# MATH 4310
# PARTIAL DIFFERENTIAL EQUATIONS

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## Contents

# 1. First-Order Equations

## 1.1. A model for traffic flow.

We will begin our course by deriving and analyzing a model for traffic flow on an idealized road. This is a worthwhile object of study for at least three reasons.

**1.** We are, for better or for worse, all familiar with car traffic!

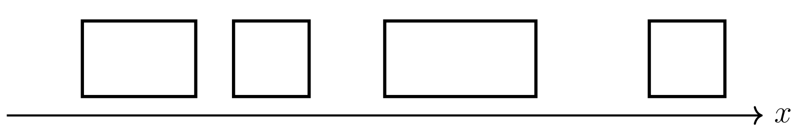
**2.** This model, in its broadest form, will lead to several fundamental PDE, which we will solve in fairly general contexts, and whose solutions we can then sensibly interpret in the context of our lived experience with traffic.

**3.** The derivation and solution of these PDE will involve a number of auxiliary techniques and ideas, some of which will provide a good review of prior calculus, and others that are probably new, and quite worth learning.

From our points of view as individual drivers, traffic is fundamentally a "discrete" phenomenon — my car, your car, the cars immediately in front of, behind, and to the left and right of us. Zooming out, traffic can appear more like a fluid — imagine the flow of traffic along a busy road that you see when descending in an airplane. While the discrete viewpoint, which we discuss briefly, is quite worthwhile for separate study, we will ultimately be less concerned with the behavior of individual cars and more with the "flow" of traffic in general. For example, if you're thinking about the traffic on your drive home, is it more useful to know where every single car is, or to have a less precise, but more transparent, notion of the "density" of traffic, so that you know when to expect "heavy" traffic or "light"? I vote for the latter.

We will make a number of simplifying assumptions so that we can work in a "continuous" setting (and thereby derive PDE) and so that our models are tractable. This is a classical dilemma of applied mathematics, and one that you probably met when studying ODE: our models should be complicated enough to capture real-world phenomena, but simple enough so that we can solve them in a reasonable amount of time.

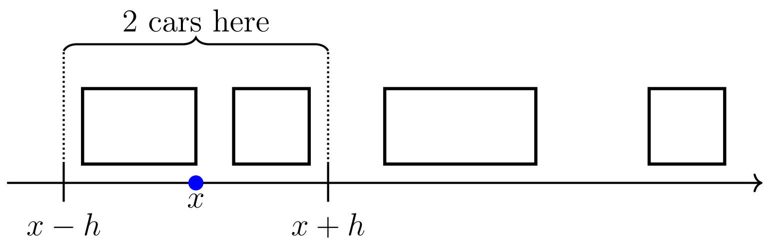
Here is our first simplifying assumption: we will study a single-lane road on which all traffic travels in the same direction, to the right. In particular, cars cannot pass each other, and cars do not enter or exit the road at any finite point; there are no intersections and no on-/off-ramps. We will denote position along the road relative to some fixed point of reference by $x$. (If we need units for $x$, we'll use miles.) Here is a sketch of this road with some (rectangular) cars along it.




As we said, we will not track the behavior of individual cars, in part because this information is not as useful as what we *will* track, and in part because individual measurements will not give us the PDE that we want.

The starring quantity in our analysis will be *density*, which should measure, roughly, "how many cars are on the road at a particular point in time and near a particular location on the road per unit length." This is a lot of data. Let's introduce some notation to keep track of everything. People usually employ $\rho$ for density, and so we'll say that a function $\rho$ of the variables $x$ and $t$ measures the density of traffic at the point $x$ along the road and the time $t$ for a *suitable spatial length* $h > 0$ if

$$\frac{\text{the number of cars on the road between points } x-h \text{ and } x+h \text{ at time } t}{2h} \approx \rho(x,t).$$

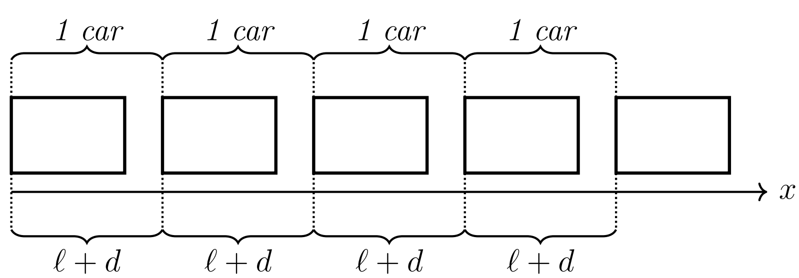



Maybe the key word here is "suitable": what makes for a "suitable" length $h$? The number $h$ should not be too small, like 1 foot, or else we won't be able to fit even one car between $x - h$ and $x + h$. But $h$ should not be too large, like (perhaps) 1 mile, or else we won't be tracking the number of cars reasonably close to position $x$. Qualitatively, we expect that if $\rho(x,t)$ is "large," then there are "many" cars around point $x$ at time $t$, and the opposite if $\rho(x,t)$ is "small."

**1.1.1 Example.**

*Suppose every car on the road is $\ell$ miles long and $d$ miles apart from its nearest neighbor ahead and behind. (Using miles to measure these lengths probably feels weird, but we're using miles already to track position along the road, so let's be consistent.) Then it is reasonable to say that there is 1 car per $\ell + d$ miles along the road, and so the traffic density should be*

$$\rho(x,t) = \frac{1}{\ell + d}.$$




This is where we finished on Monday, August 15, 2022.

Density alone does not tell us everything about traffic; we are also probably concerned with how quickly traffic flows. After all, heavy traffic can move quickly, and light traffic can move slowly (say, in bad weather). One way to capture traffic speed (other than sampling the velocities of individual cars) is to count how many cars pass a given point along the road over an interval of time; this quantity is the *flux* of traffic. We say that a function $q$ of the variables $x$ and $t$ measures the flux of traffic at the point $x$ along the road and the time $t$ for a *suitable time length* $k > 0$ if

$$\frac{\text{the number of cars that pass through point } x \text{ between times } t-k \text{ and } t+k}{2k} \approx q(x,t).$$

As before, what counts as a "suitable" time length $k$ can be neither too large nor too small — too small, and no cars can get by; too large, and we might incorporate too many different traffic patterns (early morning traffic, mid-morning doldrums) into our model simultaneously.

**1.1.2 Example.**

*Say that all cars along the road move at the constant speed of $v_0 > 0$ miles per hour and that the traffic density is $\rho_0 > 0$ cars per mile. Fix a position $x$ along the road, a time $t$, and a "suitable" time length $k > 0$. Then between times $t-k$ and $t+k$, all cars will travel*

$$\frac{v_0 \text{ miles}}{\text{hour}} \times (t+k) - (t-k) \text{ hours} = 2kv_0 \text{ miles}.$$

*In particular, any car within $2kv_0$ miles to the left of position $x$ will pass through position $x$ between times $t-k$ and $t+k$. (Remember that all cars are traveling to the right.) And there are*

$$2kv_0 \text{ miles} \times \frac{\rho_0 \text{ cars}}{\text{mile}} = 2kv_0\rho_0$$

*cars over this stretch of $2kv_0$ miles to the left of position $x$. Thus the flux should be*

$$\frac{2kv_0\rho_0 \text{ cars}}{2k \text{ hours}} = \frac{v_0\rho_0 \text{ cars}}{\text{hour}}.$$

The preceding calculation suggests that if we have a way of defining[1] "average velocity" $v(x,t)$ of the cars at position $x$ along the road and time $t$, then density, flux, and velocity might be related by

$$q(x,t) = v(x,t)\rho(x,t).$$

You should take a moment to check that all the units make sense in the equality above. This relationship is not really something that we can *prove* — I mean, how can we really "prove" any of these claims about traffic? — but rather it is an identity that we will *assume*. We will see that this assumption leads to good results!

[1] One way to do this is to fix a position $x$ along the road and a time $t$ and a "suitable" (that word again) spatial length $h$, count the number of cars between points $x-h$ and $x+h$ along the road, and take the average of the velocities of those cars:

$$v(x,t) \approx \frac{1}{\text{the number of cars between points } x-h \text{ and } x+h} \times \text{the sum of the velocities of these cars.}$$

So, now we have three quantities that tell us meaningful stuff about traffic. We have yet to see any PDE, though. We need one more quantity first. Fix positions $a$ and $b$ along the road with $a < b$. We claim that

$$\int_a^b \rho(x,t)\ dx$$

tells us how many cars, on average, are between points $a$ and $b$. This is something we might have learned in calculus[2], but let's briefly review why.

Divide the interval $[a, b]$ into $n$ small subintervals of the form $[x_k, x_{k+1}]$ for $k = 0, \ldots, n-1$, where

$$x_k = a + \left(\frac{b-a}{n}\right) k.$$

You just know an integral is going to pop out of this, right? Then $n$ is large, $[x_k, x_{k+1}]$ is a small interval, and so we expect

$$\rho(x,t) \approx \rho(x_k,t)$$

for $x_k \leq x \leq x_{k+1}$. Then there are roughly $\rho(x_k, t)$ cars/mile between points $x_k$ and $x_{k+1}$ on the road at time $t$, and so there are

$$\frac{\rho(x_k,t)\ \text{cars}}{\text{mile}} \times (x_{k+1} - x_k)\ \text{miles} = \rho(x_k,t)(x_{k+1} - x_k)\ \text{cars}$$

on the road between points $x_k$ and $x_{k+1}$ at time $t$. Hence there are roughly

$$\sum_{k=0}^{n-1} \rho(x_k,t)(x_{k+1} - x_k)$$

cars on the road between points $a$ and $b$ at time $t$, and this sum is a left-endpoint approximation to the Riemann integral $\int_a^b \rho(x,t)\ dx$.

Just to be clear, note that $x$ is the "dummy" variable of integration, and we really have a mapping

$$t \mapsto \int_a^b \rho(x,t)\ dx$$

of time $t$ into the number of cars between points $a$ and $b$ at time $t$. You know what they say: if it moves, differentiate it. Thus the rate of change of the number of cars on the road between points $a$ and $b$ is

$$\frac{d}{dt} \int_a^b \rho(x,t)\ dx.$$

We are now going to do something that is wholly unjustified but mathematically very useful; we will discuss when this is the right thing to do later, in great (and painful) detail. The Riemann integral is basically a sum:

$$\int_a^b \rho(x,t)\ dx = \sum_{k=0}^{n-1} \rho(x_k,t)(x_{k+1} - x_k).$$

[2] The amount of stuff in a region $= \displaystyle\int_{\text{that region}}$ density of stuff $d$space.

And derivatives pass through sums:

$$\frac{d}{dt}\sum_{k=0}^{n-1}\rho(x_k,t)(x_{k+1}-x_k)=\sum_{k=0}^{n-1}\frac{d}{dt}[\rho(x_k,t)(x_{k+1}-x_k)]=\sum_{k=0}^{n-1}\rho_t(x_k,t)(x_{k+1}-x_k).$$

Thus we might expect that

$$\frac{d}{dt}\int_a^b \rho(x,t)\ dx=\int_a^b \rho_t(x,t)\ dx.$$

This is called **DIFFERENTIATING UNDER THE INTEGRAL**, and it is a Really Big Deal. (This is also one of those new calculus concepts that I said we'd learn.) For now, trust me that it's the right thing to do, because we're going to switch focus for a moment.

When you're trying to learn something about a certain quantity, it's sometimes worthwhile to try to compute it in two very different ways and then equate the two results. We are trying to figure out the rate of change of the number of cars between points $a$ and $b$ on the road. Remember our (over)simplifying assumptions: cars move to the right on the road, they never leave the road, and they never enter the road. So, the only way that the number of cars between points $a$ and $b$ can change is when cars enter the interval $[a,b]$ at point $a$ and leave the interval at point $b$. This is what flux tells us: there are $q(a,t)$ cars passing point $a$ per hour at time $t$, and there are $q(b,t)$ cars passing point $b$ per hour at time $t$. Employing the general formula

$$\text{Rate of change of a quantity} \ = \ \text{Rate in} - \text{Rate out},$$

we find that the rate of change of the number of cars between points $a$ and $b$ at time $t$ is

$$q(a,t)-q(b,t)$$

cars per hour. Thus

$$\int_a^b \rho_t(x,t)\ dx=q(a,t)-q(b,t).$$

Next we will employ a little trick called the fundamental theorem of calculus (FTC):

$$q(a,t)-q(b,t)=\int_b^a q_x(x,t)\ dx=-\int_a^b q_x(x,t)\ dx.$$

We conclude

$$\int_a^b \rho_t(x,t)\ dx=-\int_a^b q_x(x,t)\ dx,$$

and this rearranges to

$$\int_a^b \big[\rho_t(x,t)+q_x(x,t)\big]\ dx=0.$$

Now we're getting somewhere: we have an equation and some partial derivatives. But there is an integral in that equation. Note, however, that we said nothing about what $a$ and

$b$ were; we just took them to be arbitrary positions along the road. So, let's fix $a = 0$ and let $b$ be arbitrary. Then the function

$$F(b) := \int_0^b \big[\rho_t(x,t) + q_x(x,t)\big]\, dx$$

is constant, with $F(b) = 0$ for all $b$. Hence

$$F'(b) = 0$$

for all $b$. But the *other* part of the FTC says

$$F'(b) = \frac{d}{db}\int_0^b \big[\rho_t(x,t) + q_x(x,t)\big]\, dx = \rho_t(b,t) + q_x(b,t).$$

Thus

$$\rho_t(b,t) + q_x(b,t) = 0$$

for all $b$. If we write $x$ instead of $b$, then we get an honest-to-goodness PDE:

$$\rho_t(x,t) + q_x(x,t) = 0,$$

or, more succinctly,

$$\rho_t + q_x = 0.$$

There is just one problem: this is *one* equation featuring *two* unknowns, $\rho$ and $q$, and as such this equation is overdetermined. But remember we intuited the relationship $q = v\rho$ among flux, velocity, and $\rho$. Substituting this in, we find

$$\rho_t + (v\rho)_x = 0.$$

Now, there is still the matter of the function $v$. We will not consider velocity $v$ as an unknown but instead as a *coefficient*. In other words, if we prescribe a velocity function for the road, can we figure out traffic density? This seems like a reasonable question to ask.

In the following we will study a variety of conditions on $v$ that lead to different PDE; we will solve them and interpret their solutions in the context of traffic density.

## 1.2. Some notation and terminology.

### *1.2.1. Notation.*

It will be convenient to agree upon certain notation, abbreviations, and turns of phrase now and forever.

• We will say something like "Let $u = u(x,t)$" to indicate that we are studying a function $u$ of the two variables $x$ and $t$.

• If $A$ and $B$ are sets, then we will abbreviate the sentence "$f$ is a function from $A$ to $B$" by the symbol $f\colon A \to B$. Most of the time, our functions will go from (a subset of) $\mathbb{R}^2$ to $\mathbb{R}$, or from $\mathbb{R}$ to $\mathbb{C}$, where $\mathbb{C}$ is the set of all complex numbers. Thus the symbol $u\colon \mathbb{R}^2 \to \mathbb{R}$ means that $u$ is a function from $\mathbb{R}^2$ to $\mathbb{R}$, while $f\colon [-\pi,\pi] \to \mathbb{C}$ means that $f$ is a function from the interval $[-\pi,\pi]$ to $\mathbb{C}$.

• I will try to rig things so that $u$ is the unknown function for which we are solving. (The introduction with traffic density and $\rho$ was an exception.) The letters $f$ and $g$ will typically denote given functions in a problem.

• The independent variables will most often be $x$ and $t$. You should try to see a connection between the behavior of $x$ and "space" and the behavior of $t$ and "time" in the physical context of a given problem.

• I will usually write partial derivatives with a subscript, so if $u = u(x,t)$, then $u_x$ is the partial derivative with respect to $x$. For example, if $u(x,t) = x^2 + 2\sin(xt)$, then

$$u_t = 2x\cos(xt).$$

We will also use the "$\partial$" notation to indicate the action of taking a partial derivative, e.g.,

$$\frac{\partial}{\partial x}[x^2 + 2\sin(xt)] = 2x + 2t\cos(xt).$$

• If $A$ is a set and $x$ is an element of $A$, then we write $x \in A$. For example, with $\mathbb{R}$ as the set of all real numbers, 1 is a real number, and so $1 \in \mathbb{R}$.

• If $A$ is a set, and every element of $A$ is also an element of a set $B$, then we write $A \subseteq B$. For example, every element of the interval $[0,1]$ is a real number, so $[0,1] \subseteq \mathbb{R}$.

• We denote $d$-dimensional space by $\mathbb{R}^d = \{(x_1, \ldots, x_d) \mid x_k \in \mathbb{R}\}$. We will denote vectors in typed text by bold letters, but I will *underline* handwritten vectors. So, in the notes I will write something like $\mathbf{x} = (1,0)$, but on the board or on paper I'd say $\underline{x} = (1,0)$. You are welcome to use "arrows" for your vectors, e.g., $\vec{x}$, but I encourage you to reserve "hat" notation like $\widehat{x}$ for an extremely important concept yet to come. We will not make a big deal of distinguishing row vectors from column vectors.

• A function is **$r$-TIMES CONTINUOUSLY DIFFERENTIABLE** if all of its (mixed) partial derivatives exist up to and including order $r$. We will denote the set of all $r$-times continuously differentiable functions defined on a set $U \subseteq \mathbb{R}^d$ by $\mathcal{C}^r(\mathbb{R}^d)$. For example, if $u \in \mathcal{C}^2(\mathbb{R}^2)$, then all of the following functions would have to be defined and continuous on $\mathbb{R}^2$:

$$u, \qquad u_x, \qquad u_t, \qquad u_{xx}, \qquad u_{xt}, \qquad u_{tt}, \qquad \text{and} \qquad u_{tx}.$$

We will introduce other symbols and notational language as we need them. While they can be strange at first, and sometimes very idiosyncratic to this class, the goal is always to reduce confusion and promote a common vocabulary for easier communication.

This is where we finished on Wednesday, August 17, 2022.

*1.2.2. Terminology.*

The traffic density equation

$$\rho_t + (v\rho)_x = 0$$

is one example of a **PARTIAL DIFFERENTIAL EQUATION (PDE)**: it's *an equation involving a function of at least two variables and (some of) its partial derivatives.* This is not really a "definition" of a PDE, maybe more of an "*un*definition" — that word "involving" is awfully vague. We won't attempt to define a PDE in general, but do pay attention to patterns in the structures of the PDE that we do study.

Here is one important structural property of a PDE. The **ORDER** of a PDE is the order of the highest derivative that appears in that PDE. For example, the traffic density problem is first-order because only first-order partial derivatives appear in it. We will make somewhat less of a fuss about order in PDE than you probably saw in ODE; in particular, we will meet three second-order PDE that have dramatically different behaviors.

Here's another structural property that will resemble something we learned quite a bit about in ODE. Let's expand the traffic density equation a bit. The product rule says[3]

$$(v\rho)_x = v_x\rho + v\rho_x,$$

and so the traffic equation is

$$\rho_t + v_x\rho + v\rho_x = 0.$$

Remember that we are assuming that $v$ is given to us.

Now we define an *operator* $\mathcal{F}$ by

$$\mathcal{F}[\rho](x,t) := \rho_t(x,t) + v_x(x,t)\rho(x,t) + v(x,t)\rho_x(x,t).$$

That is, $\mathcal{F}$ acts on a (sufficiently differentiable function) $\rho$ of the two variables $x$ and $t$ to produce a new function $\mathcal{F}[\rho]$ of the two variables $x$ and $t$. We can condense the traffic equation as

$$\mathcal{F}[\rho] = 0.$$

Indeed, most equations in life boil down to an abstract structure like the one above, if we define $\mathcal{F}$, $\rho$, and 0 correctly.

The operator $\mathcal{F}$ has a very nice property: it is **LINEAR** in $\rho$. You can, and should, check that if $c_1$ and $c_2$ are real numbers and $\rho_1$ and $\rho_2$ are sufficiently differentiable functions of two variables, then

$$\mathcal{F}[c_1\rho_1 + c_2\rho_2] = c_1\mathcal{F}[\rho_1] + c_2\mathcal{F}[\rho_2].$$

The equality above is an equality of *functions*, which means that it has to hold when all the functions are evaluated at the same point $(x,t)$, for all points $(x,t)$. Most, though not all, of the equations that we will study in this course will be linear.

## 1.3. Some PDE philosophy.

In studying ordinary differential equations (ODE), you probably spent a great deal of time with certain broad classes of ODE, both because they are important for modeling and theory

[3] Here we are assuming that velocity $v$ is a function of time and space that is *independent* of density $\rho$. Later it will be quite worthwhile and instructive to consider the realistic case when velocity does depend on density: $v(x,t) = \mathcal{V}(\rho(x,t))$ for some function $\mathcal{V} \in \mathcal{C}^1(\mathbb{R})$. This will lead to a *nonlinear* problem.

and because they have accessible solution techniques. For example, if $a$ and $b$ are continuous functions on the interval $I$, the linear first-order equation

$$a(x)y'(x) + b(x)y(x) = f(x), \tag{1.3.1}$$

which is a nice abstraction of physical phenomena like population growth and temperature flow, can be solved explicitly (up to evaluating some antiderivatives) via the integrating factor method. We'll review that presently. However, solving the analogue of this equation in the world of PDE, even for functions of just two variables, is nontrivial, and sometimes impossible to do explicitly.

Lawrence C. Evans, in his magisterial graduate-level text *Partial Differential Equations*, captures the challenge and the orientation of PDE study quite evocatively:

> *"There is no general theory concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations that are important for applications within and outside of mathematics, with the hope that insight from the origins of these PDE can give clues as to their solutions."*

Geometry, in particular, is a challenge. In ODE, the natural domain of a solution is a subinterval of the real line $\mathbb{R}$; there just aren't really any other "interesting" kinds of domains. In PDE, where our unknown functions depend on at least two variables, we have the much broader two-dimensional structure of $\mathbb{R}^2$, whose more exotic subsets can include rectangles (roughly the analogue of a one-dimensional interval) but also circles, annuli, and infinite strips.

Initial (and boundary) conditions also change in the PDE regime. Whereas in ODE we studied initial value problems (IVP) with data taken at a single initial point, e.g.

$$\begin{cases} y'(x) + a(x)y(x) = f(x) \\ y(x_0) = y_0, \end{cases}$$

in PDE we could impose initial conditions along a whole one-dimensional object. For example, we will soon learn how to solve the "constant" velocity traffic problem

$$\begin{cases} \rho_t + v_0\rho_x = 0 \\ \rho(x,0) = g(x), \end{cases}$$

where we have specified the velocity $v(x,t) = v_0$ to be constant and prescribed the "initial" traffic density $\rho(x,0) = g(x)$ at time $t = 0$. In other words, we do not have an initial condition as a single "point" value but rather a whole function's value!

Algebraically, even stating problems is also a challenge. More variables mean more problems! For example, the analogue of the first-order linear ODE (1.3.1) in just two variables is

$$a(x,t)u_x(x,t) + b(x,t)u_t(x,t) + c(x,t)u(x,t) = f(x,t). \tag{1.3.2}$$

And this is just two variables!

For these reasons, we will narrow our focus in this course to very particular PDE that, over time, have evolved as canonical models for certain physical phenomena and canonical examples of certain solution techniques and theories. In other words, we will follow the mission statement of Peter J. Olver in his modern *Introduction to Partial Differential Equations*:

> *"[T]he primary purpose of a course in partial differential equations is to learn the principal solution techniques and to understand the underlying mathematical analysis."*

Whenever possible, we will interpret our solutions in some physical context to see how they do, or do not, reflect "real-world" behavior. We will highlight the theoretical differences in how we must handle different equations, and we will see what techniques their solutions might have in common.

## 1.4. Some PDE that are really ODE.

Some meaningful PDE turn out to contain derivatives with respect to only one variable, in which case they are really ODE with respect to that variable. We'll start with a traffic example and then do some more abstract problems.

### *1.4.1. Traffic with zero velocity.*

Consider the traffic density equation

$$\rho_t + (v\rho)_x = 0 \tag{1.4.1}$$

in the case of *zero* velocity: $v = 0$. Then (1.4.1) becomes

$$\rho_t = 0. \tag{1.4.2}$$

This equation has only one derivative in it, the $t$-derivative, and so it's really an ODE.

To see and exploit this, fix $x$ to be any number, and let

$$w(t) := \rho(x,t). \tag{1.4.3}$$

Then

$$w'(t) = \rho_t(x,t) = 0,$$

where $w'$ is the derivative of $w$ with respect to its only independent variable, $t$. Since $w'(t) = 0$, we know that $w$ must be constant: there is a number $c$ such that $w(t) = c$ for all $t$. (We didn't specify for which $t$ the function $w$ should be defined, but we may as well take the domain of $w$ to be all real numbers.)

Consequently, the function $\rho(x,t) = w(t) = c$ solves (1.4.2). Indeed,

$$\rho_t(x,t) = \frac{\partial}{\partial t}[c] = 0.$$

We can do better. Observe that when we defined $w$ in (1.4.3), we froze $x$. This suggests that we could let $c$ depend on $x$ instead of being constant. Indeed, if $\rho(x,t) = c(x)$, where $c$ is any function of $x$, we have

$$\rho_t(x,t) = \frac{\partial}{\partial t}[c(x)] = 0.$$

This calculation works if $c$ is not differentiable with respect to $x$, or even discontinuous with respect to $x$. However, it is “good manners” to *require that a solution to a differential equation (ordinary or partial) have as many continuous derivatives as there are derivatives in the problem.* The original traffic equation (1.4.1) has first derivatives with respect to $x$ and $t$. If we demand that $c$ be a continuously differentiable function, then putting $\rho(x,t) = c(x)$ gives a function whose first derivatives with respect to both $x$ and $t$ are continuous.

Let's see how our work holds up physically. If the velocity of the traffic is $v(x,t) = 0$, then the traffic isn't moving at all, and so the traffic density should not change as time goes on. That is, the traffic density should be a function just of position along the road, not time: $\rho(x,t) = c(x)$ for some function $c$.

Finally, just for the sake of practicing notation and vocabulary, let's convince ourselves that all solutions to $\rho_t = 0$ are continuously differentiable functions of $x$ alone.

**1.4.1 Theorem.**

*Suppose that $\rho \in \mathcal{C}^1(\mathbb{R}^2)$ solves $\rho_t(x,t) = 0$ for all $(x,t) \in \mathbb{R}^2$. Then there is a function $c \in \mathcal{C}^1(\mathbb{R})$ such that $\rho(x,t) = c(x)$ for all $x$.*

**Proof.** For each fixed $x \in \mathbb{R}$, the fundamental theorem of calculus (FTC) tells us

$$\rho(x,t) = \rho(x,0) + \int_0^t \rho_t(x,\tau)d\tau = \rho(x,0).$$

The second equality holds because $\rho_t = 0$; the first equality holds by using the FTC on the interval $[0,t]$. Put $c(x) = \rho(x,0)$; since $\rho \in \mathcal{C}^1(\mathbb{R}^2)$, we have $c \in \mathcal{C}^1(\mathbb{R})$. □

By the way, if we have a function $\rho = \rho(x,t)$, and if we fix $t \in \mathbb{R}$, then we will sometimes abbreviate the mapping $x \mapsto \rho(x,t)$ by $\rho(\cdot,t)$. You should be comfortable with what the following statement says:

$$\rho \in \mathcal{C}^1(\mathbb{R}^2) \Longrightarrow \rho(\cdot,t) \in \mathcal{C}^1(\mathbb{R}) \text{ for all } t \in \mathbb{R}.$$

You can imagine, I trust, what $\rho(x,\cdot)$ means.

### *1.4.2. PDE with mostly missing derivatives.*

To review some essential ODE techniques, we will do a few contrived examples.

**1.4.2 Example (A separable problem).**

*Find all functions $u = u(x,t)$ such that*

$$u_x = -2xtu^2.$$

**Solution.** This problem only involves a derivative with respect to $x$. We consider $t$ as fixed and put $w(x) = u(x,t)$ to see that $w$ must solve

$$w' = -2xtw^2.$$

We might rewrite this ODE as

$$\frac{dw}{dx} = (-2tx)w^2,$$

and we see that the right side factors as a function of $x$ (with $t$ as a constant) and a function of $w$. More precisely, put $a(x) = -2tx$ and $b(w) = w^2$ to see that our ODE is

$$\frac{dw}{dx} = a(x)b(w).$$

This is a *separable* problem, and so we separate variables: for $w \neq 0$, we must have

$$\frac{1}{w^2}\frac{dw}{dx} = -2tx.$$

Then we integrate both sides with respect to $x$:

$$\int \left(\frac{1}{w^2}\frac{dw}{dx}\right) \, dx = \int (-2t)x \, dx.$$

The integral on the right is

$$\int (-2t)x \, dx = -2t \int x \, dx = -tx^2 + c(t).$$

Here we have added the constant of integration as a function $c$ of $t$. The integral on the left is

$$\int \left(\frac{1}{w^2}\frac{dw}{dx}\right) \, dx = \int \frac{dw}{w^2} = -w^{-1}.$$

We have not bothered to add a $t$-dependent constant of integration here.

Thus $w$ satisfies

$$-w^{-1} = -tx^2 + c(t),$$

which is the same as

$$w = \frac{1}{tx^2 + c(t)}.$$

Note that since $c$ is arbitrary, we did not write a minus sign on it. And so separation of variables tells us that the solutions are

$$u(x,t) = \frac{1}{tx^2 + c(t)},$$

with $c \in \mathcal{C}^1(\mathbb{R})$ arbitrary.

However, we are not quite done. In separating variables, we assumed $w \neq 0$. If $w = 0$, that means $u = 0$. But $u = 0$ is also a solution (check it). No choice of $c \in \mathcal{C}^1(\mathbb{R})$ will turn $u(x,t) = 1/(tx^2 + c(t))$ into a function that is identically zero. And so we should be sure to report $u = 0$ as another solution, for completeness. ▲

This is where we finished on Friday, August 19, 2022.

**1.4.3 Example (A first-order linear problem).**

**(i)** *Find all functions $u = u(x,t)$ such that*

$$u_t + 2tu = xe^{-t^2}.$$

**(ii)** *Solve*

$$\begin{cases} u_t + 2u = xe^{-t^2} \\ u(x,x) = \sin(x). \end{cases}$$

**Solution.** **(i)** If it helps, abbreviate $w(t) = u(x,t)$ with $x$ fixed, so

$$w' + 2tw = xe^{-t^2}.$$

This is a first-order linear equation in $w$. We solve this by multiplying both sides by an *integrating factor* to turn the left side into a "perfect" product-rule-style derivative. Namely, take

$$\mu(t) = e^{\int 2t\ dt} = e^{t^2}$$

and then multiply both sides by $\mu$ to find

$$w'(t)e^{t^2} + 2te^{t^2}w(t) = x.$$

The left side really is

$$w'(t)e^{t^2} + 2te^{t^2}w(t) = \frac{d}{dt}[w(t)e^{t^2}],$$

and so the equation for $w$ is the same as

$$\frac{d}{dt}[w(t)e^{t^2}] = x.$$

Integrate both sides with respect to $t$ and add an $x$-dependent "constant" of integration:

$$w(t)e^{t^2} = xt + c(x).$$

Conclude

$$u(x,t) = xte^{-t^2} + c(x)e^{-t^2}. \tag{1.4.4}$$

**(ii)** The "side" condition here means that if we restrict $u$ to the "diagonal" line $t = x$, then we get back $\sin(x)$. We know that all solutions to this problem have the form of (1.4.4). So, we need the map $c$ to satisfy

$$\sin(x) = u(x,x) = x^2e^{-x^2} + c(x)e^{-x^2}.$$

We solve for $c$:

$$c(x) = e^{x^2}\sin(x) - x^2.$$

Thus

$$u(x,t) = xte^{-t^2} + [e^{x^2}\sin(x) - x^2]e^{-t^2}.$$ ▲

**1.4.4 Example (A second-order linear problem).**

*Find all functions* $u = u(x,t)$ *such that*

$$u_{xx} - 2u_x + 5u = 0.$$

**Solution.** With $t$ fixed and $w(x) = u(x,t)$, we see that $w$ solves the second-order constant-coefficient linear homogeneous problem:

$$w'' - 2w' + 5w = 0. \tag{1.4.5}$$

We solve this by considering the scalar solutions to the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0.$$

The quadratic formula and some arithmetic tell us that $\lambda = 1 \pm 2i$. This is a complex conjugate pair, and so all solutions to (1.4.5) have the form

$$w(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

As usual, when we pass back to the solution $u = u(x,t)$ of the PDE, we make the constants depend on the missing variable:

$$u(x,t) = c_1(t)e^x \cos(2x) + c_2(t)e^x \sin(2x). \qquad \blacktriangle$$

## 1.5. The transport equation (and related ideas).

As we mentioned in (1.3.2) of Section 1.3, the PDE analogue of the celebrated first-order linear ODE

$$a(x)y'(x) + b(x)y(x) = f(x)$$

is

$$a(x,t)u_x(x,t) + b(x,t)u_t(x,t) + c(x,t)u(x,t) = f(x,t).$$

In the special case that $a$ and $b$ are constant and $c$ and $f$ are identically zero, this reduces to

$$au_x + bu_t = 0. \tag{1.5.1}$$

This is the TRANSPORT EQUATION, and we will find much value in it both as a physical model and a pedagogical source.

The transport equation arises from the traffic density equation $\rho_t + (v\rho)_x = 0$ by taking the velocity $v$ to be constant: $v(x,t) = v_0$ for all $x$, $t \in \mathbb{R}$ and some $v_0 \in \mathbb{R}$. (Previously we treated the case $v_0 = 0$. Also, since we originally assumed that traffic travels only to the right, we probably should take $v_0 > 0$ to be precise.) Then

$$(v\rho)_x = (v_0\rho)_x = \partial_x[v_0]\rho + v_0\rho_x = v_0\rho_x,$$

and so the traffic equation is

$$\rho_t + v_0\rho_x = 0.$$

This is a special case of (1.5.1).

### 1.5.1. Solving the transport equation.

We first assume that $a \neq 0$ and $b \neq 0$. Otherwise, if $a = 0$, then the transport equation reduces to $bu_t = 0$, and this is really an ODE. Likewise, the transport equation is also an ODE if $b = 0$.

Here is the key insight:

$$au_x + bu_t = (a, b) \cdot (u_x, u_t) = (a, b) \cdot \nabla u.$$

Let's parse this calculation carefully. Recalling our (flexible) convention for vectors, by $(a, b)$ here we mean the vector in $\mathbb{R}^2$ whose first component is $a$ and whose second is $b$. By $\cdot$ we mean the dot product: if $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, then

$$(x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1y_2.$$

And by $\nabla u$ we mean the gradient vector of $u$:

$$\nabla u(x, t) := \big(u_x(x, t), u_t(x, t)\big).$$

So, the transport equation is equivalent to a statement about the dot product of $(a, b)$ with $\nabla u$:

$$au_x + bu_t = 0 \iff (a, b) \cdot \nabla u = 0.$$

With this definition of the dot product, we could write this as

$$\nabla u \cdot (a, b) = 0.$$

Now, hopefully this looks familiar from multivariable calculus: it says that the *directional derivative*[4] *of $u$ at any point $(x, t)$ in the direction of the vector $(a, b)$ is zero*, and consequently *$u$ is constant along lines parallel to the vector $(a, b)$*.

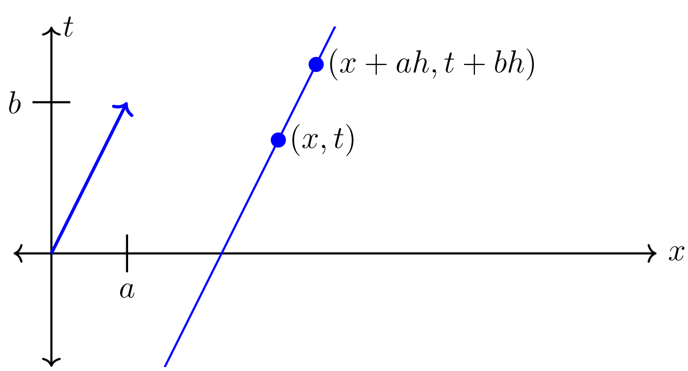
Let's build this up from scratch. Fix $(x, t) \in \mathbb{R}^2$; we won't change this point from now on. The line in $\mathbb{R}^2$ parallel to the vector $(a, b)$ that passes through the point $(x, t)$ is the set

$$\{(x + ah, t + bh) \mid h \in \mathbb{R}\}. \tag{1.5.2}$$

And so if we abbreviate

$$w(h) := u(x + ah, t + bh),$$

then $w$ is just the restriction of $u$ to this line.




[4] Usually multivariable calculus makes a big deal of the vector's being a *unit* vector. We will not make a big deal about that here.

We calculate

$$w'(h) = au_x(x + ah, t + bh) + bu_t(x + ah, t + bh) = 0. \tag{1.5.3}$$

The first equality is the multivariable chain rule. The second equality is the transport equation, which $u$ satisfies. And so the function $w$ is constant.

This is where we finished on Monday, August 22, 2022.

Incidentally,

$$w'(0) = \lim_{h \to 0} \frac{w(h) - w(0)}{h} = \lim_{h \to 0} \frac{u(x + ah, t + bh) - u(x, t)}{h},$$

and this limit is the way that we usually define the directional derivative of $u$ at the point $(x, t)$ in the direction $(a, b)$. Using the first equality in (1.5.3), we get

$$w'(0) = \nabla u(x, t) \cdot (a, b),$$

which is the usual computational shortcut for the directional derivative.

Anyway, we won't need the directional derivative any further here; what matters is that $w$ is constant. Thus, given $(x, t) \in \mathbb{R}^2$, we have

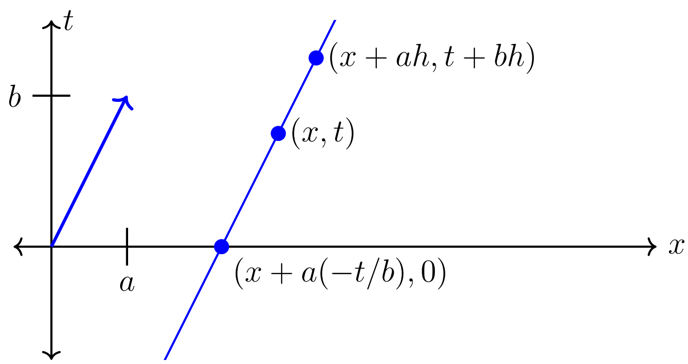
$$u(x + ah_1, t + bh_1) = w(h_1) = w(h_2) = u(x + ah_2, t + bh_2) \tag{1.5.4}$$

for any choice of $h_1$ and $h_2$.

We want to figure out what the values $u(x, t)$ should be. We can make $u(x, t)$ appear in the equality (1.5.4) by taking, say, $h_1 = 0$, and then we just drop the subscript on $h_2$. Thus if $u$ solves the transport equation, then the following identity holds for all $x$, $t$, and $h$:

$$u(x, t) = u(x + ah, t + bh). \tag{1.5.5}$$

Now, can we choose $h$ in some clever way to help us out? One approach, which may not be obvious[5], is to think about what are the "simplest" points $(x + ah, t + bh)$ that fall on this special line. Maybe some of the least complicated are those with at least one coordinate equal to 0. Note that $t + bh = 0$ if and only if $h = -t/b$.




[5] Here is another approach, which involves assuming some extra "data" in our problem. Many problems impose an initial condition of the form $u(x, 0) = g(x)$. Here we want to know the value of $u(x, t)$ for a given pair $(x, t)$, and we know the two identities $u(x, t) = u(x + ah, t + bh)$ for all $h \in \mathbb{R}$ and $u(X, 0) = g(X)$ for all $X \in \mathbb{R}$. Note that I have suddenly written $X$, not $x$, because we are thinking that $x$ is fixed, along with $t$. If $h$ satisfies $t + bh = 0$, then $u(x, t) = u(x + ah, 0) = g(x + ah)$. So we take $h = -t/b$, and we find $u(x, t) = g(x - at/b) = g((bx - at)/b)$.

Then with this choice of $h$, we have

$$u(x,t) = u\left(x + a\left(-\frac{t}{b}\right), t + b\left(-\frac{t}{b}\right)\right) = u\left(x - \frac{at}{b}, 0\right).$$

It will make some notation nicer if we get a common denominator here:

$$u(x,t) = u\left(\frac{bx - at}{b}, 0\right).$$

This means that $u$ depends on what happens in its $x$-coordinate alone! Specifically, set

$$p(\xi) := u(\xi/b, 0).$$

Then we have shown that if $u$ solves the transport equation, then

$$u(x,t) = p(bx - at).$$

Furthermore, $p$ is continuously differentiable on $\mathbb{R}$, since we are (tacitly) assuming that $u$ is continuously differentiable on $\mathbb{R}^2$.

Since we worked backward, we should check our work: let $p$ be a differentiable function and define $u(x,t) := p(bx - at)$. Then $u$ should solve the transport equation $au_x + bu_t = 0$. I'll let you work through that.

**1.5.1 Theorem (Transport equation).**

*Let $a$, $b \in \mathbb{R}$. A function $u \in \mathcal{C}^1(\mathbb{R}^2)$ solves the transport equation $au_x + bu_t = 0$ if and only if there exists a function $p \in \mathcal{C}^1(\mathbb{R})$ such that*

$$u(x,t) = p(bx - at). \tag{1.5.6}$$

**1.5.2 Example.**

*Find all functions $u = u(x,t)$ that satisfy $u_x - 3u_t = 0$.*

**Solution.** This is a transport equation with $a = 1$ and $b = -3$, so all solutions have the form

$$u(x,t) = p(-3x + 1(t)) = p(-3x + t)$$

for some $p \in \mathcal{C}^1(\mathbb{R})$. ▲

**1.5.3 Example.**

*Find all solutions to the initial value problem*

$$\begin{cases} -2u_x + 5u_t = 0 \\ u(x,0) = e^{-x}. \end{cases}$$

**Solution.** Any solution to the transport equation $-2u_x + 5u_t = 0$ must have the form $u(x,t) = p(5x - (-2)t) = p(5x + 2t)$ for some $p \in \mathcal{C}^1(\mathbb{R})$. We also want $u(x,0) = e^{-x}$, so we need $p$ to satisfy

$$e^{-x} = u(x,0) = p(5x + 2(0)) = p(5x).$$

That is, $p$ solves

$$p(5x) = e^{-x}.$$

So what is $p$?

One way to find a formula for $p$ as a function of the independent variable $\xi \in \mathbb{R}$ alone, not as a function of $5x$, is to substitute. Put $\xi = 5x$, so $x = \xi/5$. Then

$$p(\xi) = p(5x) = e^{-x} = e^{-\xi/5}.$$

Thus

$$u(x,t) = p(5x - 2t) = e^{-(5x+2t)/5}.$$ ▲

Algebraically and analytically, what just happened? We found that the function $p$ needed to solve an equation of the form

$$p(h(x)) = g(x)$$

for given functions $h$ and $g$. (Specifically, $h(x) = 5x$ and $g(x) = e^{-x}$.) Fortunately, $h$ turned out to be invertible from $\mathbb{R}$ to $\mathbb{R}$. We put $\xi = h(x)$ and were able to solve for $x$ as $x = h^{-1}(\xi)$. Then we could define $p$ as

$$p(\xi) = p(h(h^{-1}(\xi)) = g(h^{-1}(\xi)).$$

Things aren't always so nice.

