Contents

Day 1: Monday, January 9

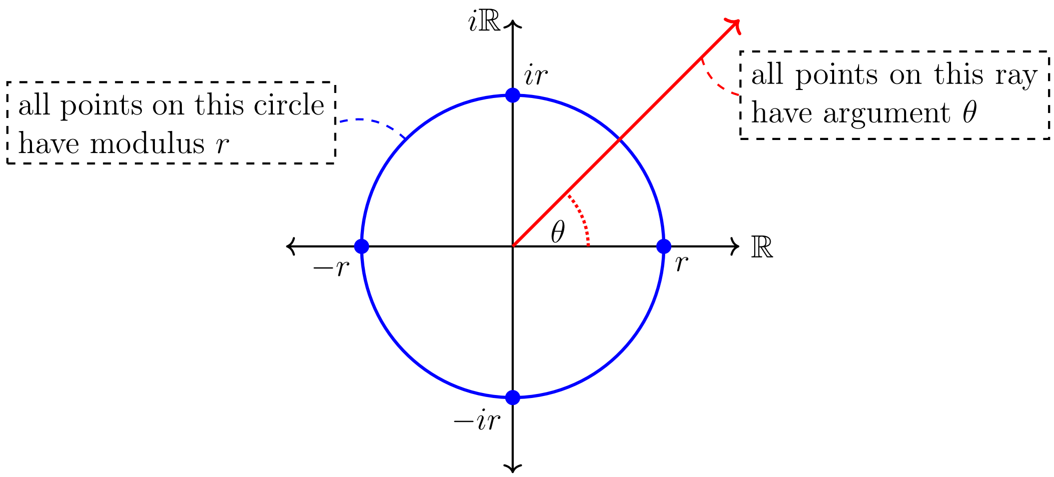
We talked about our feelings about calculus and got to know each other. We did some of the basic arithmetic on p. 4 of the textbook and thought about the complex plane ($\mathbb{C} = \mathbb{R}^2$, from a certain point of view) as on p. 13. We studied the modulus on p. 15.

Day 2: Wednesday, January 11

We studied the conjugate on p. 4 and the multiplicative inverse (= division = reciprocal) on pp. 5–6. The properties of the conjugate in Proposition 1.1.5 are very useful. We connected the conjugate and the modulus via Proposition 1.2.5. We talked about the inequalities in Proposition 1.2.8, whose lengthy proof you *don't* have to read. Last, we talked about polar coordinates, arguments, and principal arguments, as on pp. 25–30. By the way, polar coordinates are only defined for nonzero complex numbers, and I will write $\mathbb{C} \setminus \{0\} := \{z \in \mathbb{C} \mid z \neq 0\}$ to denote the set of those nonzero numbers. It is worth knowing that equation (1.3.13) on p. 30 contains a comprehensive formula for the principal argument, but you definitely don't have to memorize it. Consider working through Examples 1.1.2, 1.1.4, 1.1.6, 1.2.3, 1.2.7, 1.3.3, and 1.3.4.

Day 3: Friday, January 13

We continued talking about polar coordinates, moduli, and arguments. I think this is the key picture to keep in mind.



We did several calculations, stated (but not solved) below. Last, we actually defined the complex numbers formally, as the book did all the way back on p. 4. We have identified the complex number $x + iy$ with the ordered pair (x, y) in the past, and so in particular identify x with $(x, 0)$ and i with $(0, 1)$. Now we will make this into a definition.

Specifically, a complex number is an ordered pair of real numbers:

$$\mathbb{C} := \mathbb{R}^2 = \{(x, y) \mid x,\ y \in \mathbb{R}\}.$$

In the following, I will presume that we know all about real numbers, and that we could define (x, y) set-theoretically if we had to. The game is to define operations on complex numbers *just* in terms of operations on real numbers and ordered pairs. I think there's only one way to add ordered pairs:

$$(x, y) \oplus (u, v) := (x + u, y + v).$$

I am writing $\oplus$ to emphasize that adding ordered pairs is not the same as adding real numbers. Next, we want, after FOILing it out,

$$(x + iy)(u + iv) = xu - yv + i(xv + yu),$$

and so we *define*

$$(x, y) \odot (u, v) := (xu - yv, xv + yu).$$

If you've taken algebra, you can (and should) show that $\mathbb{C}$ is a field.

