add on an arbitrary homogeneous solution x_{h}. The homogeneous problem is $\ddot{x} - x = 0$; its characteristic equation is $\lambda^2 - 1 = 0$; the roots of the characteristic equation are $\lambda = \pm 1$; and so any homogeneous solution has the form $x_{\mathsf{h}}(t) = c_1 e^t + c_2 e^{-t}$ for some constants c_1 and c_2. (Say that five times fast.) Thus all solutions to $\ddot{x} - x = e^{2t}$ are

$$x(t) = c_1 e^t + c_2 e^{-t} + \frac{e^{2t}}{3}.$$

Finally, to solve the IVP, we just need to choose c_1 and c_2 above to meet the initial conditions. We want

$$0 = x(0) = c_1 + c_2 + \frac{1}{3},$$

and, since

$$\dot{x}(t) = c_1 e^t - c_2 e^{-t} + \frac{2e^{2t}}{3},$$

we also want

$$1 = \dot{x}(0) = c_1 - c_2 + \frac{2}{3}.$$

This gives a linear system of equations for c_1 and c_2:

$$\begin{cases} c_1 + c_2 = -1/3 \\ c_1 - c_2 = -2/3. \end{cases}$$

There are many, many ways to solve this system; one way is to get $c_1 = c_2 - 2/3$ from the second equation and plug that into the first equation to find

$$c_2 - \frac{2}{3} + c_2 = -\frac{1}{3},$$

which turns into

$$2c_2 = \frac{1}{3},$$

or $c_2 = 1/6$, and thus

$$c_1 = \frac{1}{6} - \frac{2}{3} = -\frac{1}{2}.$$

(Ugh, fractions.) We conclude

$$x(t) = -\frac{e^t}{2} + \frac{e^{-t}}{6} + \frac{e^{2t}}{3}.$$ ▲

Say that we change the situation above ever so slightly to

$$\ddot{x} - \dot{x} = e^t. \tag{3.4.5}$$

If you guess $x(t) = \alpha e^t$ and plug that in, I claim that the absurd situation

$$e^t = 0$$

will result. Go ahead and try it. Here's the problem: the function $x(t) = e^t$ solves $\ddot{x} - \dot{x} = 0$.

However, I claim two nice things. First, we don't need new methods to solve $\ddot{x} - \dot{x} = e^t$. It's really a first-order problem in disguise. Second, working through this problem will teach us something valuable.

Let's shed the disguise. The ODE under consideration involves a "perfect derivative":

$$e^t = \ddot{x} - \dot{x} = \frac{d}{dt}[\dot{x} - x].$$

Antidifferentiate both sides to get

$$\dot{x} - x = e^t + k,$$

which is a first-order linear problem forced by $b(t) = e^t + k$. Here k is an arbitrary constant of integration. We have a number of methods for solving such a problem. However you do it (and you should do it, please and thank you), the result is

$$x(t) = te^t + c_1 + c_2e^t, \tag{3.4.6}$$

where c_1 and c_2 are arbitrary constants. To be clear, we have shown that every solution to (3.4.5) has the form (3.4.6).

The structure of (3.4.6) should not be surprising. Look at the homogeneous version of (3.4.5), which is $\ddot{x} - \dot{x} = 0$. The characteristic equation is $\lambda^2 - \lambda = 0$, so its roots are $\lambda = 0, 1$, and therefore every homogeneous solution has the form $x_{\mathsf{h}}(t) = c_1 + c_2 e^t$ for some constants c_1 and c_2. Now look at (3.4.6). If we take $c_1 = c_2 = 0$, we see that a particular solution to the nonhomogeneous problem $\ddot{x} - \dot{x} = e^t$ is $x_{\mathsf{p}}(t) = te^t$. This is almost our original guess $x(t) = \alpha e^t$, except now there is an extra factor of t.

Here is the lesson.

"If your first guess fails, multiply by t and try again." $\qquad (t)$

This is where we finished on Wednesday, November 15, 2022.

3.4.3 Example.

Without doing any calculations, what do you expect will happen to solutions of

$$\ddot{x} + 4\dot{x} + 3x = 12e^{-3t}$$

over long times? Do some calculations and confirm your expectations.

Solution. The coefficient on $\dot{x}$ is nonnegative and the coefficient on x is positive, so this ODE could model the displacement of a harmonic oscillator. In particular, the coefficient on $\dot{x}$ is positive, so this oscillator is experiencing friction. We therefore expect that if we can find one particular solution x_{p} to this ODE, every other solution x will behave like x_{p} as $t \to \infty$. And since the oscillator is forced by $g(t) = 12e^{-3t}$, we expect that x_{p} will look like g. Since $\lim_{t\to\infty} 12e^{-3t} = 0$, it's a good guess that we will have $\lim_{t\to\infty} x_{\mathsf{p}}(t) = 0$, too.