**1.5.4 Example.**

*What can you say about solutions to the problem*

$$\begin{cases} -2u_x + 5u_t = 0 \\ u(x, \sin(x)) = e^{-x}? \end{cases}$$

**Solution.** This (contrived) problem is an example of the deeper geometric questions that we can ask in PDE, as compared to ODE. Here we are not given an "initial" condition at time $t = 0$. Rather, we have specified the behavior of $u$ along the "parametric" curve $\{(x, \sin(x)) \mid x \in \mathbb{R}\}$ in $\mathbb{R}^2$.

As before, we know that the solution has to have the form $u(x,t) = p(5x + 2t)$ for some continuously differentiable function $p$. Taking $t = \sin(x)$, we need $p$ to satisfy

$$e^{-x} = u(x, \sin(x)) = p(5x + 2\sin(x)).$$

Trying to substitute $\xi = 5x + 2\sin(x)$ and then solve for $x$ as a function of $\xi$ will not yield any good formulas.

However, perhaps inspired by a graph of $h(x) := 5x + 2\sin(x)$, we might note that this map is strictly increasing on $\mathbb{R}$ and therefore has an inverse. (Specifically, check that $h'(x) > 0$ for all $x$.) Then the commentary above suggests that

$$p(\xi) = e^{-h^{-1}(\xi)}.$$

To check that our solution $u(x,t) = e^{-h^{-1}(5x+2t)}$ actually works, we would need part of the **INVERSE FUNCTION THEOREM**, which states that if $h$ is differentiable and strictly increasing, then its inverse $h^{-1}$ is also differentiable with $(h^{-1})'(\xi) = 1/h(\xi)$. ▲

This is where we finished on Wednesday, August 24, 2022.

A transport equation is very easy to solve, thanks to the formula (1.5.6), which is one of those formulas that's well worth memorizing. Because of this solution formula, and the facile algebraic and differential structure of the transport equation, we will frequently return to the transport equation to illustrate new concepts.

### *1.5.2. Traveling waves.*

Previously we saw that taking traffic velocity to be constant in the traffic density problem $\rho_t + (v\rho)_x = 0$, i.e., taking $v(x,t) = v_0 \in \mathbb{R}$, led to the transport equation $\rho_t + v_0\rho_x = 0$. Now we know that all solutions to this transport equation have the form

$$\rho(x,t) = p(x - v_0 t), \tag{1.5.7}$$

where $p \in \mathcal{C}^1(\mathbb{R})$.

What could this mean physically in the context of traffic along our idealized road? Suppose that we know the *initial* traffic density:

$$\rho(x,0) = g(x) \tag{1.5.8}$$

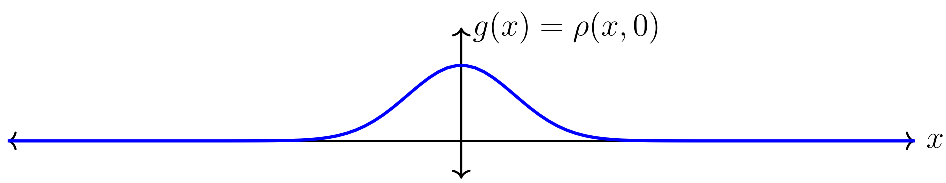
for some function $g \in \mathcal{C}^1(\mathbb{R})$; maybe we got an aerial view of the road at the start of the morning commute. Then (1.5.7) and (1.5.8) together tell us

$$g(x) = \rho(x,0) = p(x - v_0(0)) = p(x). \tag{1.5.9}$$

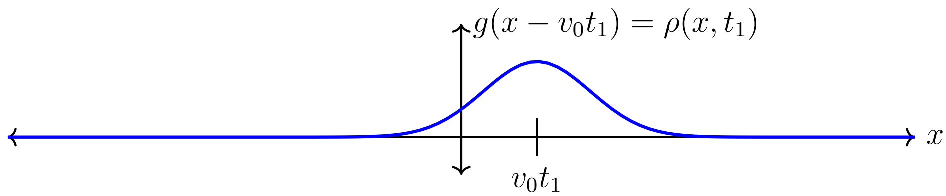
Thus for any time $t$ in the future, the density is

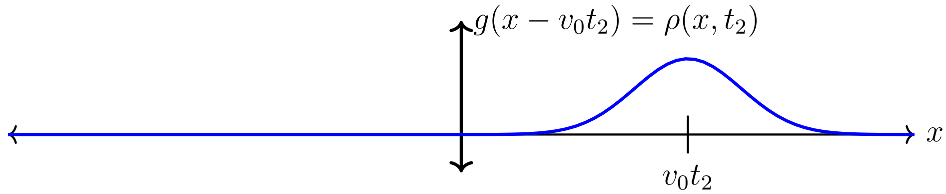
$$\rho(x,t) = p(x - v_0 t) = g(x - v_0 t). \tag{1.5.10}$$

Imagine that we continue to view the whole road at successive moments in time. What does this formula for $\rho$ say about the pattern in which traffic evolves? Say that $g = u(\cdot, 0)$ has the graph below.



If we assume that $v_0 > 0$ (recall that $v_0$ is the velocity of traffic, and traffic on this road always moves to the right) and that $t > 0$ (time marches on), then the traffic density at time $t$ is $u(\cdot, t)$, and, since $\rho(x, t) = g(x - v_0 t)$, the graph of $\rho(\cdot, t)$ is the graph of $g$ shifted to the right along the $x$-axis by $v_0 t$ units.




In other words, if traffic moves to the right at a constant positive velocity, then the initial density just gets carried forward at at that velocity. We call a solution of the form $\rho(x, t) = g(x - v_0 t)$ a **TRAVELING WAVE**. In the immortal words of G. B. Whitham from his staggering *Linear and Nonlinear Waves*,

> *"[A]* ***wave*** *is any recognizable signal that is transferred from one part of [a] medium, to another with a recognizable velocity of propagation. The signal may be any feature of the disturbance, such as a maximum or an abrupt change in some quantity, provided that it can be clearly recognized and its location at any time can be determined. The signal may distort, change its magnitude, and change its velocity provided it is still recognizable."*

And a *traveling* wave is a wave that propagates with a constant "speed" and whose "profile" maintains the same shape during this propagation.

We can capture traveling wave behavior for a PDE in the unknown $u = u(x, t)$ by making the **TRAVELING WAVE ANSATZ**

$$u(x, t) = p(x - ct).$$

Here $p$ is a function of a single real variable[6] and $c$ is a real number; we call $p$ the wave **PROFILE** and $c$ the wave **SPEED**. Making such an ansatz[7] often reduces a PDE in $u$ to an ODE in $p$ with $c$ as a parameter. We may or may not be able to solve that ODE for all values of $c$. Also, traveling wave behavior is often only one of many kinds of solution behavior that a PDE can enjoy. We will often look for traveling wave solutions to the subsequent equations that we study and interpret their behavior physically, and, whenever possible, we will contrast those traveling waves with other types of solutions.

[6] Often it is helpful to denote that variable by a different letter, like $X$ or $\xi$; if you read (1.5.9) and (1.5.10) carefully, you'll see that we were using $x$ as the independent variable of $p$, $g$, and $u$!

[7] Remember that "ansatz" is a fancy word for "guess."

Here is a "transport equation with a nonlinear term" for which we will find traveling waves.

**1.5.5 Example.**

*How would you find all traveling wave solutions to*

$$u_x + u_t + u^2 = 0?$$

**Solution.** We make the traveling wave ansatz $u(x,t) = p(x-ct)$ for a profile function $p = p(X)$ and a wave speed $c \in \mathbb{R}$. The multivariable chain rule tells us that

$$u_x(x,t) = p'(x-ct) \quad \text{and} \quad u_t(x,t) = -cp'(x-ct).$$

Thus $p$ and $c$ must satisfy

$$p'(x-ct) - cp'(x-ct) + [p(x-ct)]^2 = 0$$

for all $x$, $t \in \mathbb{R}$. If we take $x = X$ and $t = 0$, which we are free to do, we see that $p$ must satisfy

$$(1-c)p'(X) + [p(X)]^2 = 0,$$

or, more succinctly,

$$(1-c)p' + p^2 = 0.$$

This is actually a separable ODE, and we can rewrite it as

$$(1-c)\frac{dp}{dX} = -p^2.$$

This is where we finished on Friday, August 26, 2022.

We see that $p = 0$ is a solution (and, in fact, if $c = 1$, then $p = 0$ is the *only* solution, as then $p^2 = 0$ and thus $p = 0$), and otherwise we separate variables and integrate:

$$(1-c)\frac{dp}{dX} = -p^2 \Longrightarrow \frac{1}{p^2}\frac{dp}{dX} = c-1 \Longrightarrow \int p^{-2}\, dp = \int (c-1)\, dX \Longrightarrow p^{-1} = (c-1)X + K.$$

Here we are using $K$, not $c$, for the constant of integration. Thus if $1 - c \neq 0$, then

$$p(X) = \frac{1}{(c-1)X + K}.$$

We conclude that if $c \neq 1$, then all traveling waves are

$$u(x,t) = \frac{1}{(c-1)(x-ct) + K} \quad \text{and} \quad u = 0, \tag{1.5.11}$$

while the only traveling wave with speed $c = 1$ is the zero wave $u = 0$. Note that in (1.5.11), the function on the left is still defined when $c = 1$, but then it is constant and nonzero — and nonzero constants do not solve $u_x + u_t + u^2 = 0$. Thus there really is a difference between the $c = 1$ and $c \neq 1$ wave speeds. ▲

*1.5.3. Rescaling variables.*

The constant-velocity traffic density problem is $\rho_x + v_0\rho_t = 0$. This has only one non-unitary coefficient, $v_0$, unlike the more general transport equation $au_x + bu_t = 0$. As it turns out, both equations are equivalent to a simpler problem. (For the sake of your education, pretend for now that you don't know how to solve the transport equation as quickly and painlessly as you already do.)

Experience with calculus suggests that a term of the form $au_x$ could result from taking an $x$-derivative of a function whose $x$-coordinate is scaled by $a$. And the same for the term $bu_t$.

Motivated by this, we assume that $u$ solves $au_x + bu_t = 0$ and rewrite

$$u(x,t) = U(\alpha x, \beta t)$$

for some function $U = U(\xi, \tau)$ and some coefficients $\alpha$, $\beta \in \mathbb{R}$. We are intentionally calling the variables of $U$ something other than $x$ and $t$. We will find a PDE that $U$ solves, and then we will choose $\alpha$ and $\beta$ to have particular values so that this PDE is very simple.

The multivariable chain rule gives

$$u_x(x,t) = \alpha U_\xi(\alpha x, \beta t) \quad \text{and} \quad u_t(x,t) = \beta U_\tau(\alpha x, \beta t).$$

Then $u$ solves $au_x(x,t) + bu_t(x,t) = 0$ for all $(x,t) \in \mathbb{R}^2$ if and only if $U$ solves

$$\alpha a U_\xi(\alpha x, \beta t) + \beta b U_\tau(\alpha x, \beta t) = 0$$

for all $(x,t) \in \mathbb{R}^2$. Since $a \neq 0$ and $b \neq 0$, we can now choose

$$\alpha := \frac{1}{a} \quad \text{and} \quad \beta := \frac{1}{b}$$

to see that $U$ solves

$$U_\xi(\alpha x, \beta t) + U_\tau(\alpha x, \beta t) = 0$$

for all $(x,t) \in \mathbb{R}^2$.

Finally, given $(\xi, \tau) \in \mathbb{R}^2$, we can take

$$x := \frac{\xi}{\alpha} \quad \text{and} \quad t := \frac{\tau}{\beta}$$

to see that $U$ *really* solves

$$U_\xi(\xi, \tau) + U_\tau(\xi, \tau) = 0$$

for all $(\xi, \tau) \in \mathbb{R}^2$. In other words, if we can solve

$$U_\xi + U_\tau = 0,$$

and if we define $u$ by

$$u(x,t) := U(a^{-1}x, b^{-1}t),$$

then $u$ solves $au_x + bu_t = 0$.

Now, this probably feels very silly, because we know that all solutions to $U_\xi + U_\tau = 0$ have the form $U(\xi, \tau) = P(\xi - \tau)$ for some $P \in \mathcal{C}^1(\mathbb{R})$. And, rescaling, this gives

$$u(x,t) = P(a^{-1}x - b^{-1}t) = P\left(\frac{x}{a} - \frac{t}{b}\right) = P\left(\frac{bx - at}{ab}\right).$$

Since $P \in \mathcal{C}^1(\mathbb{R})$ was arbitrary, so is $P(\cdot/ab)$, and so we recover Theorem 1.5.1.

But pretend that we had no clue as to where to start solving $au_x + bu_t = 0$. Surely working with fewer arbitrary coefficients in the problem $U_\xi + U_\tau = 0$ couldn't hurt — and might even help! In the future, I may present a complicated problem to us and then work on a simplified version and leave you to chase through the algebra of rescaling that connects the two.

**1.5.6 Example.**

*The* **Korteweg–de Vries (KdV)** *equation arises in the study of water waves. (Hopefully we will see it arise later in the term.) It reads*

$$au_t + bu_{xxx} + cuu_x = 0.$$

*You can think of $u(x,t)$ as being the height of a wave at position $x$ along a one-dimensional channel and time $t$. Here $a$, $b$, and $c$ are all nonzero. How can we change variables to an equivalent problem in which all the coefficients are 1?*

**Solution.** There are three coefficients in the problem. (Since they are all nonzero, really there are two: divide everything by $a$, so that, keeping the same labels, you get $u_t + bu_{xxx} + cuu_x = 0$.) This suggests that we nondimensionalize with the change of variables

$$u(x,t) = \alpha U(\beta x, \gamma t), \qquad U = U(\xi, \tau).$$

We are scaling not just the variables of $U$ but also its output here. We calculate

$$u_t = \alpha\gamma U_\tau, \qquad u_x = \alpha\beta U_\xi, \qquad \text{and} \qquad u_{xxx} = \alpha\beta^3 U_{\xi\xi\xi}.$$

Thus $u$ solves the given KdV equation if and only if $U$ solves

$$a\alpha\gamma U_\tau + b\alpha\beta^3 U_{\xi\xi\xi} + c\alpha^2\beta U_\xi = 0.$$

Equivalently,

$$a\gamma U_\tau + b\beta^3 U_{\xi\xi\xi} + c\alpha\beta U_\xi = 0.$$

We want

$$a\gamma = b\beta^3 = c\alpha\beta = 1.$$

So, we take $\gamma = 1/a$ and $\beta = 1/b^{1/3}$. Then $\alpha$ needs to satisfy $c\alpha/b^{1/3} = 1$, and so $\alpha = b^{1/3}c$. It is not too much harder to show how to convert this KdV equation into one with nonunital coefficients, i.e., $AU_\tau + BU_{\xi\xi\xi} + CUU_\xi = 0$ for $A$, $B$, $C$ nonzero, but I'll let you do that as an exercise. ▲

This is where we finished on Monday, August 29, 2022.

*1.5.4. The nonhomogeneous transport equation.*

Our work with the transport equation has so far hinged on taking the right side to be identically zero. However, it will not be too much more work to construct solutions to the **NONHOMOGENEOUS TRANSPORT EQUATION**

$$au_x + bu_t = f(x,t). \tag{1.5.12}$$

Here $a$ and $b$ are nonzero and $f$ is an arbitrary continuous function on $\mathbb{R}^2$. We will see how an equation of this form arises in the context of traffic density, but let's defer the application for a bit to Section 1.5.5.

For now, fix $(x,t) \in \mathbb{R}^2$. Motivated by our successful analysis of the (homogeneous) transport equation, let's ask what problem the function

$$w(h) := u(x+ah, t+bh)$$

solves. The nonhomogeneous transport equation (1.5.12) says

$$au_x(\xi,\tau) + bu_t(\xi,\tau) = f(\xi,\tau)$$

for all $(\xi,\tau) \in \mathbb{R}^2$, so if we take $\xi = x+ah$ and $\tau = t+bh$, then we get

$$w'(h) = au_x(x+ah,t+bh) + bu_t(x+ah,t+bh) = f(x+ah,t+bh). \tag{1.5.13}$$

That is, $w$ must be an antiderivative of the map $h \mapsto f(x+ah,t+bh)$. (Previously, in the "homogeneous" transport equation, $w$ was an antiderivative of 0.) I think this is a situation that calls for a definite integral. Specifically, we want to make the value $w(0) = u(x,t)$ appear somewhere in our work, so let's integrate from 0 to $h$:

$$w(h) - w(0) = \int_0^h w'(s)\ ds = \int_0^h f(x+as,t+bs)\ ds. \tag{1.5.14}$$

We are running a bit low on letters, so I am using $s$ for the variable of integration.

By the way, we effectively obtained (1.5.14) when we did the homogeneous transport equation. There, $f = 0$, and so (1.5.14) just says $w(h) = w(0)$ for all $h$. I claim that the equality $w(h) = w(0)$ is exactly (1.5.5). Do you agree?

Motivated once more by our prior success, let's take $h = -t/b$ in (1.5.14). We rearrange and get

$$u(x,t) = w(0) = w(h) - \int_0^h f(x+as,t+bs)\ ds = u\left(\frac{bx-at}{b},0\right) - \int_0^{-t/b} f(x+as,t+bs)\ ds. \tag{1.5.15}$$

We are pretty much done, but I claim it will pay off to clean up the integral.

There are a lot of ways to change variables in the integral, but the best way for our purposes turns out to be substituting[8]

$$\sigma = t + bs.$$

[8] An unfortunate consequence of using $u$ as the unknown in a PDE is that it's difficult to $u$-sub anymore. I will try to use $\sigma$ when substituting here, since $\sigma$ sounds like $s$, only fancier.

I admit that there's no clear motivation for this choice, but maybe one reason is that we like the second coordinate of our functions to be "simple"; setting $h = -t/b$ gives us a 0 in the second slot of $u$, and setting $\sigma = t + bs$ makes the second coordinate in the integrand simpler.

Here's what this does:

$$d\sigma = b\ ds, \qquad \sigma(0) = t, \qquad \sigma(-t/b) = 0, \quad \text{and} \quad s = (\sigma - t)/b.$$

Thus

$$-\int_0^{-t/b} f(x + as, t + bs)\ ds = -\frac{1}{b}\int_t^0 f\left(x + a\left(\frac{\sigma - t}{b}\right), \sigma\right)\ d\sigma = \frac{1}{b}\int_0^t f\left(\left[\frac{bx - at}{b}\right] + \frac{a\sigma}{b}, \sigma\right)\ d\sigma. \quad (1.5.16)$$

I like isolating the $(bx - at)/b$ term in the first coordinate of $f$ to make things look more like the $u((bx - at)/b, 0)$ term.

As with the homogeneous transport equation, we put $p(X) = u(X/b, 0)$ to conclude the following.

**1.5.7 Theorem (Nonhomogeneous transport equation).**

*Let $a$, $b \in \mathbb{R}$ with $b \neq 0$ and let $f \in \mathcal{C}(\mathbb{R}^2)$. If a function $u \in \mathcal{C}^1(\mathbb{R}^2)$ solves the nonhomogeneous transport equation $au_x + bu_t = f(x, t)$, then there is a function $p \in \mathcal{C}^1(\mathbb{R})$ such that*

$$u(x, t) = p(bx - at) + \frac{1}{b}\int_0^t f\left(\left[\frac{bx - at}{b}\right] + \frac{as}{b}, s\right)\ ds. \quad (1.5.17)$$

This is probably *not* a formula worth memorizing, but rather one whose location is worth knowing so that we can look it up in the future. Note our phrasing of this theorem compared to the result for the homogeneous transport equation (Theorem 1.5.1): it is not an "if and only if" result. Specifically, how do we know that a function of the form (1.5.17) actually solves the nonhomogeneous transport equation? We'd have to differentiate that function. But how do we differentiate the integral term? We address this in Appendix A.

This is where we finished on Wednesday, August 31, 2022.

### *1.5.5. Traffic with generation/dissipation.*

The homogeneous and nonhomogeneous transport equations are special cases of the following PDE:

$$au_x + bu_t + c(x, t)u = f(x, t). \quad (1.5.18)$$

The homogeneous equation corresponds to $c = f = 0$, while the nonhomogeneous equation has just $c = 0$. You can solve (1.5.18) by following our now tried (tired?) and true method of putting $w(h) = u(x + ah, t + bh)$ and finding an ODE for $w$. You'll get the first-order problem

$$w'(h) + p(h)w(h) = q(h), \qquad p(h) := c(x + ah, t + bh), \qquad q(h) := f(x + ah, t + bh).$$

You can solve this with an integrating factor and then back out $u(x,t)$ by putting $h = -t/b$.

I don't think there's much to be learned by working through the details of this in class. Instead, let's see how (1.5.18) arises in the context of traffic, and how we might expect its solutions to behave *without* going right to a formula.

Suppose that traffic continues to move in one direction along a one-lane road, but now let's allow traffic both to enter and leave the road. It's been a while, so let's review our notation and ideas from Section 1.1. The traffic density is $\rho$, so $\rho(x,t)$ tells us how many cars per mile there are at point $x$ along the road and time $t$. The traffic flux is $q$, so $q(x,t)$ tells us how many cars per hour are passing point $x$ at time $t$. And the rate of change of the number of cars between points $a$ and $b$ on the road at time $t$ is (because we know what happens when we integrate density with respect to space, and because we know how to differentiate under the integral) $\int_a^b \rho_t(x,t)\ dx$.

Previously, traffic could only change within the spatial interval $[a,b]$ on the road by entering from the left at a rate of $q(a,t)$ cars per hour and leaving from the right at a rate of $q(b,t)$ cars per hour. Now traffic can enter or leave the road at any point $x$ and any time $t$. One useful way to quantify this is to fix a point $x$ and time $t$ and two numbers $h,\ k > 0$. Then we measure how many cars enter/exit the road between points $x-h$ and $x+h$ from time $t-h$ to time $t+h$. Let's say that a function $r = r(x,t)$ measures the "generation/dissipation" of traffic at point $x$ and time $t$ if

$$r(x,t) \approx \frac{\text{change in number of cars between points } x-h \text{ and } x+h \text{ over times } t-h \text{ to } t+h}{(2h)(2k)}.$$

Thus the units of $r(x,t)$ are cars per mile per hour.

I claim then that the rate of change of the number of cars on the road between points $a$ and $b$ is

$$q(a,t) - q(b,t) + \int_a^b r(x,t)\ dx = \int_a^b [r(x,t) - q_x(x,t)]\ dx. \tag{1.5.19}$$

This follows from our belief that

$$\text{Rate of change of a quantity} = \text{Rate in} - \text{Rate out}.$$

Specifically, $q(a,t)$ is the rate of cars entering the interval $[a,b]$ at point $a$ and $q(b,t)$ is the rate of traffic leaving the interval $[a,b]$ at point $b$. But now we must also account for the *cars that are entering or exiting the road* any point $x$ in $[a,b]$, and this gives rise to the integral on the left of (1.5.19).

To see how the integral enters the story, divide $[a,b]$ into $n$ small subintervals $[x_k, x_{k+1}]$. We expect that $r(x,t) \approx r(x_k,t)$ for $x_k \le x \le x_{k+1}$, and so roughly, over the interval $[x_k, x_{k+1}]$ on the road, there are $r(x_k,t)$ cars per hour per mile entering or exiting the road. Thus, over the stretch $[x_k, x_{k+1}]$, the rate of change of the number of cars on the road *due to cars entering or exiting the road* is

$$\frac{r(x_k,t)\ \text{cars}}{\text{mile} \times \text{hour}} \times (x_{k+1} - x_k)\ \text{miles} = \frac{r(x_k,t)(x_{k+1}-x_k)\ \text{cars}}{\text{hour}}$$

And so the rate of change of the number of cars in the interval $[a,b]$ *due to cars entering or exiting the road* is

$$\sum_{k=1}^{n} r(x_k,t)(x_{k+1}-x_k) \approx \int_a^b r(x,t)\ dx.$$

I've italicized the phrase *due to cars entering or exiting the road* because the number of cars within $[a, b]$ also changes when cars pass point $a$ or $b$ *but still stay on the road*; these latter changes introduce the $q(a, t) - q(b, t)$ terms in (1.5.19).

Anyway, if you believe (1.5.19), we get

$$\int_a^b [\rho_t(x, t) + q_x(x, t) - r(x, t)]\, dx = 0,$$

and since $[a, b]$ was an arbitrary interval, it follows as in Section 1.1 that

$$\rho_t + q_x - r = 0. \tag{1.5.20}$$

Previously we said $q = v\rho$, where $v(x, t)$ is the velocity of traffic at point $x$ and time $t$. We can be more precise about $r$ if we assume that cars enter/exit along the road depending on the structure of the road (the presence of on/off-ramps and/or intersections) and *exit* if the density is too high. So, I would like to say

$$r(x, t) = f(x, t) - c(x, t)\rho(x, t). \tag{1.5.21}$$

The term $f$ is an overall "generation/dissipation" term that captures how many cars per mile per hour enter/exit due to the road's "structure," while the term $c\rho$ is proportional to the current density. (So $f$ should have units cars per mile per hour, while $c$ should have units in 1/hour, right?) We might expect that as density gets higher, more cars will want to leave the road; to ensure this, we should assume $c(x, t) > 0$ for all $x$ and $t$.

All together, we can rearrange (1.5.20) and (1.5.21) into

$$\rho_t + (v\rho)_x + c(x, t)\rho = f(x, t). \tag{1.5.22}$$

If we take velocity to be constant, then we get (1.5.18).

## 1.6. Variable-coefficient problems.

Let's return, once again, to the traffic density problem

$$\rho_t + (v\rho)_x = 0.$$

Here we are back to our original model, in which cars cannot enter or exit the road (unlike the model in Section 1.5.5. Recall that $\rho(x, t)$ is the number of cars per mile on the road around point $x$ and time $t$, while $v(x, t)$ is the velocity of the cars at point $x$ and time $t$. Previously, we got results when $v$ was constant; this caused the initial traffic density to be somehow propagated along the road with "speed" $v$.

What if, however, $v$ is not constant? Then the traffic problem becomes

$$\rho_t + v_x(x, t)\rho + v(x, t)\rho_x = 0.$$

I suggested to you at the start of Section 1.5.5 how to handle the case where $v$ is constant and $v_x$ is not constant; here, however, that is trivial, as if $v$ is constant, then $v_x = 0$. If $v$ is not constant, then we need to do something new.

We will study the general variable-coefficient first-order linear PDE[9]

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = f(x,t),$$

which I previously mentioned in (1.3.2) to illustrate how quickly the number of terms in PDE can grow when we allow more variables than in ODE. These methods will not always be successful in that the coefficients $a$ and $b$ will need to be pretty "nice" in order for us to get explicit formulas for solutions.

This is where we finished on Friday, September 9, 2022.

It will be worthwhile to revisit, briefly, the transport equation.

### 1.6.1. *The transport equation revisited, now with more geometry.*

Previously, we rewrote the constant-coefficient homogeneous transport equation

$$au_x + bu_t = 0$$

as the gradient equation

$$\nabla u \cdot (a,b) = 0,$$

which told us that $u$ should be constant on lines parallel to the vector $(a,b)$. More analytically, we fixed a pair[10] $(x_0,t_0) \in \mathbb{R}^2$, introduced the function $w(h) := u(x_0 + ah, t_0 + bh)$, and saw that $w$ had to be constant. In other words, $w' = 0$. More geometrically, then, on any line parallel to the vector $(a,b)$, the PDE $au_x + bu_t = 0$ reduced to an ODE.

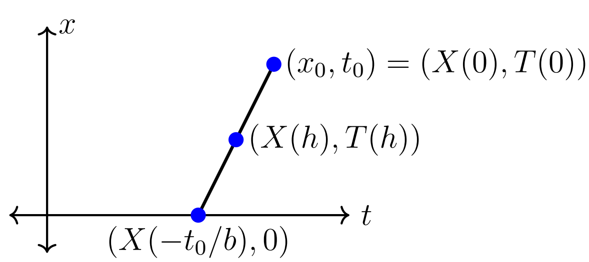
Let's combine analysis (formulas) and geometry (graphs). If, for given $x_0$ and $t_0$, we put

$$X(h) := x_0 + ah \quad \text{and} \quad T(h) := t_0 + bh, \tag{1.6.1}$$

then $(X,T)$ is really a parametric mapping of $\mathbb{R}$ into $\mathbb{R}^2$, and the composition $w(h) = u(X(h),T(h))$ is constant. (Note that the maps $X$ and $T$ are functions of $h$, but they do also depend on $x_0$ and $t_0$; we are not indicating this latter dependence.) What's also nice is that given an arbitrary pair $(x_0,t_0) \in \mathbb{R}^2$, we were able to construct the maps $(X,T)$ so that we had $X(0) = x_0$ and $T(0) = t_0$. This allowed us to equate

$$u(x_0,t_0) = u(X(0),T(0)) = w(0) = w(h) = u(X(h),T(h))$$

for all $h$, and then we could choose $h$ to be something "useful" ($h = -t_0/b$) so that we could learn more about $u(x_0,t_0)$.




[9] I am following the usual convention in ODE and PDE of writing the given coefficients and nonhomogeneities in the equation with their dependence on $(x,t)$ explicit, whereas I suppress that dependence in the unknown $u$ and its derivatives.

[10] I will want to refer to the $x$-axis from time to time here, so I don't want to call my fixed coordinates $x$ and $t$ anymore.

This reasoning extended without too much conceptual trouble to the problem

$$au_x + bu_t + c(x,t)u = f(x,t),$$

where $a$ and $b$ were still constant, which I gave to you as a practice problem. By restricting $u$ to the curves $(X,T)$, the idea was to reduce the PDE above to a first-order linear ODE — a slightly more complicated first-order linear problem than $w' = 0$, but nonetheless one that could be solved by an integrating factor.

### *1.6.2. Characteristics.*

How could we do this more generally for a variable-coefficient problem

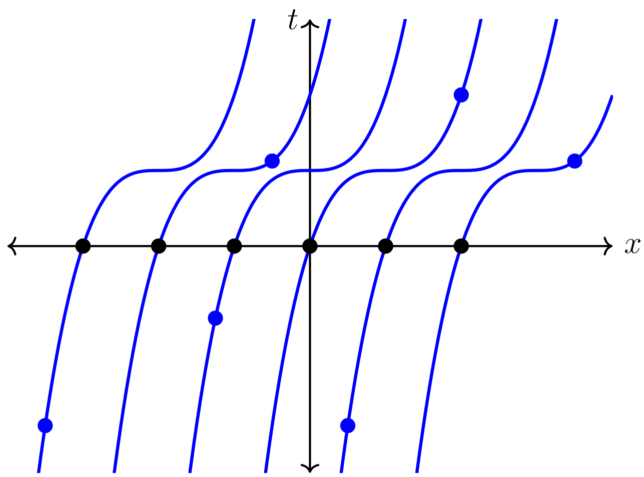
$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = f(x,t)? \tag{1.6.2}$$

As usual, we will assume, at the very least, that $a$, $b$, $c$, $f \in \mathcal{C}(\mathbb{R}^2)$. Based on our success with the transport equation, it would be great if we could find a family of parametric curves $(X,T)$ such that

**(C1)** For a given point $(x,t) \in \mathbb{R}^2$ there is a member of the family that passes through $(x,t)$;

**(C2)** When restricted to the image[11] of any $(X,T)$ in the family, the solution $u$ to (1.6.2) satisfies an ODE;

**(C3)** Each curve $(X,T)$ in the family intersects a "useful" (like the $x$-axis, but not necessarily the $x$-axis) curve exactly once, so that we could somehow learn more about $u$, perhaps in the context of a specific initial and/or side condition.




Above is a cartoon of a family of curves that seems to do all of the above if, in (C3), we want the family to intersect the $x$-axis only once. Fix your favorite blue dot in $\mathbb{R}^2$, and it

[11] The **IMAGE** of the map $(X,T)$ is the set of points $\{(X(h),T(h)) \mid h \in I\}$, where $(X,T)$ is defined on the set $I$.

looks like there is exactly one member of the family that passes through it. Then you can trace that curve back to a unique intersection with the $x$-axis.

Here is how we could achieve (C1), (C2), and (C3) for the problem (1.6.2). The condition (C1) is pretty easy: we just want

$$X(0) = x \quad \text{and} \quad T(0) = t.$$

The precise value of the parametric variable at which $(X, T)$ intersects $(x, t)$ really doesn't matter, but I think it's nice to visualize the image of $(X, T)$ as starting at $(x, t)$ and then meandering throughout the plane.

To see how we might achieve (C2), first note that the restriction of $u$ to the image of $(X, T)$ is the function

$$w(h) := u(X(h), T(h)).$$

The multivariable chain rule tells us that, if $X$ and $T$ are differentiable,

$$w'(h) = u_x(X(h), T(h))X'(h) + u_t(X(h), T(h))T'(h).$$

This looks almost like the first two terms in (1.6.2) and will look exactly like them if $X$ and $T$ satisfy

$$X'(h) = a(X(h), T(h)) \quad \text{and} \quad T'(h) = b(X(h), T(h)). \tag{1.6.3}$$

Indeed, if $u$ solves (1.6.2) and the pair $(X, T)$ solves (1.6.3), then

$$\begin{aligned} w'(h) &= u_x(X(h), T(h))a(X(h), T(h)) + u_t(X(h), T(h))b(X(t), T(h)) \\ &= f(X(h), T(h)) - c(X(h), T(h))u(X(h), T(h)) \\ &= f(X(h), T(h)) - c(X(h), T(h))w(h). \end{aligned}$$

If we put

$$\alpha(h) := c(X(h), T(h)) \quad \text{and} \quad \beta(h) := f(X(h), T(h)),$$

then $w$ solves the ODE

$$w' + \alpha(h)w = \beta(h), \tag{1.6.4}$$

which we can treat with an integrating factor. In principle, then, we could learn more about $w$ and exploit the identity $u(x, t) = w(0)$ to learn more about $u$.

And so to achieve (C1) and (C2), we really want $X$ and $T$ to solve the *system* of IVP

$$\begin{cases} X' = a(X, T) \\ T' = b(X, t) \end{cases} \quad \text{with} \quad \begin{cases} X(0) = x \\ T(0) = t. \end{cases} \tag{1.6.5}$$

By the way, this is exactly what worked with the constant-coefficient transport problem. Take a look at $(X, T)$ defined via (1.6.1) and convince yourself that

$$\begin{cases} X' = a \\ T' = b, \end{cases} \quad \text{with} \quad \begin{cases} X(0) = x_0 \\ T(0) = t_0. \end{cases}$$

Solving a system of IVP like (1.6.5) is both facile and tough. On one hand, if $a$ and $b$ are continuously differentiable on $\mathbb{R}^2$ (which asks a little more than our initial hypothesis of continuity), then ODE theory assures us that a unique solution always exists. On the other, even for very banal choices of $a$ and $b$, it may be impossible to construct formulas for the solutions. Thus in our examples and exercises, we will limit ourselves to very carefully chosen toy problems.

Last, we have not said anything about (C3). This depends greatly on the problem at hand, and we may sometimes fail here. However, our failures will always be enlightening, once we get over them.

Before proceeding, we name this family of curves associated with the problem (1.6.2).

**1.6.1 Definition.**

*A family of curves* $(X, T)$ *is a* **CHARACTERISTIC SYSTEM** *for the PDE* (1.6.2) *if, for each* $(x, t) \in \mathbb{R}^2$, *there is a pair* $(X, T)$ *in the family such that*

$$\begin{cases} X'(h) = a(X(h), T(h)) \\ T'(h) = b(X(h), T(h)) \end{cases} \quad \text{with} \quad \begin{cases} X(0) = x \\ T(0) = t. \end{cases}$$

So here is what we will do to solve (1.6.2). We will first construct the characteristic system $(X, T)$. In practice, this will require $a$ and $b$ to be fairly special functions. Then we will restrict the solution $u$ to the image of a curve $(X, T)$ in this system by putting

$$w(h) = u(X(h), T(h)),$$

so that $w$ will satisfy the ODE (1.6.4). We can solve this ODE to learn more about $w$. For any point $(x, t) \in \mathbb{R}^2$, we can then choose the characteristic curve $(X, T)$ with $X(0) = x$ and $T(0) = t$ to see that

$$u(x, t) = u(X(h), T(h)) = w(h)$$

for all $h$. Hopefully, then, the precise form of $w$ (obtained, in principle, via integrating factors), will suggest a nice value of $h$ for us to choose. Alternatively, or additionally, the underlying geometric of the characteristic curves (which is something that we could learn by plotting them) may suggest additional properties of the solution $u$.

*1.6.3. Examples (of varying difficulty and with varying success).*

**1.6.2 Example.**

*What can you say about the structure of solutions* $u = u(x, t)$ *to*

$$xu_x + u_t = 0?$$

**Solution.** Here $a(x, t) = x$ and $b(x, t) = 1$. We expect that any solution $u$ will be constant on the image of a parametric curve $(X, T)$ satisfying

$$\begin{cases} X' = X \\ T' = 1 \end{cases} \quad \text{with} \quad \begin{cases} X(0) = x \\ T(0) = t \end{cases}$$

for $(x,t) \in \mathbb{R}^2$ arbitrary. The general solution to the system of ODE is

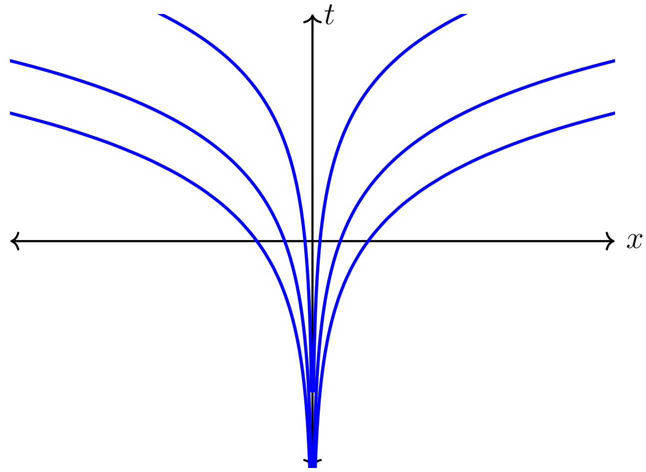
$$X(h) = c_1 e^h \quad \text{and} \quad T(h) = h + c_2$$

for arbitrary constants $c_1$ and $c_2$. Incorporating the initial conditions, we need $x = X(0) = c_1$ and $t = c_2$. Thus the characteristics are

$$X(h) = xe^h \quad \text{and} \quad T(h) = h + t.$$

This is where we finished on Monday, September 12, 2022.

Here are some plots of these characteristics.




Put $w(h) = u(X(h), T(h))$ to see that $w' = 0$ (I'll let you check that). And so

$$u(x,t) = u(X(0), T(0)) = w(0) = w(h) = u(xe^h, h + t)$$

for any $(x,t) \in \mathbb{R}^2$ and $h \in \mathbb{R}$. Motivated by our prior success with the transport equation and friends, we might choose $h = -t$ to eliminate the dependence of $u$ on its second coordinate. We find that

$$u(x,t) = u(xe^{-t}, 0)$$

for any pair $(x,t) \in \mathbb{R}^2$.

Remember that, as usual, we are working backward. We assumed that we had a solution $u = u(x,t)$ to $xu_x + u_t = 0$, and we concluded that such a solution must satisfy $u(x,t) = u(xe^{-t}, 0)$. If we put $p(\xi) := u(\xi, 0)$, then $p \in \mathcal{C}^1(\mathbb{R})$ and $u$ satisfies $u(x,t) = p(xe^{-t})$ for all $(x,t)$. Conversely, it is easy to check that if $p \in \mathcal{C}^1(\mathbb{R})$ is arbitrary, then defining $u(x,t) := p(xe^{-t})$ gives a solution to $xu_x + u_t = 0$. And so all solutions to $xu_x + u_t = 0$ have the form $u(x,t) = p(xe^{-t})$ for some $p \in \mathcal{C}^1(\mathbb{R})$. ▲

This is where we finished on Wednesday, September 14, 2022.

Here is an example where the characteristic equations are more complicated. The overall behavior of the solution is also more subtle and requires some non-obvious geometric insight.

**1.6.3 Example.**

*What can you say about solutions $u = u(x,t)$ to*

$$tu_x - xu_t = 0?$$

*Assume $x > 0$.*

**Solution.** Here $a(x,t) = t$ and $b(x,t) = -x$. I'm going to enumerate the different steps of this solution, so you know it's going to be a chore.

**1.** Fix a point $(x,t) \in \mathbb{R}^2$. The characteristics of this PDE are the solutions of the IVP

$$\begin{cases} X' = T \\ T' = -X \end{cases} \qquad \text{with} \qquad \begin{cases} X(0) = x \\ T(0) = t. \end{cases}$$

If you haven't done much with systems of ODE/IVP, then I admit that this can look intimidating; the system in Example 1.6.2 was *decoupled*, and the $X$-equation did not involve $T$, nor did the $T$-equation involve $X$. Here is the quick and dirty way[12] to solve a problem of this special form.

As usual, we work backward: if we have solutions $(X,T)$ such that $X' = T$ and $T' = -X$, then $T$ is differentiable, and so $X'$ is differentiable. We calculate

$$X'' = (X')' = T' = X,$$

and so $X'' + X = 0$. Thus

$$X(h) = c_1 \cos(h) + c_2 \sin(h)$$

for some constants $c_1$ and $c_2$. Then

$$T(h) = X'((h) = -c_1 \sin(h) + c_2 \cos(h).$$

Now we impose the initial conditions:

$$x = X(0) = c_1 \qquad \text{and} \qquad t = T(0) = c_2.$$

Thus the characteristics are

$$X(h) = x\cos(h) + t\sin(h) \qquad \text{and} \qquad T(h) = -x\sin(h) + t\cos(h). \tag{1.6.6}$$

[12] Here is a slightly more general, and maybe less quick, approach. The ODE problem here is an example of the more general linear system

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2. \end{cases}$$

If you didn't see this in an ODE class (possibly because you took my ODE class and I didn't teach it to you!), trust me that problems like this are quite worth studying. If $a_{12} = 0$, use direct integration to solve for $y_1$. Then substitute $y_1$ into the second equation, and use an integrating factor. If $a_{12} \neq 0$, differentiate both sides of the first equation to get $y_1'' = a_{11}y_1' + a_{12}y_2'$. Substitute $y_2' = a_{21}y_1 + a_{22}y_2 = a_{21}y_1 + a_{22}(y_1' - a_{11}y_1)$ to get a second-order linear equation for $y_1$ alone. Solve that, and then use $y_2 = (y_1' - a_{11}y_1)/a_{12}$.