The most important property (I think) of $\odot$ is

$$(0, 1) \odot (0, 1) = \big((0 \cdot 0) - (1 \cdot 1), (0 \cdot 1) + (1 \cdot 0)\big) = (-1, 0),$$

where $\cdot$ means multiplication of real numbers (which we presume to understand perfectly). I will leave it to you to check that

$$(x, y) = (x, 0) \oplus \big[(0, 1) \odot (y, 0)\big].$$

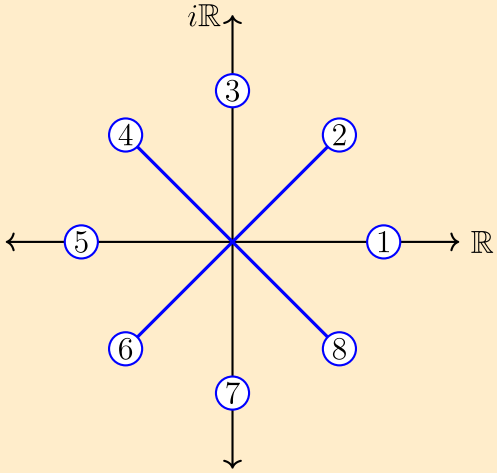
This is exactly the expansion of a complex number that we expect from the expression $x+iy$. So, from now on, if someone asks you for a rigorous definition of i, say $i = (0, 1)$.

There are just two problems. First, ordered pair notation and the $\odot$ operation are too bulky for daily use. Second, we expect that every real number is a complex number, but $\mathbb{R}$ is not a subset of $\mathbb{R}^2 = \mathbb{C}$. We get around the second issue by "identifying" each real number x with the natural relative $(x, 0)$ as an ordered pair. For the algebraically minded, $\mathbb{R}$ is isomorphic to a subfield (call it $\mathbb{C}_{\mathbb{R}}$) of $\mathbb{C}$, and we just denote elements $(x, 0)$ of this subfield $\mathbb{C}_{\mathbb{R}}$ by their "pullback" x in $\mathbb{R}$ For the first issue, we use this identification of $(x, 0) \in \mathbb{R}^2$ with $x \in \mathbb{R}$ and express any calculations with (x, y) by $x + iy$. From time to time, especially when we talk about limits and continuity, it will be very useful to import properties of $\mathbb{R}^2$ from multivariable calculus into our work, but otherwise let us never speak of this again.

Here are some exercises for you, some of which we did in class.

3.1 Problem. For $z = -1+i$, find $\operatorname{Arg}(z)$, write z in its polar form, and calculate $\operatorname{Re}(1/z)$.

3.2 Problem. Compute the principal argument of each kind of point labeled in the plane below. These are the kinds of arguments that we will use most often in the course (so long, 30–60–90 triangles).



① $\operatorname{Re}(z) > 0$, $\operatorname{Im}(z) = 0$
② $\operatorname{Re}(z) = \operatorname{Im}(z) > 0$
③ $\operatorname{Re}(z) = 0$, $\operatorname{Im}(z) > 0$
④ $\operatorname{Re}(z) = -\operatorname{Im}(z)$, $\operatorname{Im}(z) > 0$
⑤ $\operatorname{Re}(z) < 0$, $\operatorname{Im}(z) = 0$
⑥ $\operatorname{Im}(z) = \operatorname{Re}(z)$, $\operatorname{Re}(z) < 0$
⑦ $\operatorname{Re}(z) = 0$, $\operatorname{Im}(z) < 0$
⑧ $\operatorname{Im}(z) = -\operatorname{Re}(z)$, $\operatorname{Re}(z) > 0$

3.3 Problem. Explain how the values of i^n are "4-periodic" in n, i.e., $i^n = i^{n+4}$ for all $n \in \mathbb{Z}$, and use this to compute something like i^{1977} quickly.

3.4 Problem. (Wholly optional.) If you've taken abstract algebra, look up the definition of a field and then prove that $\mathbb{C}$ is a field under $\oplus$ and $\odot$. Prove that $\mathbb{R}$ is isomorphic to the subfield $\mathbb{R} \times \{0\} = \{(x, 0) \mid x \in \mathbb{R}\}$ of $\mathbb{C}$.