Let's get an analytic solution. We might want to guess $x(t) = \alpha e^{-3t}$, but our experience above should make us cautious. Could $g(t) = 12e^{-3t}$ solve the homogeneous problem $\ddot{x} + 4\dot{x} + 3x = 0$? Let's check. The characteristic equation is $\lambda^2 + 4\lambda + 3 = 0$, and this factors into $(\lambda+1)(\lambda+3) = 0$. Its roots, therefore, are $\lambda = -1$ and $\lambda = -3$, and so all homogeneous solutions are $x_{\mathsf{h}}(t) = c_1 e^{-t} + c_2 e^{-3t}$. In particular, yes, $g(t) = 12e^{-3t}$ does solve the homogeneous problem, and so guessing $x(t) = \alpha e^{-3t}$ won't work.

Instead, we guess $x(t) = \alpha t e^{-3t}$ and compute

$$\dot{x}(t) = \alpha e^{-3t} - 3\alpha t e^{-3t}$$

and

$$\ddot{x}(t) = -3\alpha e^{-3t} - 3\alpha e^{-3t} + 9\alpha t e^{-3t} = -6\alpha e^{-3t} + 9\alpha t e^{-3t}.$$

Then we want α to satisfy

$$12e^{-3t} = \big(-6\alpha e^{-3t} + 9\alpha t e^{-3t}\big) + 4\big(\alpha e^{-3t} - 3\alpha t e^{-3t}\big) + 3\alpha t e^{-3t} = -2\alpha e^{-3t}.$$

Divide through by e^{-3t} to get

$$12 = -2\alpha,$$

and thus $\alpha = -6$.

Every solution, therefore, has the form

$$x(t) = c_1 e^{-t} + c_2 e^{-3t} - 6te^{-3t},$$

and so, certainly, every solution satisfies $\lim_{t\to\infty} x(t) = 0$. ▲

3.4.3. Sinusoidal forcing.

Let's consider now ODE of the form

$$\ddot{x} + p\dot{x} + qx = A\cos(\omega t) \quad \text{or} \quad \ddot{x} + p\dot{x} + qx = A\sin(\omega t),$$

where p, q, A, and ω are all numbers. Such sinusoidal forcing functions could represent periodic forces regularly applied to the oscillator; the parameter A controls the amplitudes, or extreme values, of the forces, and ω controls their frequencies. Inspired by the dual questions of (1) *How does the forcing function show up in the solution?* and (2) *What kinds of functions x can have derivatives that add up to sines or cosines?*, a natural guess might be $x(t) = \alpha\cos(\omega t)$ or $x(t) = \alpha\sin(\omega t)$. However, this could lead to "balancing" problems with the first derivative term; if we are trying to solve $\ddot{x} + p\dot{x} + qx = A\cos(\omega t)$ by guessing $x(t) = \alpha\cos(\omega t)$, then $\dot{x}$ will kick in a $\sin(\omega t)$ term, but $\ddot{x}$ won't counterbalance that, and so we'll have a lonely sine in the algebra.

Instead, the better guess is

$$x(t) = \alpha\cos(\omega t) + \beta\sin(\omega t).$$

3.4.4 Example.

Find all solutions to $\ddot{x} + 2\dot{x} + 10x = 4\cos(2t)$.

Solution. We know that all solutions to the nonhomogeneous problem $\ddot{x} + 2\dot{x} + 10x = 4\cos(2t)$ have the form $x(t) = x_{\mathsf{h}}(t) + x_{\mathsf{p}}(t)$, where x_{h} solves the homogeneous problem $\ddot{x}_{\mathsf{h}} + 2\dot{x}_{\mathsf{h}} + 10x_{\mathsf{h}} = 0$ and x_{p} is one particular solution to the nonhomogeneous problem. Since the homogeneous problem represents an undriven, damped oscillator (since the coefficients on $\dot{x}$ and x are both positive), we expect that its solutions x_{h} vanish as $t \to \infty$. Then, over long times, the end behavior of a solution x to the nonhomogeneous problem will closely resemble that of the particular solution x_{p}.