**2.** By design, $u$ is constant on the image of $(X, T)$, and so

$$u(x, t) = u(x\cos(h) + t\sin(h), -x\sin(h) + t\cos(h))$$

for any choice of the numbers $x$, $t$, and $h$. Big deal. *What does this mean? What does this tell us about $u$?*

Previously, in Example 1.6.2 and in the long haul with the transport equation, we got very lucky: there was an obvious choice for $h$ that simplified the expression for $u$. I don't see any such obvious choice here. Maybe we could try to pick $h$ such that

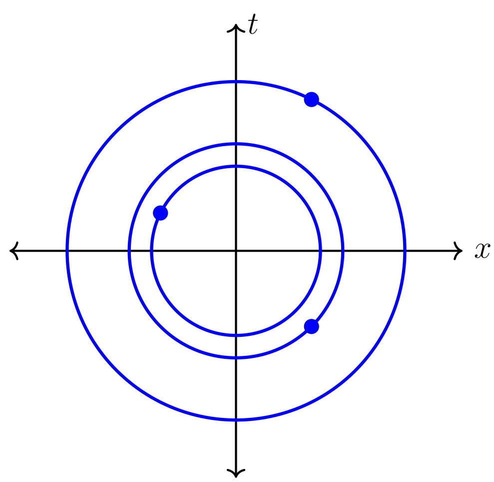
$$-x\sin(h) + t\cos(h) = 0,$$

which is the same as saying

$$\tan(h) = \frac{x}{t},$$

but that becomes problematic quickly when we try to think about the inverse tangent (and maybe dividing by $t = 0$), right?

**3.** We can learn more if we think *geometrically*, not *algebraically*. If we are so lucky as to find explicit formulas for the characteristic curves, it's worthwhile figuring out what parametric curve the characteristics actually trace out. I say we pick some values for $x$ and $t$ and stuff (1.6.6) into our favorite graphing utility. We only need to run the parameter $h$ in an interval of length $2\pi$, since $X$ and $T$ are both $2\pi$-periodic. In the pictures below, the blue dot is the "initial" point $(X(0), T(0))$ for each choice of $(X, T)$.




The pictures tell us that the set of all points $\{(X(h), T(h)) \mid h \in \mathbb{R}\}$ is just the circle of radius $x^2 + t^2$ centered at the origin. You can prove this more precisely from the formulas for $X$ and $T$.

And so $u$ is constant on any circle centered at the origin. What does this tell us? A circle centered at the origin is just the set of all points that are the same distance from $(0, 0)$. The distance between the points $(0, 0)$ and $(x, t)$ is $\sqrt{x^2 + t^2}$. And so we might think that $u(x, t)$ depends only on $\sqrt{x^2 + t^2}$, not on the particular coordinates $x$ and $t$. Then perhaps we could write

$$u(x, t) = p(\sqrt{x^2 + t^2})$$

for some nice function $p$.

**4.** Let's do all this more precisely without any of the intuitive digressions above. Suppose that $u$ solves $tu_x - xu_t = 0$. Then

$$u(x,t) = u(x\cos(h) + t\sin(h), -x\sin(h) + t\cos(h))$$

for all $(x,t) \in \mathbb{R}^2$ and $h \in \mathbb{R}$. Write the point $(x\cos(h) + t\sin(h), -x\sin(h) + t\cos(h))$ in polar coordinates[13]: there is $\Theta(h) \in \mathbb{R}$ such that

$$x\cos(h) + t\sin(h) = \sqrt{x^2+t^2}\cos(\Theta(h)) \quad \text{and} \quad -x\sin(h) + t\cos(h) = \sqrt{x^2+t^2}\sin(\Theta(h)).$$

Then

$$u(x,t) = u(\sqrt{x^2+t^2}\cos(\Theta(h)), \sqrt{x^2+t^2}\sin(\Theta(h)))$$

for all $(x,t) \in \mathbb{R}^2$ and $h \in \mathbb{R}$. Define

$$p(\xi) := u(\xi\cos(\Theta(0)), \xi\sin(\Theta(0))),$$

so $p \in \mathcal{C}^1(\mathbb{R})$. We have taken $h = 0$ purely for notational convenience. We conclude that

$$u(x,t) = u(\sqrt{x^2+t^2}\cos(\Theta(0)), \sqrt{x^2+t^2}\sin(\Theta(0))) = p(\sqrt{x^2+t^2}).$$

We can clean this up slightly to eliminate the square root by setting $q(\xi) := p(\sqrt{\xi})$, so $u(x,t) = q(x^2+t^2)$. However, we no longer have $q \in \mathcal{C}^1(\mathbb{R})$ but instead $q \in \mathcal{C}^1((0,\infty))$. Because we are working with $x > 0$, we don't have to worry about differentiability at the origin. (This is messy, technical, and interesting.) ▲

**1.6.4 Example.**

*Can you solve*

$$\begin{cases} tu_x - xu_t = 0, \ x > 0 \\ u(t^2,t) = t? \end{cases}$$

*Interpret your answer in terms of characteristics.*

**Solution.** We know that all solutions have the form $u(x,t) = p(\sqrt{x^2+t^2})$ for some $p \in \mathcal{C}^1(\mathbb{R})$. To meet the side condition, we want

$$t = u(t^2,t) = p(\sqrt{(t^2)^2+t^2}) = p(\sqrt{t^4+t^2}).$$

Does this help us find a formula for $p$? We might try to put $\xi = \sqrt{t^4+t^2}$ and then solve for $t$ in terms of $\xi$; we did something like this back in Example 1.5.3. I claim that we will fail, at least if we want to keep working with all $t$; in particular, the evenness of the map $t \mapsto \sqrt{t^4+t^2}$ should suggest a lack of one-to-oneness here.

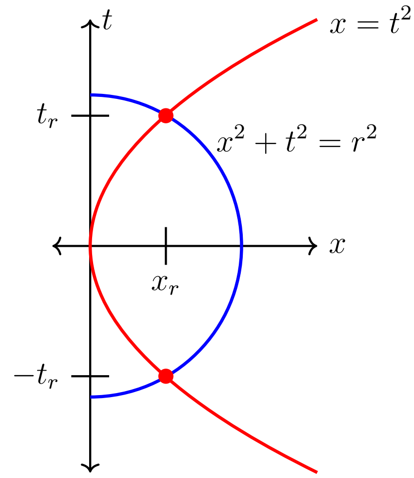
[13] Here's why I thought of polar coordinates. I wanted to relate $\sqrt{x^2+t^2}$ to the formula for $u$. I knew that $\sqrt{x^2+t^2}$ was the length of the line segment from $(0,0)$ to $(x\cos(h) + t\sin(h), -x\sin(h) + t\cos(h))$ for any $h$. And I knew that I was working with circles. Then I sat with these things for about half an hour and read the textbook, and suddenly polar coordinates popped into my head. I wish I could be more systematic about how I got the idea — maybe read the textbook?

More precisely, if we are able to solve this problem, then $p$ must satisfy

$$1 = p(\sqrt{1^4 + 1^2}) = p(\sqrt{2}) \quad \text{and} \quad -1 = p(\sqrt{(-1)^4 + (-1)^2}) = p(\sqrt{2}),$$

which is impossible. And so, no, we can't solve this problem.

Here is the snarl in terms of characteristics. The characteristics are the circles $x^2+t^2 = r^2$, where $r > 0$. The solution $u$ must be constant on these circles. But we are also requiring that the value of $u$ at a point $(x,t)$ on the curve $\{(t^2,t) \mid t \in \mathbb{R}\}$ be the second coordinate of that point.




This curve intersects every characteristic $x^2 + t^2 = r^2$ at two distinct points $(x_r, t_r^+)$ and $(x_r, t_r^-)$, where $t_r^+ \neq t_r^-$. So $u$ must take two different values on the same characteristic. ▲

This is where we finished on Monday, September 19, 2022.

**1.6.5 Example.**

*Consider an infinitely long one-lane road that starts at an initial point $x = 0$ and continues forever to right. Suppose that at point $x \geq 0$ along the road and time $t = 0$, the traffic density is $g(x)$. There is $M > 0$ such that $0 \leq g(x) \leq M$ for all $x$. At any point $x \geq 0$ along the road and time $t \geq 0$, traffic moves to the right with a constant velocity of $v(x,t) = x$ miles per hour. Finally, at any point $x \geq 0$ along the road and time $t \geq 0$, one car per mile per hour enters the road, and no cars exit the road. What happens over long times to the traffic density at any point $x \geq 0$?*

**Solution.** We should try to translate all of this (very contrived) data into the model equation (1.5.22), which says that if (1) traffic moves with velocity $v$, (2) there is no decrease in traffic density that is directly proportional to current density, and (3) a net $f(x,t)$ cars per mile per hour enter/exit the road, then density satisfies

$$\rho_t + (v\rho)_x = f(x,t).$$

The velocity is $v(x,t) = x$, while the generation/dissipation term is $f(x,t) = 1$. Thus density satisfies

$$\rho_t + (x\rho)_x = 1,$$

which is to say,

$$x\rho_x + \rho_t + \rho = 1. \tag{1.6.7}$$

We already found the characteristics for this problem in Example 1.6.2; they are

$$X(h) = xe^h \quad \text{and} \quad T(h) = h + t.$$

We now consider the ODE that a solution $\rho$ must satisfy when restricted to the image of $(X,T)$. Put $w(h) = \rho(xe^h, h+t)$ to find that[14]

$$1 = (xe^h)\rho_x(xe^h, h+t) + \rho_t(xe^h, h+t) + \rho(xe^h, h+t) = \frac{d}{dh}[\rho(xe^h, h+t)] + \rho(xe^h, h+t) = w'(h) + w(h).$$

Multiply the ODE

$$w' + w = 1$$

by the integrating factor $\mu(h) = e^h$ to find

$$\frac{d}{dh}[e^h w(h)] = e^h,$$

and integrate[15] from 0 to $h$ to obtain

$$e^h w(h) - w(0) = e^h - 1.$$

We solve for $w(0) = \rho(x,t)$:

$$\rho(x,t) = w(0) = e^h w(h) - e^h + 1 = e^h \rho(xe^h, h+t) - e^h + 1.$$

Vast prior experience suggests taking $h = -t$, and that is particularly useful here, since it will allow us to use the initial condition:

$$\rho(x,t) = e^{-t}\rho(xe^{-t}, 0) - e^{-t} + 1 = e^{-t}g(xe^{-t}) - e^{-t} + 1.$$

To determine the long-time behavior of the density, we want to calculate $\lim_{t\to\infty} \rho(x,t)$ for $x$ fixed. Since $g$ is bounded with $0 \le g(x) \le M$ for all $x$, we have

$$0 \le \lim_{t\to\infty} e^{-t} g(xe^{-t}) \le M \lim_{t\to\infty} e^{-t} = 0,$$

[14] If the variable-chasing in the first equality is giving you grief, observe that our original problem (1.6.7) is the same as

$$\xi\rho_x(\xi,\tau) + \rho_t(\xi,\tau) + \rho(\xi,\tau) = 1$$

for all $\xi$ and $\tau$. The first equality below then follows by taking $\xi = xe^h$ and $\tau = h + t$. The subscripts of $x$ and $t$ on the partial derivatives $\rho_x$ and $\rho_t$ are purely symbolic.

[15] I am writing this in words rather than symbols. If I were doing it in symbols, to avoid overworking $h$, I might say

$$\frac{d}{d\eta}[e^\eta w(\eta)] = e^\eta \Longrightarrow \int_0^h \frac{d}{d\eta}[e^\eta w(\eta)]\, d\eta = \int_0^h e^\eta\, d\eta \Longrightarrow e^h w(h) - w(0) = e^h - 1.$$

and so

$$\lim_{t\to\infty} e^{-t}g(xe^{-t}) = 0.$$

Thus

$$\lim_{t\to\infty} \rho(x,t) = \lim_{t\to\infty} [e^{-t}g(xe^{-t}) - e^{-t} + 1] = 0 - 0 + 1 = 1.$$

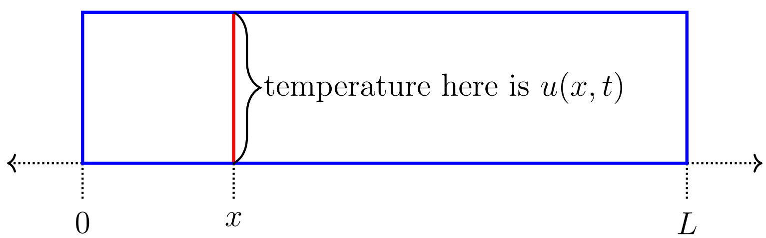
And so, over very long times, the density boils down to one lonely car per mile. The "dispersion" of density due to increased velocity as position along the road increases (if Car 1 is behind Car 2, then both Car 1 and Car 2 are speeding up, but Car 2 is speeding up faster) balances perfectly with the influx of cars. ▲

# 2. The Heat Equation

## 2.1. Physical perspectives.

The heat equation will be our first second-order PDE that is not really an ODE in disguise. Our techniques for analyzing the heat equation will be considerably different from those that we used for the transport equation and its ilk. They will lead to many of the important questions that have driven modern real and complex analysis. We'll find lots of formulas for solutions, but we'll also pose and answer those important questions (although we won't exactly *solve* all of them).

Many of these important questions arise from some fairly simple physical sources. Consider a rod that is $L$ units long and placed along a horizontal axis like the following sketch. The rod is sufficiently uniform that all vertical cross-sections have the same temperature. That is, at a given point $x$ with $0 \leq x \leq L$ and time $t$, the temperature anywhere in the rod along the vertical cross-section at $x$ is the same value, which we call $u(x,t)$.




It can be shown (and our textbook does a very good job of this) that if the sides of the rod (except possibly the sides at $x = 0$ and $x = L$) are insulated, so no heat energy passes through those sides, then the rod's temperature obeys the **heat equation**

$$u_t = u_{xx}.$$

The book allows for a more general heat equation of the form

$$u_t = ku_{xx}.$$

I claim that you can, and should, rescale this latter heat equation into the first version. For simplicity, we will work mostly with $k = 1$.

We will obtain our first solutions to the heat equation via a new technique that reduces the problem to, big surprise, an issue of ODE. Before that, however, we will look at some important auxiliary data that naturally accompanies the heat equation.

This is where we finished on Wednesday, September 21, 2022.

One piece of data, which I will mention only briefly now, is the **initial temperature distribution**. Assume that we know the temperature of the whole rod at the start: when $t = 0$, we have

$$u(x,0) = f(x), \ 0 \leq x \leq L,$$

for some function $f$ defined on $[0, L]$. The precise form of $f$ will motivate much of our later analysis.

More varied are the possible **BOUNDARY CONDITIONS** for the ends of the rod at $x = 0$ and $x = L$. Remember that the surface of the rod is insulated *except possibly at the ends*, so no heat energy enters or exits the rod *except possibly at the ends*. Here are three non-exhaustive possibilities.

**1.** *The ends are maintained at some (possible time-dependent) temperatures.* There are functions $a$ and $b$ defined on $[0, \infty)$ such that

$$u(0, t) = a(t),\ t \geq 0, \quad \text{and} \quad u(L, t) = b(t),\ t \geq 0.$$

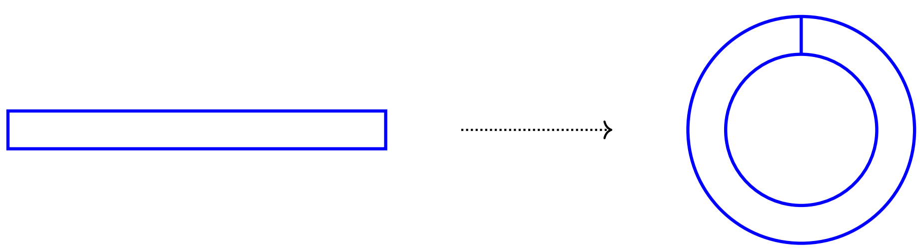
For example, on the left end you could be holding a candle against the rod, and on the right I'm holding a baby dragon. We will particularly work with the case of constant, identical temperatures: $a(t) = b(t) = b(0)$ for all $t$. In that case, we might as well declare the constant temperature at the ends to be the zero temperature (i.e., whatever zero degrees are in our units of temperature, zero is $b(0)$) and assume

$$u(0, t) = u(L, t) = 0,\ t \geq 0.$$

**2.** *The ends are also insulated.* Insulation means that heat energy cannot transfer through the ends. Without a transfer of heat energy, the temperature cannot change. As soon as you hear the words "cannot" and "change" in the same sentence, you should think of a zero derivative. For insulated ends, then, we put[16]

$$u_x(0, t) = u_x(L, t) = 0.$$

**3.** *The rod is really a flexible wire, and we bend it into a circle.* If we bend just right, then the parts of the wire that were formerly at $x = 0$ and $x = L$ are now touching.



In my drawing above, the vertical blue line in the concentric circles indicates where the $x = 0$ and $x = L$ endpoints are now touching. We can still identify points around this bent wire with their counterparts in the interval $[0, L]$ for the unbent wire, and so the heat equation still governs the wire's temperature. However, because the physical behavior of the wire at the "touching" point should be the same whether you approach that point from the left or the right, we impose the **PERIODIC BOUNDARY CONDITIONS**

$$u(0, t) = u(L, t),\ t \geq 0, \quad \text{and} \quad u_x(0, t) = u_x(L, t),\ t \geq 0. \tag{2.1.1}$$

[16] See p. 157 in the book for a better physical motivation of this choice than I'm giving here.

When we impose both initial and boundary conditions, we get an initial value problem-boundary value problem, which I will abbreviate as an IVP-BVP. For example, the problem

$$\begin{cases} u_t = u_{xx}, \ 0 \le x \le L, \ t \ge 0 \\ u(x,0) = f(x) \\ u(0,t) = u(L,t) = 0 \end{cases}$$

describes the behavior of the temperature of a rod whose initial temperature distribution is $f$ and whose ends are maintained at the constant temperature 0. There are lots of ways to mix and match the boundary conditions — say, insulate the left end but maintain the right end at a constant temperature — and our goal will be less to understand all of them and more to see how a select few choices of boundary conditions lead to interesting mathematics.

## 2.2. Separation of variables.

You are familiar with the notion of making an **ANSATZ** for a differential equation: a (more or less) educated guess about the solution's form. When we learned ODE, we made the "exponential" ansatz $y(x) = e^{rx}$ for the constant-coefficient second-order linear homogeneous problem $ay'' + by' + cy = 0$. There $r$ was a fixed real number, and we substituted this ansatz into the ODE and tried to learn more about $r$; we found that $r$ had to satisfy the quadratic equation $ar^2 + br + c = 0$. In PDE (in no small part due to my own interests), we've made the traveling wave ansatz $u(x,t) = p(x - ct)$ on a PDE for $u = u(x,t)$, and this results in an ODE that $p$ satisfies with $c$ as a parameter.

Here we will make a different kind of ansatz for the heat equation; this ansatz will work for many other PDE, and we'll return to it often. Suppose that a solution $u = u(x,t)$ to $u_t = u_{xx}$ has the form

$$u(x,t) = X(x)T(t) \tag{2.2.1}$$

for some functions $X = X(x)$ and $T = T(t)$. Since we have posed the heat equation for a rod, we are assuming $0 \le x \le L$ and $t \ge 0$. What can we learn about $X$ and $T$?

The ansatz (2.2.1) is called a **PRODUCT ANSATZ** for $u$. The rationale behind this ansatz (which goes back centuries, by the way) is that (2.2.1) is a transparent way of "sorting out" the dependence of $u$ on both of its variables. With experience, the convenience of associating the function $X$ with the independent variable $x$ tends to outweigh the annoyance of writing both lowercase $x$ and uppercase $X$ in the same problem. After some work with our heat equation in one spatial dimension, we'll see how to adapt product ansatzes to PDE with more variables. The process of solving a PDE via a product ansatz is sometimes called **SEPARATION OF VARIABLES**, not to be confused with the ODE technique.

### *2.2.1. The product ansatz for the heat equation.*

Again, we assume that the solution $u = u(x,t)$ to $u_t = u_{xx}$ has the form $u(x,t) = X(x)T(t)$. Then

$$u_t(x,t) = X(x)T'(t), \qquad u_x(x,t) = X'(x)T(t), \qquad \text{and} \qquad u_{xx}(x,t) = X''(x)T(t).$$

Hence $X$ and $T$ must satisfy

$$X(x)T'(t) = X''(x)T(t).$$

Here is a key, and maybe not obvious, idea. For any pair $(x, t)$ at which $X(x)T(t) \neq 0$, let's divide to find

$$\frac{X(x)T'(t)}{X(x)T(t)} = \frac{X''(x)T(t)}{X(x)T(t)},$$

and thus

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}. \tag{2.2.2}$$

We have now "separated" the variables from each other onto either side of the equals sign.

What gives us the right to assume $X(x)T(t) \neq 0$? Nothing, at first, but when one makes an ansatz one enters the land of wishful thinking — we can assume any restrictions we want on the components of the ansatz (e.g., in a traveling wave ansatz, we might want to assume $u(x, t) = p(x - ct)$ with $c > 1$ and $p$ an even function). That doesn't guarantee that the ansatz will be successful, or useful, but it does limit the scope of our subsequent search.

Here is the second key idea. The equality (2.2.2) has to hold for all $x$ and $t$ (specifically, all $0 \leq x \leq L$ and all $t \geq 0$). Let's freeze $x = 0$ and let $t \geq 0$ be arbitrary. Then

$$\frac{T'(t)}{T(t)} = \frac{X''(0)}{X(0)}.$$

Abbreviate

$$c := \frac{X''(0)}{X(0)}.$$

Then $T$ satisfies

$$\frac{T'(t)}{T(t)} = c$$

for all $t$, and so $T$ really solves the familiar ODE

$$T' = cT.$$

Thus there is a constant $a$ such that

$$T(t) = ae^{ct}$$

for all $t$.

Returning to (2.2.2), we now freeze $t = 0$ to find

$$\frac{X''(x)}{X(x)} = \frac{T'(0)}{T(0)} = c$$

for all $x$. Then $X$ must solve

$$X'' = cX,$$

which we usually package as

$$X'' - cX = 0.$$

Before proceeding, we note that the parameter $c$ in the ODE for $X$ is the same as the parameter $c$ in the ODE for $T$; the two ODE are not wholly independent of each other.

Experience suggests that the sign of $c$ determines the form of solutions to $X'' - cX = 0$. Here are the three key cases.

**1.** $c > 0$. Then $X$ really solves $X'' - (\sqrt{c})^2 X = 0$, and the solutions to this ODE all have the form

$$X(x) = b_1 e^{\sqrt{c}x} + b_2 e^{-\sqrt{c}x}$$

for some constants $b_1$ and $b_2$.

**2.** $c = 0$. Then we have the direct integration problem $X'' = 0$, and the solutions are

$$X(x) = b_1 x + b_2.$$

**3.** $c < 0$. Then $-c = |c|$, and so $X$ solves $X'' + |c|X = 0$, i.e., $X'' + (\sqrt{|c|})^2 X = 0$. The solutions are

$$X(x) = b_1 \cos(\sqrt{|c|}x) + b_2 \sin(\sqrt{|c|}x).$$

We put everything together to find three families of product solutions:

$$u(x,t) = X(x)T(t) = \begin{cases} ae^{ct}\big[b_1 e^{\sqrt{c}x} + b_2 e^{-\sqrt{c}x}\big], \ c > 0 \\ b_1 x + b_2 \\ ae^{-ct}\big[b_1 \cos(\sqrt{|c|}x) + b_2 \sin(\sqrt{|c|}x)\big], \ c < 0. \end{cases}$$

This is correct, but it could be a lot better. In particular, it looks like there are up to four free constants in the formula for $u$: $a$, $b_1$, $b_2$, and $c$. This is not exactly true.

First, keeping track of $\sqrt{c}$ and $\sqrt{|c|}$ is annoying. A better idea, which has evolved over the centuries along with the general notion of separation of variables in PDE, is to rewrite $c$ in the following forms. Any real number $c > 0$ can be written as $c = \lambda^2$, where $\lambda > 0$, while any real number $c < 0$ can be written as $c = -\lambda^2$, where for convenience we demand $\lambda > 0$. This eliminates the square roots and absolute values and gets us

$$u(x,t) = \begin{cases} ae^{\lambda^2 t}\big[b_1 e^{\lambda x} + b_2 e^{-\lambda x}\big], \ \lambda > 0 \\ b_1 x + b_2 \\ ae^{-\lambda^2 t}\big[b_1 \cos(\lambda x) + b_2 \sin(\lambda x)\big], \ \lambda > 0. \end{cases}$$

This is cleaner, but it can still be better. One thing for the future — we will assume without making a big fuss about it that, if necessary, arbitrary real constants $c$ can be expressed in the form $c = \lambda^2$, $c = 0$, and $c = -\lambda^2$ to cover all possibilities.

Here is how we can make things even better. Look at the case $\lambda > 0$ and distribute $a$:

$$ae^{\lambda^2 t}\big[b_1 e^{\lambda x} + b_2 e^{-\lambda x}\big] = e^{\lambda^2 t}[ab_1 e^{\lambda x} + ab_2 e^{-\lambda x}\big].$$

Since $a$, $b_1$, and $b_2$ can be arbitrary, the products $ab_1$ and $ab_2$ can also be arbitrary. (Just like when you combine constants of integration when you're collecting/adding antiderivatives.) We'll just write $c_1 = ab_1$ and $c_2 = ab_2$. (I hope it's okay to reintroduce the letter $c$ here.)

Thus the cleanest way to write the product solutions are as follows.

**2.2.1 Theorem (Product solutions for the heat equation).**

*For given real numbers $\lambda > 0$ and arbitrary $c_1$, $c_2$, defining $u = u(x,t)$ in the following ways yields a solution to the heat equation $u_t = u_{xx}$:*

$$u(x,t) = \begin{cases} e^{\lambda^2 t}\big[c_1 e^{\lambda x} + c_2 e^{-\lambda x}\big], \ \lambda > 0 \\ c_1 x + c_2, \\ e^{-\lambda^2 t}\big[c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\big], \ \lambda > 0. \end{cases}$$

This is where we finished on Friday, September 23, 2022.

*2.2.2. The general method of separation of variables.*

If you want to find product solutions to a PDE involving a function of $n$ independent variables, express the solution as a product of $n$ functions, where each function is a function of exactly one of those independent variables. Substitute this ansatz into the PDE, divide through by the original product, and obtain a family of ODE that the various factor functions solve. This is best illustrated with a PDE that depends on more than two variables.

**2.2.2 Example.**

*The* **TWO-DIMENSIONAL HEAT EQUATION** *reads*

$$u_t = u_{xx} + u_{yy},$$

*where $u = u(x,y,t)$. Make a product ansatz*

$$u(x,y,t) = X(x)Y(t)T(t).$$

*What ODE do $X$, $Y$, and $T$ solve?*

**Solution.** With $u(x,y,t) = X(x)Y(y)T(t)$, we calculate

$$u_t(x,y,t) = X(x)Y(y)T'(t), \qquad u_{xx}(x,y,t) = X''(x)Y(y)T(t), \\ \text{and} \qquad u_{yy}(x,y,t) = X(x)Y''(y)T(t).$$

So, we need

$$XYT' = X''YT + XY''T.$$

We divide both sides by the original product $XYT$ to find

$$\frac{XYT'}{XYT} = \frac{X''YT}{XYT} + \frac{XY''T}{XYT},$$

and thus

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}.$$

This equality has to hold for all $x$, $y$, and $t$ in the respective domains of $X$, $Y$, and $T$. Let's fix two of the three variables at a time.

First, fix $x_0$ and $y_0$ and let $t$ be arbitrary. Then

$$\frac{T'(t)}{T(t)} = \frac{X''(x_0)}{X(x_0)} + \frac{Y''(y_0)}{Y(y_0)} =: c. \tag{2.2.3}$$

So, $T$ must satisfy

$$T' = cT.$$

Next, fix $t_0$ and $y_0$ and let $x$ be arbitrary. Then

$$\frac{X''(x)}{X(x)} = \frac{T'(t_0)}{T(t_0)} - \frac{Y''(y_0)}{Y(y_0)} =: d. \tag{2.2.4}$$

Then $X$ must satisfy

$$X'' = dX.$$

Finally, fix $t_0$ and $x_0$ and let $y$ be arbitrary. Then

$$\frac{T'(t_0)}{T(t_0)} = \frac{X''(x_0)}{X(x_0)} + \frac{Y''(y)}{Y(y)},$$

and so $Y$ satisfies

$$\frac{Y''(y)}{Y(y)} = \frac{T'(t_0)}{T(t_0)} - \frac{X''(x_0)}{X(x_0)}.$$

We should try to relate the fixed quantity on the right to $c$ and $d$. By (2.2.3), we have

$$\frac{T'(t_0)}{T(t_0)} = c,$$

and by (2.2.4), we have

$$\frac{X''(x_0)}{X(x_0)} = d.$$

Thus $Y$ must satisfy

$$\frac{Y''(y)}{Y(y)} = c - d.$$

All together, $X$, $Y$, and $T$ must solve

$$\begin{cases} X'' - dX = 0 \\ Y'' - (c-d)Y = 0 \\ T' = cT. \end{cases} \tag{2.2.5}$$

This is a linear system of decoupled ODE, but the equations are coupled, in a certain sense, via the parameters $c$ and $d$.

The solutions to (2.2.5) will depend on the signs of $d$ and $c + d$. Let's consider just one case: $d > 0$ and $c - d > 0$. Write $d = \lambda^2$ with $\lambda > 0$ and $c - d = \mu^2$, with $\mu > 0$. Note that, then, $c = \mu^2 + d = \mu^2 + \lambda^2$. Then (2.2.5) reads

$$\begin{cases} X'' - \lambda^2 X = 0 \\ Y'' - \mu^2 Y = 0 \\ T' = (\mu^2 + \lambda^2)T, \end{cases}$$

and so solutions are

$$X(x) = a_1 e^{\lambda x} + a_2 e^{-\lambda x}, \qquad Y(y) = a_3 e^{\mu y} + a_4 e^{-\mu y}, \qquad \text{and} \qquad T(t) = a_5 e^{(\mu^2+\lambda^2)t}.$$

Thus one kind of product solution to the 2D heat equation is

$$u(x, y, t) = X(x)Y(y)T(t) = \left[a_1 e^{\lambda x} + a_2 e^{-\lambda x}\right]\left[a_3 e^{\mu y} + a_4 e^{-\mu y}\right] a_5 e^{(\mu^2+\lambda^2)t}.$$

Here $a_1, \ldots, a_5$ are arbitrary; as before, we could multiply through by $a_5$ to reduce down to four arbitrary constants (along with $\mu$ and $\lambda$). Then the product solution has the slightly simpler form

$$u(x, y, t) = \left[b_1 e^{\lambda x} + b_2 e^{-\lambda x}\right]\left[b_3 e^{\mu y} + b_4 e^{-\mu y}\right] e^{(\mu^2+\lambda^2)t}.$$ ▲

### *2.2.3. The heat equation with constant end-temperature.*

The IVP-BVP

$$\begin{cases} u_t = u_{xx}, \ 0 \le x \le L, \ t \ge 0 \\ u(x, 0) = f(x) \\ u(0, t) = u(L, t) = 0 \end{cases} \tag{2.2.6}$$

models the temperature of a rod of length $L > 0$ with initial temperature distribution $f$ and boundaries maintained at the constant zero temperature. What product solutions can this problem have?

Theorem 2.2.1 gives all possible product solutions to the heat equation. For convenience, I'll repeat them here:

$$u(x, t) = \begin{cases} e^{\lambda^2 t}\left[c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right], \ \lambda > 0 \\ c_1 x + c_2, \\ e^{-\lambda^2 t}\left[c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\right], \ \lambda > 0. \end{cases} \tag{2.2.7}$$

One way of proceeding is to see what the initial temperature distribution demands. This is actually pretty restrictive. For example, if (2.2.6) has a product solution of the first form, where $\lambda > 0$, then we must have

$$f(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}.$$

In the other two cases, a similar restriction on $f$ results. So, one thing that we learn at once is that product solutions only work for very specific initial temperature distributions.

What effect do the boundary conditions $u(0, t) = u(L, t) = 0$ have? Can the free constants $c_1$ and $c_2$ be chosen to help us meet them? Let's check.

**1.** $u(x,t) = e^{\lambda^2 t}\big[c_1 e^{\lambda x} + c_2 e^{-\lambda x}\big]$. Then we need

$$0 = u(0,t) = e^{\lambda^2 t}(c_1 + c_2) \quad \text{and} \quad 0 = u(L,t) = e^{\lambda^2 t}\big[c_1 e^{\lambda L} + c_2 e^{-\lambda L}\big]$$

for all $t \geq 0$. Dividing through by $e^{\lambda t}$, this gives a linear system of equations for $c_1$ and $c_2$:

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0. \end{cases}$$

There are a lot of ways to solve this kind of problem, or to prove that no solution exists. (I like looking at matrix determinants if you know linear algebra.)

Here's one such way. The first equation forces $c_1 = -c_2$, and so the second equation becomes

$$c_2\big[e^{-\lambda L} - e^{\lambda L}\big] = 0.$$

If $c_2 = 0$, then $c_1 = 0$, and so $u = 0$. This is the **TRIVIAL SOLUTION** to the IVP-BVP, and it only works if $f = 0$.

If $c_2 \neq 0$, then we need

$$e^{-\lambda L} - e^{\lambda L} = 0.$$

There are lots of ways to proceed here, too, but my favorite is to turn this into requiring

$$e^{\lambda L} = e^{-\lambda L}.$$

The exponential is strictly increasing, and therefore one-to-one, so $\lambda L = -\lambda L$. Since $\lambda \neq 0$, we have $L = -L$, and so $2L = 0$. But then $L = 0$. This is forbidden.

We conclude that the only possibility is to have $c_1 = c_2 = 0$ and $u = 0$. Thus only the trivial product solution works, and that only in the case $f = 0$.

This is where we finished on Monday, September 26, 2022.

**2.** $u(x,t) = c_1 x + c_2$. We need

$$0 = u(0,t) = c_2 \quad \text{and} \quad 0 = u(L,t) = c_1 L + c_2 = c_1 L.$$

Since $L \neq 0$, we must have $c_1 = 0$, and so, again, only the trivial product solution works.

**3.** $u(x,t) = e^{-\lambda^2 t}\big[c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\big]$. We need

$$0 = u(0,t) = e^{-\lambda^2 t} c_1 \quad \text{and} \quad 0 = u(L,t) = e^{-\lambda^2 t}\big[c_1 \cos(\lambda L) + c_2 \sin(\lambda L)\big].$$

The boundary condition at $x = 0$ forces $c_1 = 0$, and then the condition at $x = L$ requires

$$c_2 \sin(\lambda L) = 0.$$

If we take $c_2 = 0$, then, yet again, we arrive at the trivial solution. However, we could also require $\lambda$ to satisfy

$$\sin(\lambda L) = 0.$$

Recall that the zeros of the sine are the integer[17] multiples of $\pi$:

$$\sin(X) = 0 \iff X = k\pi, \quad \text{for an integer } k.$$

Thus if $\lambda L = k\pi$ for some integer $k$, then we can meet the boundary conditions without resorting to the trivial solution. That is, we take

$$\lambda = \frac{k\pi}{L},$$

and so the product solution has the form

$$u(x,t) = ce^{-(k\pi/L)^2 t} \sin\left(\frac{k\pi x}{L}\right).$$

Here $c$ can be any real number. Note that if we take $c = 0$, then we recover the trivial solution.

We can summarize the work above as follows.

**2.2.3 Lemma.**

*Suppose that $u = u(x,t)$ solves the heat equation $u_t = u_{xx}$ for $0 \le x \le L$ and $t \ge 0$ and meets the boundary conditions*

$$u(0,t) = u(L,t) = 0, \ \ t \ge 0.$$

*Suppose also that $u$ has the product structure $u(x,t) = X(x)T(t)$ for some functions $X$ and $T$. Then there is a constant $c$ and an integer $k$ such that*

$$u(x,t) = ce^{-(k\pi/L)^2 t} \sin\left(\frac{k\pi x}{L}\right). \tag{2.2.8}$$

This result should feel underwhelming. The only product solutions to the heat equation that meet the boundary conditions $u(0,t) = u(L,t) = 0$ for all $t$ have a very restrictive form. In particular, such solutions only work with very special initial temperature distributions. Indeed, if $u$ has the form (2.2.8) and $u(x,0) = f(x)$ for some function $f$, then $f$ is just

$$f(x) = c\sin\left(\frac{k\pi x}{L}\right).$$

We can improve on this result slightly if we note that the heat equation is linear. That is, if $u_1$ and $u_2$ both solve the heat equation[18] for $0 \le x \le L$ and $t \ge 0$, and if $a_1$ and $a_2$ are real numbers, then the linear combination $a_1u_1 + a_2u_2$ also solves the heat equation. Here is what we obtain (more precisely, what you obtain — I'll leave you to prove this).

[17] Recall that the integers are the numbers $k$ of the form $k = 0, \pm 1, \pm 2, \ldots$, and we denote the set of all integers b y $\mathbb{Z}$.

[18] When you check this, as you will no doubt diligently do on your own, I suggest that you write their partial derivatives either as $(u_1)_t$ or $\partial_t[u_1]$ to avoid overworking the subscripts.

**2.2.4 Theorem.**

*Fix $L > 0$ and let $n \geq 1$ be an integer. Let $a_0, \ldots, a_n$ be real numbers and define*

$$f(x) := \sum_{k=1}^{n} a_k \sin\left(\frac{k\pi x}{L}\right). \tag{2.2.9}$$

*Then the function*

$$u(x,t) := \sum_{k=1}^{n} a_k e^{-(k\pi/L)^2 t} \sin\left(\frac{k\pi x}{L}\right)$$

*solves the IVP-BVP* (2.2.6).

Just to make sure we're reading this result correctly, let's do a quick little problem.

**2.2.5 Example.**

*Find a solution to the IVP-BVP*

$$\begin{cases} u_t = u_{xx}, \ 0 \leq x \leq 1, \ t \geq 0 \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = f(x), \ 0 \leq x \leq 1, \end{cases}$$

*where*

$$f(x) = \sin(\pi x) - 3\sin(6\pi x) + 9\sin(12\pi x).$$

**Solution.** Here we are working with a rod of length $L = 1$. To use Theorem 2.2.4, we need to put the initial temperature distribution into the form (2.2.14). We can do this by "adding zero" in a number of places. Put $a_1 = 1$, $a_6 = -3$, $a_{12} = 9$, and $a_k = 0$ for $k \neq 1, 6, 12$. Then

$$f(x) = \sum_{k=1}^{12} a_k \sin(k\pi x),$$

and so the solution $u$ to the IVP-BVP has the form

$$u(x,t) = \sum_{k=1}^{12} a_k e^{-(k\pi)^2 t} \sin(n\pi x) = e^{-\pi^2 t}\sin(\pi x) - 3e^{-(6\pi)^2 t}\sin(6\pi x) + 9e^{-(12\pi)^2 t}\sin(12\pi x). \quad \blacktriangle$$

Theorem 2.2.4 is still a very limited result. We can only solve the heat IVP-BVP (for rods with ends at the same constant temperature) if the initial temperature distribution has a very special form. In particular, note that $f$ as defined by (2.2.14) is odd, infinitely differentiable, bounded, and periodic, and $f$ also satisfies $f(0) = f(L) = 0$. Certainly there should be many reasonable initial temperature distributions that are not so nice! (For contrast, when we solved the nonhomogeneous transport equation $au_x + bu_t = f(x,t)$, to invoke Leibniz's rule it is more than sufficient for that $f$ to be just continuously differentiable. So, our limited prior experience with PDE should suggest that we should be able to work with much more general initial temperature distributions.)

There is one condition on $f$ above that is really necessary. Namely, if $u$ solves (2.2.6), then

$$f(0) = u(0,0) = 0 \quad \text{and} \quad f(L) = u(L,0) = 0.$$

The first equality in each pair above is the initial condition, and the second is a boundary condition. Thus if the problem (2.2.6) can have a solution, at the very least we need $f(0) = f(L) = 0$. But do we really need $f$ to be odd, infinitely differentiable, bounded, and periodic? Probably not.

### *2.2.4. Toward the idea of Fourier series.*

Here is how we are going to get around our heretofore severe limitations on the initial temperature distribution $f$. We are successful if we assume that $f$ has the form

$$f(x) = \sum_{k=1}^{n} a_k \sin\left(\frac{k\pi x}{L}\right).$$

Such a function is called a **TRIGONOMETRIC POLYNOMIAL**, and it looks something like a **(CLASSICAL) POLYNOMIAL**, which we usually write as

$$g(x) = \sum_{k=0}^{n} a_k x^k.$$

Classical polynomials are ideal for calculus manipulations; you can integrate and differentiate them with ease, and computing their values is just a matter of multiplying and adding. Of course, not all functions are polynomials, but some of the nicest functions have Taylor expansions; we can sometimes write a function $g$ as

$$g(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then $g$ is still easy to manipulate from the point of view of calculus, and all of the "data" of $g$ is effectively encoded in the sequence of coefficients $(a_k)$. If you don't remember all the theory of infinite series, we'll review the essential highlights shortly.

Now, here's the idea. What if, given an initial temperature distribution $f$, we could write

$$f(x) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) \tag{2.2.10}$$

for some sequence of coefficients $(a_k)$? Then, by analogy with Theorem 2.2.4, we might expect the solution of the IVP-BVP (2.2.6) to be

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-(k\pi/L)^2 t} \sin\left(\frac{k\pi x}{L}\right).$$

There are any number of problems and challenges with this idea. How can we find such $a_k$ in the expansion of $f$? What does "=" mean in both series? Should $u$ defined in this way be differentiable? We will discuss at length some of these questions and some of their answers, although we will not rigorously prove all of our results. Indeed, asking and answering the right questions about these trigonometric series — these *Fourier* series — has spurred the development of much of modern analysis.

This is where we finished on Wednesday, September 28, 2022.

*2.2.5. The heat equation on a circular wire.*

We discussed the case of a thin rod or wire bent into a circle in Case 3 of Section 2.1. It turns out to be convenient (trust me) to assume that the unbent wire now has length $2L$ with its ends placed at $x = -L$ and $x = L$, not $x = 0$ and $x = L$ as before. Bend the wire into a circle and join the ends at $x = \pm L$. Then the boundary conditions (2.1.1) are

$$u(-L, t) = u(L, t) \quad \text{and} \quad u_x(-L, t) = u_x(L, t). \tag{2.2.11}$$

That is, we now want to solve the IVP-BVP

$$\begin{cases} u_t = u_{xx}, \ -L \le x \le L, \ t \ge 0 \\ u(x, 0) = f(x), \ -L \le x \le L \\ u(-L, t) = u(L, t), \ u_x(-L, t) = u_x(L, t). \end{cases} \tag{2.2.12}$$

Before we attempt to find any solutions, our experience suggests that we should study the initial temperature distribution $f$ further. Can we solve this IVP-BVP for any $f$? I say no: if $u$ is a solution, then the initial condition forces $f(L) = u(L, 0)$, and the boundary condition forces $u(L, 0) = u(-L, 0)$, and the initial condition again forces $u(-L, 0) = f(-L)$. Thus we need $f$ to satisfy $f(-L) = f(L)$, and a similar argument, using $f'(x) = u_x(x, 0)$, shows the necessity of $f'(-L) = f(L)$.

Now we build on our prior experience in a different way: we again attempt to find solutions that are product solutions. The three kinds of product solutions for $u_t = u_{xx}$ are listed in (2.2.7). I claim that the first two kinds will either give only the trivial solution $u = 0$ if we fit them to the boundary conditions (2.2.11), or they will only give special cases of what we could figure out using the third kind. (I'll leave you to make sense of that.)