3.5 Problem. (Wholly optional.) If you've taken linear and abstract algebra, check that $\odot$ is really matrix-vector multiplication:

$$(x, y) \odot (u, v) = (xu - yv, xv + yu) = \begin{bmatrix} xu - yv \\ xv + yu \end{bmatrix} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now put

$$\mathbb{R}_{\mathbb{C}}^{2\times 2} := \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \in \mathbb{R}^{2\times 2} \;\middle|\; x,\ y \in \mathbb{R} \right\}$$

and check that $\mathbb{R}^2$ and $\mathbb{R}_{\mathbb{C}}^{2\times 2}$ are isomorphic as fields, where addition in both is componentwise, multiplication in $\mathbb{R}^2$ is $\odot$, and multiplication in $\mathbb{R}_{\mathbb{C}}^{2\times 2}$ is the usual multiplication of 2×2 matrices. And so if you want to think of $\mathbb{C}$ as a special set of matrices, you can.

Day 4: Wednesday, January 18

We started talking about functions, and we'll continue to do so for the rest of the course. See pp. 41–42 for some definitions.

In my notation, the string of five symbols $f \colon \mathcal{D} \to \mathbb{C}$ should be read as "the function f from $\mathcal{D}$ to $\mathbb{C}$." Here $\mathcal{D} \subseteq \mathbb{C}$ is the domain of f. (We will use the word "domain" with at least one other meaning later on, so watch the context.) By $\mathcal{D} \subseteq \mathbb{C}$ I mean that if $z \in \mathcal{D}$, then $z \in \mathbb{C}$.

Informally, a function $f\colon \mathcal{D} \to \mathbb{C}$ is a rule that pairs each number $z \in \mathcal{D}$ with a unique number $w \in \mathbb{C}$, and we write $w = f(z)$. The set $f(\mathcal{D}) := \{f(z) \mid z \in \mathbb{C}\}$ is the range or image of f (image starts with "im," which already has another meaning in this course, so we'll say "range"). The set-theoretic meaning of $f\colon \mathcal{D} \to \mathbb{C}$ (which you should understand, but which in practice we won't invoke too much, if at all) is that f is a set of ordered pairs of complex numbers such that for each $z \in \mathcal{D}$, there is a unique $w \in \mathbb{C}$ such that $(z, w) \in f$, and of course we write $w = f(z)$.

We know a handful of functions already, like integer powers: for $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, we put

$$z^n = \begin{cases} 1,\ n = 0 \text{ (even for })z = 0 \\ zz^{n-1},\ n \geq 1 \\ \left(\dfrac{1}{z}\right)^{|n|},\ n \leq -1 \text{ and } z \neq 0, \text{ with } \dfrac{1}{z} := \dfrac{\overline{z}}{|z|^2}. \end{cases}$$

Polynomials are linear combinations of nonnegative integer powers: $f(z) = \sum_{k=0}^{n} a_k z^k$ with $a_0, \ldots, a_n \in \mathbb{C}$. We have the "projections" onto real and imaginary parts, like $f(z) = \mathrm{Re}(z)$; in that case, we might use the notation

$$f\colon \mathbb{C} \to \mathbb{R}\colon z \mapsto \mathrm{Re}(z)$$

to emphasize (1) that the range of f is a subset of $\mathbb{R}$ and (2) that the "formula" for f is $f(z) = \mathrm{Re}(z)$.

Here is something that is not a function: for $z \in \mathbb{C} \setminus \{0\}$, if $z = |z|(\cos(\theta) + i\sin(\theta))$, let $f(z) = \theta$. The problem is that we could use θ or $\theta + 2\pi$. This function is not "well-defined." (Set-theoretically, it is a *relation*: all functions are relations, but not all relations are functions.) However, if we specify θ more precisely, we do get a function, say, $\mathrm{Arg}\colon \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$. Are you comfortable with every piece of notation in the string of symbols

$$\mathrm{Arg}\colon \mathbb{C} \setminus \{0\} \to (-\pi, \pi]?$$

See p. 42 for sketches of how to visualize maps from $\mathbb{C}$ to $\mathbb{C}$ and draw a picture like that on p. 42 for $f(z) = \overline{z}$ with the domain as $\mathcal{D} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$, i.e., as Quadrant I. What is the range?