We can confirm this by solving the characteristic equation $\lambda^2 + 2\lambda + 10 = 0$ for

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2(1)} = -1 \pm 3i.$$

All solutions to the homogeneous problem are therefore $x_{\mathsf{h}}(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$, and certainly these satisfy $\lim_{t\to\infty} x_{\mathsf{h}}(t) = 0$. Then every solution to the nonhomogeneous problem has the form

$$x(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + x_{\mathsf{p}}(t)$$

for one particular solution $x_{\mathsf{p}}(t)$, and so $\lim_{t\to\infty} x(t) - x_{\mathsf{p}}(t) = 0$.

This is where we finished on Monday, November 28, 2022.

We now find this particular solution by guessing that

$$x(t) = \alpha\cos(2t) + \beta\sin(2t)$$

solves $\ddot{x} + 2\dot{x} + 10x = 4\cos(2t)$ for some α and β. We will figure out these two undetermined coefficients. First, we compute

$$\dot{x}(t) = -2\alpha\sin(2t) + 2\beta\cos(2t) \quad \text{and} \quad \ddot{x}(t) = -4\alpha\cos(2t) - 4\beta\sin(2t).$$

Then we need

$$\begin{aligned}
4\cos(2t) &= \ddot{x}(t) + 2\dot{x}(t) + 10x(t) \\
&= -4\alpha\cos(2t) - 4\beta\sin(2t) \\
&\quad + 2\big[-2\alpha\sin(2t) + 2\beta\cos(2t)\big] \\
&\quad + 10\big[\alpha\cos(2t) + \beta\sin(2t)\big] \\
&= \big[-4\alpha + 4\beta + 10\alpha\big]\cos(2t) + \big[-4\beta - 4\alpha + 10\beta\big]\sin(2t).
\end{aligned}$$

If we subtract $4\cos(2t)$ from both sides and simplify, we find that we need

$$\big[6\alpha + 4\beta - 4\big]\cos(2t) + \big[6\beta - 4\alpha\big]\sin(2t) = 0.$$

This has to be true for all t.

Here is a wonderful auxiliary fact, which I invite you to prove: if A, B, and ω are such that

$$A\cos(\omega t) + B\sin(\omega t) = 0$$

for all t, then $A = 0$ and $B = 0$. Taking this fact for granted, it must then be the case that α and β satisfy the system

$$\begin{cases} 6\alpha + 4\beta - 4 = 0 \\ 6\beta - 4\alpha = 0. \end{cases}$$

There are, as always, lots of ways to do this. One approach is to find $\alpha = 6\beta/4 = 3\beta/2$ from the second equation and plug that into the first to get

$$6\left(\frac{3\beta}{2}\right) + 4\beta - 4 = 0,$$

which is the same as

$$13\beta = 4,$$

or $\beta = 4/13$. Then $\alpha = 6/13$, and so a particular solution to the nonhomogeneous problem is

$$x_{\mathsf{p}}(t) = \frac{6\cos(2t)}{13} + \frac{4\sin(2t)}{13}.$$

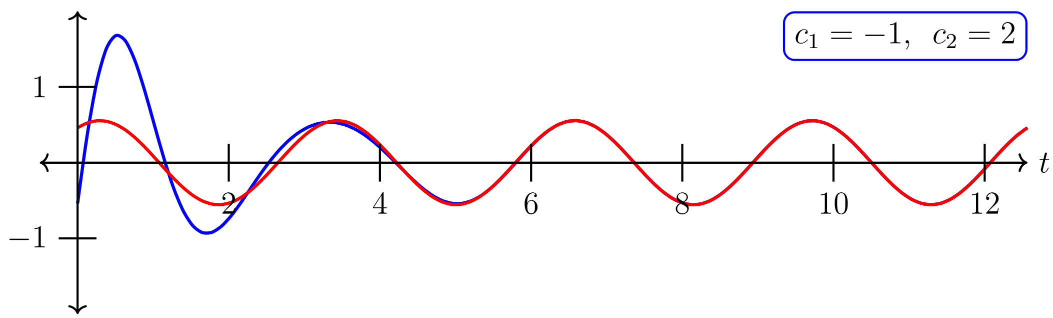
Thus all solutions to the nonhomogeneous problem are