So, we will only discuss together product solutions of the third kind:

$$u(x, t) = e^{-\lambda^2 t}\big[c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\big].$$

We want to learn as much as we can about $\lambda$, $c_1$, and $c_2$, and we start by seeing what the boundary conditions demand.

First, $u(-L, t) = u(L, t)$ says

$$e^{-\lambda^2 t}\big[c_1 \cos(-\lambda L) + c_2 \sin(-\lambda L)\big] = e^{-\lambda^2 t}\big[c_1 \cos(\lambda L) + c_2 \sin(\lambda L)\big].$$

We can cancel the factor of $e^{-\lambda^2 t}$ from both sides; the precise value of $t$ plays no role here. Then we use the evenness of the cosine and the oddness of the sine to find

$$c_1 \cos(\lambda L) - c_2 \sin(\lambda L) = c_1 \cos(\lambda L) + c_2 \sin(\lambda L).$$

This rearranges to

$$2c_2 \sin(\lambda L) = 0.$$

Of course, we could take $c_2 = 0$, but our experience suggests that this would put us perilously close to a trivial solution; if we want $\sin(\lambda L) = 0$, we need $\lambda L = k\pi$ for an integer $k$. In other words, we can meet the first boundary condition by taking

$$\lambda = \frac{k\pi}{L}$$

for an integer $k$. Note that since $\sin(0) = 0$, taking $k = 0$ gives the same effect as taking $c_2 = 0$.

To address the other boundary condition, we first calculate

$$u_x(x,t) = e^{-\lambda^2 t}\big[-c_1\lambda\sin(\lambda x) + c_2\lambda\cos(\lambda x)\big].$$

Then the condition $u_x(-L,t) = u_x(L,t)$ forces

$$e^{-\lambda^2 t}\big[-c_1\lambda\sin(-\lambda L) + c_2\lambda\cos(-\lambda L)\big] = e^{-\lambda^2 t}\big[-c_1\lambda\sin(\lambda L) + c_2\lambda\cos(\lambda L)\big].$$

Canceling the $e^{-\lambda^2 t}$ factors and using the parity of sine and cosine, we arrive at

$$c_1\lambda\sin(\lambda L) + c_2\lambda\cos(\lambda L) = -c_1\lambda\sin(\lambda L) + c_2\lambda\cos(\lambda L),$$

and thus

$$2c_1\lambda\sin(\lambda L) = 0,$$

which looks almost like what we had with the first boundary condition. Taking $c_1 = 0$ certainly works, but we can also take $\lambda = 0$ or $\sin(\lambda L) = 0$. Note that we can take $\lambda = 0$ by assuming $\lambda = k\pi/L$ with $k = 0$. And so we haven't really learned anything new from this boundary condition.

Assuming that what I said about the other two kinds of product solutions is true, our work on the third kind produces the following "structural" result, very much like Lemma 2.2.3.

**2.2.6 Lemma.**

*Suppose that $u = u(x,t)$ solves the heat equation $u_t = u_{xx}$ for $0 \le x \le L$ and $t \ge 0$ and meets the boundary conditions*

$$u(-L,t) = u(L,t),\ t \ge 0 \quad \textit{and} \quad u_x(-L,t) = u_x(L,t),\ t \ge 0.$$

*Suppose also that $u$ has the product structure $u(x,t) = X(x)T(t)$ for some functions $X$ and $T$. Then there are constants $a$ and $b$ and an integer $k$ such that*

$$u(x,t) = e^{-(k\pi/L)^2 t}\left[a\cos\left(\frac{k\pi x}{L}\right) + b\sin\left(\frac{k\pi x}{L}\right)\right]. \tag{2.2.13}$$

Just to be clear, if we don't want to have $\lambda = k\pi/L$ for some integer $k$, then the work above requires $a = b = 0$ in (2.2.13), in which case $u = 0$. And, once again, linearity allows us to extend this result to slightly more complicated initial temperature distributions.

**2.2.7 Theorem.**

*Fix $L > 0$ and let $n \geq 1$ be an integer. Let $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ be real numbers and define*

$$f(x) := \sum_{k=0}^{n} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right]. \tag{2.2.14}$$

*Then the function*

$$u(x,t) := \sum_{k=0}^{n} e^{-(k\pi/L)^2 t} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right]$$

*solves the IVP-BVP* (2.2.12).

This (loosely) motivates us to seek a different expansion of a more complicated initial temperature distribution $f$ as a series of the form

$$f(x) = \sum_{k=0}^{\infty} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right], \tag{2.2.15}$$

so that we could, perhaps, solve the associated IVP-BVP (2.2.12) via

$$u(x,t) = \sum_{k=0}^{\infty} e^{-(k\pi/L)^2 t} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right].$$

It turns out that learning how to write functions in this more complicated form (2.2.15) will actually help us write them as well as sums of just sines, like (2.2.10), or even just cosines, and to that lengthy endeavor we will now turn.

# 3. Fourier Series

## 3.1. Some (very) finite-dimensional linear algebra.

Our brief experience with the heat equation has induced us to study how we might write a function $f$ defined on $[-L, L]$ in the form

$$f(x) = \sum_{k=0}^{\infty} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right].$$

How can we possibly define the "data" $a_k$ and $b_k$ so that such an equality has a chance of working (regardless of what exactly the convergence of this series means)? One way to motivate what could otherwise be some arbitrary definitions and conventions to come is to take a brief excursion into finite-dimensional linear algebra. Essentially everything that follows could be covered in a multivariable calculus class (and probably was, in some way, for you).

We denote by $\mathbb{C}$ the set of all complex numbers; informally[19], this is the set

$$\mathbb{C} := \{x + iy \mid x,\ y \in \mathbb{R}\}, \qquad i^2 = -1.$$

For $z = x + iy \in \mathbb{C}$, we define its **complex conjugate** as

$$\overline{z} := x - iy$$

and its **modulus** as

$$|z| := \sqrt{x^2 + y^2} = z\overline{z},$$

the second equality being something that you should check.

We denote by $\mathbb{C}^n$ all $n$-tuples of complex numbers:

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) \mid z_k \in \mathbb{C},\ k = 1, \ldots, n\}.$$

On $\mathbb{C}^n$, we define the **dot product**: for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{C}^n$, we put

$$\mathbf{x} \cdot \mathbf{y} := \sum_{k=1}^{n} x_k \overline{y_k}.$$

And we have the **norm**

$$\|\mathbf{x}\| := \left( \sum_{k=1}^{n} |x_k|^2 \right)^{1/2} = \sqrt{\mathbf{x} \cdot \mathbf{x}},$$

again, the second equality being something that you should check.

[19] A good part of a rigorous complex analysis course is making sense of what the juxtaposition $iy$ really means.

The dot product is a great tool for extracting the "data" of a vector. Assuming that vector addition and scalar multiplication proceed componentwise[20] and writing, for ease of reading, the vectors below as column vectors, here is a nice little calculation:

$$\mathbf{x} := \begin{pmatrix}1\\9\\7\\7\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 9\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 7\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 7\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Put

$$\mathbf{e}_1 := \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \mathbf{e}_2 := \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \mathbf{e}_3 := \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad \text{and} \qquad \mathbf{e}_4 := \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

and note that

$$\mathbf{e}_j \cdot \mathbf{e}_\ell = \begin{cases} 1, & j = \ell \\ 0, & j \neq \ell. \end{cases}$$

Assuming that the dot product obeys the expected rules

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z}) \quad \text{and} \quad (\alpha \mathbf{x}) \cdot \mathbf{y} = \alpha(\mathbf{x} \cdot \mathbf{y}),$$

we can calculate

$$\mathbf{x} \cdot \mathbf{e}_2 = \big(\mathbf{e}_1 + 9\mathbf{e}_2 + 7\mathbf{e}_3 + 7\mathbf{e}_4\big) \cdot \mathbf{e}_2 = 9,$$

and more generally obtain

$$\mathbf{x} = \sum_{j=1}^{4} (\mathbf{x} \cdot \mathbf{e}_j)\mathbf{e}_j.$$

All the essential "data" of $\mathbf{x} = (1, 9, 7, 7)$ are its four entries. The dot product extracts that data and allows us to write $\mathbf{x}$ in the transparent form above. As we look toward expressing functions as (infinite) sums of sines and cosines, we should seek a tool that will allow us to extract the coefficients on those sines and cosines easily from those functions.

This is where we finished on Friday, September 30, 2022.

## 3.2. Some (very brief) remarks about series.

Let $L > 0$. We are thinking that if

$$f(x) := \sum_{k=0}^{\infty} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right], \tag{3.2.1}$$

[20] If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ and $\alpha \in \mathbb{C}$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$ and $\alpha \mathbf{x} = (\alpha x_1, \ldots, \alpha x_n)$

then we can get a solution $u$ to the IVP-BVP

$$\begin{cases} u_t = u_{xx}, \ -L \le x \le L, \ t \ge 0 \\ u(x,0) = f(x) \\ u(-L,t) = u(L,t), \ u_x(-L,t) = u_x(L,t), \ t \ge 0 \end{cases} \tag{3.2.2}$$

by defining

$$u(x,t) = \sum_{k=0}^{\infty} e^{-(k\pi/L)^2 t}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right]. \tag{3.2.3}$$

We have already agreed that learning how to represent a given initial temperature distribution $f$ in the form (3.2.1) is key. After that, the formula above for $u$ is. . .just that, a formula. But what exactly do these series *mean*? In particular, will defining $u$ as such a series yield a function that is sufficiently differentiable?

We will therefore take a moment to review some essential properties of series. Let $(z_k)$ be a sequence in $\mathbb{C}$ starting from $k = 0$. That is, for each integer $k \ge 0$, we have chosen a complex number $z_k$. The symbol

$$\sum_{k=0}^{\infty} z_k$$

has (at least) two meanings.

**1.** For an integer $n \ge 0$, put

$$S_n := \sum_{k=0}^{n} z_k.$$

Then the symbol $\sum_{k=0}^{\infty} z_k$ is just another name for the sequence of partial sums $(S_n)$. In symbols,

$$\sum_{k=0}^{\infty} z_k = \left(\sum_{k=0}^{n} z_k\right).$$

Since each partial sum only involves adding finitely many numbers together, each term $S_n$ of this sequence is always defined.

**2.** Suppose there is a complex number $L$ such that the limit $\lim_{n\to\infty} S_n$ exists and equals $L$. Then we write $\sum_{k=0}^{\infty} z_k = L$ and call $L$ the SUM of the series. In symbols,

$$\sum_{k=0}^{\infty} z_k = \lim_{n\to\infty} \sum_{k=0}^{n} z_k.$$

**3.2.1 Example.**

*What does the symbol*

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

*mean?*

**Solution.** The symbol

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

denotes both the sequence

$$\left(\sum_{k=0}^{n} \frac{1}{2^k}\right)$$

and its limit as $n \to \infty$, if this limit exists. The formula for the $n$th partial sum of the geometric series[21] gives

$$\sum_{k=0}^{n} \frac{1}{2^k} = \frac{1-(1/2)^{n+1}}{1-1/2} = 2\left(1 - \frac{1}{2^{n+1}}\right) = \frac{2^{n+2}-1}{2^{n+1}},$$

and so

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \left(\sum_{k=0}^{n} \frac{1}{2^k}\right) = \left(\frac{2^{n+2}-1}{2^{n+1}}\right).$$

Moreover, since the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^k} = \lim_{n\to\infty} 2\left(1 - \frac{1}{2^{n+1}}\right) = 2$$

exists, the symbol $\sum_{k=0}^{\infty}(1/2)^k$ also means

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2.$$

▲

Thus, given any pair of sequences $(a_k)$ and $(b_k)$, a real number $L > 0$, and an arbitrary real number $x$, it always makes sense to speak of the sequence of partial sums

$$\left(\sum_{k=0}^{n} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right]\right).$$

The convergence of this sequence, i.e., the existence of the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right],$$

is more ticklish.

[21] Let $r \in \mathbb{C}$ with $|r| < 1$. Then

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

for any integer $n \geq 0$, and so

$$\lim_{n\to\infty} \sum_{k=0}^{n} r^k = \frac{1}{1-r}.$$

## 3.3. Trigonometric polynomials.

Since our first goal is to express a given function $f$ in the form

$$f(x) = \sum_{k=0}^{\infty}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right], = \lim_{n\to\infty}\sum_{k=0}^{n}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right], \tag{3.3.1}$$

If we are going to write $f$ in the form (3.3.1), we need to know how to construct the appropriate sequences $(a_k)$ and $(b_k)$ of coefficients. One way to learn this is to consider when $f$ is not merely a limit of such sums but when $f$ is genuinely such a finite sum of sines and cosines. Such functions have a special name.

**3.3.1 Definition.**

*Fix $L > 0$ and let $n \geq 0$ be an integer. Let $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$. A* **TRIGONOMETRIC POLYNOMIAL** *is a function of the form*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right].$$

We have split the "$k = 0$" term out as $a_0/2$ for reasons of convenience that will not be easy to describe right now. Our experience with "classical" polynomials, i.e., functions of the form

$$p(z) = \sum_{k=0}^{n}\alpha_k z^k, \tag{3.3.2}$$

for $\alpha_0, \ldots, \alpha_n$, $z \in \mathbb{C}$ and $n \geq 0$ an integer suggests that the coefficients of a polynomial are uniquely determined. Indeed, for $p$ above we have

$$\alpha_k = \frac{p^{(k)}(0)}{k!}. \tag{3.3.3}$$

Say that we are given a function $f$ and know that it is a trigonometric polynomial, but we don't know the value of its coefficients. Can we reconstruct those coefficients just from more general knowledge about $f$? This is similar to saying that we are given a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$, and we want to determine the entries $x_k$. We saw earlier that the dot product lets us do that:

$$\mathbf{x} = \sum_{k=1}^{n}(\mathbf{x}\cdot\mathbf{e}_k)\mathbf{e}_k,$$

where $\mathbf{e}_k$ is the $k$th standard basis vector, i.e., the $j$th entry of $\mathbf{e}_k$ is 0 if $j \neq k$ and 1 if $j = k$. In other words, $x_k = \mathbf{x}\cdot\mathbf{e}_k$.

The right, but maybe not at all obvious, tool to extract the coefficients of a trigonometric polynomial is something very similar to the dot product.

**3.3.2 Definition.**

*Let $L > 0$ and let $f$ and $g$ be integrable functions from $[-L, L]$ to $\mathbb{C}$. (Hereafter, the expression $f\colon [-L, L] \to \mathbb{C}$ means that $f$ is a function from $[-L, L]$ to $\mathbb{C}$, and likewise for $g$.) Define*

$$\langle f, g \rangle := \int_{-L}^{L} f(x)\overline{g(x)}\ dx.$$

*The complex number $\langle f, g \rangle$ is the $(L^2)$-***INNER PRODUCT** *of $f$ and $g$.*

This tool deserves a bit of discussion before we use it.

**3.3.3 Remark.**

**(i)** *We are, unfortunately, overworking $L$ here: the symbol $L$ appears both in the interval $[-L, L]$ and in the name of the inner product. As part of the interval, $L$ calls to mind the "length" of the rod whose heat distribution we are studying. As part of the inner product's name, $L$ refers to the mathematician Henri Lebesgue.*

**(ii)** *The word "integrable" is one of the most mysterious in undergraduate mathematics, and now here we are talking about integrable functions that can be complex-valued. Since any complex number $z$ can be written uniquely in the form $z = u + iv$, where $u$ and $v$ are real, any function $f$ on $[-L, L]$ that takes values in $\mathbb{C}$ can be written uniquely in the form $f(x) = u(x) + iv(x)$, where $u$ and $v$ are real-valued functions on $[-L, L]$. We then say that $f$ is integrable on $[-L, L]$ if both $u$ and $v$ are integrable on $[-L, L]$ in the usual sense of "integrable" from real-valued calculus. Whatever that means. We define*

$$f = u + iv \Longrightarrow \int_{-L}^{L} f(x)\ dx := \int_{-L}^{L} u(x)\ dx + i\int_{-L}^{L} v(x)\ dx.$$

*Essentially* **every** *important property of real-valued integrals (linearity, substitution, integration by parts) holds for complex-valued integrals.*

**(iii)** *The notation $\langle f, g \rangle$ for an inner product is not absolutely universal. Be aware of context and notational conventions when consulting other sources.*

The inner product inherits a number of useful algebraic properties from the complex conjugate and the familiar definite integral. Proving these properties is a good exercise in definitions, which you should undertake on your own.

**3.3.4 Theorem.**

*Assume that all functions $f$, $f_1$, $f_2$, $g$, $g_1$, and $g_2$ below are integrable on the same interval $[-L, L]$.*

**(i)** $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$.

**(ii)** $\langle f, g_1 + g_2 \rangle = \langle f, g_1 \rangle + \langle f, g_2 \rangle$.

**(iii)** $\langle \alpha f, g\rangle = \alpha \langle f, g\rangle$ *for any* $\alpha \in \mathbb{C}$.

**(iv)** $\langle f, \beta g\rangle = \overline{\beta} \langle f, g\rangle$ *for any* $\beta \in \mathbb{C}$.

**(v)** $\overline{\langle f, g\rangle} = \langle g, f\rangle$.

This is where we finished on Monday, October 3, 2022.

Now that we possess the $L^2$-inner product, we can determine, once and for all, the coefficients of a trigonometric polynomial. In the following, the number $L > 0$ will be fixed, and we will not keep track of it in our notation. For integers $k$, put

$$\mathsf{c}_k(x) := \cos\left(\frac{k\pi x}{L}\right) \quad \text{and} \quad \mathsf{s}_k(x) := \sin\left(\frac{k\pi x}{L}\right). \tag{3.3.4}$$

We will only work with $k \geq 0$, but these functions are certainly defined at all (real) $k$. Then

$$\frac{a_0}{2} + \sum_{k=1}^{n}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right] = \frac{a_0}{2}\mathsf{c}_0(x) + \sum_{k=1}^{n}\big[a_k\mathsf{c}_k(x) + b_k\mathsf{s}_k(x)\big].$$

Note that $\mathsf{c}_0(x) = 1$ for all $x$.

We have the following "orthogonality" relations.

**3.3.5 Theorem.**

*The following identities hold:*

$$\langle \mathsf{c}_k, \mathsf{c}_\ell\rangle = \begin{cases} 2L, & k = \ell = 0 \\ L, & k = \ell \geq 1 \\ 0, & k \neq \ell, \end{cases} \qquad \langle \mathsf{s}_k, \mathsf{s}_\ell\rangle = \begin{cases} L, & k = \ell \geq 1 \\ 0, & k \neq \ell, \end{cases} \qquad \text{and} \qquad \langle \mathsf{c}_k, \mathsf{s}_\ell\rangle = 0.$$

**Proof.** This is a long exercise in trig identities and trig integrals. We do just two easy cases. First, suppose $k \geq 1$. Then

$$\langle \mathsf{c}_k, \mathsf{c}_0\rangle = \int_{-L}^{L} \mathsf{c}_k(x)\overline{\mathsf{c}_0(x)}\, dx = \int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\, dx = \int_{(k\pi(-L))/L}^{(k\pi(L))/L} \frac{L}{k\pi}\cos(u)\, du$$
$$= \frac{L}{k\pi}\int_{-k\pi}^{k\pi}\cos(u)\, du = \frac{L}{k\pi}\sin(u)\Big|_{u=-k\pi}^{k\pi} = \frac{L}{k\pi}\big[\sin(k\pi) - \sin(-k\pi)\big] = 0.$$

Here we substituted $u = k\pi x/L$ and used the identity $\sin(j\pi) = 0$, valid for any integer $j$.

Next, take $k = 0$. Then

$$\langle \mathsf{c}_0, \mathsf{c}_0\rangle = \int_{-L}^{L} \mathsf{c}_0(x)\overline{\mathsf{c}_0(x)}\, dx = \int_{-L}^{L} dx = 2L. \qquad \square$$

Put

$$f(x) := \frac{a_0}{2} + \sum_{k=1}^{n} \big[a_k \mathsf{c}_k(x) + b_k \mathsf{s}_k(x)\big].$$

Motivated by the expansion $\mathbf{x} = \sum_{k=1}^{n} (\mathbf{x} \cdot \mathbf{e}_k)\mathbf{e}_k$ of a vector $\mathbf{x} \in \mathbb{C}^n$, for which the dot product is a key tool and for which the identity $\mathbf{e}_k \cdot \mathbf{e}_\ell = 0$ for $k \neq \ell$ is so crucial, we start calculating:

$$\begin{aligned}
\langle f, \mathsf{c}_0 \rangle &= \left\langle \frac{a_0}{2}\mathsf{c}_0 + \sum_{k=1}^{n} \big[a_k \mathsf{c}_k + b_k \mathsf{s}_k\big], \mathsf{c}_0 \right\rangle \\
&= \left\langle \frac{a_0}{2}\mathsf{c}_0, \mathsf{c}_0 \right\rangle + \left\langle \sum_{k=1}^{n} \big[a_k \mathsf{c}_k + b_k \mathsf{s}_k\big], \mathsf{c}_0 \right\rangle \text{ by part (i) of Theorem 3.3.4} \\
&= \left\langle \frac{a_0}{2}\mathsf{c}_0, \mathsf{c}_0 \right\rangle + \sum_{k=1}^{n} \langle a_k \mathsf{c}_k + b_k \mathsf{s}_k, \mathsf{c}_0 \rangle \ \text{ again by part (i) of Theorem 3.3.4} \\
&= \left\langle \frac{a_0}{2}\mathsf{c}_0, \mathsf{c}_0 \right\rangle + \sum_{k=1}^{n} \big[\, \langle a_k \mathsf{c}_k, \mathsf{c}_0 \rangle + \langle b_k \mathsf{s}_k, \mathsf{c}_0 \rangle \,\big] \text{ by part (i) of Theorem 3.3.4, now used } n \text{ times} \\
&= \frac{a_0}{2} \langle \mathsf{c}_0, \mathsf{c}_0 \rangle + \sum_{k=1}^{n} \big[a_k \langle \mathsf{c}_k, \mathsf{c}_0 \rangle + b_k \langle \mathsf{s}_k, \mathsf{c}_0 \rangle \,\big] \text{ by part (iii) of Theorem 3.3.4} \\
&= \left(\frac{a_0}{2}\right) 2L \text{ since } \langle \mathsf{c}_0, \mathsf{c}_0 \rangle = 2L \text{ and } \langle \mathsf{c}_k, \mathsf{c}_0 \rangle = \langle \mathsf{s}_k, \mathsf{c}_0 \rangle = 0 \text{ for } k \geq 1 \text{ by Theorem 3.3.5} \\
&= a_0 L.
\end{aligned}$$

Thus we can solve for $a_0$ in terms of $f$ and an inner product:

$$a_0 = \frac{1}{L} \langle f, \mathsf{c}_0 \rangle .$$

More generally, we can express all the coefficients of a trigonometric polynomial using the $L^2$-inner product and a calculation exactly like the preceding one.

**3.3.6 Theorem.**

*Let*

$$f(x) := \frac{a_0}{2} + \sum_{k=1}^{n} \big[a_k \mathsf{c}_k(x) + b_k \mathsf{s}_k(x)\big]. \tag{3.3.5}$$

*Then*

$$a_k = \frac{1}{L} \langle f, \mathsf{c}_k \rangle \quad \text{and} \quad b_k = \frac{1}{L} \langle f, \mathsf{s}_k \rangle .$$

The result above says that the coefficients of a trigonometric polynomial are uniquely determined: if we know the polynomial, then we have only one choice for the coefficients. This is exactly like a classical polynomial's coefficients in (3.3.2) and (3.3.3). Another way to paraphrase this result is that if

$$\frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right] = \frac{\widetilde{a}_0}{2} + \sum_{k=1}^{n} \left[\widetilde{a}_k \cos\left(\frac{k\pi x}{L}\right) + \widetilde{b}_k \sin\left(\frac{k\pi x}{L}\right)\right] \tag{3.3.6}$$

for all $x$, then $a_k = \widetilde{a}_k$ and $b_k = \widetilde{b}_k$.

## 3.4. Real Fourier series.

Our goal is to express a function $f$ defined on $[-L, L]$ in the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right]. \tag{3.4.1}$$

We need to determine the coefficients $a_k$ and $b_k$.

Theorem 3.3.6 tells us how to do this in the special case that $f$ is a trigonometric polynomial: put

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) dx \quad \text{and} \quad b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) dx.$$

When $k$ is sufficiently large, eventually $a_k$ and $b_k$ will always be 0. Specifically, if $f$ has the form (3.3.5) for some finite $n$, then the integrals above are 0 for $k > n$.

When $f$ is not a trigonometric polynomial, the work above gives us an idea of how to define $a_k$ and $b_k$: integrate $f$ against appropriately scaled sines or cosines. For such integrals to exist, we need $f$ to be integrable on $[-L, L]$, which we will assume from now on.

**3.4.1 Definition.**

*Let $f\colon [-L, L] \to \mathbb{C}$ be integrable. The* REAL FOURIER COEFFICIENTS OF $f$ ON $[-L, L]$ *are*

$$\mathsf{a}_k[f] := \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) dx \quad \textit{and} \quad \mathsf{b}_k[f] := \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) dx. \tag{3.4.2}$$

The notation above for the real Fourier coefficients is idiosyncratic and possibly unique to our book and/or this course. I like the square brackets because they indicate that a function's Fourier coefficients depend on that function! Note that, as with $\mathsf{c}_k$ and $\mathsf{s}_k$ in (3.3.4), we are not indicating the dependence on $L$. Nonetheless, the interval $[-L, L]$ is critical to the definition, and the same function may have different Fourier coefficients on different intervals. It is wholly possible to define Fourier coefficients on a "half" interval like $[0, L]$ or even on a general interval $[a, b]$; we will not really need such tools, though. Finally, the "real" coefficients may be "complex" valued if $f$ takes nonreal values; the point is that in (3.4.2), we are integrating against the real-valued cosine and sine. Later we will consider "complex" coefficients that arise from integrating against a complex exponential.

Now our question is if, given an integrable function $f\colon [-L, L] \to \mathbb{C}$, the series

$$\frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} \left[ \mathsf{a}_k[f] \cos\left(\frac{k\pi x}{L}\right) + \mathsf{b}_k[f] \sin\left(\frac{k\pi x}{L}\right) \right]$$

converges for a given $x$, and, in particular, if it converges to $f(x)$.

This is where we finished on Wednesday, October 5, 2022.

We name this series.

**3.4.2 Definition.**

*Let $f\colon [-L, L] \to \mathbb{C}$ be integrable. The* **FORMAL REAL FOURIER SERIES OF** $f$ **ON** $[-L, L]$ *is the series*

$$\mathsf{FS}[f](x) := \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} \left[\mathsf{a}_k[f]\cos\left(\frac{k\pi x}{L}\right) + \mathsf{b}_k[f]\sin\left(\frac{k\pi x}{L}\right)\right],$$

*where $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$ are defined in* (3.4.2).

This is a "real" Fourier series because we are using the "real" coefficients $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$. Presently we will meet a "complex" Fourier series with coefficients defined differently.

This is a "formal" Fourier series because we are not saying anything about convergence right now. Per Section 3.2, you should think of the symbol $\mathsf{FS}[f](x)$ as denoting a sequence of partial sums:

$$\mathsf{FS}[f](x) = \left(\frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{n} \left[\mathsf{a}_k[f]\cos\left(\frac{k\pi x}{L}\right) + \mathsf{b}_k[f]\sin\left(\frac{k\pi x}{L}\right)\right]\right).$$

Certainly the terms of this sequence are always defined for any $x$ in $[-L, L]$ and any integer $n \ge 1$.

The notation $\mathsf{FS}[f](x)$ is once again idiosyncratic and specialized to, more or less, our book. I like it because it indicates a double dependency: the Fourier series depends on both the function $f$ and the value $x$ at which we are evaluating it.

The calculation of a formal Fourier series is essentially an exercise in calculus: do some integrals and slap them together into a series. In this course we will worry very much about properties of Fourier series and not stress too much over the integration techniques that yield their formulas.

**3.4.3 Example.**

*Let $f(x) = x$. Find the real Fourier series for $f$ on $[-L, L]$.*

**Solution.** We need to evaluate the integrals

$$\mathsf{a}_k[f] = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{k\pi x}{L}\right)\,dx \quad \text{and} \quad \mathsf{b}_k[f] = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{k\pi x}{L}\right)\,dx.$$

We need to compute $\mathsf{a}_k[f]$ for $k \ge 0$ and $\mathsf{b}_k[f]$ for $k \ge 1$. When $k = 0$, things are simpler, so we first compute

$$\mathsf{a}_0[f] = \frac{1}{L}\int_{-L}^{L} x\,dx = 0.$$

A recurring theme in Fourier series calculation is *do the $k = 0$ work separately from $k \neq 0$.*

Integrating by parts for $k \neq 0$, we have the antiderivatives

$$\int_{-L}^{L} x \cos\left(\frac{k\pi x}{L}\right)\ dx = \frac{L^2}{k^2\pi^2}\left[\cos\left(\frac{k\pi x}{L}\right) + \frac{k\pi x}{L}\sin\left(\frac{k\pi x}{L}\right)\right]\Bigg|_{x=-L}^{L}$$

and

$$\int_{-L}^{L} x \sin\left(\frac{k\pi x}{L}\right)\ dx = \frac{L^2}{k^2\pi^2}\left[\sin\left(\frac{k\pi x}{L}\right) - \frac{k\pi x}{L}\cos\left(\frac{k\pi x}{L}\right)\right]\Bigg|_{x=-L}^{L}.$$

Note that these antiderivatives are only valid for $k \neq 0$.

Using the facts that

$$\sin(\pm k\pi) = 0 \quad \text{and} \quad \cos(\pm k\pi) = \cos(k\pi) = (-1)^k,$$

we have

$$\begin{aligned}
\mathsf{a}_k[f] &= \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{k\pi x}{L}\right)\ dx \\
&= \frac{1}{L}\left(\frac{L^2}{k^2\pi^2}\right)[\cos(k\pi) + k\pi\sin(k\pi) - \cos(-k\pi) - k\pi(-1)\sin(-k\pi)] \\
&= \frac{L}{k^2\pi^2}[(-1)^k + k\pi\cdot 0 - (-1)^k + k\pi\cdot 0] \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{b}_k[f] &= \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{k\pi x}{L}\right)\ dx \\
&= \frac{1}{L}\left(\frac{L^2}{k^2\pi^2}\right)[\sin(k\pi) - k\pi\cos(k\pi) - \sin(-k\pi) + k\pi(-1)\cos(-k\pi)] \\
&= \frac{L}{k^2\pi^2}[0 - k\pi(-1)^k - 0 + k\pi(-1)(-1)^k] \\
&= \frac{L}{k^2\pi^2}[(-1)^{k+1}k\pi + k\pi(-1)^{k+1}] \\
&= (-1)^{k+1}\frac{2L}{k\pi}.
\end{aligned}$$

By the way, we could have computed $\mathsf{a}_k[f]$ more quickly by noting that we were integrating an odd function over the symmetric interval $[-L, L]$, and such an integral is always 0.

We conclude

$$\mathsf{FS}[f](x) = \frac{2L}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\left(\frac{k\pi x}{L}\right).$$

We are saying nothing right now about the convergence of this series. Indeed, none of the familiar convergence tests from calculus look immediately applicable; the factor $(-1)^{k+1}$ might make us think that this is an alternating series, but we still have to contend with the factor $\sin(k\pi x/L)$. ▲

**3.4.4 Example.**

*Let $f(x) = |x|$. What is $\mathsf{FS}[f](x)$ on $[-\pi, \pi]$?*

**Solution.** We need to compute

$$\mathsf{a}_k[f] = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(kx)\ dx \quad \text{and} \quad \mathsf{b}_k[f] = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(kx)\ dx.$$

This time we will use a little less brute force calculus and a little more cleverness.

Observe that $g_k(x) := |x|\cos(kx)$ is even and $h_k(x) := |x|\sin(kx)$ is odd:

$$g_k(-x) = |-x|\cos(-kx) = |x|\cos(kx) = g_k(x)$$

and

$$h_k(-x) = |-x|\sin(-kx) = -|x|\sin(kx) = -h_k(x).$$

Properties of integrals of even and odd functions over symmetric intervals then give

$$\int_{-\pi}^{\pi}|x|\cos(kx)\ dx = 2\int_0^{\pi}|x|\cos(kx)\ dx = 2\int_0^{\pi}x\cos(kx)\ dx$$

and

$$\int_{-\pi}^{\pi}|x|\sin(kx)\ dx = 0.$$

Thus

$$\mathsf{b}_k[f] = 0$$

and, for $k \neq 0$,

$$\mathsf{a}_k[f] = \frac{2}{\pi}\int_0^{\pi}x\cos(kx)\ dx$$

$$= \frac{2\pi k\sin(\pi k) + 2\cos(\pi k) - 2}{\pi k^2}$$

$$= \frac{2(-1)^k - 2}{\pi k^2}$$

$$= \begin{cases} 0, & k \text{ is even} \\ -\dfrac{4}{\pi k^2}, & k \text{ is odd.} \end{cases}$$

Last, for $k = 0$, we have

$$\mathsf{a}_0[f] = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi.$$

Thus on $[-\pi, \pi]$, the Fourier series of $f(x) = |x|$ is

$$\mathsf{FS}[f](x) = \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} \mathsf{a}_k[f]\cos(kx) + \mathsf{b}_k[f]\sin(kx) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \mathsf{a}_k[f]\cos(kx).$$

It is wholly reasonable to rewrite this series as

$$\mathsf{FS}[f](x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2}\cos((2j+1)x),$$

where we are now summing over only the odd integers. However, this representation may be less transparent if we are trying to solve the heat equation with initial temperature distribution $f(x) = |x|$; recall that to solve the IVP-BVP (3.2.2) with the initial temperature in the form (3.2.1), we want to use the formal solution (3.2.3). It may be harder to read off this formal solution from $\mathsf{FS}[f]$ in the second form above, where we are only summing over the odd integers. ▲

This is where we finished on Friday, October 7, 2022.

## 3.5. Complex Fourier series.

There is another way of writing Fourier series that often provides a cleaner and more compact expression. Before delving into the issue of Fourier series convergence, it is worth exploring this additional means of expression. This approach hinges on the notion of the complex exponential.

### *3.5.1. The complex exponential.*

**3.5.1 Definition.**

*Let $y$ be a real number. The* COMPLEX EXPONENTIAL *of $y$ is*

$$e^{iy} := \cos(y) + i\sin(y).$$

The motivation for this definition comes from the power series definition of the exponential, which reads

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Using the identity $i^2 = -1$ and the Taylor series for sine and cosine, one can rearrange the power series

$$\sum_{k=0}^{\infty} \frac{(iy)^k}{k!}$$

into the sum $\cos(y) + i\sin(y)$. The following are essential properties of the complex exponential; proving them will be good practice for you.

**3.5.2 Lemma.**

**(i)** $e^{in\pi} = (-1)^n$ *for all integers* $n$.

**(ii)** *The function* $g(y) := e^{i\alpha y}$ *is differentiable for all real* $\alpha$ *and* $g'(y) = i\alpha \mathsf{e}_\alpha(y)$. *That is,*

$$\frac{d}{dy}[e^{i\alpha y}] = i\alpha e^{i\alpha y}.$$

**(iii)** $\cos(y) = \dfrac{e^{iy} + e^{-iy}}{2}$

**(iv)** $\sin(y) = \dfrac{e^{iy} - e^{-iy}}{2i}$

**(v)** $\overline{e^{iy}} = e^{-iy}$

*3.5.2. Complex trigonometric polynomials.*

With these essential facts about the complex exponential in hand, we will rewrite a trigonometric polynomial

$$f(x) := \frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \mathsf{c}_k(x) + b_k \mathsf{s}_k(x)\right], \qquad \mathsf{c}_k(x) = \cos\left(\frac{k\pi x}{L}\right), \qquad \mathsf{s}_k(x) = \sin\left(\frac{k\pi x}{L}\right), \tag{3.5.1}$$

as a sum of complex exponentials. Put

$$\mathsf{e}_k(x) := e^{ik\pi x/L},$$

so that

$$\mathsf{c}_k(x) = \frac{\mathsf{e}_k(x) + \mathsf{e}_{-k}(x)}{2} \quad \text{and} \quad \mathsf{s}_k(x) = \frac{\mathsf{e}_k(x) - \mathsf{e}_{-k}(x)}{2i}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n} \left[a_k \mathsf{c}_k + b_k \mathsf{s}_k\right] &= \sum_{k=1}^{n} \left[a_k \left(\frac{\mathsf{e}_k + \mathsf{e}_{-k}}{2}\right) + b_k \left(\frac{\mathsf{e}_k - \mathsf{e}_{-k}}{2i}\right)\right] \\ &= \sum_{k=1}^{n} \left[\left(\frac{a_k}{2} + \frac{b_k}{2i}\right)\mathsf{e}_k + \left(\frac{a_k}{2} - \frac{b_k}{2i}\right)\mathsf{e}_{-k}\right] = \sum_{k=1}^{n} \left(\frac{a_k}{2} + \frac{b_k}{2i}\right)\mathsf{e}_k + \sum_{k=1}^{n} \left(\frac{a_k}{2} - \frac{b_k}{2i}\right)\mathsf{e}_{-k}.\end{aligned} \tag{3.5.2}$$

We will rewrite the second sum using the following "change of variables."

**3.5.3 Lemma.**

*Let $n \ge 0$ be an integer and let $z_k \in \mathbb{C}$ for each integer $k$ satisfying $-n \le k \le -1$. Then*

$$\sum_{j=-n}^{-1} z_j = \sum_{k=1}^{n} z_{-k}.$$

**Proof.** A rigorous proof uses induction on $n$, which we will not pursue. Instead, consider the following formal proof:

$$\sum_{j=-n}^{-1} z_j = z_{-n} + z_{-n+1} + \cdots + z_{-2} + z_{-1} = z_{-1} + z_{-2} + \cdots + z_{-(n-1)} + z_{-n} = \sum_{k=1}^{n} z_{-k}.$$

The purpose of induction is to make the expression $+\cdots+$ above precise.

Last, consider the following "moral" proof. If $f\colon [a,b] \to \mathbb{C}$ is integrable, then we can substitute $u = -x$ to calculate

$$\int_a^b f(x)\ dx = -\int_{-a}^{-b} f(-u)\ du = \int_{-b}^{-a} f(-u)\ du.$$

The sum identity of this lemma is just a "discrete" version of this "continuous" change of variables. ☐

To use this lemma on the sum

$$\sum_{k=1}^{n} \left(\frac{a_k}{2} - \frac{b_k}{2i}\right) \mathsf{e}_{-k},$$

we want to have

$$z_k = \left(\frac{a_k}{2} - \frac{b_k}{2i}\right) \mathsf{e}_{-k},$$

so

$$z_j = \left(\frac{a_{-j}}{2} - \frac{b_{-j}}{2i}\right) \mathsf{e}_{j}.$$

Thus

$$\sum_{j=-n}^{-1} \left(\frac{a_{-j}}{2} - \frac{b_{-j}}{2i}\right) \mathsf{e}_{j} = \sum_{j=-n}^{-1} z_j = \sum_{k=1}^{n} z_k = \left(\frac{a_k}{2} - \frac{b_k}{2i}\right) \mathsf{e}_{-k}, \tag{3.5.3}$$

which was our original sum.

Put (3.5.1), (3.5.2), and (3.5.3) together to conclude

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(\frac{a_k}{2} + \frac{b_k}{2i}\right) \mathsf{e}_k + \sum_{k=-n}^{-1} \left(\frac{a_k}{2} - \frac{b_k}{2i}\right) \mathsf{e}_{-k}.$$

We can compress this into one sum by introducing new coefficients. Put

$$\gamma_k := \begin{cases} \dfrac{a_k}{2} + \dfrac{b_k}{2i}, \ 1 \le k \le n \\ \\ \dfrac{a_0}{2}, \ k = 0 \\ \\ \dfrac{a_k}{2} - \dfrac{b_k}{2i}, \ -n \le k \le -1. \end{cases}$$

We conclude

$$f(x) = \sum_{k=-n}^{n} \gamma_k \mathsf{e}_k(x).$$

Here is a summary of our work.

**3.5.4 Definition.**

*Let $L > 0$ and let $n \ge 0$ be an integer. A function of the form*

$$f(x) = \sum_{k=-n}^{n} \gamma_k e^{ik\pi x/L} \tag{3.5.4}$$

*for $\gamma_k \in \mathbb{C}$ is a* **COMPLEX TRIGONOMETRIC POLYNOMIAL**.

Along the lines of what we remarked about "real" Fourier coefficients, what is "complex" about this complex trigonometric polynomial is not that the coefficients $\gamma_k$ can be complex numbers, but that we are expanding the sum in (3.5.4) as a product of those coefficients against the complex exponential.

**3.5.5 Lemma.**

*Let $n \ge 0$ be an integer and let $a_0, \ldots, a_n$, $b_1, \ldots, b_n$, $\gamma_{-n}, \ldots, \gamma_n \in \mathbb{C}$. Suppose that*

$$\frac{a_0}{2} + \sum_{k=1}^{n} \left[ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right] = \sum_{k=-n}^{n} \gamma_k e^{ik\pi x/L}$$

*for $-L \le x \le L$. Then*

$$\gamma_k = \begin{cases} (a_k - ib_k)/2, \ 1 \le k \le n \\ a_0/2, \ k = 0 \\ (a_k + ib_k)/2, \ -n \le k \le -1 \end{cases}$$

*and*

$$a_k = \begin{cases} 2\gamma_0, \ k = 0 \\ \gamma_k + \gamma_{-k}, \ 1 \le k \le n \end{cases} \qquad \text{and} \qquad b_k = i(\gamma_k - \gamma_{-k}).$$

Given a complex trigonometric polynomial, we can use the $L^2$-inner product to extract its coefficients. We will need the following orthogonality relations.

**3.5.6 Theorem.**

*Let $j$ and $k$ be integers. Then*

$$\langle \mathsf{e}_j, \mathsf{e}_k \rangle = \begin{cases} 2L, & j = k \\ 0, & j \neq k. \end{cases}$$

**Proof.** We compute

$$\langle \mathsf{e}_j, \mathsf{e}_k \rangle = \int_{-L}^{L} \mathsf{e}_j(x)\overline{\mathsf{e}_k(x)}\ dx = \int_{-L}^{L} e^{ij\pi x/L} e^{-ik\pi x/L}\ dx = \int_{-L}^{L} e^{i(j-k)\pi x/L}\ dx.$$

If $j = k$, then this reduces to

$$\langle \mathsf{e}_j, \mathsf{e}_k \rangle = \int_{-L}^{L} 1\ dx = 2L.$$

For $j \neq k$, we antidifferentiate the complex exponential:

$$\langle \mathsf{e}_j, \mathsf{e}_k \rangle = \int_{-L}^{L} e^{i(j-k)\pi x/L}\ dx = \frac{e^{i(j-k)\pi x/L}}{i(j-k)\pi/L}\bigg|_{x=-L}^{x=L} = 0.$$

I will leave the last calculation of that difference as an exercise for you. □

Note how much easier this proof was than that of Theorem 3.3.5 (which, to be fair, we didn't really do). This is the power of the complex exponential: many calculations are much more transparent than with sines and cosines!

Now suppose that $f = \sum_{k=-n}^{n} \gamma_k \mathsf{e}_k$ is a complex trigonometric polynomial. Fix an integer $j$ satisfying $-n \leq j \leq n$. We calculate

$$\langle f, \mathsf{e}_j \rangle = \left\langle \sum_{k=-n}^{n} \gamma_k \mathsf{e}_k, \mathsf{e}_j \right\rangle = \sum_{k=-n}^{n} \langle \gamma_k \mathsf{e}_k, \mathsf{e}_j \rangle = \sum_{k=-n}^{n} \gamma_k \langle \mathsf{e}_k, \mathsf{e}_j \rangle = \gamma_j (2L).$$

Thus we have uniquely determined the coefficients of $f$ as

$$\gamma_j = \frac{1}{2L} \langle f, \mathsf{e}_j \rangle,$$

which we package as the following result.

**3.5.7 Theorem.**

*Let*

$$f(x) = \sum_{k=-n}^{n} \gamma_k \mathsf{e}_k(x).$$

*Then*

$$\gamma_k = \frac{1}{2L}\langle f, \mathsf{e}_k \rangle .$$

*3.5.3. Complex Fourier series.*

The equivalence of real and complex trigonometric polynomials in Lemma 3.5.5 and the expression for the coefficients of a complex trigonometric polynomial in Theorem 3.5.7 motivate the following definition for more general functions.

**3.5.8 Definition.**

*Let* $f\colon [-L, L] \to \mathbb{C}$ *be integrable. The* COMPLEX FOURIER COEFFICIENTS OF $f$ ON $[-L, L]$ *are*

$$\widehat{f}(k) := \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}\,dx. \tag{3.5.5}$$

Just to be sure, the “complex” part of these coefficients is that they arise by integrating against the complex exponential. Also, be careful in that (1) we have $2L$, not $L$, in the denominator, unlike with the real Fourier coefficients and (2) there is a negative sign in the exponential. Don’t forget either of those parts of the formula!

This is where we finished on Monday, October 10, 2022.

The symmetric nature of a complex trigonometric polynomial might encourage us to consider the series

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ik\pi x/L} \tag{3.5.6}$$

as the analogue of a Fourier series but with complex Fourier coefficients. This is exactly the right idea, but we should explore some properties of this series. . .starting with what it means.

Recall the real Fourier series $\mathsf{FS}[f](x)$ of an integrable $f\colon [-L, L] \to \mathbb{C}$ means at least one, possibly two, things: the sequence of partial sums of this series and the limit of that sequence, if the limit exists. For an integer $n \geq 1$, we denote the $n$th partial sum of the real Fourier series by

$$\mathsf{S}_n[f](x) := \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{n}\big[\mathsf{a}_k[x]\mathsf{c}_k(x) + \mathsf{b}_k(x)\mathsf{s}_k(x)\big].$$

The functions $\mathsf{c}_k$ and $\mathsf{s}_k$ were defined in (3.3.4). So, the symbol $\mathsf{FS}[f](x)$ always means the sequence

$$\mathsf{FS}[f](x) = \big(\mathsf{S}_n[f](x)\big)$$

and maybe, if things go right, the limit

$$\mathsf{FS}[f](x) = \lim_{n\to\infty}\mathsf{S}_n[f](x).$$

The whole point of Lemma 3.5.5 is that

$$\mathsf{S}_n[f](x) = \sum_{k=-n}^{n} \widehat{f}(k) e^{ik\pi x/L}. \tag{3.5.7}$$

So, $\lim_{n\to\infty} \mathsf{S}_n[f](x)$ exists if and only if

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \widehat{f}(k) e^{ik\pi x/L}$$

exists, in which case the two limits are the same. This suggests a definition for the series in (3.5.6). Naturally, we'll give it in a more abstract context first.

**3.5.9 Definition.**

*For each integer $k = 0, \pm 1, \pm 2, \ldots$, let $z_k \in \mathbb{C}$. The symbol*

$$\sum_{k=-\infty}^{\infty} z_k$$

*has the following dual meanings.*

**(i)** *The symbol $\sum_{k=-\infty}^{\infty} z_k$ always denotes the sequence of "symmetric" partial sums $\sum_{k=-n}^{n} z_k$ for integers $n \geq 1$, i.e.,*

$$\sum_{k=-\infty}^{\infty} z_k = \left( \sum_{k=-n}^{n} z_k \right).$$

**(ii)** *The symbol $\sum_{k=-\infty}^{\infty} z_k$ also denotes the limit of this sequence of "symmetric" partial sums, if this limit exists. That is,*

$$\sum_{k=-\infty}^{\infty} z_k = \lim_{n\to\infty} \sum_{k=-n}^{n} z_k, \tag{3.5.8}$$

*if the limit on the right exists; if the limit exists, we call its value the* SUM *of the series.*

The "symmetry" in the limit (3.5.8) can lead to some weird results of "doubly infinite" series converging when we might expect them to diverge based on our calculus knowledge of series. This limit should also remind you of one way of thinking about the improper integral of a function $f$ on $(-\infty, \infty)$: sometimes it is the case that

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx,$$

and sometimes not. You will have the pleasure of exploring some of these technical details in forthcoming exercises.

Now we can return our attention to Fourier series.

**3.5.10 Definition.**

*Let $f\colon [-L, L] \to \mathbb{C}$ be integrable. The* **formal complex Fourier series of $f$ on** $[-L, L]$ *is the series*

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ik\pi x/L},$$

*where $\widehat{f}(k)$ is defined in (3.5.5). We interpret this series according to the meanings in Definition 3.5.9.*

There is no need to introduce new notation for the complex Fourier series. By (3.5.7), the "symmetric" partial sums of the complex Fourier series are exactly the partial sums of the real Fourier series. Thus, as sequences, the two kinds of series agree, and the discussion preceding Definition 3.5.9 explains why one series converges if and only if the other does, in which case both series converge to the same sum.

Incidentally, note the mismatch of sign in the complex exponential between the complex Fourier coefficient and the complex Fourier series. The complex Fourier coefficient requires us to integrate against $e^{-ik\pi x/L}$; this is an artifact of the complex conjugation in the $L^2$-inner product. The complex Fourier series requires us to multiply against $e^{ik\pi x/L}$; this is a consequence of the conversion from a real to a complex trigonometric polynomial in Lemma 3.5.5.

**3.5.11 Example.**

*Find the complex Fourier series of $f(x) = e^x$ on $[-\pi, \pi]$.*

**Solution.** Of course, we could find the real Fourier series first, but let's work just with the complex Fourier coefficients as a learning exercise. For the complex series, we need to evaluate

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ik\pi x/\pi}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ik)x}\, dx.$$

We may antidifferentiate with the complex exponential just as we do with the real one to find

$$\int_{-\pi}^{\pi} e^{(1-ik)x}\, dx = \frac{e^{(1-ik)x}}{1-ik}\bigg|_{x=-\pi}^{\pi} = \frac{e^{(1-ik)\pi} - e^{(1-ik)(-\pi)}}{1-ik}.$$

Here it is important that $1 - ik \neq 0$ for any integer $k$. This is in fact true for any real number $k$, for if $1 - ik = 0$, then we can solve for $k = 1/i = -i$, which is definitely not real. In fact, note that we calculated $\widehat{f}(k)$ without singling out the case $k = 0$, as was necessary for the real series examples.

We simplify

$$e^{(1-ik)\pi} = e^{\pi}e^{-ik\pi} = e^{\pi}(-1)^k$$

and likewise $e^{(1-ik)(-\pi)} = e^{-\pi}(-1)^k$. Thus

$$\widehat{f}(k) = \frac{(-1)^k[e^{\pi} - e^{-\pi}]}{2\pi(1-ik)},$$

and so the complex Fourier series of $f$ on $[-\pi, \pi]$ is

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k[e^{\pi} - e^{-\pi}]}{2\pi(1-ik)} e^{ikx}. \qquad \blacktriangle$$

## 3.6. Convergence of Fourier series.

We now have two expressions for the Fourier series of an integrable function $f\colon [-L, L] \to \mathbb{C}$ on the interval $[-L, L]$, the real series and the complex series. At a given $x \in [-L, L]$, one of these series converges if and only if the other does, in which case both series converge to the same number. There's no hard and fast rule for when you should use one of these series instead of the other. The real series probably gives a more transparent (candidate for a) solution to the heat equation IVP-BVP (3.2.2), but the complex series only requires us to calculate one integral and to keep track of one kind of coefficient, not two.

Now that we know how to calculate Fourier series, we will ask questions about their convergence. For a given function $f$, we want to determine two things in particular. Does $\mathsf{FS}[f](x)$ converge as a series for a given $x$? If so, does $\mathsf{FS}[f](x)$ converge to $f(x)$? If we are going to extract the solution to an IVP-BVP from the Fourier series for the initial temperature distribution, we want to be sure that this Fourier series is a legitimate representation for that distribution!

Here is one necessary condition to have $\mathsf{FS}[f](x) = f(x)$ on all of $[-L, L]$. I claim that

$$\mathsf{FS}[f](-L) = \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} (-1)^k \mathsf{a}_k[f] = \mathsf{FS}[f](L).$$

By "$=$" here I mean equality as a sequence of partial sums and not necessarily anything about convergence. So, if $\mathsf{FS}[f](x)$ does converge and equal $f(x)$ for any $x \in [-L, L]$, then we must have

$$f(-L) = \mathsf{FS}[f](-L) = \mathsf{FS}[f](L) = f(L).$$

The restriction $f(-L) = f(L)$ means that there are many functions with nicely defined Fourier coefficients that cannot always be equal to their Fourier series.

On the other hand, this restriction is less burdensome than it might appear. In fact, we argued right after (2.2.12) that if we can solve the IVP-BVP (3.2.2) with initial temperature distribution $f$, then $f$ is differentiable with

$$f(L) = f(-L) \quad \text{and} \quad f'(L) = f'(-L). \tag{3.6.1}$$

So, we will first restrict our discussion of Fourier series convergence to functions satisfying the "periodic" conditions above.

### *3.6.1. Riemann–Lebesgue lemmas.*

You can (and will) show that the following is true. If $f\colon [-L, L] \to \mathbb{C}$ is integrable and if its real Fourier coefficients $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$ are nice enough that the series

$$\sum_{k=1}^{\infty} |\mathsf{a}_k[f]| \quad \text{and} \quad \sum_{k=1}^{\infty} |\mathsf{b}_k[f]| \tag{3.6.2}$$

converge, then $\mathsf{FS}[f](x)$ converges for each $x \in [-L, L]$. This will not tell us *what* the sum of $\mathsf{FS}[f](x)$ is, in particular whether or not it's $f(x)$, but it's a start.

The next natural question is how we might guarantee the convergence of the series (3.6.2). We learned many tests for series convergence in calculus; I claim that most (though not all) of the time we really use one test above all others. Here it is; it works just as well for complex numbers as it does real ones.

**3.6.1 Theorem (Comparison test).**

*For each integer $k \geq 1$, let $z_k \in \mathbb{C}$ and let $w_k \geq 0$ such that $|z_k| \leq w_k$ and $\sum_{k=1}^{\infty} w_k$ converges. Then $\sum_{k=1}^{\infty} z_k$ also converges.*

To what shall we compare the Fourier coefficients $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$? Hopefully some numbers that add up to a convergent series! Working with both real coefficients $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$ will be tiresome, so let's instead try to estimate the complex coefficients $\widehat{f}(k)$. Whatever we learn about $\widehat{f}(k)$ could be translated to results on $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$ using the formulas in Lemma 3.5.5. Moreover, for simplicity, I will assume $L = \pi$ in the following.

We have

$$2\pi\widehat{f}(k) = \int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

I put the factor of $2\pi$ on the left just to keep it off the integral. Let's assume that $f$ is differentiable and satisfies the "periodic" conditions (3.6.1). To study the integral above, a good idea is to integrate by parts; after all, we know about $f'$ (but not the antiderivative of $f$), and the exponential is easy to integrate (or differentiate). So, assume $k \neq 0$ for now and take

$$u = f(x) \qquad dv = e^{-ikx}\,dx$$

$$du = f'(x)\,dx \qquad v = \frac{e^{-ikx}}{-ik}.$$

Then we have

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = \frac{f(x)e^{-ikx}}{-ik}\bigg|_{x=-\pi}^{x=\pi} - \int_{-\pi}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}\,dx.$$

This is where we finished on Wednesday, October 12, 2022.

We first use the identities

$$f(-\pi) = f(\pi) \quad \text{and} \quad e^{-ik\pi} = e^{ik\pi} = (-1)^k$$

to simplify

$$\frac{f(x)e^{-ikx}}{-ik}\bigg|_{x=-\pi}^{x=\pi} = \left(-\frac{1}{ik}\right)\left[f(\pi)e^{-ik\pi} - f(-\pi)e^{ik\pi}\right] = \left(-\frac{1}{ik}\right)\left[f(\pi)(-1)^k - f(\pi)(-1)^k\right] = 0.$$

Thus

$$2\pi\widehat{f}(k) = \frac{1}{ik}\int_{-\pi}^{\pi} f'(x)e^{-ikx}\, dx.$$

Suppose we add the assumption (meh, why not?) that $f$ is twice-continuously differentiable, i.e., $f''$ exists and is continuous. We haven't talked about it, but if the heat IVP-BVP (3.2.2) has a solution, then the initial temperature distribution actually has to be twice-continuously differentiable. Exactly the same integration by parts argument as above, just using $f'$ and not $f$ now, yields

$$2\pi\widehat{f}(k) = \frac{1}{(ik)^2}\int_{-\pi}^{\pi} f''(x)e^{-ikx}\, dx. \tag{3.6.3}$$

Now let us estimate. The triangle inequality for integrals gives

$$\left|2\pi\widehat{f}(k)\right| = \left|\frac{1}{(ik)^2}\int_{-\pi}^{\pi} f''(x)e^{-ikx}\, dx\right| \le \frac{1}{|ik|^2}\int_{-\pi}^{\pi} |f''(x)e^{-ikx}|\, dx = \frac{1}{k^2}\int_{-\pi}^{\pi}|f''(x)|\, dx.$$

Since $f''$ is continuous, the integral $\int_{-\pi}^{\pi}|f''(x)|\, dx$ exists. Also, we have used the identities

$$|ik|^2 = |i|^2|k|^2 = k^2 \quad \text{and} \quad |e^{ikx}| = 1,$$

valid for all real numbers $x$ and $k$.

We have proved an important result for $L = \pi$; I will state this for $L > 0$ in general and encourage you to make the (mostly algebraic) changes in the arguments above to get the following.

**3.6.2 Lemma.**

*Let $f \in \mathcal{C}^2([-L, L])$ with $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Then for all integers $k \ne 0$, the complex Fourier coefficient $\widehat{f}(k)$ satisfies*

$$\left|\widehat{f}(k)\right| \le \left(\frac{L}{2\pi}\int_{-\pi}^{\pi}|f''(x)|\, dx\right)\frac{1}{k^2}. \tag{3.6.4}$$

This result is an example of a variety of estimates that are all called the **Riemann–Lebesgue lemma**. Broadly, people use this terminology to refer to a result that implies the *vanishing* of the Fourier coefficients. The estimate (3.6.4) implies

$$\lim_{k\to\infty}\widehat{f}(k) = \lim_{k\to\infty}\widehat{f}(-k) = 0$$

and, moreover (and more importantly), gives a sense of the "rate" of that decay: the Fourier coefficients go to 0 like $k^{-2}$ as $k \to \pm\infty$.

We can use the estimate (3.6.4) and the formulas in Lemma 3.5.5 to show a similar estimate for the real coefficients. Namely, if $f \in \mathcal{C}^2([-L, L])$ with $f(-L) = f(L)$ and $f'(-L) = f'(L)$, then there is $C > 0$ such that

$$|\mathsf{a}_k[f]| \le \frac{C}{k^2} \quad \text{and} \quad |\mathsf{b}_k[f]| \le \frac{C}{k^2} \tag{3.6.5}$$

for $k \geq 1$. This is exactly the sort of estimate that we want to use in conjunction with the comparison test; recall that the "$p$-series"

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$. By (3.6.5), we can take $p = 2$ to see that the series $\sum_{k=1}^{\infty} |\mathsf{a}_k[f]|$ and $\sum_{k=1}^{\infty} |\mathsf{b}_k[f]|$ converge.

All our work "sums up" to the following result.

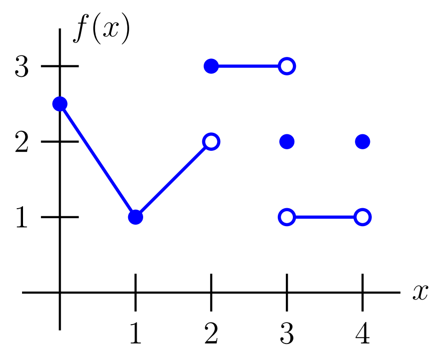
**3.6.3 Theorem.**

*Let $f \in \mathcal{C}^2([-L, L])$ with $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Then $\mathsf{FS}[f](x)$ converges for all real numbers $x$.*

### *3.6.2. Pointwise convergence.*

The Riemann–Lebesgue lemma gives us very helpful conditions on a function $f$ that yield the convergence of its Fourier series $\mathsf{FS}[f](x)$. However, knowing *that* a series converges is not the same as knowing *what* a series converges to; indeed, most of the calculus tests that imply convergence do not give formulas for the sum of the series. It turns out that the Fourier series of much more general functions than those in Theorem 3.6.3 do converge for all $x$, and they converge to "something" that closely resembles the original function. We will state this result very carefully but not prove it; my opinion is that the proof (while not difficult) does not teach us many new things about Fourier series.

The following picture describes the class of functions whose Fourier series always converge.




Our experience in calculus teaches us that we should call the function $f$ drawn above "piecewise continuous." Let's tease out some nice properties of $f$.

**1.** $f$ is defined on the entire interval $[0, 4]$.

**2.** $f$ is continuous at all points in $[0, 4]$ except 2, 3, and 4.

**3.** $f$ is differentiable at all points in $[0, 4]$ except 1, 2, 3, and 4.

**4.** $f$ is "well-behaved" at its jump discontinuities. The left and right limits

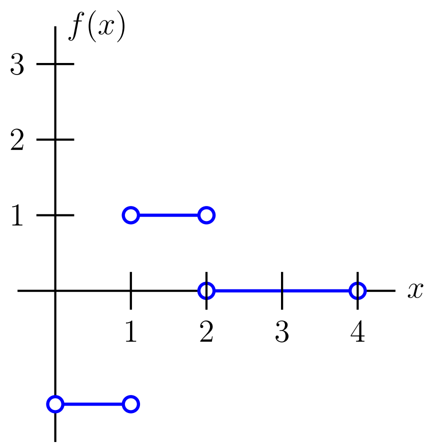
$$\lim_{\xi \to x^-} f(\xi) \quad \text{and} \quad \lim_{\xi \to x^+} f(\xi)$$

exist at all points $x$ in $(0,4)$. Of course, they are not equal at the points of discontinuity. The limits at the endpoints,

$$\lim_{\xi\to 0^+} f(\xi) \quad \text{and} \quad \lim_{\xi\to 4^-} f(\xi),$$

also exist.

**5.** The derivative $f'$ is defined at all points in $[0,4]$ except 1, 2, 3, and 4. Here's its graph.




**6.** The derivative is "mostly" continuous and has good left and right limits everywhere. That is, $f'$ is continuous on all of its domain except at finitely many points, and at all points $x$ in its domain, the limits

$$\lim_{\xi\to x^-} f'(\xi) \quad \text{and} \quad \lim_{\xi\to x^+} f(\xi)$$

exist.

We can distill these specific properties of $f$ into a general definition. First, it will be nice to have notation for left and right limits:

$$f(x^\pm) := \lim_{\xi\to x^\pm} f(\xi).$$

For example, with $f$ graphed above, we have

$$f(1^+) = f(1^-) = 1, \qquad f(2^-) = 2, \quad \text{and} \quad f(2^+) = 3.$$

**3.6.4 Definition.**

*A function $f\colon [a,b] \to \mathbb{C}$ is* **PIECEWISE CONTINUOUSLY DIFFERENTIABLE** *on $[a,b]$ if the following hold.*

**(i)** *$f$ is continuous and differentiable at all but finitely many points of its domain.*

**(ii)** *The limits $f(x^\pm)$ exist for all $x$ in $(a,b)$.*

**(iii)** *The limits $f(a^+)$ and $f(b^-)$ exist.*

**(iv)** *The derivative $f'$ is continuous at all but finitely many points of its domain (the domain of $f'$ need not be all of $[a, b]$). Since $f$ is differentiable at all but finitely many points of $[a, b]$, this means that $f'$ is defined and continuous at all but finitely many points of $[a, b]$.*

**(v)** *The limits $f'(x^{\pm})$ exist for all $x$ in $(a, b)$.*

**(vi)** *The limits $f'(a^+)$ and $f'(b^-)$ exist.*

We can state a very precise result on the convergence of the Fourier series of a piecewise differentiable function.

This is where we finished on Friday, October 14, 2022.

**3.6.5 Theorem.**

*Suppose that $f\colon [-L, L] \to \mathbb{C}$ is piecewise continuously differentiable. Then $\mathsf{FS}[f](x)$ converges for all $x \in [L, L]$ with*

$$\mathsf{FS}[f](x) = \begin{cases} \dfrac{f(x^+) + f(x^-)}{2}, & -L < x < L \\ \\ \dfrac{f(-L^+) + f(L^-)}{2}, & x = \pm L. \end{cases} \tag{3.6.6}$$

In other words, if $f$ is piecewise continuously differentiable on $[-L, L]$, then $\mathsf{FS}[f](x)$ converges to the average of the left and right limits of $f$ at $x$ for $x$ in the open interval $(-L, L)$. At the endpoints, $\mathsf{FS}[f](\pm L)$ converges to the average of the limit of $f$ from the right as $x \to -L$ and the limit of $f$ from the left as $x \to L$. (Incidentally, a piecewise continuously differentiable function is integrable, so it makes sense to talk about its Fourier coefficients and Fourier series.) We will not prove this theorem; its proof is neither terribly difficult nor very quick (otherwise, we *would* prove it).

The first consequence of this theorem is that we can both relax the hypotheses of Theorem 3.6.3 and improve its result. I will let you prove this sometime soon.

**3.6.6 Corollary.**

*Let $f$ be continuous and piecewise continuously differentiable on $[-L, L]$ and suppose that $f(-L) = f(L)$. Then $\mathsf{FS}[f](x) = f(x)$ for all $x \in [-L, L]$.*

This corollary is a relaxation of Theorem 3.6.3 because we no longer need $f$ to be twice-continuously differentiable; we are not even assuming that $f$ is differentiable on all of $[-L, L]$. It also strengthens Theorem 3.6.3 because we get more than just convergence: we get a formula for the sum of the Fourier series. (And *that* is something that most of our calculus convergence tests never provide.) Finally, note that the phrase "continuous and piecewise continuously differentiable" is not redundant — take $f(x) = |x|$ to see that $f$ is continuous everywhere but only piecewise continuously differentiable due to the corner at $x = 0$.

**3.6.7 Example.**

*Let*

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

*What is* $\mathsf{FS}[f](x)$ *on* $[-1, 1]$*? Graph* $\mathsf{FS}[f]$.

**Solution.** We see "by inspection" that $f$ is piecewise continuously differentiable on $[-1, 1]$. Specifically, $f$ is continuous on $[-1, 0)$ and $(0, 1]$ with

$$f(0^-) = \lim_{x \to 0^-} f(x) = 0 \quad \text{and} \quad f(0^+) \lim_{x \to 0^+} f(x) = 1.$$

Hence

$$\mathsf{FS}[f](0) = \frac{f(0^+) + f(0^-)}{2} = \frac{1+0}{2} = \frac{1}{2}.$$

For $-1 < x < 0$ and $0 < x < 1$, continuity gives $f(x^+) = f(x^-) = f(x)$, thus

$$\mathsf{FS}[f](x) = \frac{f(x^+) + f(x^-)}{2} = \frac{2f(x)}{2} = f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

Last, for the endpoints, we have

$$f(-1^+) = \lim_{x \to -1^+} f(x) = 0 \quad \text{and} \quad f(-1^-) = \lim_{x \to -1^-} f(x) = 1,$$

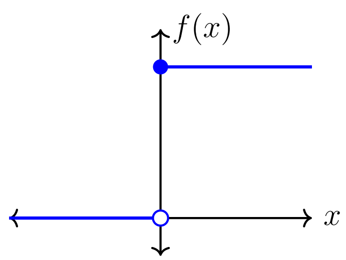
thus

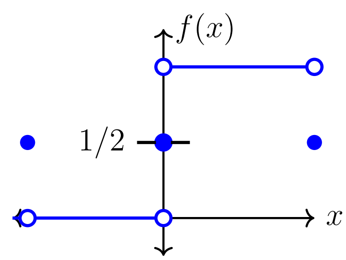
$$\mathsf{FS}[f](\pm 1) = \frac{f(-1^+) + f(1^-)}{2} = \frac{0+1}{2} = \frac{1}{2}.$$

All together,

$$\mathsf{FS}[f](x) = \begin{cases} 1/2, & x = -1 \\ 0, & -1 < x < 0 \\ 1/2, & x = 0 \\ 1, & 0 < x < 1 \\ 1/2, & x = 1. \end{cases}$$

Here are graphs of $f$ and $\mathsf{FS}[f]$.




▲

If the Fourier series for $f$ converges on $[-L, L]$, then the series necessarily converges on $\mathbb{R}$, since the series is $2L$-periodic. Thus we sometimes speak of $\mathsf{FS}[f]$ as a function on $\mathbb{R}$ even though $f$ is only defined on $[-L, L]$.

The pointwise convergence of Fourier series can turn out quite badly if we relax the "differentiable" part of the "piecewise continuously differentiable" requirement of Theorem 3.6.5. Deep results of twentieth-century Fourier analysis (I emphasize the century because Fourier series have been around a lot longer than that, so it took a long time to figure this out) show the following.

**Bad Thing 1.** A function can be integrable on $[-L, L]$ and yet its Fourier series diverges[22] at all $x$ in $[-L, L]$. Obviously, such a function cannot be piecewise continuously differentiable.

**Bad Thing 2.** Given any sequence of points $(x_k)$ in $[-L, L]$, it is possible to find a continuous function whose Fourier series diverges at each $x_k$.

Thus the (piecewise continuous) differentiability condition really is essential to maintain control over pointwise convergence.

This is where we finished on Monday, October 17, 2022.

*3.6.3. Convergence in the mean.*

In the following, suppose that $f\colon [-L, L] \to \mathbb{C}$ is integrable. As before, we denote the $n$th partial sum of the Fourier series $\mathsf{FS}[f](x)$ for $f$ on $[-L, L]$ by $\mathsf{S}_n[f](x)$. This means three things.

**1.** $\mathsf{FS}[f](x)$ always means the sequence $\big(\mathsf{S}_n[f](x)\big)$ of partial sums.

**2.** $\mathsf{FS}[f](x)$ converges as a series if and only if the limit $\lim_{n\to\infty} \mathsf{S}_n[f](x)$ exists, in which case $\mathsf{FS}[f](x)$ also denotes the (complex) number $\lim_{n\to\infty} \mathsf{S}_n[f](x)$.

**3.** $\mathsf{FS}[f](x) = f(x)$ if and only if $\lim_{n\to\infty} \mathsf{S}_n[f](x) = f(x)$.

We know that it is easier for $\lim_{n\to\infty} \mathsf{S}_n[f](x)$ to exist than for $\lim_{n\to\infty} \mathsf{S}_n[f](x)$ to exist and equal $f(x)$. For the existence, we just need $f$ to be piecewise continuously differentiable on $[-L, L]$; for existence and equality, we need $f$ to be piecewise continuously differentiable on $[-L, L]$ and continuous on $(-L, L)$ with $f(-L) = f(L)$. Perhaps, though, we should not be too attached to "pointwise" convergence. If we are thinking about the heat equation on a rod or circular wire, does it really matter if we are "off" in our temperature predictions for one or two (or finitely many) points on the object, so long as "in general" or "on average" our predictions are correct? Here we will relax our standards and our definition of $=$ to see some other ways of comparing a Fourier series to the underlying function.

Instead of (or in addition to) working pointwise, $x$ by $x$, a good question to ask is whether the entire *function* $\mathsf{S}_n[f]$ is, "on average," a good approximation to the entire *function* $f$.

[22] That is, $\mathsf{FS}[f](x)$ fails to converge as a series.

Recall from calculus that the **AVERAGE VALUE** of an integrable function $g\colon [a,b] \to \mathbb{C}$ is

$$\frac{1}{b-a}\int_a^b g(x)\ dx. \tag{3.6.7}$$

I say that $\mathsf{S}_n[f]$ is, on average, a good approximation to $f$ if, on average, $|\mathsf{S}_n[f] - f|$ is close to 0. And so $\mathsf{S}_n[f]$ approximates, on average, $f$ well on $[-L, L]$ if

$$\frac{1}{2L}\int_{-L}^{L} |f(x) - \mathsf{S}_n[f](x)|\ dx \tag{3.6.8}$$

is close to 0. Since we will always work on the same interval $[-L, L]$, and since $1/2L$ is a constant factor, the integral above in (3.6.8) will be small if and only if the quantity

$$\int_{-L}^{L} |f(x) - \mathsf{S}_n[f](x)|\ dx$$

is small.

We surely cannot expect this integral to be zero even for large $n$, and so we will have to accept *some* errors. Hopefully it is reasonable that we can allow ourselves to be less concerned with *small* errors and more concerned with *large* errors. Note that if $z \in \mathbb{C}$ with $|z| < 1$, then $|z|^r < |z| < 1$ for any $r > 1$. That is, if $|z|$ is "small," then $|z|^r$ will be "smaller" for $r > 1$. Perhaps the simplest $r > 1$ for the purposes of multiplication is the integer $r = 2$. (I mean, would you rather calculate $|z|^{1.5}$ or $|z|^2$?)

So, one way to measure how well $\mathsf{S}_n[f]$ approximates, on average, $f$ over $[-L, L]$ and give more "weight" to large errors and less to small errors is to measure the difference $\mathsf{S}_n[f] - f$ by the quantity

$$\int_{-L}^{L} |\mathsf{S}_n[f](x) - f(x))|^2\ dx.$$

We will actually make one change to this quantity. Scaling (multiplying) $f$ be a constant $\alpha$ just scales the Fourier coefficients of $f$ by $\alpha$ and likewise the Fourier series. But

$$\int_{-L}^{L} |\mathsf{S}_n[\alpha f](x) - \alpha f(x))|^2\ dx = \alpha^2 \int_{-L}^{L} |\mathsf{S}_n[f](x) - f(x))|^2\ dx.$$

For this reason, we introduce a square root.

**3.6.8 Definition.**

*Let $g\colon [-L, L] \to \mathbb{C}$ be integrable. The $L^2$*-**NORM OF** *$g$ on $[-L, L]$ is*

$$\|g\| := \left(\int_{-L}^{L} |g(x)|^2\ dx\right)^{1/2}. \tag{3.6.9}$$

Incidentally, it is a property of the Riemann integral, which we take for granted here, that if $g\colon [-L, L] \to \mathbb{C}$ is integrable, then so is $|g|^r$ for any $r > 0$, so the integral in (3.6.9) is always defined. As with the $L^2$-inner product, the $L^2$ in the name of the norm is not connected to the $L$ in the interval $[-L, L]$. The $L^2$-inner product and norm are, however, intimately related:

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

This relation is a key step in establishing the following two facts, which we will not prove here. The first says that measured in the $L^2$-norm, the Fourier series for $f$ is an excellent approximation "on average" to $f$.

**3.6.9 Theorem.**

*Let $f\colon [-P, P] \to \mathbb{C}$ be integrable. Then*

$$\lim_{n\to\infty} \|f - \mathsf{S}_n[f]\| = 0. \tag{3.6.10}$$

The chief advantage of Theorem 3.6.9 over our prior pointwise results is that it requires only integrability, no continuity, differentiability, or periodicity. After all, Fourier coefficients are defined for any integrable function $f$, so it should be satisfying that we can *somehow* measure Fourier convergence with only integrability as a hypothesis.

The convergence in the limit (3.6.10) often goes by a special name.

**3.6.10 Definition.**

*If $(f_k)$ is a sequence of integrable functions on $[-L, L]$ and $f\colon [-L, L] \to \mathbb{C}$ is also integrable with*

$$\lim_{k\to\infty} \|f - f_k\| = 0,$$

*then we say that $(f_k)$* **CONVERGES TO $f$ IN THE MEAN**.

So, per Theorem 3.6.9, the partial sums of the (real or complex) Fourier series of an integrable function $f$ always converge to $f$ in the mean. Nonetheless, convergence in the mean of a Fourier series to its does not imply pointwise convergence. First, it may be the case that the series $\mathsf{FS}[f](x)$ diverges for some $x$. Second, it may be the case that the series $\mathsf{FS}[f](x)$ converges but not to $f(x)$; this certainly happens at $x = \pm L$ whenever $\mathsf{FS}[f](\pm L)$ converge and $f$ is not $2L$-periodic. A third nasty situation is that there exist integrable functions $f$ and $g$ on $[-L, L]$ such that $\widehat{f}(k) = \widehat{g}(k)$ for all $k$, $\mathsf{FS}[f](x)$ converges for all $x$ to $f(x)$, but $f(x) \neq g(x)$ for some $x$. Thus the Fourier series representation of a function will not uniquely determine that function! I'll ask you to suss this out as an exercise. So, not only can we lose all control over the behavior of a Fourier series without the assumption of piecewise continuity and differentiability (and even then the Fourier series may not converge pointwise back to the original function at all points), we do not even have a one-to-one and onto pairing of functions and Fourier coefficients.

We conclude on a more optimistic note. Theorem 3.6.9 tells us that, "on average" and "penalizing large errors more than small ones," the $n$th partial sums of the Fourier series of $f$ are very good approximations to $f$ when $n$ is large. In fact, they are the *best* possible approximations of $f$ by trigonometric polynomials, at least if "best" is measured by "how small the difference is with respect to the $L^2$-norm."

**3.6.11 Theorem.**

*Let $f\colon [-L,L] \to \mathbb{C}$ be integrable. The nth partial sum $\mathsf{S}_n[f]$ of the Fourier series for $f$ is the best approximation to $f$ by an nth degree trigonometric polynomial in the $L^2$-norm on $[-L,L]$. That is, if $\gamma_0, \gamma_{\pm 1}, \ldots, \gamma_{\pm n} \in \mathbb{C}$ and $g(x) := \sum_{k=-n}^{n} \gamma_k e^{ik\pi x/L}$, then*

$$\|f - \mathsf{S}_n[f]\| \le \|f - g\|.$$

This is where we finished on Wednesday, October 19, 2022.

### *3.6.4. Summing up.*

So what? Given all the ways that Fourier series can converge, or fail to converge, or only sort-of converge, what should we do? It depends. My personal opinion is that you should use the type of convergence that (1) solves your problem and (2) gives you the answer that you want while working (3) in the most convenient way possible. Sometimes you will only get (1) and not (2), or (2) and not (1). Competence with (3) comes with experience.

## 3.7. Solving differential equations with Fourier series.

### *3.7.1. The heat equation on the circular wire.*

We return to the heat equation on a circular wire:

$$\begin{cases} u_t = u_{xx}, \ -L \le x \le L, \ t \ge 0 \\ u(x,0) = f(x), \ -L \le x \le L \\ u(-L,t) = u(L,t), \ u_x(-L,t) = u_x(L,t), \ t \ge 0. \end{cases} \tag{3.7.1}$$

If $f$ is integrable, we know how to compute its formal Fourier series:

$$\mathsf{FS}[f](x) = \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} \left[\mathsf{a}_k[f]\cos\left(\frac{k\pi x}{L}\right) + \mathsf{b}_k[f]\sin\left(\frac{k\pi x}{L}\right)\right],$$

where, as always,

$$\mathsf{a}_k[f] = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx \quad \text{and} \quad \mathsf{b}_k[f] = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx.$$

Since $f$ is only integrable right now, we can't say anything about the convergence of $\mathsf{FS}[f](x)$, and so we think of it just as a sequence of partial sums.

Our experience when $f$ was a trigonometric polynomial (in which case $\mathsf{FS}[f](x) = f(x)$ for all $x$) suggests that we try to construct a solution to (3.7.1) as

$$u(x,t) = \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} e^{-(k\pi/L)^2 t}\left[\mathsf{a}_k[f]\cos\left(\frac{k\pi x}{L}\right) + \mathsf{b}_k[f]\sin\left(\frac{k\pi x}{L}\right)\right]. \tag{3.7.2}$$

Right now, the expression above is just a formal series, i.e., a sequence of partial sums. We need to figure out if this sequence converges for a given pair $(x,t)$. Then, if $u(x,t)$ is defined, is $u$ sufficiently differentiable, and does $u$ actually solve (3.7.1)?

For the remainder of this discussion, we are going to make several big assumptions that will cut down on the amount that we need to write but that won't actually affect any of the deeper math. Assume that $L=\pi$ and $f$ is odd. Then $\mathsf{a}_k[f]=0$ for all $k$. Also, write $b_k$ instead of $\mathsf{b}_k[f]$. Then

$$\mathsf{FS}[f](x)=\sum_{k=1}^{\infty} b_k \sin(kx) \quad \text{and} \quad u(x,t)=\sum_{k=1}^{\infty} b_k e^{-k^2t}\sin(kx).$$

Let's talk about the convergence of the formal series $u(x,t)$. At $t=0$ we have

$$u(x,0)=\sum_{k=1}^{\infty} b_k \sin(kx)=\mathsf{FS}[f](x).$$

So, convergence of the series $u(x,0)$ is the same as convergence of the series $\mathsf{FS}[f](x)$. Theorem 3.6.5 gives us sufficient conditions for $\mathsf{FS}[f](x)$ to converge: we want $f$ to be piecewise continuously differentiable on $[-\pi,\pi]$. But we also want $u$ to meet the boundary condition $u(x,0)=f(x)$ for $-\pi\le x\le\pi$. Thus we really want $f(x)=\mathsf{FS}[f](x)$ for all $x\in[-\pi,\pi]$. To get this equality for $x\in(-\pi,\pi)$, Theorem 3.6.5 tells us that $f$ should be continuous as well as piecewise continuously differentiable[23] on $[-\pi,\pi]$. Since $\mathsf{FS}[f](-\pi)=\mathsf{FS}[f](\pi)$, we also want $f(-\pi)=f(\pi)$. So, we add that condition to our wish list.

To summarize, $u$ is defined at $(x,0)$ and satisfies $u(x,0)=f(x)$ if $f$ is continuous and piecewise continuously differentiable on $[-\pi,\pi]$ with $f(\pi)=f(-\pi)$. This is a lot more than what we started out assuming, which was that $f$ was just integrable. Here's the life lesson: it's a lot easier to write down a formal solution to (3.7.1) than it is to check that it converges!

What happens for $t>0$? Here the factor of $e^{-k^2t}$ comes into play. To check that the series $\sum_{k=1}^{\infty} b_k e^{-k^2t}\sin(kx)$ converges, we'll start with the comparison test. We estimate

$$|b_k e^{-k^2t}\sin(kx)|\le Ce^{-k^2t}, \qquad C:=\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|\;dx.$$

Here we used $|\sin(kx)|\le 1$ and the triangle inequality for integrals:

$$|b_k|=\left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\;dx\right|\le\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)\sin(kx)|\;dx\le\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|\;dx. \tag{3.7.3}$$

The first inequality above is the triangle inequality, and it is valid, by the way, when $f$ is just integrable; we don't need continuity or piecewise continuous differentiability to get (3.7.3).

So, it suffices to show that the series $\sum_{k=1}^{\infty} e^{-k^2t}$ converges. I claim that it does, as long as $t>0$. You can check that with the ratio test or maybe the comparison test combined with a geometric series. This is due to the fact that $e^{-k^2t}$ gets *really* small as $k\to\infty$.

[23] This is not redundant! Think about $f(x)=|x|$, which is continuous and piecewise continuously differentiable. However, this $f$ is not continuously differentiable.

We conclude that $u(x,t)$ is defined for all $x \in [-\pi,\pi]$ and $t > 0$, and we only needed $f$ to be integrable for this to work out. The $t = 0$ case is harder, and demands more of $f$, because we don't have that really small factor of $e^{-k^2 t}$ to help.

Now that $u(x,t)$ is always defined, is it differentiable, and, in particular, does it satisfy the heat equation? Here is what we probably want to say:

$$u_t(x,t) = \frac{\partial}{\partial t} \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin(kx) = \sum_{k=1}^{\infty} \frac{\partial}{\partial t} \left[ b_k e^{-k^2 t} \sin(kx) \right] = \sum_{k=1}^{\infty} -k^2 b_k e^{-k^2 t} \sin(kx)$$

and

$$u_{xx}(x,t) = \frac{\partial^2}{\partial x^2} \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin(kx) = \sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2} \left[ b_k e^{-k^2 t} \sin(kx) \right] = \sum_{k=1}^{\infty} -k^2 b_k e^{-k^2 t} \sin(kx).$$

The last equalities in each chain above were easy to compute since the terms of the sum, $b_k e^{-k^2 t} \sin(kx)$, are artifacts of the original separation of variables approach from long ago, and so their $x$- and $t$-derivatives should be straightforward. The sticky point is the interchange of derivative and sum. Why are

$$\frac{\partial}{\partial t} \sum_{k=1}^{\infty} = \sum_{k=1}^{\infty} \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial^2}{\partial x^2} \sum_{k=1}^{\infty} = \sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2}$$

valid? If this is permissible, then we do have $u_t = u_{xx}$, as desired.

This should remind you of differentiating under the integral, which took some work but more or less ended nicely for all involved. The integral is, from a certain point of view, an infinite sum, and if the integrand is nice enough, we can pass a derivative through the integral. The same is true for series, and it's a tool called the Weierstrass $M$-test. Unfortunately, that tool is a bit more technical than Leibniz's rule (at least, I think so), and I don't think any deep discussion of it here will convince you that interchanging sum and derivative is valid unless you were already convinced. I'll do it in an appendix, though.

So, let me just tell you the good news: $u$ as defined above *is* sufficiently differentiable for all $x \in [-\pi,\pi]$ and all $t > 0$, and so $u$ solves the heat equation on the region

$$\mathcal{D} := \left\{ (x,t) \in \mathbb{R}^2 \;\middle|\; -\pi \le x \le \pi,\ t > 0 \right\}.$$

That is, interchanging sum and derivative is valid at all $(x,t) \in \mathcal{D}$. Everything hinges on that incredibly small factor $e^{-k^2 t}$, which facilitates the invocation of the Weierstrass $M$-test, from which so many good things flow. This is a remarkable "smoothing" property of the series defining $u$ at $t > 0$; even though the initial data may not be differentiable anywhere (integrable $\not\Longrightarrow$ differentiable), nonetheless as soon as time jumps beyond $t = 0$, the series $u$ converges and is differentiable!

This is where we finished on Monday, October 24, 2022.

What happens at $t = 0$? This is a bit worse. The problem is that the factor of $e^{-k^2 t}$ is no longer present to make things really small. There are two ways of proceeding here.