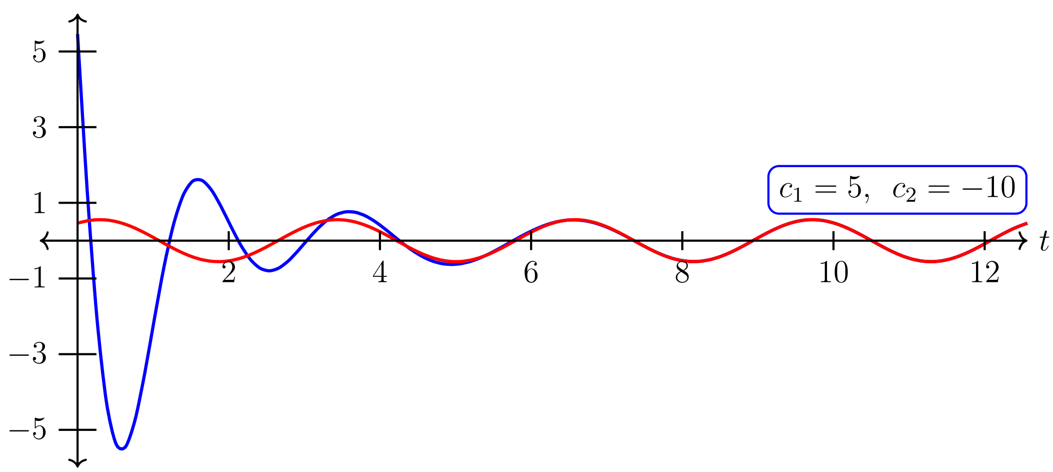
$$x(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \frac{6\cos(2t)}{13} + \frac{4\sin(2t)}{13}.$$

We see analytically that

$$\lim_{t\to\infty} x(t) - \left(\frac{6\cos(2t)}{13} + \frac{4\sin(2t)}{13}\right) = \lim_{t\to\infty} c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) = 0,$$

so, indeed, over long times any solution x to the nonhomogeneous problem behaves like the particular solution x_{p}. It is instructive to consult several plots for particular choices of c_1 and c_2 (which determine the initial conditions).





These plots indicate that, indeed, the general solution always "settles down" into the particular solution. However, the "settling down" appears to happen more quickly in the first plot than in the second, which suggests that choosing the initial conditions (i.e., choosing c_1 and c_2) has an effect on the "intermediate" dynamics of the solution (and the "initial" dynamics, too!). ▲

3.4.4. Resonance in the undamped oscillator.

Nothing too exciting can happen in a damped harmonic oscillator. If $p > 0$ and $q > 0$, then if we know one solution x_{p} to

$$\ddot{x} + p\dot{x} + qx = g(t),$$

we know the long-time dynamics of all solutions to this ODE: any solution x ultimately behaves like x_{p}. Initial conditions are irrelevant!

The *undamped* oscillator has more exciting possibilities. Consider the following IVP:

$$\begin{cases} \ddot{x} + x = A\cos(\omega t) \\ x(0) = 0 \\ \dot{x}(0) = 0. \end{cases}$$

Let's describe what this problem means physically. The mass and the spring constant of the oscillator have the same numerical value ($m = \kappa$); the oscillator is forced by $g(t) = A\cos(\omega t)$, where the parameter A controls the "amplitude" or maximum value of this force and the parameter ω allows us to control the frequency of this force (since the cosine is even, we may as well take $\omega > 0$); and the oscillator starts moving from its rest position with no initial velocity.

Let's find an analytic solution. The homogeneous problem is $\ddot{x}+x = 0$, so its characteristic equation is $\lambda^2 + 1 = 0$. I really hope that by now we see immediately $\lambda = \pm i$, and so the homogeneous solutions are

$$x_{\mathsf{h}}(t) = c_1\cos(t) + c_2\sin(t).$$

Then all solutions to the nonhomogeneous problem have the form

$$x(t) = c_1\cos(t) + c_2\sin(t) + x_{\mathsf{p}}(t)$$

for some particular solution x_{p}.

The method of undetermined coefficients then suggests that we guess

$$x(t) = \alpha\cos(\omega t) + \beta\sin(\omega t),$$

compute

$$\dot{x}(t) = -\alpha\omega\sin(\omega t) + \beta\omega\cos(\omega t) \quad \text{and} \quad \ddot{x}(t) = -\alpha\omega^2\cos(\omega t) - \beta\omega^2\sin(\omega t),$$

and stuff everything into the ODE to find that α and β must satisfy

$$A\cos(\omega t) = -\alpha\omega^2\cos(\omega t) - \beta\omega^2\sin(\omega t) + \alpha\cos(\omega t) + \beta\sin(\omega t) = \alpha(1-\omega^2)\cos(\omega t) + \beta(1-\omega^2)\sin(\omega t).$$

Rearranged, this reads

$$\big[\alpha(1-\omega^2) - A\big]\cos(\omega t) + \beta(1-\omega^2)\sin(\omega t) = 0. \tag{3.4.7}$$

Since this has to be true for all t, we need

$$\alpha(1-\omega^2) - A = 0 \quad \text{and} \quad \beta(1-\omega^2) = 0.$$

If $\omega \neq \pm 1$ (and really this means $\omega \neq 1$, since above we parenthetically agreed $\omega > 0$), then the second equation yields $\beta = 0$ and the first equation allows us to solve for α as

$$\alpha = \frac{A}{1-\omega^2}.$$

Then the particular solution is

$$x_{\mathsf{p}}(t) = \frac{A\cos(\omega t)}{1-\omega^2}.$$

Nothing too exotic, right?