The first option is *overkill*. If we take the initial data $f$ to be *exceptionally* nice, then we can ensure that $u$ is differentiable on the larger region

$$\mathcal{D}_0 := \left\{(x,t) \in \mathbb{R}^2 \;\middle|\; -\pi \le x \le \pi, t \ge 0\right\},$$

which is to say, that interchanging sum and derivative is valid for all $(x,t) \in \mathcal{D}_0$. Here is what "exceptionally nice" means: $f$ should be four times continuously differentiable, i.e., $f \in \mathcal{C}^4([-\pi,\pi])$, and satisfy the "periodic" conditions $f^{(s)}(-\pi) = f^{(s)}(\pi)$ for $s = 0$, 1, 2, 3. If $f$ has these properties, then the same integration by parts argument that gave the Riemann–Lebesgue estimate in Lemma 3.6.2, now repeated more times, will show

$$|\widehat{f}(k)| \le \frac{C}{k^4}, \ k \ne 0.$$

This leads to an estimate of the form $|b_k| \le C/k^4$, and so the terms in the series $u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t}\sin(kx)$ all satisfy

$$|b_k e^{-k^2 t}\sin(kx)| \le \frac{C}{k^4}$$

for all $(x,t) \in \mathcal{D}_0$. The point is that now we have better control at time $t = 0$, not just $t > 0$. For subtle reasons, the factor of $k^{-4}$ is really important; $k^{-3}$ just won't cut it.

However, these requirements on $f$ are excessive. If you believe my nebulous references to the Weierstrass $M$-test, we could get convergence and differentiability of $u$ on $\mathcal{D}$ just by having $f$ continuous and piecewise continuously differentiable with $f(-\pi) = f(\pi)$. Should we really require so much more of $f$ and rule out initial temperature distributions that worked before just to control time $t = 0$? Maybe not.

Here is the second option: *give up and ask a different question*. Instead of demanding that $u$ solve the IVP-BVP on all of $\mathcal{D}_0$, we will just ask that $u$ solve it on $\mathcal{D}$ but that an extra "continuity condition" of the form

$$\lim_{(\xi,t)\to(x,0)} u(\xi,t) = u(x,0)$$

hold. This limit says that $u$ evolves from its initial state $u(x,0) = f(x)$ in a continuous manner in space and time, but maybe not in a *differentiable* manner when we pass from time $t = 0$ to time $t > 0$. To get this limit, remarkably, all we need is that $f$ be piecewise continuously differentiable, which is what we were assuming before. This sort of thing happens a lot in math: we want to solve a problem in a very specific way, we discover that we can get part of the solution easily and the rest with more difficult, and so we change our goals and compromise.

The following theorem is a summary of all our hard work on the heat equation.

**3.7.1 Theorem.**

*Let $f\colon [-L,L] \to \mathbb{C}$ be integrable and let $\mathsf{a}_k[f]$ and $\mathsf{b}_k[f]$ be its real Fourier coefficients. For $-L \le x \le L$ and $t \ge 0$, define the formal series $u(x,t)$ by*

$$u(x,t) := \frac{\mathsf{a}_0[f]}{2} + \sum_{k=1}^{\infty} e^{-(k\pi/L)^2 t}\left[\mathsf{a}_k[f]\cos\left(\frac{k\pi x}{L}\right) + \mathsf{b}_k[f]\sin\left(\frac{k\pi x}{L}\right)\right]. \tag{3.7.4}$$

*Then the following hold.*

**(i)** *The series* (3.7.4) *converges to an infinitely differentiable function* $u$ *on the region* $\mathcal{D} = \{(x,t) \in \mathbb{R}^2 \mid -L \le x \le L,\ t > 0\}$ *and satisfies*

$$\begin{cases} u_t = u_{xx},\ -L \le x \le L,\ t > 0 \\ u(-L,t) = u(L,t),\ u_x(-L,t) = u_x(L,t),\ t > 0. \end{cases} \tag{3.7.5}$$

**(ii)** *If* $f$ *is continuous and piecewise continuously differentiable on* $[-L, L]$ *with* $f(-L) = f(L)$*, then the series* (3.7.4) *converges to a continuous function on the larger region* $\mathcal{D}_0 = \{(x,t) \in \mathbb{R}^2 \mid -L \le x \le L,\ t \ge 0\}$ *and satisfies*

$$u(-L,0) = u(L,0), \qquad u(x,0) = f(x),\ -L \le x \le L,$$

*and*

$$\lim_{(\xi,t)\to(x,0)} u(\xi,t) = u(x,0),\ -L \le x \le L.$$

*This function* $u$ *is still infinitely differentiable on* $\mathcal{D}$ *and satisfies* (3.7.5)*. However,* $u$ *may not be differentiable on* $\mathcal{D}_0$*.*

### *3.7.2. The complex Fourier series for the circular heat equation.*

We tended to work with real Fourier series when solving the heat equation's IVP-BVP, as those real series lend themselves naturally to "series solutions" for $u$. We could have worked just as well with complex Fourier series, except the series solution for $u$ would involve summing from $-\infty$ to $\infty$, and summing from 1 to $\infty$ is bad enough. Nonetheless, the complex series is, as we've noticed, sometimes more useful (and sometimes less) than the real series, so let's revisit the formal solution to the heat equation as a complex series. Take $L = \pi$ for simplicity. Then a reasonable candidate for the solution to

$$\begin{cases} u_t = u_{xx},\ -\pi \le x \le \pi,\ t \ge 0 \\ u(x,0) = f(x),\ -\pi \le x \le \pi \\ u(-\pi,t) = u(\pi,t),\ u_x(-\pi,t) = u_x(\pi,t),\ t \ge 0 \end{cases} \tag{3.7.6}$$

is

$$u(x,t) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{-k^2 t} e^{ikx}. \tag{3.7.7}$$

If you think *carefully* about this series just as a formal series (= sequence of partial sums), you should see that it's the same as the "real" series (3.7.2). And as with that real series, the convergence and differentiability of (3.7.7) is fraught with peril.

Let's not think about convergence anymore. One advantage of (3.7.7), I think, is that it looks a lot like a Fourier series. Say we freeze time $t$ and put

$$v(x) := u(x,t).$$

We can think of $t$ as a parameter in $v$. Then (3.7.7) says that

$$v(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{-k^2t}e^{ikx}. \tag{3.7.8}$$

If the formal series above actually converges, then $v$ is a function and (3.7.8) is its Fourier series expansion, and in particular

$$\widehat{v}(k) = \widehat{f}(k)e^{-k^2t}.$$

I think we can see this from (3.7.7). If $t$ is fixed, then the Fourier coefficients of $u(\cdot, t)$ should be $\widehat{f}(k)e^{-k^2t}$. Let's make this precise.

**3.7.2 Definition.**

*Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $u$ is a function defined on the region*

$$\left\{(x,t) \in \mathbb{R}^2 \;\middle|\; x \in [-L, L],\ t \in I\right\}.$$

*Suppose that for some $t \in I$, the map $x \mapsto u(x,t)$ is integrable on $[-L, L]$; call this map $u(\cdot, t)$. Then the* $k$**TH SPATIAL FOURIER COEFFICIENT** *of $u(\cdot, t)$ is*

$$\widehat{u}(k,t) := \frac{1}{2L}\int_{-L}^{L} u(x,t)e^{-ik\pi x/L}\,dx.$$

*We say "spatial" because we are integrating in the spatial variable $x$, not the temporal variable $t$.*

Then our work above says the following. If $u$ solves the IVP-BVP (3.7.6), then

$$\widehat{u}(k,t) = \widehat{f}(k)e^{-k^2t}. \tag{3.7.9}$$

Here it should be understood that $L = \pi$. It took us a *lot* of work to reach this point. Could we have gotten there faster? I say yes, if we ask the right question: what can we learn about the Fourier coefficients $\widehat{u}(k,t)$ of a solution $u$ to (3.7.6)?

The first thing to note is that if $u_t = u_{xx}$, then we can take Fourier coefficients of $u_t$ and $u_{xx}$ according to Definition 3.7.2. We get

$$\widehat{u_t}(k,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_t(x,t)e^{-ikx}\,dx \quad \text{and} \quad \widehat{u_{xx}}(k,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_{xx}(x,t)e^{-ikx}\,dx, \tag{3.7.10}$$

and since $u_t = u_{xx}$, we have

$$\widehat{u_t}(k,t) = \widehat{u_{xx}}(k,t). \tag{3.7.11}$$

We want to use this equality learn about $\widehat{u}(k,t)$.

Look at the structure of the integrals in (3.7.10). The variable of integration is always $x$, but we are taking different kinds of derivatives. In the integral for $\widehat{u_t}$, it's, of course, a

$t$-derivative. This should call to mind Leibniz's rule. Let's assume[24] that the solution $u$ to (3.7.6) is nice enough that we can interchange integral and derivative to get

$$\int_{-\pi}^{\pi} u_t(x,t)e^{-ikx}\,dx = \frac{\partial}{\partial t}\left[\int_{-\pi}^{\pi} u(x,t)e^{-ikx}\,dx\right]$$

and thus

$$\widehat{u_t}(k,t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x,t)e^{-ikx}\,dx\right] = \frac{\partial}{\partial t}[\widehat{u}(k,t)]. \tag{3.7.12}$$

We have made $\widehat{u}(k,t)$ appear!

Now look at the other integral: $\int_{-\pi}^{\pi} u_{xx}(x,t)e^{-ikx}\,dx$. We are differentiating and integrating with respect to the same variable, $x$. We have done this before, specifically in obtaining Lemma 3.6.2. If we put, again, $v(x) = u(x,t)$ for time $t$ fixed, then

$$\widehat{u_{xx}}(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v''(x)e^{-ikx}\,dx,$$

and if we play our cards right, we can integrate by parts here.

Specifically, we need the following result, which is really just a generalization of the work preceding Lemma 3.6.2.

**3.7.3 Lemma.**

*Let $g\colon [-\pi,\pi] \to \mathbb{C}$ be $r$-times continuously differentiable and suppose that*

$$g^{(s)}(-\pi) = g^{(s)}(\pi) \quad \textit{for} \quad s = 0,\ldots,r-1.$$

*Then*

$$\widehat{g^{(s)}}(k) = (ik)^s\widehat{g}(k) \quad \textit{for each} \quad s = 0,\ldots,r.$$

This lemma can be generalized to functions on $[-L,L]$, but then a factor of $L$ would appear in the expression for $\widehat{g^{(s)}}$.

We are assuming that $u$ solves (3.7.6). The boundary conditions allow us to apply this lemma with $r = 2$ (and $t$ fixed) to get

$$\widehat{u_{xx}}(k,t) = (ik)^2\widehat{u}(k,t) = -k^2\widehat{u}(k,t). \tag{3.7.13}$$

Again, $\widehat{u}(k,t)$ appears!

This is where we finished on Wednesday, October 26, 2022.

Now, (3.7.11) says that we can equate (3.7.12) and (3.7.13). We get

$$\frac{\partial}{\partial t}[\widehat{u}(k,t)] = -k^2\widehat{u}(k,t).$$

[24] We are working backwards, as one often does in solving equations. We have assumed that $u$ solves (3.7.6) and we want to learn about its Fourier series. We may as well assume anything else that helps; what's the worst that's going to happen?

Believe it or not, this is an *ODE for* $\widehat{u}(k,t)$ *in the independent variable* $t$ *with* $k$ *as a parameter*. To see this more clearly, put $w(t) := \widehat{u}(k,t)$. Then we have

$$w'(t) = -k^2 w(t).$$

This is an exponential growth (or decay) kind of equation, and so its solution is

$$w(t) = w(0)e^{-k^2 t}.$$

I have intentionally put $w(0)$ as a factor above, not an arbitrary constant $C$. And so

$$\widehat{u}(k,t) = w(t) = w(0)e^{-k^2 t} = \widehat{u}(k,0)e^{-k^2 t}.$$

To calculate $\widehat{u}(k,0)$, we use the initial condition $u(x,0) = f(x)$ from the IVP-BVP (3.7.6) to get $\widehat{u}(k,0) = \widehat{f}(k)$. And so

$$\widehat{u}(k,t) = \widehat{f}(k)e^{-k^2 t},$$

exactly as we predicted in (3.7.9).

Here is why this is so important. Before, we took an incredibly long time to come up with the formal solution (3.7.7). Now, by using Definition 3.7.2, Lemma 3.7.3, and a little algebra and ODE, we got the Fourier coefficients of $u$ very quickly. Let's strip away all the narration above and symbol-push just to see how it all came together:

$$\begin{aligned} u_t = u_{xx} &\Longrightarrow \widehat{u_t}(k,t) = \widehat{u_{xx}}(k,t) \\ &\Longrightarrow \partial_t[\widehat{u}](k,t) = (ik)^2\widehat{u}(k,t) \\ &\Longrightarrow \partial_t[\widehat{u}](k,t) = -k^2\widehat{u}(k,t) \\ &\Longrightarrow \widehat{u}(k,t) = \widehat{u}(k,0)e^{-k^2 t} \\ &\Longrightarrow \widehat{u}(k,t) = \widehat{f}(k)e^{-k^2 t} \text{ if } u(x,0) = f(x). \end{aligned}$$

Here I hope it's okay that I'm writing

$$\partial_t = \frac{\partial}{\partial t}.$$

### *3.7.3. Formal Fourier series solutions to PDE and ODE.*

If all we care about is getting a formula for the Fourier coefficients of a solution to a differential equation, we can do that pretty quickly now.

**3.7.4 Example.**

*Suppose that* $u = u(x,t)$ *solves*

$$\begin{cases} u_t = u_{xx} - u, \ -\pi \le x \le \pi, \ t \ge 0 \\ u(x,0) = f(x), \ -\pi \le x \le \pi \\ u(-\pi,t) = u(\pi,t), \ u_x(-\pi,t) = u_x(\pi,t), \ t \ge 0. \end{cases}$$

*Find the Fourier coefficients of* $u$ *and write* $u$ *as a formal Fourier series* $\sum_{k=-\infty}^{\infty}\widehat{u}(k,t)e^{ikx}$.

**Solution.** Grind. It. Out.

$$\begin{aligned} u_t = u_{xx} - u &\Longrightarrow \widehat{u_t}(k,t) = \widehat{u_{xx} - u}(k,t) \\ &\Longrightarrow \partial_t[\widehat{u}](k,t) = \widehat{u_{xx}}(k,t) - \widehat{u}(k,t) \\ &\Longrightarrow \partial_t[\widehat{u}](k,t) = (ik)^2\widehat{u}(k,t) - \widehat{u}(k,t) \\ &\Longrightarrow \partial_t[\widehat{u}](k,t) = -k^2\widehat{u}(k,t) - \widehat{u}(k,t) \\ &\Longrightarrow \partial_t[\widehat{u}](k,t) = -(k^2+1)\widehat{u}(k,t) \\ &\Longrightarrow \widehat{u}(k,t) = \widehat{u}(k,0)e^{-(k^2+1)t} \\ &\Longrightarrow \widehat{u}(k,t) = \widehat{f}(k)e^{-(k^2+1)t}. \end{aligned}$$

The formal Fourier series for $u$ is therefore

$$\sum_{k=-\infty}^{\infty} \widehat{u}(k,t)e^{ikx} = \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{-(k^2+1)t}e^{ikx}.$$

For $t > 0$, we probably expect this sum to converge due to the nice and small factor of $e^{-(k^2+1)t}$. At $t = 0$, matters become more delicate and we would need further knowledge of $f$ (which here, I guess, we are presuming is integrable, but maybe not much more). ▲

The work above is *formal*, which in mathematics means "doing manipulations that look right but may be unjustified." Formal analysis is highly useful; it shows you what your solution probably looks like, if a solution exists. However, we worked backwards above: we assumed the problem had a solution, and we found its Fourier series. We would have to do more work to explain why this Fourier series converges.

While we have many methods available to solve ODE (and definitely do not need Fourier series to do so), we can deploy the formal techniques above on problems involving functions of only one variable.

**3.7.5 Example.**

*Use Fourier series to find all (formal) solutions to the ODE*

$$y'' + y = g(x).$$

*What do you learn about $g$ along the way?*

**Solution.** We take the Fourier coefficients of both sides and see what happens:

$$\begin{aligned} y'' + y = g(x) &\Longrightarrow \widehat{y'' + y}(k) = \widehat{g}(k) \\ &\Longrightarrow \widehat{y''}(k) + \widehat{y}(k) = \widehat{g}(k) \\ &\Longrightarrow (ik)^2\widehat{y}(k) + \widehat{y}(k) = \widehat{g}(k) \\ &\Longrightarrow -k^2\widehat{y}(k) + \widehat{y}(k) = \widehat{g}(k) \\ &\Longrightarrow (1-k^2)\widehat{y}(k) = \widehat{g}(k). \end{aligned}$$

Of course, now we want to divide by $1 - k^2$ and solve for $\widehat{y}(k)$ in terms of $\widehat{g}(k)$. However, something new enters the problem: $1 - k^2 = 0$ if $k = \pm 1$.

This leads to a requirement on $g$, or a "solvability condition" for the ODE: we need

$$\widehat{g}(\pm 1) = (1 - (\pm 1)^2)\widehat{y}(\pm 1) = 0.$$

This should feel weird, possibly unfair. In ODE, we learned that we could solve $y'' + y = g(x)$ for many $g$, possibly using undetermined coefficients or annihilators if $g$ had a special form, and otherwise throwing ourselves on the mercy of the dreaded method of variation of parameters. We surely did not talk about Fourier coefficients and the need for $\widehat{g}(\pm 1) = 0$. So why is this restriction showing up here? Recall Lemma 3.7.3: to have $\widehat{y''}(k) = (ik)^2$, we need

$$y(-\pi) = y(\pi) \quad \text{and} \quad y'(-\pi) = y'(\pi). \tag{3.7.14}$$

That is, if we want our solution $y$ to be nice enough for these Fourier manipulations to be valid, $y$ needs to satisfy the *boundary conditions* (3.7.14). This is assuming rather more on $y$ than just that $y$ solves an ODE; in fact, an ODE with boundary conditions is "harder" to solve than an ODE with initial conditions!

In any case, for $y \neq \pm 1$, we have

$$\widehat{y}(k) = \frac{\widehat{g}(k)}{1-k^2},$$

while for $k = \pm 1$, we have no information about $\widehat{y}(\pm 1)$. The formal Fourier series for $y$ is then

$$\sum_{k=-\infty}^{\infty} \widehat{y}(k)e^{ikx} = \sum_{k=-\infty}^{\infty} \frac{\widehat{g}(k)}{1-k^2}e^{ikx} + \widehat{y}(-1)e^{-ix} + \widehat{y}(1)e^{ix}.$$

Here I am assuming $\widehat{g}(\pm 1) = 0$ and taking the controversial position that $0/0 = 0$ so that

$$\frac{\widehat{g}(\pm 1)}{1-(\pm 1)^2} = \frac{0}{0} = 0,$$

purely for notational convenience.

With this in mind, put

$$y_0(x) := \sum_{k=-\infty}^{\infty} \frac{\widehat{g}(k)}{1-k^2}e^{ikx},$$

and let me reiterate that this is a formal sum that may or may not converge. Then we expect that the solution $y$ is

$$y(x) = y_0(x) + \widehat{y}(-1)e^{-ix} + \widehat{y}(1)e^{ix}.$$

This should look very familiar. We know that taking $e^{\pm ix}$ solve the homogeneous problem $y'' + y = 0$, and that all solutions to $y'' + y = g(x)$ are the sum of a particular solution to the nonhomogeneous problem and a linear combination of solutions to the homogeneous problem. Here $\widehat{y}(-1)e^{-ix} + \widehat{y}(1)e^{ix}$ is a linear combination of homogeneous solutions, with $\widehat{y}(\pm 1)$ as arbitrary coefficients. ▲

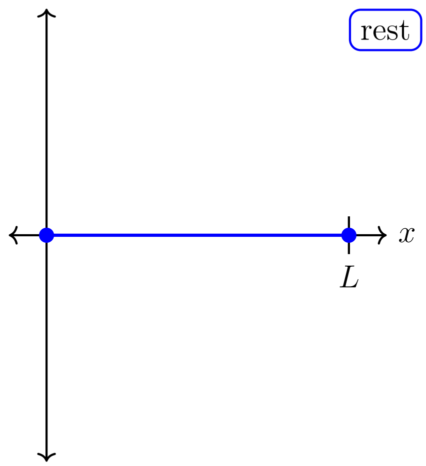
This is where we finished on Friday, October 28, 2022.

## 4. The Wave Equation

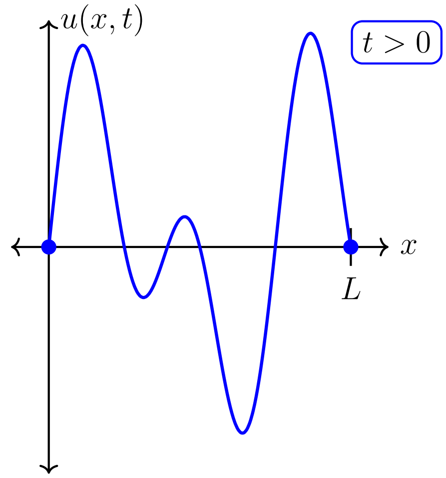
The **wave equation** for a function $u = u(x,t)$ reads

$$u_{tt} = c^2 u_{xx}.$$

This equation models, among other things, the displacement of a plucked one-dimensional string from equilibrium. Imagine that the string is $L$ units long and at rest is stretched horizontally as in the picture below.




For $0 \le x \le L$, the value $u(x,t)$ will be the vertical displacement at time $t$ of the point on the string that was originally at position $x$ on the horizontal axis. Here I have drawn the plucked string at some time $t > 0$; I've imagined that the left and right ends ($x = 0$ and $x = L$) have remained fixed on the horizontal axis, as often happens when you're plucking strings.




I am also imagining that the string moves only vertically, i.e., its motion is constrained to a two-dimensional plane that passes through the horizontal axis. The string can move "up and down" but not "in and out" or "left and right."

The parameter $c > 0$ that appears in the wave equation is the result of a rescaling and nondimensionalization of the "real" problem that the displacement of the string solves, per

Newton's law. We could further rescale the problem so that we just study $u_{tt} = u_{xx}$, much like we did with the heat equation (which often reads $u_t = \kappa u_{xx}$, not $u_t = u_{xx}$). However, I think it will be more evocative later for us to keep $c$ in play and see how it appears in solutions.

## 4.1. Formal solutions to the wave equation via Fourier series.

A reasonable boundary value problem for the wave equation reads

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ 0 \le x \le L, \ t \ge 0 \\ u(0,t) = u(L,t) = 0. \end{cases}$$

Here the plucked string has length $L > 0$, and its ends at $x = 0$ and $x = L$ are fixed throughout its motion. We could solve this problem with separation of variables and get product solutions, and then the linearity of the PDE and the boundary conditions would allow us to add finite linear combinations of those product solutions and get new solutions. I won't do this, but you should; it's good practice with separation of variables.

Instead, let's use all of our hard work on Fourier series to find a formal solution to the somewhat different problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ -\pi \le x \le \pi, \ t \ge 0 \\ u(-\pi,t) = u(\pi,t), \ u_x(-\pi,t) = u_x(\pi,t). \end{cases}$$

Also, we added boundary conditions on $u_x$. I am less concerned right now with getting *all* solutions to the wave equation as I am with getting *some* and with practicing Fourier series.

So here we go:

$$\begin{aligned} u_{tt} = c^2 u_{xx} &\Longrightarrow \widehat{u_{tt}}(k,t) = c^2 \widehat{u_{xx}}(k,t) \\ &\Longrightarrow \partial_t^2[\widehat{u}](k,t) = c^2 (ik)^2 \widehat{u}(k,t) \\ &\Longrightarrow \partial_t^2[\widehat{u}](k,t) + c^2 k^2 \widehat{u}(k,t) = 0. \end{aligned}$$

If it helps, put $w(t) = \widehat{u}(k,t)$ to see that $w$ solves $w'' + c^2k^2 w = 0$. When $k \ne 0$, of course we should think $w(t) = \alpha\cos(ckt) + \beta\sin(ckt)$, and when $k = 0$, we want[25] $w(t) = \beta t + \alpha$. We should not neglect the role of the parameter $k$ but rather allow $\alpha$ and $\beta$ to depend on $k$. So, we conclude

$$\widehat{u}(k,t) = \begin{cases} \alpha(k)\cos(ckt) + \beta(k)\sin(ckt), \ k \ne 0 \\ \beta(0)t + \alpha(0), \ k = 0 \end{cases} \tag{4.1.1}$$

for some functions $\alpha$ and $\beta$ defined on all the integers. (It is likely that $\alpha$ and $\beta$ will also depend on $c$, but I just don't feel like writing $\alpha_c(k)$ and $\beta_c(k)$. You don't either, I bet.)

[25] I peeked ahead and saw that it is better to write $\beta t + \alpha$ than $\alpha t + \beta$. We are going to want a "continuity in $k$" property here — yes, I know $k$ is an integer, not any real number, but bear with me — and so as $k \to 0$ we expect $w(t) \to \alpha$. Thus for $k = 0$ we want $\beta t + \alpha$, so we can evaluate $\beta \cdot 0 + \alpha = \alpha$. Make sense?

Experience with second-order linear ODE tells us that we need two initial conditions to solve for $\alpha$ and $\beta$. That is, we really want to be working with a problem of the form

$$\begin{cases} w'' + c^2k^2w = 0 \\ w(0) = \,? \\ w'(0) = \,??. \end{cases}$$

But $w(t) = \widehat{u}(k,t)$ and $w'(t) = \partial_t[\widehat{u}](k,t)$, so $w(0) = \widehat{u}(k,0)$ and $w'(0) = \partial_t[\widehat{u}](k,0)$. If we think about it, this means we want two initial conditions on $u$, and these are conditions on $u(\cdot,0)$ and $u_t(\cdot,0)$. Let's amend our original problem to

$$\begin{cases} u_{tt} = c^2u_{xx}, \ -\pi \le x \le \pi, \ t \ge 0 \\ u(x,0) = f(x), \ -\pi \le x \le \pi \\ u_t(x,0) = g(x), \ -\pi \le x \le \pi \\ u(-\pi,t) = u(\pi,t) = 0, \ u_x(-\pi,t) = u_x(\pi,t) = 0. \end{cases}$$

Then we want $\widehat{u}$ to satisfy (4.1.1) along with

$$\widehat{u}(k,0) = \widehat{f}(k) \quad \text{and} \quad \partial_t[\widehat{u}](k,0) = \widehat{g}(k).$$

In terms of (4.1.1), this means that for $k \neq 0$,

$$\widehat{f}(k) = \widehat{u}(k,0) = \big[\alpha(k)\cos(ckt) + \beta(k)\sin(ckt)\big]\big|_{t=0} = \alpha(k)$$

and

$$\widehat{g}(k) = \partial_t[\widehat{u}](k,0) = \frac{\partial}{\partial t}\big[\alpha(k)\cos(ckt) + \beta(k)\sin(ckt)\big]\bigg|_{t=0} = \beta(k)ck,$$

and so

$$\beta(k) = \frac{\widehat{g}(k)}{ck}, \ k \neq 0.$$

At $k = 0$, (4.1.1) says that we need

$$\widehat{f}(0) = \widehat{u}(0,0) = \big[\beta(0)t + \alpha(0)\big]\big|_{t=0} = \alpha(0)$$

and

$$\widehat{g}(0) = \partial_t[\widehat{u}](0,0) = \frac{\partial}{\partial t}\big[\beta(0)t + \alpha(0)\big]\bigg|_{t=0} = \beta(0).$$

Thus

$$\widehat{u}(k,t) = \begin{cases} \widehat{f}(k)\cos(ckt) + \dfrac{\widehat{g}(k)}{ck}\sin(ckt), \ k \neq 0 \\ \\ \widehat{f}(0) + \widehat{g}(0)t, \ k = 0. \end{cases}$$

We can compress this further and make both pieces look more alike if we rewrite

$$\frac{\sin(ckt)}{ck} = \left(\frac{\sin(ckt)}{ckt}\right)t,$$

which is valid for $k \neq 0$ and $t \neq 0$. (We've had $c \neq 0$ all along.) Now, the factor in parentheses is something you know from calculus already. Put

$$\operatorname{sinc}(\xi) := \begin{cases} \dfrac{\sin(\xi)}{\xi}, & \xi \neq 0 \\ \\ 1, & \xi = 0. \end{cases}$$

L'Hospital's rule, or Taylor series, or whatever will tell you that $\operatorname{sinc}(\cdot)$ is infinitely differentiable. Thus for $k \neq 0$ and $t \neq 0$, we have

$$\begin{aligned} \widehat{u}(k,t) = \widehat{f}(k)\cos(ckt) + \frac{\widehat{g}(k)}{ck}\sin(ckt) &= \widehat{f}(k)\cos(ckt) + \widehat{g}(k)\left(\frac{\sin(ckt)}{ckt}\right)t \\ &= \widehat{f}(k)\cos(ckt) + \widehat{g}(k)\operatorname{sinc}(ckt)t. \quad (4.1.2) \end{aligned}$$

It turns out that for $k = 0$ or $t = 0$, the equality above is still true. I will let you check that. We can therefore conclude

$$\widehat{u}(k,t) = \widehat{f}(k)\cos(ckt) + \widehat{g}(k)\operatorname{sinc}(ckt)t,$$

and so we have proved our first result about the wave equation.

**4.1.1 Theorem.**

*Let $f$ and $g$ be integrable on $[-\pi, \pi]$ and let $c > 0$. The formal solution to the wave IVP-BVP*

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\pi \leq x \leq \pi, \ t \geq 0 \\ u(x,0) = f(x), & -\pi \leq x \leq \pi \\ u_t(x,0) = g(x), & -\pi \leq x \leq \pi \\ u(-\pi,t) = u(\pi,t), & u_x(-\pi,t) = u_x(\pi,t) \end{cases}$$

*is*

$$u(x,t) = \sum_{k=-\infty}^{\infty} \left[\widehat{f}(k)\cos(ckt) + \widehat{g}(k)\operatorname{sinc}(ckt)t\right]e^{ikx}. \quad (4.1.3)$$

*We should interpret this sum as a formal series, i.e., a sequence of partial sums, unless we have more information about $f$ and $g$.*

Convergence of (4.1.3) will depend greatly on properties of $f$ and $g$; in particular, there is no factor in (4.1.3) that is nicely small in $k$ for all $t > 0$, unlike our friend $e^{-k^2 t}$ in the heat equation.

This is where we finished on Monday, October 31, 2022.

## 4.2. D'Alembert's formula.

It is probably difficult to visualize from the formal Fourier series solution (4.1.3) for the wave equation what that solution is doing. Shouldn't it have some, well, wave-like behavior? A different approach is to abandon Fourier series and boundary conditions and just think about the pure wave equation $u_{tt} = c^2 u_{xx}$. We will assume now that both $x$ and $t$ can be any real number; after all, if we can solve the problem on $\mathbb{R}^2$, then surely we can solve it for $x \in [0, L]$ or any bounded interval.

### *4.2.1. Formulaic derivations.*

The right idea is to factor. The wave equation $u_{tt} = c^2 u_{xx}$ also reads

$$0 = u_{tt} - c^2 u_{xx} = \partial_t^2[u] - c^2\partial_x^2[u] = (\partial_t^2 - c^2\partial_x^2)[u] = (\partial_t - c\partial_x)(\partial_t + c\partial_x)[u].$$

Huh? That is a nice bit of symbol-pushing, based on our long experience factoring differences of squares, but what does it mean? First, I am abbreviating

$$\partial_t = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_x = \frac{\partial}{\partial x}.$$

Next,

$$(\partial_t + c\partial_x)[u] = u_t + cu_x,$$

and if $v := (\partial_t + c\partial_x)[u]$, then

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)[u] = (\partial_t - c\partial_x)[v] = v_t - cv_x.$$

So, if $u$ solves the wave equation $u_{tt} = c^2 u_{xx}$, then $v$ as defined above solves the transport equation

$$v_t - cv_x = 0,$$

which we might write in the more familiar form

$$-cv_x + v_t = 0.$$

Of course, this has the solution

$$v(x, t) = p(x - (-c)t) = p(x + ct)$$

for some arbitrary function $p \in \mathcal{C}^1(\mathbb{R})$. This is Theorem 1.5.1.

This means that the actual solution $u$ to the wave equation has to satisfy

$$(\partial_t + c\partial_x)[u] = v,$$

which is to say

$$cu_x + u_t = p(x + ct).$$

This is a nonhomogeneous transport equation, which we solved in Theorem 1.5.7. That theorem says that the general solution to

$$aw_x + bw_t = f(x, t)$$

has the form

$$w(x,t) = q(bx-at) + \frac{1}{b}\int_0^t f\left(\left[\frac{bx-at}{b}\right] + \frac{as}{b}, s\right)\ ds,$$

where $q \in \mathcal{C}^1(\mathbb{R})$ is arbitrary.

Here $a = c$, $b = 1$, and $f(x,t) = p(x+ct)$. Thus the solution $u$ to the wave equation has the form

$$u(x,t) = q(x-ct) + \int_0^t p\big((x-ct)+cs+cs\big)\ ds = q(x-ct) + \int_0^t p(x-ct+2cs)\ ds. \quad (4.2.1)$$

It will be worthwhile to clean up the integral a bit. Substitute $\xi = x - ct + 2cs$, so $d\xi = 2c\ ds$, and

$$\int_0^t p(x-ct+2cs)\ ds = \int_{x-ct}^{x-ct+2ct} p(\xi)\ d\xi = \int_{x-ct}^{x+ct} p(\xi)\ d\xi.$$

Then the solution $u$ to the wave equation is[26]

$$u(x,t) = q(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} p(s)\ ds. \quad (4.2.2)$$

This is a pretty clean formula for the solution, much better than our stab at Fourier series in Theorem 4.1.1. Also, note that because the independent variables $x$ and $t$ only appear in the limits of integration of the integral in (4.2.2), as opposed to within the integrand of the integral in (4.2.1), we can differentiate (4.2.2) without Leibniz's rule.

If we stare at (4.2.2) and second-guess ourselves in the right way, we'll notice that we only took $q$ to be continuously differentiable, yet two derivatives of $u$ with respect to $x$ and $t$ need to exist. So, we want to be sure that we can differentiate $q$ twice. Is this allowed? Yes: remember that we are working backward. We started out assuming that $u$ solved the wave equation $u_{tt} = c^2 u_{xx}$; per our conventions, this means that $u \in \mathcal{C}^2(\mathbb{R}^2)$. Now, notice that we can get $q$ to show up in (4.2.2) by itself if we take $t = 0$:

$$u(x,0) = q(x) + \frac{1}{2c}\int_x^x p(s)\ ds = q(x), \quad (4.2.3)$$

since $\int_x^x p(s)\ ds = 0$ for all $x$. Thus since $u \in \mathcal{C}^2(\mathbb{R}^2)$, we have $u(\cdot,0) \in \mathcal{C}^2(\mathbb{R})$, and so $q \in \mathcal{C}^2(\mathbb{R})$. (Note that we cannot say more about $p$ than $p \in \mathcal{C}^1(\mathbb{R})$, since the integral on $p$ "consumes" the first derivatives in $x$ and $t$.)

Here is what we have shown.

**4.2.1 Lemma.**

*Let $c > 0$ and suppose that $u \in \mathcal{C}^2(\mathbb{R}^2)$ solves the wave equation $u_{tt} = c^2 u_{xx}$ on $\mathbb{R}^2$. Then there exist $p \in \mathcal{C}^1(\mathbb{R})$ and $q \in \mathcal{C}^2(\mathbb{R})$ such that*

$$u(x,t) = q(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} p(s)\ ds. \quad (4.2.4)$$

[26] I am going back to $s$ as the variable of integration here.

We just saw that the initial values $u(x,0)$ completely determine the choice of $q$ in this formula for $u$. We might expect that a choice of values for $u_t(x,0)$ will help determine $p$, and indeed it does. Specifically, we will solve the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x,\ t < \infty \\ u(x,0) = f(x), & -\infty < x < \infty \\ u_t(x,0) = g(x), & -\infty < x < \infty. \end{cases}$$

Assuming that $u$ has the form (4.2.4), the calculation in (4.2.3) shows $q = f$, and so

$$u(x,t) = f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} p(s)\ ds.$$

We can then calculate $u_t$:

$$u_t(x,t) = f'(x-ct)(-c) + \frac{1}{2c}\frac{\partial}{\partial t}\left[\int_{x-ct}^{x+ct} p(s)\ ds\right] = -cf'(x-ct) + \frac{p(x+ct)+p(x-ct)}{2}. \tag{4.2.5}$$

More precisely, to differentiate the integral, we rewrote

$$\int_{x-ct}^{x+ct} p(s)\ ds = \int_{x-ct}^{0} p(s)\ ds + \int_{0}^{x+ct} p(s)\ ds = -\int_{0}^{x-ct} p(s)\ ds + \int_{0}^{x+ct} p(s)\ ds$$

and used the fundamental theorem of calculus[27] to find

$$\begin{aligned}\frac{\partial}{\partial t}\left[\int_{x-ct}^{x+ct} p(s)\ ds\right] = -\frac{\partial}{\partial t}\left[\int_{0}^{x-ct} p(s)\ ds\right] + \frac{\partial}{\partial t}\left[\int_{0}^{x+ct} p(s)\ ds\right] &= -p(x-ct)(-c)+p(x+ct)(c) \\ &= p(x+ct)c + p(x-ct)c.\end{aligned}$$

Then (4.2.5) gives

$$g(x) = u_t(x,0) = -cf'(x) + \frac{p(x)+p(x)}{2} = -cf'(x) + p(x),$$

and so

$$p(x) = g(x) + cf'(x).$$

Thus

$$u(x,t) = f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \left[cf'(s) + g(s)\right]\ ds,$$

and we can integrate $f'$ to conclude

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds.$$

We have now proved the following result.

---

[27] $\dfrac{\partial}{\partial t}\left[\int_0^t v(s)\ ds\right] = v(t)$ and $\dfrac{\partial}{\partial t}\left[\int_0^{w(t)} v(s)\ ds\right] = v(w(t))w'(t)$.

**4.2.2 Lemma.**

*Let $c > 0$ and take $f \in \mathcal{C}^2(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$. Then the function*

$$u(x,t) := \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds \tag{4.2.6}$$

*is the unique solution to the wave equation IVP*

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ -\infty < x, \ t < \infty \\ u(x,0) = f(x), \ -\infty < x < \infty \\ u_t(x,0) = g(x), \ -\infty < x < \infty. \end{cases} \tag{4.2.7}$$

This is vastly more explicit and general than anything we've said yet about the heat equation. Our success here arises from the extra $t$-derivative in the wave equation and the near-miracle of how factoring the difference of squares transcends arithmetic to the level of differential operators.

This is where we finished on Friday, November 4, 2022.

Let's do a couple of explicit IVP.

**4.2.3 Example.**

*Let $c = 1$. Solve the IVP (4.2.7) for the following initial data. Graph the solutions as functions $u(\cdot, t)$ of $x$ with time $t$ fixed at various points. What do you observe?*

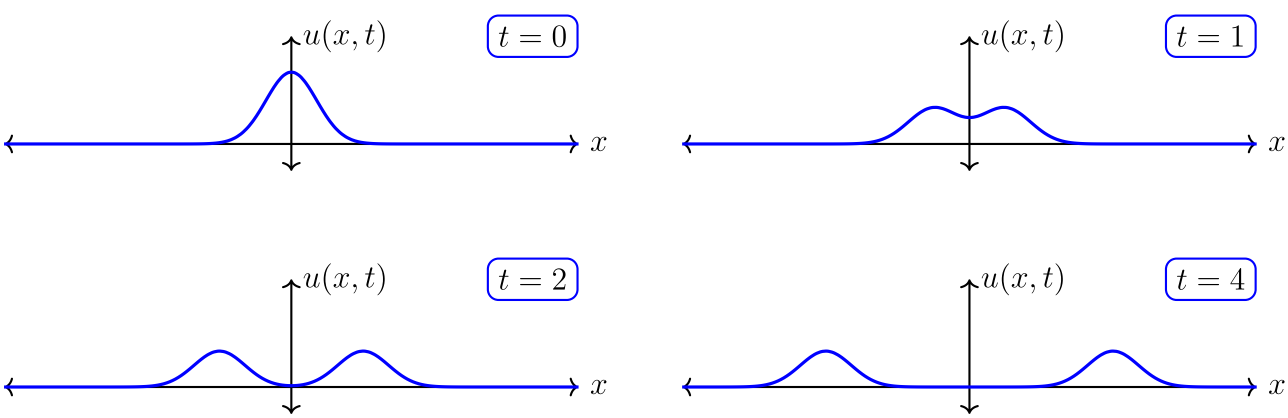
**(i)** $f(x) = 2e^{-x^2}$, $g(x) = 0$

**(ii)** $f(x) = 10e^{-x^2}$, $g(x) = \cos(x)$

**Solution.** **(i)** The formula (4.2.6) tells us that the solution is

$$u(x,t) = \frac{2e^{-(x+t)^2} - 2e^{-(x-t)^2}}{2} + \frac{1}{2}\int_{x-ct}^{x+ct} 0\ ds = e^{-(x+t)^2} + e^{-(x-t)^2}.$$

Here are some plots.

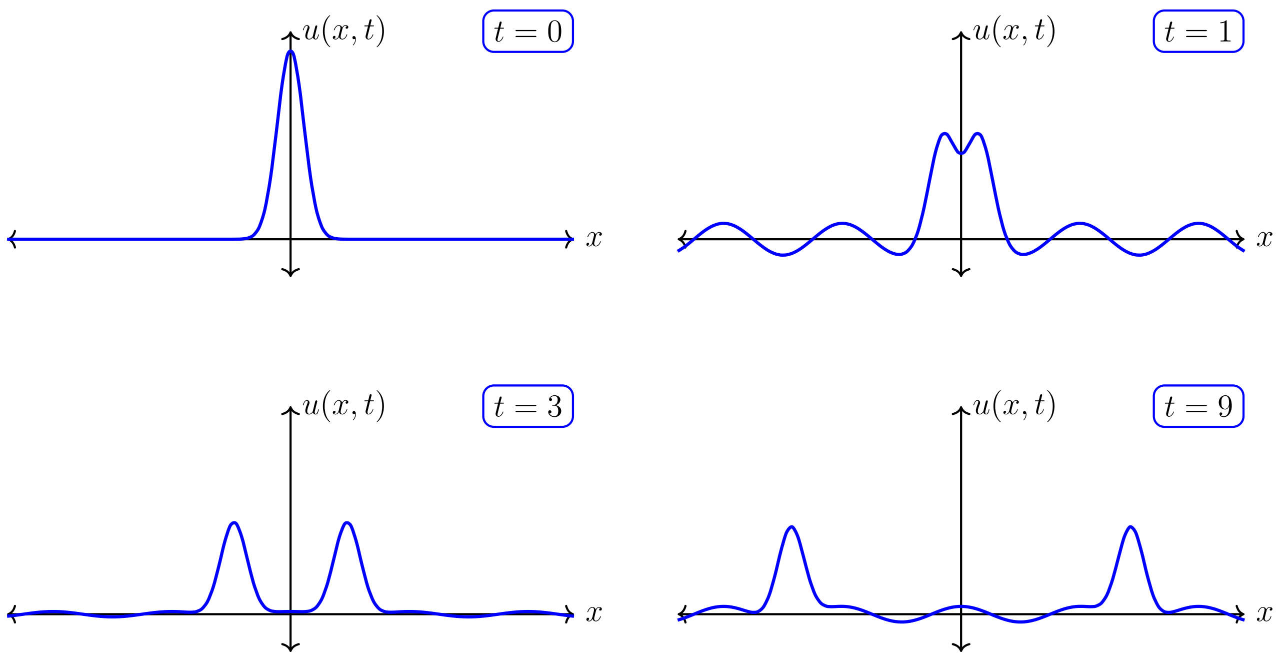


It looks like the initial condition $u(x,0) = 2e^{-x^2}$ has split into two smaller "pulses," one moving to the right and the other to the left. This is exactly what the formula $u(x,t) = e^{-(x+t)^2} - e^{-(x-t)^2}$ says: as $t$ increases, the graph of $x \mapsto e^{-(x+t)^2}$ moves to the left, while $x \mapsto e^{-(x-t)^2}$ moves to the right. However, the graph of $u(\cdot,t)$ is not really just the graph of $x \mapsto e^{-(x+t)^2}$ superimposed on the graph of $x \mapsto e^{-(x-t)^2}$; there is an interaction between the two graphs due to the sum in the definition of $u$. Nonetheless, this interaction is very "weak" for $x$ or $t$ large because $e^{-\xi^2}$ is very small when $\xi$ is very large.

**(ii)** The formula (4.2.6) tells us that the solution is

$$u(x,t) = \frac{5e^{-(x+t)^2} + 5e^{-(x-t)^2}}{2} + \frac{1}{2}\int_{x-ct}^{x+ct} \cos(s)\,ds = 5\big[e^{-(x+t)^2} + e^{-(x-t)^2}\big] + \frac{\big[\sin(x+ct) - \sin(x-ct)\big]}{2}.$$

Here are some graphs.




Again, it looks like the initial condition "splits" into two "smaller" pulses that travel to the right and left; now there is more "noise" between them due to the nonzero initial condition on $u_t$. In particular, the pulses are not nearly as "identical" as they were for the previous initial data; contrast times 1, 3, and 9 with the previous pulses for times 1, 2, and 4. ▲

*4.2.2. Traveling wave interpretations.*

Between the analytic results above and the numerically produced graphs, I think we have more than enough information to answer the question *Where is the wave behavior in the wave equation?* First, we know the very tractable formulas (4.2.4) for the general solution to the wave equation on $\mathbb{R}^2$ and (4.2.6) for the wave IVP. Both formulas look alike in that they depend on the expressions $x \pm ct$, and the only real difference is that (4.2.6) explicitly incorporates the initial data for $u(\cdot,0)$ and $u_t(\cdot,0)$. We know from our nontrivial experience with traveling waves so far that if $p = p(\xi)$ is a function of a single variable, then the functions

$(x,t) \mapsto p(x \pm ct)$ represent traveling waves. We saw "sort of" traveling wave behavior in Example 4.2.3, except the graphs weren't exactly the horizontal translates of a particular profile.

Here is how we can make analytic sense of this behavior. Go back to (4.2.6), which tells us that if $u_{tt} = c^2 u_{xx}$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, then

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds.$$

Rewrite this as

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(s)\ ds\right) + \frac{1}{2}\left(\frac{f(x-ct)}{2} - \frac{1}{c}\int_0^{x-ct} g(s)\ ds\right).$$

Put

$$\phi(\xi) := \frac{1}{2}\left(f(\xi) + \frac{1}{c}\int_0^{\xi} g(s)\ ds\right) \quad \text{and} \quad \psi(\xi) := \frac{1}{2}\left(f(\xi) - \frac{1}{c}\int_0^{\xi} g(s)\ ds\right).$$

Then

$$u(x,t) = \phi(x+ct) + \psi(x-ct). \tag{4.2.8}$$

This is our traveling wave behavior: the solution $u$ to the wave equation is the superposition of a left-moving traveling wave $(x,t) \mapsto \phi(x+ct)$ and a right-moving traveling wave $(x,t) \mapsto \psi(x-ct)$. (Here "left" and "right" assume $t > 0$; otherwise, the propagation directions are reversed.) Since the graphs of $\phi$ and $\psi$ move in opposite directions, we might call the structure of (4.2.8) a "counterpropagating" traveling wave.

We conclude the following general formula for the solution to the wave equation.

**4.2.4 Theorem (D'Alembert's formula).**

*A function $u \in \mathcal{C}^2(\mathbb{R}^2)$ solves the wave equation $u_{tt} = c^2 u_{xx}$ on $\mathbb{R}^2$ if and only if there are functions $\phi$, $\psi \in \mathcal{C}^2(\mathbb{R})$ such that*

$$u(x,t) = \phi(x-ct) + \psi(x+ct).$$

Our arguments above proved D'Alembert's formula using results about the initial value problem, but we don't need initial data to get it. Indeed, D'Alembert's formula pops out of Lemma 4.2.1, too; you should check that.

## 4.3. The method of images.

We have discussed two problems for the wave equation: an IVP-BVP posed on the spatial interval $[-\pi, \pi]$, which we solved (opaquely) with Fourier series, and an IVP posed for $x \in \mathbb{R}$, which we solved quite transparently with, among other things, D'Alembert's formula. A bounded spatial domain probably makes more physical sense, if the wave equation is supposed to model the displacement of a string (strings usually have finite length, right?), but the infinite spatial domain made a lot of the analysis nicer — we didn't have to worry about boundary conditions.

Now we do. Here is a reasonable IVP-BVP for a string of length $L > 0$ when its left and right ends are fixed (i.e., they don't move, so their displacement is always 0):

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ 0 \le x \le L, \ -\infty < t < \infty \\ u(x,0) = f(x), \ 0 \le x \le L \\ u_t(x,0) = g(x), \ 0 \le x \le L \\ u(0,t) = u(L,t) = 0, \ -\infty < t < \infty. \end{cases} \tag{4.3.1}$$

We will solve this through the classical mathematical method of "faking it." The problem is that our initial data $f$ and $g$ are only defined on $[0, L]$. Suppose that we knew their values on all of $\mathbb{R}$. Then we could use any of the results of the previous section to find a formula for $u$, which we would then test against the boundary conditions.

This raises the mathematical question of *extending* a function. For our purposes here, an **EXTENSION** of a function $h\colon [0, L] \to \mathbb{R}$ is a function $\widetilde{h}\colon \mathbb{R} \to \mathbb{R}$ such that $\widetilde{h}(x) = h(x)$ for all $x \in [0, L]$. However, there are infinitely many ways to extend a function; just take your favorite real number $c$ and put $\widetilde{h}(x) = c$ for $x < 0$ and $x > L$. We want to find the *right* extensions of our initial data. The questions that we raise along the way will be no less valuable than the actual solution that we find to our IVP-BVP (4.3.1).

Let's say that we have extended $f$ and $g$ to all of $\mathbb{R}$ as $\widetilde{f}$ and $\widetilde{g}$, respectively. Just to be clear, this means that

$$\widetilde{f}(x) = f(x) \quad \text{and} \quad \widetilde{g}(x) = g(x) \quad \text{for} \quad 0 \le x \le L.$$

If $\widetilde{f} \in \mathcal{C}^2(\mathbb{R})$ and $\widetilde{g} \in \mathcal{C}^1(\mathbb{R})$, then we can use Lemma 4.2.2 to write the solution to

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ -\infty < x, \ t < \infty \\ u(x,0) = \widetilde{f}(x), \ -\infty < x < \infty \\ u_t(x,0) = \widetilde{g}(x), \ -\infty < x < \infty \end{cases}$$

as

$$u(x,t) = \frac{\widetilde{f}(x+ct) + \widetilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(s)\ ds. \tag{4.3.2}$$

So, however we extend $f$ and $g$, we need the extensions to be sufficiently differentiable so that we can invoke Lemma 4.2.2. At the very least we will need $f \in \mathcal{C}^2([0, L])$ and $g \in \mathcal{C}^1([0, L])$ if their extensions to $\mathbb{R}$ are to be sufficiently differentiable.

Let's proceed on faith and hope with the assumption that we *have* found suitable extensions $\widetilde{f}$ and $\widetilde{g}$. Then we know that $u$ given by the formula (4.3.2) solves $u_{tt} = c^2 u_{xx}$ on $\mathbb{R}^2$ and meets the initial conditions on $\mathbb{R}$ so in particular $u$ solves the wave equation for $x \in [0, L]$ and meets the initial conditions on $[0, L]$. More precisely, for $0 \le x \le L$,

$$u(x,0) = \widetilde{f}(x) = f(x) \quad \text{and} \quad u_t(x,0) = \widetilde{g}(x) = g(x).$$

This does not teach us anything useful about what $\widetilde{f}$ and $\widetilde{g}$ should be. What do the boundary conditions have to say? Let's check. We need

$$0 = u(x,t) = \frac{\widetilde{f}(ct) + \widetilde{f}(-ct)}{2} + \frac{1}{2c}\int_{-ct}^{ct} \widetilde{g}(s)\ ds \tag{4.3.3}$$

and

$$0 = u(L,t) = \frac{\widetilde{f}(L+ct) + \widetilde{f}(L-ct)}{2} + \frac{1}{2c}\int_{L-ct}^{L+ct} \widetilde{g}(s)\ ds. \qquad (4.3.4)$$

If we stare at (4.3.3) for a while, we might see that we can achieve it if the separate equalities

$$\widetilde{f}(ct) + \widetilde{f}(-ct) = 0 \quad \text{and} \quad \int_{-ct}^{ct} \widetilde{g}(s)\ ds = 0$$

hold for all $t$. And staring at *these* equalities for some further time might suggest that we can achieve *them* by taking $\widetilde{f}$ and $\widetilde{g}$ to be odd. That gives

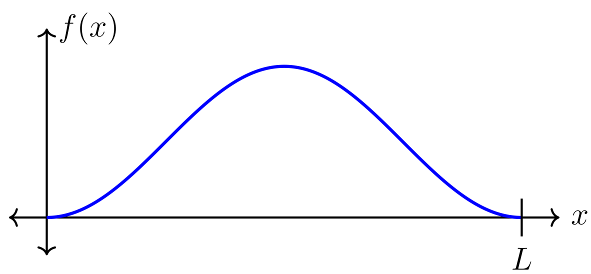
$$\widetilde{f}(x) = -\widetilde{f}(-x) \quad \text{and} \quad \int_{-x}^{x} \widetilde{g}(s)\ ds = 0$$

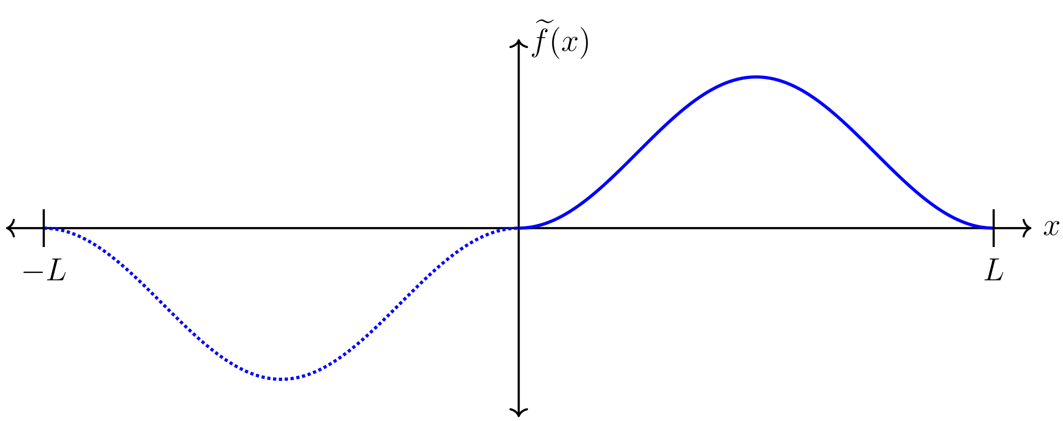
for all $x \in \mathbb{R}$.

Now, we only know what $f$ and $g$ are on $[0,L]$. So, the best we can do so far is to say

$$\widetilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x < 0 \end{cases} \quad \text{and} \quad \widetilde{g}(x) = \begin{cases} g(x), & 0 \le x \le L \\ -g(-x), & -L \le x < 0. \end{cases} \qquad (4.3.5)$$

Below I'll draw one possible graph for $f$ on $[0,L]$ and then the odd extension (you might say, "reflection") $\widetilde{f}$ on $[-L,L]$. We'll use these graphs later, too.




This is where we finished on Monday, November 7, 2022.

Watch out! We need to check the continuity and differentiability of $\widetilde{f}$ and $\widetilde{g}$. Since $f$ and $g$ are sufficiently differentiable on $[0, L]$, I hope it is clear that $\widetilde{f}$ and $\widetilde{g}$ are sufficiently differentiable on $[-L, 0)$ and $(0, L]$, and so we only need to worry about $x = 0$. Odd functions must be 0 at 0, so we need $\widetilde{f}(0) = 0$ and $\widetilde{g}(0) = 0$. Also, we want $\widetilde{f}$ and $\widetilde{g}$ to be continuous at $x = 0$, and so we need

$$\lim_{x \to 0} \widetilde{f}(x) = 0 \quad \text{and} \quad \lim_{x \to 0} \widetilde{g}(x) = 0. \tag{4.3.6}$$

You can use the definitions of $\widetilde{f}$ and $\widetilde{g}$ in (4.3.5) and the continuity of $f$ and $g$ on $[0, L]$ to conclude that the limits (4.3.6) force

$$f(0) = 0 \quad \text{and} \quad g(0) = 0.$$

This is actually a completely reasonable condition on the initial data $f$ and $g$ for the IVP-BVP (4.3.1). You should show that if there is a solution $u$ to this problem, then the following must be true:

$$f(0) = f''(0) = f(L) = f''(L) = 0 \quad \text{and} \quad g(0) = g(L). \tag{4.3.7}$$

Thus if we assume (4.3.7), then $\widetilde{f}$ and $\widetilde{g}$ defined by (4.3.5) are continuous.

To check the requisite differentiability of $\widetilde{f}$ and $\widetilde{g}$ at $x = 0$, we would need to do some annoying (but worthwhile) work with left and right limits of difference quotients using the piecewise formulas in (4.3.5). I don't think anyone wants to see that right now, but it's a good exercise, and its success hinges on the assumptions (4.3.7).

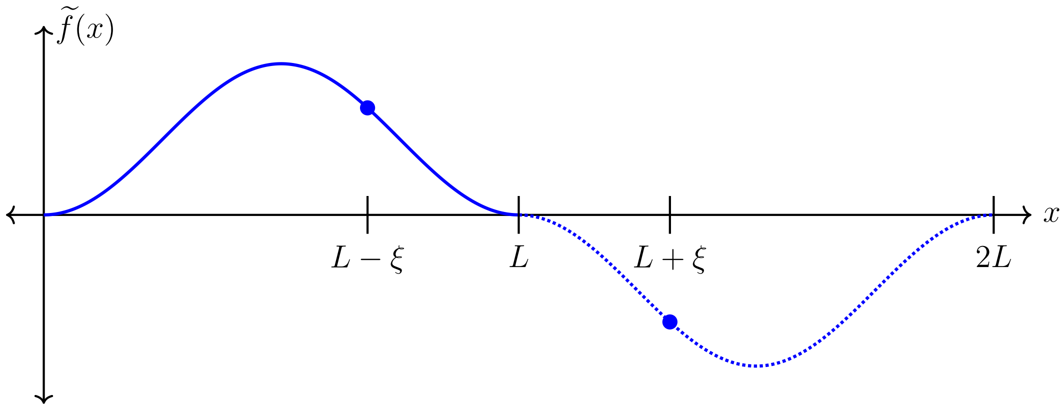
All of this work has followed from the boundary condition (4.3.3). We still need to meet the other condition, (4.3.4). That will follow if

$$\widetilde{f}(L + ct) + \widetilde{f}(L - ct) = 0 \quad \text{and} \quad \int_{L-ct}^{L+ct} \widetilde{g}(s)\ ds = 0$$

for all $t$. Equivalently, we need

$$\widetilde{f}(L + x) = -\widetilde{f}(L - x) \quad \text{and} \quad \int_{L-x}^{L+x} \widetilde{g}(s)\ ds = 0$$

for all $x \in \mathbb{R}$. This is, euphemistically, an "oddness about $L$" condition; if $L$ were 0 (which it isn't), then the condition $\widetilde{f}(0 + x) = -\widetilde{f}(0 - x)$ just says that $\widetilde{f}$ is odd. In the case at hand, with $L > 0$, here is what the condition $\widetilde{f}(L + x) = -\widetilde{f}(L - x)$ suggests for the graph of $\widetilde{f}$ beyond the interval $[-L, L]$.




I think it looks like we should just copy and paste the graph of $\widetilde{f}$ from $[-L, L]$ to the whole real line in a $2L$-periodic fashion. Indeed, I challenge you to show (this is not a hard challenge) that if $\widetilde{f}$ is odd with $\widetilde{f}(L+x) = -\widetilde{f}(L-x)$ for all $x$, then $\widetilde{f}$ must be $2L$-periodic. Conversely, if we start with the odd function $\widetilde{f}$ defined on $[-L, L]$ by (4.3.5) and extend $\widetilde{f}$ to all of $\mathbb{R}$ in a $2L$-periodic manner, then you can show that $\widetilde{f}(L+x) = -\widetilde{f}(L-x)$ always holds. We will need to worry about continuity and differentiability at integer multiples of $L$; this is where the conditions $f(L) = f''(L) = 0$ come into play.

Checking all of these details is good math and essential practice for you, but it's best done at your own pace, not in a mind-numbing presentation in class. I'll give you some particular guidance on this in forthcoming practice. Let's wrap up with a theorem summarizing all of our work.

**4.3.1 Theorem.**

*Let $c$, $L > 0$. Let $f \in \mathcal{C}^2([0, L])$ and $g \in \mathcal{C}^1([0, L])$ satisfy*

$$f(0) = f''(0) = f(L) = f''(L) = g(0) = g(L).$$

*Define $\widetilde{f}$ and $\widetilde{g}$ on $[-L, L]$ by*

$$\widetilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x < 0 \end{cases} \qquad \text{and} \qquad \widetilde{g}(x) = \begin{cases} g(x), & 0 \le x \le L \\ -g(-x), & -L \le x < 0, \end{cases}$$

*and otherwise extend $\widetilde{f}$ and $\widetilde{g}$ to all of $\mathbb{R}$ by defining $\widetilde{f}(x+2L) = \widetilde{f}(x)$ and $\widetilde{g}(x+2L) = \widetilde{g}(x)$. Then $\widetilde{f} \in \mathcal{C}^2(\mathbb{R})$, $\widetilde{g} \in \mathcal{C}^1(\mathbb{R})$, and the function*

$$u(x,t) = \frac{\widetilde{f}(x+ct) + \widetilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(s)\ ds$$

*solves the IVP-BVP*

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 \le x \le L, \ -\infty < t < \infty \\ u(x,0) = f(x), & 0 \le x \le L \\ u_t(x,0) = g(x), & 0 \le x \le L \\ u(0,t) = u(L,t) = 0, & -\infty < t < \infty. \end{cases}$$

I think the journey to the theorem, with the many questions that we asked and answered about extending $f$ and $g$, is much more valuable than the actual result. By the way, this result is sometimes called the **METHOD OF IMAGES**: we took the "images" (graphs) of the initial data on $[0, L]$ and extended/reflected those images to define functions on all of $\mathbb{R}$.

# 5. The Fourier Transform

We have really only studied three PDE in this course: the transport equation, the heat equation, and the wave equation. We had complete success with the first and the third; success with the transport equation hinged on its special structure as a "gradient" relationship, and success with the wave equation hinged on factoring it as a "product" of transport equations. Our success with the heat equation was much more limited, outside of very special initial data.

What else can we do? If we meet a problem posed on a finite spatial domain (i.e., the $x$ coordinate is constrained to an interval of the form $[a, b]$), and if we have reasonable boundary conditions, then we could possibly use Fourier series. Unfortunately, that raises a number of convergence questions, and the resulting formulas aren't all that transparent. However, our recent work with the wave equation suggests that we might be able to use results for a problem posed on an infinite spatial domain to solve a problem posed on a finite spatial interval. Moreover, we might just start with an infinite spatial domain — mathematically, why not? Physically, an infinite spatial domain just means that the object under consideration is "really long."

One of the best tools for handling problems posed on an infinite spatial domain is the Fourier transform. To understand it, we first need to talk briefly about certain kinds of improper integrals.

## 5.1. Improper integrals.

Denote by $L^1_{\text{loc}}$ the set of all functions $f \colon \mathbb{R} \to \mathbb{C}$ such that $f$ is integrable on any closed, bounded interval $[a, b] \subseteq \mathbb{R}$. In other words, for all real numbers $a$ and $b$, the integral $\int_a^b f(x)\, dx$ always exists. Every constant function, for example, is locally integrable. More generally, since a continuous function is always integrable, we have $\mathcal{C}(\mathbb{R}) \subseteq L^1_{\text{loc}}$; even more generally, if $f$ is piecewise continuous on $\mathbb{R}$, then $f \in L^1_{\text{loc}}$.

Let $f \in L^1_{\text{loc}}$ and suppose that the limits

$$\lim_{R\to\infty} \int_0^R f(x)\, dx \quad \text{and} \quad \lim_{R\to\infty} \int_{-R}^0 f(x)\, dx \tag{5.1.1}$$

exist. (The assumption $f \in L^1_{\text{loc}}$ guarantees that the integral $\int_0^R f(x)\, dx$ exists for all $R > 0$ and likewise that $\int_R^0 f(x)\, dx$ exists for all $R < 0$.) Then we define the **improper integral** of $f$ over $(-\infty, \infty)$ as

$$\int_{-\infty}^{\infty} f(x)\, dx := \lim_{R\to\infty} \int_0^R f(x)\, dx + \lim_{R\to\infty} \int_{-R}^0 f(x)\, dx. \tag{5.1.2}$$

We also say that $\int_{-\infty}^{\infty} f(x)\, dx$ **converges** if the two limits (5.1.1) exist. Of course, the two limits in (5.1.1) are the improper integrals $\int_0^{\infty} f(x)\, dx$ and $\int_{-\infty}^0 f(x)\, dx$, but we will not work all that much with these "one-sided" improper integrals.

Now denote by $L^1$ the set of all functions $f \in L^1_{\text{loc}}$ such that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges. To be clear, $f \in L^1$ if and only if $|f|$ is improperly integrable on $(-\infty, \infty)$. For $f \in L^1$, we will write

$$\|f\|_{L^1} := \int_{-\infty}^{\infty} |f(x)|\, dx,$$

and we call the value $\|f\|_{L^1}$ the $L^1$-**NORM** of $f$.

To determine if a given function $f \in L^1_{\text{loc}}$ is also in $L^1$, we can always try to compute the improper integral of $|f|$ over $(-\infty, \infty)$. Often, however, it is more convenient to use the comparison test for improper integrals; in our language, this test reads as follows.

**5.1.1 Theorem (Comparison test for improper integrals).**

*Let $f \in L^1_{\text{loc}}$ and $g \in L^1$ with $|f(x)| \le |g(x)|$ for all $x$. Then $f \in L^1$ and $\|f\|_{L^1} \le \|g\|_{L^1}$.*

If $f \in L^1$, then we can simplify the computation of $\int_{-\infty}^{\infty} f(x)\, dx$ as follows:

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx.$$

That is, if we know that $\int_{-\infty}^{\infty} f(x)\, dx$ converges, then we can compute it via the "symmetric" limit above, instead of as the sum (5.1.2).

## 5.2. Definition of the Fourier transform.

We will not give any attempt at motivating the definition of the Fourier transform. Our textbook provides both a comprehensive and accessible overview of integral transforms (pp. 415–418) and a nice connection of the Fourier transform to complex Fourier series (pp. 448–449). We will view the Fourier transform as a powerful instrument for analyzing both ODE and PDE.

Suppose that $f \in L^1$ and let $k \in \mathbb{R}$. Since the function $e_k(x) := e^{ikx}$ is continuous on $\mathbb{R}$, properties of integrals tell us that $fe_k \in L^1_{\text{loc}}$. Moreover, we can estimate $|f(x)e_k(x)| = |f(x)|$, so since $f \in L^1$, the comparison test tells us $fe_k \in L^1$ as well. Thus the following (incredibly important) integral is defined.

**5.2.1 Definition.**

*Let $f \in L^1$ and $k \in \mathbb{R}$. The* **FOURIER TRANSFORM** *of $f$ at $k$ is*

$$\mathfrak{F}[f](k) = \widehat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx.$$

The notation $\widehat{f}(k)$ of course resembles the notation for the complex Fourier coefficient. The factor of $1/\sqrt{2\pi}$ is present for reasons of convenience (and confusion) that will be apparent later; it is also present for the sake of controversy, as there are multiple natural ways that a factor of $2\pi$, or something like it, can appear in the Fourier transform.

This is where we finished on Wednesday, November 9, 2022.

The Fourier transform of a function in $L^1$ is itself a function. What kind of function the Fourier transform is depends greatly on the function being transformed.

**5.2.2 Example.**

*Let*

$$f(x) := \begin{cases} 0, & x < 0 \\ e^{-x}, & x \geq 0. \end{cases}$$

*Check that* $f \in L^1$ *and calculate* $\|f\|_{L^1}$ *and* $\widehat{f}$.

**Solution.** By inspection, $f$ is piecewise continuous on $(-\infty, \infty)$, and so $f \in L^1_{\text{loc}}$. For $R < 0$, we have $\int_R^0 f(x)\ dx = 0$, and so $\lim_{R\to\infty} \int_R^0 |f(x)|\ dx = 0$, while for $R > 0$, we have

$$\int_0^R |f(x)|\ dx = \int_0^R e^{-x}\ dx = -(e^{-R} - 1) = 1 - e^{-R}.$$

Then

$$\lim_{R\to\infty} \int_0^R |f(x)|\ dx = \lim_{R\to\infty} (1 - e^{-R}) = 1,$$

and so

$$\|f\|_{L^1} = \int_{-\infty}^{\infty} |f(x)|\ dx = \lim_{R\to-\infty} \int_{-R}^{0} |f(x)|\ dx + \lim_{R\to\infty} \int_0^R |f(x)|\ dx = 1.$$

Thus $f \in L^1$.

Now we calculate the Fourier transform. We have

$$\mathfrak{F}[f](k) = \widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\ dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_0^R e^{-x} e^{-ikx}\ dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_0^R e^{-(1+ik)x}\ dx.$$

Since $1 + ik \neq 0$ for all real $k$, we can integrate

$$\int_0^R e^{-(1+ik)x}\ dx = -\frac{e^{-(1+ik)x}}{1+ik}\bigg|_{x=0}^{x=R} = \frac{1}{1+ik} - \frac{e^{-(1+ik)R}}{1+ik}.$$

To calculate the limit as $R \to \infty$, we use the squeeze theorem:

$$0 \leq \left|\frac{e^{-(1+ik)R}}{1+ik}\right| = \frac{|e^{-R}e^{-ikR}|}{|1+ik|} = \frac{e^{-R}}{\sqrt{1+k^2}}.$$

Since

$$\lim_{R\to\infty} \frac{e^{-R}}{\sqrt{1+k^2}} = 0,$$

we have

$$\lim_{R\to\infty} \int_0^R e^{-(1+ik)x}\ dx = \frac{1}{1+ik},$$

and thus

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}(1+ik)}.$$

It may (or may not) be interesting a few properties of this transform. First, although $f \in L^1$, $\widehat{f} \notin L^1$; you can, and should, show that

$$\int_{-\infty}^{\infty} |\widehat{f}(k)| \ dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{1+k^2}}$$

diverges. However, $\widehat{f}$ is smoother than $f$ in the sense that $f$ has a jump discontinuity at $x = 0$, but $\widehat{f}$ is infinitely differentiable on $\mathbb{R}$; effectively, $\widehat{f}$ is a rational function whose denominator is never 0 on $\mathbb{R}$. We will explore this later in a more general context via Leibniz's rule; for now, imagine differentiating the integral defining $\widehat{f}$ with respect to $k$. ▲

The Fourier transform has many properties that we will explore in due course. Let's start with one of its most exciting features: what it does to derivatives.

**5.2.3 Theorem (Fourier transform of derivative).**

*Let $f \in L^1$ be differentiable and suppose $f' \in L^1$ as well. Then*

$$\widehat{f'}(k) = (ik)\widehat{f}(k).$$

**Proof.** With $e_k(x) = e^{-ikx}$, the comparison test gives $f'e_k \in L^1$, since $f' \in L^1$. Thus

$$\widehat{f'}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-ikx} \ dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} f'(x)e^{-ikx} \ dx.$$

We are using the "symmetric" limit here since $f'e_k \in L^1$. For $R > 0$ fixed, we can compute the integral with integration by parts. Take

$$u = e^{-ikx} \qquad dv = f'(x) \ dx$$
$$du = -ike^{-ikx} \ dx \qquad v = f(x).$$

Then

$$\int_{-R}^{R} f'(x)e^{-ikx} \ dx = e^{-ikx}f(x)\big|_{x=-R}^{x=R} - \int_{-R}^{R} f(x)(-ik)e^{-ikx} \ dx$$
$$= e^{-ikR}f(R) - e^{ikR}f(-R) + ik \int_{-R}^{R} f(x)e^{-ikx} \ dx.$$

It is here that we need a nice auxiliary result, whose proof I will guide you through as an exercise later: if $f \in L^1$ is differentiable with $f' \in L^1$, then $\lim_{x\to\pm\infty} f(x) = 0$. Since

$$0 \le \left|e^{-ikR}f(R) - e^{ikR}f(-R)\right| \le |f(R)| + |f(-R)|,$$

the squeeze theorem gives

$$\lim_{R\to\infty} \left[e^{-ikR}f(R) - e^{ikR}f(-R)\right] = 0,$$

and therefore

$$\lim_{R\to\infty}\int_{-R}^{R} f'(x)e^{-ikx}\,dx = \lim_{R\to\infty} ik\int_{-R}^{R} f(x)e^{-ikx}\,dx = ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \sqrt{2\pi}ik\widehat{f}(k). \quad \square$$

This is where we finished on Friday, November 11, 2022.

The transparent interaction of the Fourier transform with derivatives is one of the transform's most valuable properties. Another is its linearity, which I invite you to prove using familiar properties of integrals.

**5.2.4 Theorem (Linearity of the Fourier transform).**

*Let* $f$, $g \in L^1$ *and* $\alpha$, $\beta \in \mathbb{C}$. *Then* $\mathfrak{F}[\alpha f + \beta g] = \alpha\mathfrak{F}[f] + \beta\mathfrak{F}[g]$.

This allows us to consider $\mathfrak{F}$ as a linear operator whose domain is the vector space $L^1$. (That $L^1$ is a vector space follows from the triangle inequality for integrals. Do you see why?) The range of $\mathfrak{F}$ is…complicated. The function $f$ from Example **??** satisfies $f \in L^1$ but $\mathfrak{F}[f] \notin L^1$.

## 5.3. Solving a toy ODE with the Fourier transform.

We can solve the ODE

$$f' + 2f = g$$

using various methods from our background in ODE, including integrating factors and variation of parameters (guess $f(x) = q(x)e^{-2x}$) and, if $g$ is one of certain special functions, undetermined coefficients. We definitely don't need new techniques to solve this problem. Nevertheless, we will solve it using the Fourier transform to demonstrate how to use the transform to solve differential equations and to motivate various additional properties of the transform that are worth knowing. (The alternative is that I simply give you an endless list of transform properties without any context of why they're important or how we might use them to solve differential equations.)

### *5.3.1. Applying the Fourier transform.*

Let's start by taking the Fourier transform of both sides:

$$\mathfrak{F}[f' + 2f](k) = \widehat{g}(k).$$

What are we assuming here? First, that a solution $f$ exists. Second, that $f'+2f$ has a Fourier transform. Third, that $g$ has a Fourier transform. Thus we probably want $f' + 2f \in L^1$ and $g \in L^1$.

Let's work on the left side. Linearity tells us that

$$\mathfrak{F}[f' + 2f](k) = \widehat{f'}(k) + 2\widehat{f}(k),$$

and then the derivative property of transforms says

$$\widehat{f'}(k) = ik\widehat{f}(k).$$

Now we need $f' \in L^1$.

Here is what we have shown. If there exists a function $f \in L^1$ such that $f$ solves $f' + 2f = g$ with $f' \in L^1$ and $g \in L^1$, then

$$(ik+2)\widehat{f}(k) = \widehat{g}(k).$$

Since $k \in \mathbb{R}$, we have $ik + 2 \neq 0$, and so we may solve for $\widehat{f}$:

$$\widehat{f}(k) = \frac{\widehat{g}(k)}{ik+2}.$$

### *5.3.2. Inverting the Fourier transform.*

We have thus completely determined $\widehat{f}$. Can we reconstruct $f$ from $\widehat{f}$? Remarkably, and reassuringly, the answer is yes, and it is not too painful to state (although proving it is another matter).

**5.3.1 Theorem (Fourier inversion).**

*Let $f \in L^1$ be continuous with $\widehat{f} \in L^1$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx} \, dk.$$

The integral in the Fourier inversion theorem is effectively a Fourier transform itself, and it appears often enough that it has its own name and notation.

**5.3.2 Definition.**

*Let $f \in L^1$. The* INVERSE FOURIER TRANSFORM *of $f$ is*

$$\check{f}(x) = \mathfrak{F}^{-1}[f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx} \, dk.$$

I hope it is apparent from the definition that for $f \in L^1$ we have $\widehat{f}(k) = \check{f}(-k)$. Equivalently, $\check{f}(x) = \widehat{f}(-x)$. And, if $f \in L^1$ with $\widehat{f} \in L^1$, then the Fourier inversion theorem reads

$$f = \check{\widehat{f}}.$$

This justifies the use of the notation $\mathfrak{F}^{-1}$, which is meant to signify the inverse of $\mathfrak{F}$ (in the sense that the inverse of $f(x) = 2x + 1$ is $f^{-1}(y) = (y-1)/2$).

We can now interpret our work with the toy ODE as follows. If we have a solution $f$ to $f' + 2f = g$ with $f \in L^1$ and $g \in L^1$, then $\widehat{f}$ must satisfy

$$\widehat{f}(k) = \frac{\widehat{g}(k)}{ik+2}.$$

If we also assume[28] that $\widehat{f} \in L^1$, then since $f$ solves an ODE, $f$ is continuous, and so the Fourier inversion theorem tells us $f = \widecheck{\widehat{f}}$. That is,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\widehat{g}(k)e^{ikx}}{ik+2}\ dk.$$

While true, this is not as helpful as we'd like. Can we find a more transparent formula for $f$?

### 5.3.3. Rescaling the Fourier transform.

It would be nice if we could express $\widehat{f}$ as a "pure" Fourier transform, i.e., as the Fourier transform of some "known" function (probably involving $g$, but since $g$ is given in the statement of the ODE, $g$ is "known"). The way out is to pay attention to the structure of $\widehat{f}(k)$: it is the product of $\widehat{g}(k)$ and $1/(ik+2)$. How can we recognize this product as a "pure" or "known" transform?

The second factor might make us recall Example 5.2.2. That example told us that if

$$\mathsf{E}(k) := \begin{cases} 0, & x < 0 \\ e^{-x}, & x \ge 0, \end{cases}$$

then

$$\widehat{\mathsf{E}}(k) = \frac{1}{\sqrt{2\pi}(ik+1)}.$$

We can rewrite this expression using the classical technique of multiplying and dividing by the same number:

$$\frac{1}{ik+2} = \frac{1}{2\left[i\left(\frac{k}{2}\right)+1\right]} = \frac{1}{2}\left(\frac{1}{i\left(\frac{k}{2}\right)+1}\right) = \sqrt{2\pi}\left(\frac{1}{2}\left[\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{i\left(\frac{k}{2}\right)+1}\right)\right]\right)$$
$$= \sqrt{2\pi}\left(\frac{1}{2}\widehat{\mathsf{E}}\left(\frac{k}{2}\right)\right).$$

In other words, up to some constant factors, $1/(ik+2)$ is essentially the Fourier transform of $\widehat{\mathsf{E}}$ evaluated at $k/2$.

It turns out that there is a straightforward connection between "rescaling" a function and Fourier transforms. You can obtain the following result from the definition of the Fourier transform and changing variables in definite integrals.

[28] We do not have to assume this. If instead we assume $\widehat{g} \in L^1$, then put $h(k) := \widehat{g}(k)/(ik+2)$ and use the comparison test to show that $h \in L^1$. Then $\widehat{f} \in L^1$, too.

**5.3.3 Theorem (Rescaling the Fourier transform).**

*Let $h \in L^1$ and let $a \in \mathbb{R}$ with $a \neq 0$. Denote by $h(a\cdot)$ the map $x \mapsto h(ax)$. Then $h(a\cdot) \in L^1$ and*

$$\widehat{h(a\cdot)}(k) = \frac{1}{|a|}\widehat{h}\left(\frac{k}{a}\right).$$

The rescaling theorem tells us

$$\frac{1}{2}\widehat{\mathsf{E}}\left(\frac{k}{2}\right) = \widehat{\mathsf{E}(2\cdot)}(k),$$

and so

$$\frac{1}{ik+2} = \sqrt{2\pi}\widehat{\mathsf{E}(2\cdot)}(k).$$

We have now shown that if $f \in L^1$ satisfies $f' + 2f = g$ with $f' \in L^1$ and $g \in L^1$, then

$$\widehat{f}(k) = \sqrt{2\pi}\widehat{g}(k)\widehat{\mathsf{E}(2\cdot)}(k).$$

We now have more control over the structure of $\widehat{f}$: it is (a constant multiple of) the product of two "known" Fourier transforms.

*5.3.4. Multiplication of Fourier transforms and convolution.*

And so we should ask ourselves how to express a product like $\widehat{\phi}(k)\widehat{\psi}(k)$ for $\phi$, $\psi \in L^1$ as the Fourier transform of a single function. This is one of those times where we simply must go back to the definition. Write

$$\widehat{\phi}(k)\widehat{\psi}(k) = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x)e^{-ikx}\ dx\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(y)e^{-iky}\ dy\right).$$

Whenever you have to work with the product of two integrals, it's a good idea to use different letters for the variables of integration, thus $x$ and $y$ here. Since $\int_{-\infty}^{\infty}\psi(y)e^{-iky}\ dy$ is just a number, we can multiply it *inside* the $x$-integral:

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \psi(y)e^{-iky}\ dy\right)\phi(x)e^{-ikx}\ dx.$$

Next, for $x$ fixed, $\phi(x)e^{-ikx}$ is also just a number, so we can multiply it inside the $y$-integral this time:

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \psi(y)e^{-iky}\phi(x)e^{-ikx}\ dy\right)\ dx.$$

Let's combine the exponentials and write this as an *iterated* integral (albeit one with infinite limits of integration everywhere):

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)\psi(y)e^{-ik(x+y)}\ dy\ dx.$$

Substitution works just as well for improper integrals as it does for definite integrals. Fix $x$ and substitute $u = x + y$, $y = u - x$ in the inner integral:

$$\int_{-\infty}^{\infty} \phi(x)\psi(y)e^{-ik(x+y)}\ dy = \int_{-\infty+x}^{\infty+x} \phi(x)\psi(u-x)e^{-iku}\ du = \int_{-\infty}^{\infty} \phi(x)\psi(u-x)e^{-iku}\ du.$$

Then

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)\psi(u-x)e^{-iku}\ du\ dx.$$

Now suppose that we can change the order of integration via Fubini's theorem[29] so that

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)\psi(u-x)e^{-iku}\ dx\ du.$$

This is actually rather delicate and requires some careful thought when improper integrals are involved. If it works, we might observe that the inner $x$-integral has the factor $e^{-iku}$, which is independent of $x$. Then

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \phi(x)\psi(u-x)\ dx\right)e^{-iku}\ du.$$

The inner integral is now a function of $u$, and the outer integral contains the factor $e^{-iku}$. This might remind us of a Fourier transform. If we put

$$\omega(u) := \int_{-\infty}^{\infty} \phi(x)\psi(u-x)\ dx,$$

then we have shown

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega(u)e^{-iku}\ du,$$

and the integral on the right should look *very* much like a Fourier transform indeed.

This is where we finished on Monday, November 14, 2022.

The change of variables $v = u - x$ should allow us to write

$$\omega(u) = \int_{-\infty}^{\infty} \phi(u-v)\psi(v)\ dv$$

as well. Let's give $\omega$ its more formal name.

**5.3.4 Definition.**

*The* **CONVOLUTION** *of* $\phi$, $\psi \in L^1$ *is the function*

$$(\phi * \psi)(x) := \int_{-\infty}^{\infty} \phi(x-s)\psi(s)\ ds.$$

[29] The technical term for this sort of procedure is **Fubination**. Just kidding.

We repackage all of our prior work as

$$\widehat{\phi}(k)\widehat{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\phi * \psi)(x) e^{-ikx} \, dx = \frac{\widehat{\phi * \psi}(k)}{\sqrt{2\pi}}.$$

This meets our goal of expressing the product $\widehat{\phi}\widehat{\psi}$ as a "pure" Fourier transform of a single function. That "single function" here is the convolution $\phi * \psi$.

However, the Fubination above is a bit ticklish, and merely having $\phi$, $\psi \in L^1$ is not enough to guarantee its validity. We need to assume a bit more, but we get a bit more in return.

**5.3.5 Theorem.**

*Suppose that $\phi$, $\psi \in L^1$ are bounded: there is $M > 0$ such that $|\phi(x)| \le M$ and $|\psi(x)| \le M$ for all $x$. Then the convolution $(\phi * \psi)(x)$ is defined for all $x$, and the following hold.*

**(i)** $\phi * \psi \in L^1$.

**(ii)** $\phi * \psi \in \mathcal{C}(\mathbb{R})$.

**(iii)** $\widehat{\phi * \psi}(k) = \sqrt{2\pi}\widehat{\phi}(k)\widehat{\psi}(k)$.

### 5.3.5. *Solving the toy problem.*

It has been some time, but we have shown the following. If $f$, $g \in L^1$ with $f$ differentiable, $f' \in L^1$, and $\widehat{g} \in L^1$, and if $f' + 2f = g$, then $\widehat{f}$ satisfies

$$\widehat{f}(k) = \sqrt{2\pi}\widehat{\mathsf{E}(2\cdot)}(k)\widehat{g}(k).$$

Here $\mathsf{E}(x) = 0$ for $x < 0$ and $\mathsf{E}(x) = e^{-x}$ for $x \ge 0$, and $\mathsf{E}(2\cdot)$ means the function $x \mapsto \mathsf{E}(2x)$. I hope it is clear that $\mathsf{E}$ is bounded, indeed $|\mathsf{E}(x)| \le 1$ for all $x$. If we also assume that $g$ is bounded, then the convolution theorem allows us to write

$$\sqrt{2\pi}\widehat{\mathsf{E}(2\cdot)}(k)\widehat{g}(k) = \widehat{\mathsf{E}(2\cdot) * g}(k).$$

We are approaching the limits of how wide a "hat" can be in typesetting, so maybe it's better to write

$$\widehat{\mathsf{E}(2\cdot) * g}(k) = \mathfrak{F}[\mathsf{E}(2\cdot) * g](k).$$

Then

$$\widehat{f}(k) = \mathfrak{F}[\mathsf{E}(2\cdot) * g](k).$$

Since $f \in L^1$ with $\widehat{f} \in L^1$, and since $f$ solves an ODE and therefore is continuous, we have

$$f = \check{\widehat{f}} = \mathfrak{F}^{-1}\big[\mathfrak{F}[\mathsf{E}(2\cdot) * g]\big].$$

We want to use the inversion theorem to say, of course,

$$\mathfrak{F}^{-1}\big[\mathfrak{F}[\mathsf{E}(2\cdot) * g]\big] = \mathsf{E}(2\cdot) * g.$$

To do this, we need to know (1) $\mathsf{E}(2\cdot) * g \in L^1$, (2) $\mathfrak{F}[\mathsf{E}(2\cdot) * g] \in L^1$, and (3) $\mathsf{E}(2\cdot) * g$ is continuous. We get (1) and (3) directly from the convolution theorem. For (2), we have shown that $\widehat{f} = \mathfrak{F}[\mathsf{E}(2\cdot) * g]$, and we were assuming[30] $\widehat{f} \in L^1$ so that we could apply the inversion theorem. Thus (2) holds.