This is where we finished on Wednesday, November 30, 2022.

Since we excluded $\omega = \pm 1$, let's think about what happens (1) when ω is close to ± 1 and (2) when $\omega = \pm 1$. Taking ω close to ± 1 should make us think of $\lim_{\omega \to \pm 1}$. What happens to, say,

$$\lim_{\omega \to 1} \frac{A\cos(\omega t)}{1-\omega^2}?$$

The numerator is tame: $\lim_{\omega \to 1} A\cos(\omega t) = A\cos(t)$. But the denominator explodes:

$$\lim_{\omega \to 1^-} \frac{1}{1-\omega^2} = \infty \quad \text{and} \quad \lim_{\omega \to 1^+} \frac{1}{1-\omega^2} = -\infty.$$

This suggests that something very strange will happen if we allow $\omega = 1$.

Indeed, if we try to solve $\ddot{x} + x = \cos(t)$ by guessing $x(t) = \alpha\cos(t) + \beta\sin(t)$, then (3.4.7) with $\omega = 1$ implies

$$-A\cos(t) = 0$$

for all t. But then $A = 0$, and so our ODE reduces to $\ddot{x} + x = 0$, which no longer represents a driven oscillator. And so we learn nothing about $\ddot{x} + x = \cos(t)$ with this guess for x.

However, our myriad prior failures and the subsequent lesson (t) point to a way out: guess $x(t) = \alpha t\cos(t) + \beta t\sin(t)$ instead. If we do so, we can calculate (after a moderate amount of work, which I am going to suppress)

$$\ddot{x}(t) + x(t) = 2\beta\cos(t) - 2\alpha\sin(t)$$

Since we want $\ddot{x} + x = A\cos(t)$, we need

$$2\beta\cos(t) - 2\alpha\sin(t) = A\cos(t),$$

and this means

$$(2\beta - A)\cos(t) - 2\alpha\sin(t) = 0,$$

and so we want

$$2\beta - A = 0 \quad \text{and} \quad 2\alpha = 0.$$

Thus $\alpha = 0$ and $\beta = A/2$, so a particular solution to $\ddot{x} + x = \cos(t)$ is

$$x_{\mathsf{p}}(t) = \frac{At\cos(t)}{2}.$$

All solutions to $\ddot{x} + x = \cos(t)$, then, are

$$x(t) = c_1\cos(t) + c_2\sin(t) + \frac{At\cos(t)}{2}.$$

Here is the fascinating thing about x_{p}: it is unbounded, but $\lim_{t\to\infty} x_{\mathsf{p}}(t)$ does not exist. As time progresses, the oscillations of this oscillator become wilder (even though the oscillator returns to its equilibrium position infinitely many times). This is the phenomenon of **RESONANCE**.

This is where we finished on Friday, December 2, 2022.

3.4.5. Even worse forcing.

The analytic methods above can be generalized significantly (and horribly). Essentially, we can solve any ODE of the form

$$\ddot{x} + p\dot{x} + qx = \phi(t)e^{rt}\psi(\omega t),$$

where $\phi(t) = a_n t^n + \cdots + a_1 t + a_0$ is a polynomial, α and ω are real numbers, and $\psi(\tau) = \cos(\tau)$ or $\psi(\tau) = \sin(\tau)$ by guessing a solution that "looks like"

$$\big[\text{polynomial} \times e^{rt} \times \big(\cos(\omega t)\big] + \big[\text{another polynomial} \times e^{rt} \times \sin(\omega t)\big].$$

The "undetermined coefficients" in this guess are the coefficients of the two polynomials. The rational behind such a guess is that a forcing function of the form

$$\text{polynomial} \times \text{exponential} \times \text{sinusoid}$$

should manifest itself in a solution of the same form.

In my opinion, spending too much time on ODE like these is exactly what gives the subject a bad name and a reputation as a "cookbook" class. If your life depends on solving an ODE forced by a product of a polynomial, an exponential, and a sinusoid, you can go to a book or the internet and look up the details and then go to a computer algebra system to carry out all the derivative calculations and algebra simplifications. At that rate, though, you might as well go to a computer algebra system to find the solution in the first place.

Deep breath, inhale, exhale, calm down, recite the Analyst's Creed (AC).