We conclude

$$f = \mathsf{E}(2\cdot) * g.$$

This is where we finished on Wednesday, November 16, 2022.

More precisely, here are all of our necessary hypotheses and conclusion in one nice package.

**5.3.6 Theorem.**

*Suppose that $f$, $g\colon \mathbb{R} \to \mathbb{C}$ are functions with the following properties.*

**(i)** *$f$, $g \in L^1$ with either $\widehat{f} \in L^1$ or $\widehat{g} \in L^1$.*

**(ii)** *$g$ is bounded: there is $M > 0$ such that $|g(x)| \le M$ for all $x$.*

**(iii)** *$f$ is differentiable on $\mathbb{R}$ with $\widehat{f} \in L^1$ and $f' + 2f = g$.*

*Then $f = \mathsf{E}(2\cdot) * g$.*

A good exercise for you is to prove this theorem. We did all of the heavy lifting above, but there was a lot of interstitial commentary. You should think about how to cut out all the digressions and just cite the inversion theorem, the scaling theorem, and the convolution theorem as necessary. Footnotes 28 and 30 will help if you want to use the condition $\widehat{g} \in L^1$ instead of $\widehat{f} \in L^1$.

This is a *uniqueness* theorem. It says that if a solution $f$ to $f' + 2f = g$ exists under the hypotheses (i), (ii), and (iii), then that solution $f$ must be $\mathsf{E}(2\cdot) * g$. This is *not* an *existence* theorem. It does not say that $\mathsf{E}(2\cdot) * g$ actually solves the ODE! We would have to check that with direct calculations (and note that we have not yet talked about differentiating a convolution). And these are pretty restrictive hypotheses. We certainly never met them in our first ODE course (I think/hope). Let's unpack this a bit more.

### 5.3.6. *Interpreting the results.*

Let's be clear: *we did not need the Fourier transform to solve $f' + 2f = g$.* That's why we took an ODE class before this one. Rather, this toy problem brought up the need for

[30] Recall Footnote 28, in which we showed that we could assume $\widehat{g} \in L^1$ and deduce $\widehat{f} \in L^1$ from that. The same applies here. Do not assume $\widehat{f} \in L^1$ but instead require $\widehat{g} \in L^1$. Example 5.2.2 tells us that $\widehat{\mathsf{E}}$ is bounded, with $|\widehat{\mathsf{E}}(k)| = 1/\sqrt{1+k^2} \le 1$. Then scaling tells us $|\widehat{\mathsf{E}(2\cdot)}(k)| = |\widehat{\mathsf{E}}(k/2)|/2 \le 1/2$. Then the convolution theorem tells us

$$|\mathfrak{F}[\mathsf{E}(2\cdot) * g](k)| = |\widehat{\mathsf{E}(2\cdot)}(k)\widehat{g}(k)| \le \frac{|\widehat{g}(k)|}{2}.$$

Since $\widehat{g} \in L^1$, the comparison test says $\mathsf{E}(2\cdot) * g \in L^1$ as well, and this is (2).

a number of useful properties of the transform in what I hope was a safe, friendly setting. We know what the answer to $f' + 2f = g$ should be, and so we can compare our solution $f = \mathsf{E}(2\cdot) * g$ to it. Actually, you'll do that, too.

Instead, let's explore the formula $f = \mathsf{E}(2\cdot) * g$. By definition of the convolution,

$$\big(\mathsf{E}(2\cdot) * g\big)(x) = \int_{-\infty}^{\infty} \mathsf{E}(2(x-s))g(s)\,ds. \tag{5.3.1}$$

Recall that

$$\mathsf{E}(\xi) = \begin{cases} 0, \ \xi < 0 \\ e^{-\xi}, \ \xi \geq 0. \end{cases}$$

Thus

$$\mathsf{E}(2(x-s)) = \begin{cases} 0, \ 2(x-s) < 0 \\ e^{-2(x-s)}, \ 2(x-s) \geq 0 \end{cases} = \begin{cases} 0, \ x < s \\ e^{-2x+2s}, \ x \geq s. \end{cases}$$

So, in the integral (5.3.1), we only have to integrate over $(-\infty, x)$. This gives

$$f(x) = \int_{-\infty}^{x} e^{-2x+2s} g(s)\,ds = e^{-2x} \int_{-\infty}^{x} e^{2s} g(s)\,ds. \tag{5.3.2}$$

Here is the amazing thing. We do not need all those stringent hypotheses on $g$ for the integral

$$G(x) := \int_{-\infty}^{x} e^{2s} g(s)\,ds$$

to be defined. Indeed, since we can write

$$\int_{-\infty}^{x} e^{2s} g(s)\,ds = \int_{-\infty}^{0} e^{2s} g(s)\,ds + \int_{0}^{x} e^{2s} g(s)\,ds,$$

we just need one improper integral to exist and then the tamer definite integral to exist. I claim that the comparison test ensures the existence of $\int_{-\infty}^{0} e^{2s} g(s)\,ds$ if $g \in L^1_{\text{loc}}$ and if $g$ is bounded; the local integrability of $g$ then guarantees the existence of the definite integral $\int_0^x e^{2s} g(s)\,ds$ for all $x$. If we require instead that $g$ be bounded and continuous (and continuity thus ensures $g \in L^1_{\text{loc}}$), then we can differentiate $G$:

$$G'(x) = \frac{d}{dx}\left[\int_{-\infty}^{0} e^{2s} g(s)\,ds + \int_{0}^{x} e^{2s} g(s)\,ds\right] = 0 + \frac{d}{dx}\left[\int_{0}^{x} e^{2s} g(s)\,ds\right] = e^{2x} g(x).$$

From there we could show that $f(x) = e^{2x} G(x)$ does indeed solve $f' + 2f = g$.

So, what was the point of all our work? We didn't need the Fourier transform to solve the toy problem, and our solution turned out to be valid under much more general hypotheses than those that permit the Fourier transform analysis. Imagine, though, that we didn't know how to solve the toy problem but we knew everything about the Fourier transform. We could have *assumed* that the problem had a solution $f$ such that the Fourier transforms of $f$ and its derivatives were defined, applied the transform to the problem, solved for $\widehat{f}$, and then inverted to see what the solution $f$ *should* be. We could then have checked directly (i.e., using calculus) that $f$ as given by this inverse transform was indeed a solution. In the process, we would have shown a certain uniqueness result: if a solution exists and is sufficiently nice to have a transform, then the only such solution is the one that we just found.

The true value of the Fourier transform in solving differential equations is that it gives a *formal candidate* for the solution. We can always check a solution candidate using calculus. For me, the value of this particular toy problem is that it demonstrates the majority of the techniques that we typically use in conjunction with the Fourier transform in a "friendly" setting; we knew what to expect, so we could focus on technique. Now we'll see how to use the transform on more complicated PDE, for whose solutions we probably don't have much intuition.

## 5.4. Solving PDE with the Fourier transform.

We will use the Fourier transform and its myriad properties to obtain "formal" solutions to PDE posed spatially on $(-\infty, \infty)$. That is, our PDE will involve the unknown function $u = u(x,t)$, where $-\infty < x < \infty$. We will take the Fourier transform "in $x$" as follows.

**5.4.1 Definition.**

*Suppose that $u \colon \mathcal{D} \to \mathbb{C}$ is a function defined on a region $\mathcal{D} \subseteq \mathbb{R}^2$ of the form*

$$\mathcal{D} = \{(x,t) \mid x \in \mathbb{R},\ t \in I\}$$

*for some interval $I \subseteq \mathbb{R}$. For $t \in I$, denote by $u(\cdot,t)$ the map $x \mapsto u(x,t)$. Then the* **SPATIAL FOURIER TRANSFORM** *of $u$ on $\mathcal{D}$ is*

$$\widehat{u}(k,t) = \mathfrak{F}[u(\cdot,t)](k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\, dx.$$

The most important properties (arguably) of the spatial Fourier transform are its different interactions with respect to $x$- and $t$-derivatives. First, we can generalize Theorem 5.2.3 as follows.

**5.4.2 Corollary.**

*Let $f \in L^1$ be $r$-times continuously differentiable, with $f', \ldots, f^{(r)} \in L^1$. Then $\widehat{f^{(r)}}(k) = (ik)^r \widehat{f}(k)$.*

Thus if $u \colon \mathcal{D} \to \mathbb{C}$ is sufficiently differentiable and if $\partial_x^s[u](\cdot,t) \in L^1$ for $s = 0, \ldots, r$, its spatial Fourier transform satisfies

$$\widehat{\partial_x^r[u]}(k,t) = (ik)^r \widehat{u}(k,t). \tag{5.4.1}$$

Next, if $u$ is continuously differentiable and $u(\cdot,t) \in L^1$ for all $t$, then a version of Leibniz's rule for differentiating under the integral for improper integrals gives

$$\widehat{u_t}(k,t) = \mathfrak{F}[u_t(\cdot,t)](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(x,t)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[u(x,t)e^{-ikx}\right]\,dx$$
$$= \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx\right] = \partial_t[\widehat{u}](k,t). \quad (5.4.2)$$

*5.4.1. The heat equation.*

We will solve the heat IVP

$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty,\ t \geq 0 \\ u(x,0) = f(x) \end{cases} \tag{5.4.3}$$

with the Fourier transform. Our work will be *formal* in the sense that we will assume that a solution to this IVP exists and that $u$ is sufficiently "nice" so that any and all Fourier transforms exist. At the end, we will obtain an explicit formula for $u$, involving the initial data $f$, and then we will examine this formula carefully to see where exactly it is defined and differentiable.

We start by taking the Fourier transform of the PDE $u_t = u_{xx}$, so

$$\widehat{u_t}(k,t) = \widehat{u_{xx}}(k,t).$$

On the left, we use Leibniz's rule (5.4.2) and on the right we use (5.4.1), so

$$\partial_t[\widehat{u}](k,t) = (ik)^2\widehat{u}(k,t).$$

Then $\widehat{u}(k,\cdot)$ satisfies the IVP

$$\begin{cases} \partial_t[\widehat{u}](k,t) = -k^2\widehat{u}(k,t) \\ \widehat{u}(k,0) = \widehat{f}(k). \end{cases}$$

This is really an ODE for the function $t \mapsto \widehat{u}(k,t)$ of the single real variable $t$; now $k \in \mathbb{R}$ is a parameter. More precisely, it is an exponential growth (or decay) problem, so its solution is

$$\widehat{u}(k,t) = \widehat{f}(k)e^{-k^2t}.$$

We could then hope that the Fourier inversion theorem gives a formula for $u$:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{u}(k,t)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)e^{-k^2t}e^{ikx}\,dk.$$

We can probably get convergence of this integral for $t > 0$ using the comparison test, thanks to the all-absorbing, rapidly decaying factor of $e^{-k^2t}$. Then we would have to think about differentiating under the integral with Leibniz's rule. This, morally, resembles how we studied the convergence and differentiability of Fourier series for the heat IVP-BVP.

However, this approach will not yield the most general (and generous) result. It requires $f \in L^1$ so that $\widehat{f}$ is defined and bounded. Instead, a better approach turns out to be rewriting the formula for $\widehat{u}$ as a convolution. Specifically, if we can write $e^{-k^2t}$ as the Fourier transform of some function, then $\widehat{u}(k,t)$ will be the product of two Fourier transforms, and so we might write that as the transform of a convolution. Let's pursue this further.

We want to find $g \in L^1$ such that $\widehat{g}(k) = e^{-k^2 t}$. Naturally, $g$ will somehow depend on $t$, too; now $t$ is a parameter, and all the Fourier "action" is happening in $k$. (This is typical of what happens when we apply the Fourier transform to PDE: sometimes $t$ is the independent variable, sometimes a parameter, and the same for $x$ and $k$.) If $g$ is a function such that $\widehat{g}(k) = e^{-k^2 t}$, then we expect $g = \check{\widehat{g}}$, and so we should consider the inverse transform integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 t} e^{ikx}\ dk.$$

This integral only converges for $t > 0$, so we will assume $t > 0$ from now on. After all, we know what the solution $u$ is doing at $t = 0$: $u(x,0) = f(x)$.

Let's abbreviate

$$\mathcal{I}(x,t) := \int_{-\infty}^{\infty} e^{-k^2 t} e^{ikx}\ dk.$$

If you don't know what else to do when considering an integral of a product, try integrating by parts. Let me use uppercase letters here so as not to overwork $u$; I think the only natural choices for $U$ and $dV$ are

$$U = e^{-k^2 t} \qquad dV = e^{ikx}\ dk$$
$$dU = -2kte^{-k^2 t}\ dk \qquad V = e^{ikx}/ix.$$

Of course, this is only valid for $x \neq 0$. So, we obtain

$$\mathcal{I}(x,t) = \frac{e^{-k^2 t} e^{ikx}}{ix}\bigg|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} \frac{e^{ikx}}{ix}(-2kte^{-k^2 t})\ dk = \frac{2t}{ix}\int_{-\infty}^{\infty} ke^{-k^2 t} e^{ikx}\ dk.$$

This may not look any more useful, but let me rewrite it a little:

$$\mathcal{I}(x,t) = \frac{2t}{i^2 x}\int_{-\infty}^{\infty} e^{-k^2 t}(ike^{ikx})\ dk = -\frac{2t}{x}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[e^{-k^2 t} e^{ikx}\right] dk = -\frac{2t}{x}\frac{\partial}{\partial x}\left[\int_{-\infty}^{\infty} e^{-k^2 t} e^{ikx}\ dk\right]$$
$$= -\frac{2t}{x}\mathcal{I}_x(x,t).$$

Here I used Leibniz's rule for differentiating under an (improper) integral.

This is where we finished on Monday, November 28, 2022.

We may rewrite this as an(other) exponential growth/decay problem:

$$\mathcal{I}_x(x,t) = -\frac{x}{2t}\mathcal{I}(x,t).$$

I think it's unlikely that we would have stumbled on this PDE-that's-really-an-ODE from the start, but fooling around with integration by parts helped. In fact, this PDE for $\mathcal{I}$ is valid even at $x = 0$, as a direct calculation of $\mathcal{I}_x(0,t)$ via Leibniz's rule will show. (You should do that.) So, we have

$$\mathcal{I}(x,t) = \mathcal{I}(0,t)e^{-x^2/4t}.$$

I claim that we already know the value $\mathcal{I}(0,t)$. Indeed,

$$\mathcal{I}(0,t) = \int_{-\infty}^{\infty} e^{-k^2 t}\, dk = \int_{-\infty}^{\infty} e^{-(\sqrt{t}k)^2}\, dk = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-\ell^2}\, d\ell,$$

and calculus tells us

$$\int_{-\infty}^{\infty} e^{-\ell^2}\, d\ell = \sqrt{\pi}.$$

Thus

$$\mathcal{I}(x,t) = \frac{\sqrt{\pi}}{\sqrt{t}}e^{-x^2/4t},$$

and so

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2 t}e^{ikx}\, dk = \frac{\mathcal{I}(x,t)}{\sqrt{2\pi}} = \frac{e^{-x^2/4t}}{\sqrt{2t}} =: \widetilde{\mathcal{H}}(x,t). \tag{5.4.4}$$

The relevance of the strange notation $\widetilde{\mathcal{H}}$ will be apparent soon. The function[31] $\widetilde{\mathcal{H}}(\cdot,t)$ is infinitely differentiable on $\mathbb{R}$ and $\widetilde{\mathcal{H}}(\cdot,t) \in L^1$. If we put $\phi(k,t) := e^{-k^2 t}$, then (5.4.4) says

$$\check{\phi}(x,t) = \mathfrak{F}^{-1}[\phi(\cdot,t)](x) = \widetilde{\mathcal{H}}(x,t).$$

Since $\phi(\cdot,t) \in L^1$ and $\phi(\cdot,t)$ is continuous (in fact, $\phi$ is continuous on $\mathbb{R}^2$), the Fourier inversion theorem gives

$$\widehat{\widetilde{\mathcal{H}}}(k,t) = \mathfrak{F}[\widetilde{\mathcal{H}}(\cdot,t)](k) = \widehat{\check{\phi}}(x,t) = \phi(x,t) = e^{-k^2 t}.$$

We worked really hard at this, so let's reward ourselves with a formal little lemma.

**5.4.3 Lemma.**

*Let $\phi(x,t) := e^{-x^2 t}$. Then, for $t > 0$,*

$$\widehat{\phi}(k,t) = \frac{e^{-k^2/4t}}{\sqrt{2t}} \quad \text{and} \quad \check{\phi}(x,t) = \frac{e^{-x^2/4t}}{\sqrt{2t}}.$$

Thus the solution $u$ to the long-ago heat IVP (5.4.3) must satisfy

$$\widehat{u}(k,t) = e^{-k^2 t}\widehat{f}(k) = \mathfrak{F}[\widetilde{\mathcal{H}}(\cdot,t)](k)\widehat{f}(k) = \frac{\mathfrak{F}[\widetilde{\mathcal{H}}(\cdot,t) * f](k)}{\sqrt{2\pi}}.$$

Here $\widetilde{\mathcal{H}}(\cdot,t) * f$ is the "spatial" convolution of $\widetilde{\mathcal{H}}$ and $f$, i.e.,

$$(\widetilde{\mathcal{H}}(\cdot,t) * f)(x) = \int_{-\infty}^{\infty} \mathcal{H}(x-s)f(s)\, ds = \frac{1}{\sqrt{2t}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4t}f(s)\, ds.$$

[31] As always, $\widetilde{\mathcal{H}}(\cdot,t)$ means the map $x \mapsto \widetilde{H}(x,t)$.

Now let's combine some factors and drop the tilde and say

$$\mathcal{H}(x,t) := \frac{\widetilde{\mathcal{H}}(x,t)}{\sqrt{2\pi}} = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}.$$

We call this function $\mathcal{H}$ the **HEAT KERNEL**; in general, a kernel[32] is "the thing you convolve with."

All the analysis above has led to the following idea: the *formal* solution to the heat IVP is, for $t > 0$,

$$u(x,t) = (\mathcal{H}(\cdot,t) * f)(x) = \int_{-\infty}^{\infty} \frac{e^{-(x-s)^2/4t} f(s)}{\sqrt{4\pi t}} \, ds. \tag{5.4.5}$$

That is, for $t > 0$, the formal solution to the heat IVP is the spatial convolution of the heat kernel with the initial temperature distribution.

Let me be disappointingly clear: we have not verified that defining $u$ via (5.4.5) actually solves the heat IVP! Let's think about how that would proceed. First, we need to be sure that the integral in (5.4.5) converges for all $x \in \mathbb{R}$ and all $t > 0$; the integral is definitely not defined at $t = 0$, but we might be able to analyze it in the *limit* as $t \to 0$ anyway.

Look closely at this integral. The variable of integration is $s$, and so, from the point of view of the integral, $x$ and $t$ are fixed parameters. The question is really if the integral

$$\int_{-\infty}^{\infty} e^{-(x-s)^2/4t} f(s) \, ds$$

converges, since the denominator $\sqrt{4\pi t}$ is constant with respect to $s$. Here is the great news: the exponential decays *really* fast.

Say that either $f \in L^1$ or $f \in L^1_{\text{loc}}$ and $f$ is bounded, i.e., there is $M > 0$ such that $|f(s)| \le M$ for all $s$. Then, for $x$ and $t$ fixed, the map $s \mapsto e^{-(x-s)^2/4t} f(s)$ is locally integrable (with respect to $s$), since the exponential is continuous in $s$. Moreover, if $f \in L^1$, we have

$$|e^{-(x-s)^2/4t} f(s)| \le |f(s)|,$$

in which case the comparison test says that the integral $\int_{-\infty}^{\infty} e^{-(x-s)^2/4t} f(s) \, ds$ converges. If $f \in L^1_{\text{loc}}$ is bounded, then

$$|e^{-(x-s)^2/4t} f(s)| \le M e^{-(x-s)^2/4t},$$

and since the integral $\int_{-\infty}^{\infty} e^{-(x-s)^2/4t} \, ds$ converges for all $x \in \mathbb{R}$ and all $t > 0$ (being just a rescaled version of our old friend $\int_{-\infty}^{\infty} e^{-x^2} \, dx$), the comparison test again applies to establish the convergence of $\int_{-\infty}^{\infty} e^{-(x-s)^2/4t} f(s) \, ds$.

Thus $u(x,t) = (\mathcal{H}(\cdot,t) * f)(x)$ is always defined for $f \in L^1$ and for bounded $f \in L^1_{\text{loc}}$. To check the derivatives of $u$, we would need to differentiate under the integral. Here is more

[32] Suppose that $\mathcal{K}$ is a function of two variables and define the integral transform

$$\mathfrak{K}[f](x) := \int_a^b \mathcal{K}(x,\xi) f(\xi) \, d\xi.$$

Then $\mathcal{K}$ is the kernel of $\mathfrak{K}$.

great news: any $x$- or $t$-derivatives that would pass under the integral would only hit $\mathcal{H}$, since $f$ does not depend on $x$ or $t$. You can check that $\mathcal{H}$ is infinitely differentiable on the region $\mathcal{D} := \{(x,t) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ t > 0\}$, and, remarkably,

$$\mathcal{H}_t = \mathcal{H}_{xx}.$$

That is, the heat kernel itself satisfies the heat equation! For this reason, and for its role in the convolution (5.4.5), the heat kernel is sometimes called the **FUNDAMENTAL SOLUTION** of the heat equation And so a careful application of Leibniz's rule for differentiating under improper integrals shows that this $u$ solves the heat equation.

Things are more delicate with the initial condition. The problem is that $\mathcal{H}(x,0)$ is undefined. However, if $f$ is also continuous (which automatically implies $f \in L^1_{\text{loc}}$), it is possible to show

$$\lim_{(\xi,t)\to(x,0^+)} (\mathcal{H}(\cdot,t) * f)(\xi) = f(x)$$

for all $x \in \mathbb{R}$.

Here is everything that we have figured out.

**5.4.4 Theorem.**

*Let $f \in \mathcal{C}(\mathbb{R})$. Then the function*

$$u(x,t) := \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-(x-s)^2/4t} f(s}{\sqrt{4\pi t}}\, ds, & t > 0 \\ f(x), & t = 0 \end{cases}$$

*is infinitely differentiable on $\mathcal{D} := \{(x,t) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ t > 0\}$, continuous on $\mathcal{D}_0 := \{(x,t) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ t \geq 0\}$, and satisfies $u_t = u_{xx}$ on $\mathcal{D}$.*

This is where we finished on Friday, December 2, 2022.

*5.4.2. The wave equation.*

We already know how to solve the wave IVP

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x,\ t < \infty \\ u(x,0) = f(x), & -\infty < x < \infty \\ u_t(x,0) = g(x), & -\infty < x < \infty, \end{cases}$$

thanks to (among other things) Lemma 4.2.2. However, it is a good exercise to solve this IVP using the Fourier transform; we will learn some new properties of the transform in the process.

To apply the transform, we need $f$, $g \in L^1$, but of course we don't need that to solve the IVP, per Lemma 4.2.2. This is a reminder that restricting ourselves to the Fourier point of view will limit how general our solutions can be. Anyway, if we take the transform of the IVP, we find, using exactly the same reasoning as with Fourier series in Section 4.1, that

$$\widehat{u}(k,t) = \widehat{f}(k)\cos(ckt) + \widehat{g}(k)\operatorname{sinc}(ckt)t, \qquad \operatorname{sinc}(\xi) = \frac{\sin(\xi)}{\xi}.$$

Our task is to see how properties of the transform can help us solve for $u$ from this expression for $\widehat{u}$ and in particular recover something like Lemma 4.2.2.

Let's look at each of the two terms in the expression for $\widehat{u}$ separately. Both terms are products. We might expect to manage $\widehat{f}(k)\cos(ckt)$ via convolution, except $k \mapsto \cos(ckt)$ is not the Fourier transform of any function in $L^1$; indeed, Fourier transforms of functions in $L^1$ have to vanish as $k \to \pm\infty$, and the cosine definitely does not vanish at $\pm\infty$. Rather, we can use the complex form of the cosine to write

$$\widehat{f}(k)\cos(ckt) = \widehat{f}(k)\left(\frac{e^{ickt}+e^{-ickt}}{2}\right) = \frac{\widehat{f}(k)e^{ickt}+\widehat{f}(k)e^{-ickt}}{2}.$$

Since the definition of $\widehat{f}(k)$ involves an exponential of the form $e^{ikx}$, we might wonder how the exponentials $e^{\pm ickt}$ could interact with $\widehat{f}(k)$.

Quite nicely.

**5.4.5 Theorem (Shifting the Fourier transform).**

*Let $f \in L^1$ and $d \in \mathbb{R}$ and denote by $f(\cdot + d)$ the map $x \mapsto f(x+d)$. Then $f(\cdot + d) \in L^1$ and*

$$\widehat{f(\cdot + d)}(k) = \mathfrak{F}[f(\cdot + d)](k) = e^{ikd}\widehat{f}(k).$$

Thus

$$\widehat{f}(k)\cos(ckt) = \frac{\widehat{f}(k)e^{ik(ct)}+\widehat{f}(k)e^{ik(-ct)}}{2} = \mathfrak{F}\left[\frac{f(\cdot + ct)+f(\cdot - ct)}{2}\right](k).$$

As for the second term $\widehat{g}(k)\operatorname{sinc}(ckt)t$ in $\widehat{u}$, it turns out that the factor $\operatorname{sinc}(ckt)t$ is a Fourier transform. Specifically, you can calculate the following.

**5.4.6 Lemma.**

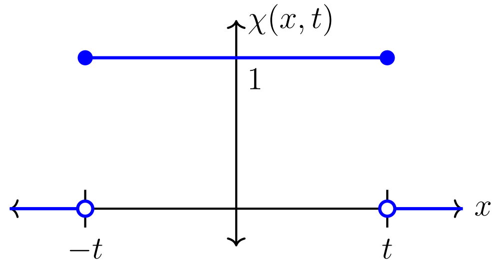
*For $t \ge 0$, define*

$$\chi(x,t) := \begin{cases} 1, & |x| \le t \\ 0, & |x| > t. \end{cases}$$

*Then*

$$\widehat{\chi}(k,t) = \mathfrak{F}[\chi(\cdot,t)](k) = \sqrt{\frac{2}{\pi}}\operatorname{sinc}(kt)t.$$

The function $\chi(\cdot,t)$ is sometimes called the **CHARACTERISTIC** or **INDICATOR** function for $[-t, t]$. Here is a graph of $\chi(\cdot, t)$ as a function of $x$ for fixed $t$.




We should therefore be inclined to rewrite $\operatorname{sinc}(ckt)t$ as the Fourier transform of something involving $\chi$. There are probably several ways to proceed here depending on how we want to treat $c$; we could consider $c$ as scaling $k$ and therefore use the preceding lemma and scaling properties of the transform. However, I think it's ultimately easier to keep $c$ with $t$; after all, that is how all of our counterpropagating waves work. So, after some casting about, we might find

$$\operatorname{sinc}(ckt)t = \frac{1}{c}\operatorname{sinc}(k(ct))(ct) = \frac{1}{c}\sqrt{\frac{\pi}{2}}\left[\sqrt{\frac{2}{\pi}}\operatorname{sinc}(k(ct)(ct))\right] = \frac{1}{c}\sqrt{\frac{\pi}{2}}\widehat{\chi}(k, ct).$$

Thus

$$\widehat{g}(k)\operatorname{sinc}(ckt)t = \frac{1}{c}\sqrt{\frac{\pi}{2}}\big(\widehat{\chi}(k, ct)\widehat{g}(k)\big).$$

Now we would like to say that $\widehat{\chi}(k, ct)\widehat{g}(k)$ is the transform of a convolution:

$$\widehat{g}(k)\widehat{\chi}(k, ct) = \frac{\mathfrak{F}[\chi(\cdot, ct) * g](k)}{\sqrt{2\pi}}.$$

This is indeed valid by the convolution theorem, if we assume that $g$ is bounded, since $\chi(\cdot, ct) \in L^1$ is already bounded. (You might notice that I flipped the order of the factors from $\widehat{\chi}(k, ct)\widehat{g}(k)$ to $\widehat{\chi}(k, ct)\widehat{g}(k)$, so that the convolution would be $\chi(\cdot, ct) * g$. Of course $\chi(\cdot, ct) * g = g * \chi(\cdot, ct)$, but I think it's more polite to put the "arbitrary" function $g$ second in the multiplication, so that in the convolution integral the $x - s$ thing doesn't go inside $g$.) And so we have

$$\widehat{u}(k, t) = \mathfrak{F}\left[\frac{f(\cdot + ct) + f(\cdot - ct)}{2} + \frac{\chi(\cdot, ct) * g}{2c}\right](k).$$

Thus our formal solution to the wave IVP is

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{(\chi(\cdot, ct) * g)(x)}{2c}.$$

A good exercise for you is to massage the convolution $\chi(\cdot, ct) * g$ until you get the integral term in Lemma 4.2.2.

I don't think we learned anything new about the wave equation here, but I hope it was good practice with the Fourier transform, and we learned about transforms of shifts and indicator functions to boot.

# A. Differentiating Under the Integral

We all know from the fundamental theorem of calculus that

$$\frac{d}{dt}\left[\int_0^t f(s)\ ds\right] = f(t)$$

if $f$ is continuous on $[0,t]$. But what happens if the integrand also depends on $t$? How do you calculate things like

$$\frac{d}{dt}\left[\int_0^t f(t,s)\ ds\right],$$

where the independent variable appears both as a limit of integration and within the integrand? More generally, how do you calculate partial derivatives like

$$\frac{\partial}{\partial x}\left[\int_0^t f(\phi(x,t,s),s)\ ds\right] \quad \text{and} \quad \frac{\partial}{\partial t}\left[\int_0^t f(\phi(x,t,s),s)\ ds\right],$$

where $f = f(X,s)$ and $\phi = \phi(x,t,s)$ are given functions? This situation arises when checking the formula for the solution to the nonhomogeneous transport equation; see (1.5.17), which has us take

$$\phi(x,t,s) = \frac{bx-at}{b} - \frac{as}{b}.$$

## A.1. Leibniz's rule for differentiating under the integral.

It turns out to be simpler to address first the question of calculating a derivative like

$$\frac{d}{dX}\left[\int_0^1 f(X,s)\ ds\right],$$

where the limits of integration are constant. So, we'll start with that.

Let's fool around. Put

$$F(X) := \int_0^1 f(X,s)\ ds.$$

Then, if $F$ is differentiable, we have

$$\begin{aligned}F'(X) = \lim_{h\to 0}\frac{F(X+h)-F(X)}{h} &= \lim_{h\to 0}\frac{1}{h}\left[\int_0^1 f(X+h,s)\ ds - \int_0^1 f(X,s)\ ds\right] \\ &= \lim_{h\to 0}\int_0^1 \frac{f(X+h,s)-f(X,s)}{h}\ ds.\end{aligned}$$

We would just love to interchange $\lim_{h\to 0}$ and $\int_0^1$, right? Then we would find

$$\lim_{h\to 0}\int_0^1 \frac{f(X+h,s)-f(X,s)}{h}\ ds = \int_0^1 \lim_{h\to 0}\frac{f(X+h,s)-f(X,s)}{h}\ ds = \int_0^1 f_X(X,s)\ ds.$$

So, in a just world we should have

$$\frac{d}{dX}\left[\int_0^1 f(X,t)\ ds\right] = \int_0^1 f_X(X,t)\ ds. \tag{A.1.1}$$

That is, we should be able to "differentiate under the integral": to pass the $X$-derivative through the integral and onto the integrand.

**A.1.1 Example.**

*Calculate*

$$\frac{d}{dX}\left[\int_0^1 s\cos(s^2+X)\ ds\right]$$

*by evaluating the integral with the fundamental theorem of calculus and then differentiating the result. Compare your answer to what* (A.1.1) *optimistically predicts.*

**Solution.** If we fix $X$ and integrate with respect to $s$, then we have

$$\int_0^1 s\cos(s^2+X)\ ds = \left.\frac{\sin(s^2+X)}{2}\right|_{s=0}^{s=1} = \frac{\sin(1+X)-\sin(X)}{2}.$$

Consequently,

$$\frac{d}{dX}\left[\int_0^1 s\cos(s^2+X)\ ds\right] = \frac{d}{dX}\left[\frac{\sin(1+X)-\sin(X)}{2}\right] = \frac{\cos(1+X)-\cos(X)}{2}.$$

Now let's compare this to (A.1.1). We have $f(X,s) = s\cos(s^2+X)$. We calculate

$$f_X(X,s) = -s\sin(s^2+X).$$

We integrate

$$\int_0^1 f_X(X,s)\ ds = -\int_0^1 s\sin(s^2+X)\ ds = \left.\frac{\cos(s^2+X)}{2}\right|_{s=0}^{s=1} = \frac{\cos(1+X)-\cos(X)}{2}.$$

And so

$$\frac{d}{dX}\left[\int_0^1 s\cos(s^2+X)\ ds\right] = \frac{\cos(1+X)-\cos(X)}{2} = \int_0^1 \frac{\partial}{\partial X}[s\cos(s^2+X)]\ ds. \qquad \blacktriangle$$

In this example differentiating under the integral worked perfectly; whether the "integrate with respect to $s$ and then differentiate with respect to $X$" method at the start of the solution was harder or easier is a matter of personal opinion. The deeper question is *when is* (A.1.1) *valid?* The interchange of $\lim_{h\to 0}$ and $\int_0^1$ is one of those "interchange of limits" operations that plagues advanced analysis and drives a lot of the stuff that you might learn about sequences and series of functions.

The world, unfortunately, is not just, and (A.1.1) can fail in a variety of circumstances. The good news, however, is that all of those circumstances are somewhat contrived and require, if not pathological, then at least "naughty" behavior of the integrand $f$. You can, and should, read several spicy examples in Appendix A.3 of our textbook for situations where differentiating under the integral fails. In my experience, though, one *can* differentiate under the integral in most situations that *arise in practice*. Here is the most benign of these situations.

**A.1.2 Theorem (Leibniz's rule for differentiating under the integral).**

*Fix numbers $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{R}$ with $\alpha < \beta$ and $\gamma < \delta$. Let*

$$\mathcal{U} = \{(X, s) \in \mathbb{R}^2 \mid \alpha \le X \le \beta,\ \gamma \le s \le \delta\}.$$

*Suppose that $f \in \mathcal{C}(\mathcal{U})$ and that $f_X$ exists on $\mathcal{U}$ with $f_X \in \mathcal{C}(\mathcal{U})$. (We do not require anything about the existence or continuity of $f_s$, and so we do not require $f \in \mathcal{C}^1(\mathcal{U})$.) Then differentiating under the integral is permissible:*

$$\frac{d}{dX}\left[\int_\gamma^\delta f(X, s)\ ds\right] = \int_\gamma^\delta f_X(X, s)\ ds$$

*for all $\alpha \le X \le \beta$.*

Observe that in Example A.1.1, we took $f(X, s) = s\cos(s^2 + X)$, and so $f \in \mathcal{C}^1(\mathbb{R}^2)$. In particular, $f$ satisfies the hypotheses of Leibniz's rule for $[\gamma, \delta] = [0, 1]$ and $[\alpha, \beta]$ arbitrary.

Leibniz's rule becomes more interesting, and more challenging, if we want to work with improper integrals over unbounded intervals. There are very good reasons for wanting to calculate something like

$$\frac{d}{dX}\left[\int_{-\infty}^{\infty} f(X, s)\ ds\right].$$

In those cases, to control for the unboundedness of the domain of integration, we would need to add extra hypotheses to Leibniz's rule.

## A.2. Two variants of Leibniz's rule.

Let's return to our motivating question: how do you calculate partial derivatives like

$$\frac{\partial}{\partial x}\left[\int_0^t f(\phi(x, t, s), s)\ ds\right] \quad \text{and} \quad \frac{\partial}{\partial t}\left[\int_0^t f(\phi(x, t, s), s)\ ds\right], \tag{A.2.1}$$

where $f = f(X, s)$ and $\phi = \phi(x, t, s)$ are given functions such that the composition $(x, t, s) \mapsto f(\phi(x, t, s), s)$ is defined?

It will help to extend Leibniz's rule to a slightly more complicated integrand.

**A.2.1 Corollary (Leibniz's rule with composition in the integrand).**

*Assume the notation and hypotheses of Theorem A.1.2. Now let $\widetilde{\alpha}$, $\widetilde{\beta} \in \mathbb{R}$ with $\widetilde{\alpha} < \widetilde{\beta}$,* *and let*

$$\widetilde{\mathcal{U}} = \left\{(\xi, s) \in \mathbb{R}^2 \;\middle|\; \widetilde{\alpha} \le \xi \le \widetilde{\beta},\ \gamma \le s \le \delta\right\}.$$

*Suppose that* $\psi \in \mathcal{C}(\widetilde{\mathcal{U}})$ *with* $\psi_\xi \in \mathcal{C}(\widetilde{\mathcal{U}})$ *and* $\alpha \le \psi(\xi) \le \beta$ *for all* $\xi$. *Then*

$$\frac{d}{d\xi}\left[\int_\gamma^\delta f(\psi(\xi, s), s)\ ds\right] = \int_\gamma^\delta f_X(\psi(\xi, s), s)\psi_\xi(\xi, s)\ ds$$

*for* $\widetilde{\alpha} \le \xi \le \widetilde{\beta}$.

**Proof.** Put $g(\xi, s) := f(\psi(\xi, s), s)$. The hypotheses and properties of continuity and derivatives under function composition imply $g \in \mathcal{C}(\widetilde{\mathcal{U}})$ and $g_\xi \in \mathcal{C}(\widetilde{\mathcal{U}})$. Leibniz's rule then says that

$$\frac{d}{d\xi}\left[\int_\gamma^\delta f(\psi(\xi, s), s)\ ds\right] = \frac{d}{d\xi}\left[\int_\gamma^\delta g(\xi, s)\ ds\right] = \int_\gamma^\delta g_\xi(\xi, s)\ ds.$$

And $g_\xi(\xi, s) = f_X(\psi(\xi, s), s)\psi_\xi(\xi, s)$ by the chain rule. □

Calculating the $x$-partial in (A.2.1) isn't too bad; when we take the partial with respect to $x$, we're assuming that $t$ is constant, and so the limits of integration are constant. If $\phi$ is continuous and $\phi_x$ exists and is continuous, then we can use the corollary to Leibniz's rule with $\xi = x$ and $\psi(x, s) = \phi(x, t, s)$ to find

$$\frac{\partial}{\partial x}\left[\int_0^t f(\phi(x, t, s), s)\ ds\right] = \int_0^t \frac{\partial}{\partial x}[f(\phi(x, t, s), s)]\ ds = \int_0^t f_X(\phi(x, t, s), s)\phi_x(x, t, s)\ ds.$$

Calculating the $t$-derivative in (A.2.1), I'm sorry to say, is harder. The problem is that $t$ appears as a limit of integration as well as in the integrand. We need one more trick.

**A.2.2 Corollary (Leibniz's rule with a variable limit of integration).**

*Assume the notation and hypotheses of Theorem A.1.2. Then*

$$\frac{d}{dX}\left[\int_\gamma^X f(X, s)\ ds\right] = f(X, X) + \int_\gamma^X f_X(X, s)\ ds$$

*for* $\alpha \le X \le \beta$.

**Proof.** We introduce the auxiliary function

$$H(X, \xi) := \int_\gamma^\xi f(X, s)\ ds.$$

Then

$$\int_\gamma^X f(X, s)\ ds = H(X, X).$$

The multivariable chain rule then tells us that

$$\frac{d}{dX}\left[\int_\gamma^X f(X,s)\ ds\right] = H_X(X,X) + H_\xi(X,X). \tag{A.2.2}$$

When $\xi$ is fixed, Leibniz's rule tells us that

$$H_X(X,\xi) = \frac{\partial}{\partial X}\left[\int_\gamma^\xi f(X,s)\ ds\right] = \int_\gamma^\xi f_X(X,s)\ ds.$$

Thus in the particular case $\xi = X$, we have

$$H_X(X,X) = \int_\gamma^X f_X(X,s)\ ds. \tag{A.2.3}$$

And when $X$ is fixed, the fundamental theorem of calculus tells us

$$H_\xi(X,\xi) = \frac{\partial}{\partial \xi}\left[\int_\gamma^\xi f(X,s)\ ds\right] = f(X,\xi).$$

Thus in the particular case $\xi = X$, we have

$$H_\xi(X,X) = f(X,X). \tag{A.2.4}$$

If we put (A.2.2), (A.2.3), and (A.2.4) together, we get the desired derivative. □

We use Leibniz's rule with a variable limit of integration and, once more, the chain rule to calculate the $t$-partial from (A.2.1):

$$\begin{aligned}\frac{\partial}{\partial t}\left[\int_0^t f(\phi(x,t,s),s)\ ds\right] &= f(\phi(x,t,t),t) + \int_0^t \frac{\partial}{\partial t}\big[f(\phi(x,t,s),s)\big]\ ds \\ &= f(\phi(x,t,t),t) + \int_0^t f_X(\phi(x,t,s),s)\phi_t(x,t,s)\ ds.\end{aligned}$$

## A.3. Application to the nonhomogeneous transport equation.

Let's conclude by going back to the transport equation, which started us down this dark path. Can you calculate

$$\frac{\partial}{\partial x}\left[\int_0^t f\left(\left[\frac{bx-at}{b}\right] + \frac{as}{b}, s\right)\ ds\right] \quad \text{and} \quad \frac{\partial}{\partial t}\left[\int_0^t f\left(\left[\frac{bx-at}{b}\right] + \frac{as}{b}, s\right)\ ds\right]?$$

And what hypotheses on $f$ do you need to do it? (Hint: you need more than $f \in \mathcal{C}(\mathbb{R}^2)$, as we stated in Theorem 1.5.7 but less than $f \in \mathcal{C}^1(\mathbb{R}^2)$.)