We often express the action of a function on the real and imaginary parts of its input by first writing $f(z) = \mathrm{Re}[f(z)] + i\,\mathrm{Im}[f(z)]$ and then $z = x+iy$, so $f(x+iy) = u(x, y) + iv(x, y)$. It is helpful to view u and v as functions from (subsets of) $\mathbb{R}^2$ to $\mathbb{R}$, so we can use multivariable calculus, but remember $\mathbb{R}^2 = \mathbb{C}$ anyway. Example 1.4.4 explores this with both formulas and pictures.

Last, we started talking about sequences. See the first two paragraphs of Section 1.5 on p. 52. Sequences are the building blocks of "interesting" functions, which we mostly express via series. Sequences are also phenomenal tools for analysis—they allow us to express "continuous" concepts in "discrete" language. We'll see both of these uses of sequences throughout the term.

Initially, a sequence for us will be a function from the positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$ to $\mathbb{C}$. That is, a sequence is a function $f\colon \mathbb{N} \to \mathbb{C}$. We usually write $z_n := f(n)$ and then say $f = (z_n)$. To be absolutely clear: the symbol (z_n) is the function from $\mathbb{N}$ to $\mathbb{C}$ that pairs n with z_n, i.e., $(n, z_n) \in (z_n)$. You should think of the string of three (maybe four?) symbols (z_n) as representing a single function. The book writes $\{z_n\}_{n=1}^{\infty}$ with curly braces; emphasizing the starting index at $n = 1$ may be helpful, but writing curly braces makes me think of a set. I won't use curly braces.

For example, if $z_n = (-1)^n$, then (z_n) is the function $f: \mathbb{N} \to \mathbb{C}$ such that $f(n) = (-1)^n$, but the range of this function is the set $\{1, -1\}$. Don't confuse ranges with functions!

Day 5: Friday, January 20

We spent the day talking about sequences. My hope is that we will develop a more precise understanding of sequences than we might have seen in calculus, but not so precise and technical an understanding as in real analysis. See Definition 1.5.1 and Figure 1.31 for the definition of convergence and the notation at the top of p. 53. Proposition 1.5.2 should be comforting. Example 1.5.3 is essentially a different version of what we did in class. Theorem 1.5.7 contains the "stability" properties of limits with respect to standard algebraic operations in $\mathbb{C}$.

We finished by talking about Theorem 1.5.8. The proof in the book is a little different from my argument, which I reproduce here. Suppose $z_n \to z$. We want to show $\operatorname{Re}(z_n) \to \operatorname{Re}(z)$ and $\operatorname{Im}(z_n) \to \operatorname{Im}(z)$. We compute

$$|\operatorname{Re}(z_n) - \operatorname{Re}(z)| = |\operatorname{Re}(z_n - z)| \le |z_n - z| \to 0.$$

Here we used the properties $\operatorname{Re}(v) - \operatorname{Re}(w) = \operatorname{Re}(v - w)$ and $|\operatorname{Re}(w)| \le |w|$, valid for all v, $w \in \mathbb{C}$.

Day 6: Monday, January 23

We continued working with Theorem 1.5.8. We showed that if $\operatorname{Re}(z_n) \to \operatorname{Re}(z)$ and $\operatorname{Im}(z_n) \to \operatorname{Im}(z)$, then $z_n \to z$. We compute

$$\begin{aligned} |z_n - z| = |[\operatorname{Re}(z_n) + i\operatorname{Im}(z_n)] - [\operatorname{Re}(z) + i\operatorname{Im}(z)]| &= |[\operatorname{Re}(z_n) - \operatorname{Re}(z)] + i[\operatorname{Im}(z_n) - \operatorname{Im}(z)]| \\ &\le |\operatorname{Re}(z_n) - \operatorname{Re}(z)| + |i[\operatorname{Im}(z_n) - \operatorname{Im}(z)]|. \end{aligned}$$

Since $z_n \to z$, we have $|\operatorname{Re}(z_n) - \operatorname{Re}(z)| \to 0$. And $|i[\operatorname{Im}(z_n) - \operatorname{Im}(z)]| = |\operatorname{Im}(z_n) - \operatorname{Im}(z)| \to 0$. Here we used the triangle inequality $|v + w| \le |v| + |w|$ and the identity $|iv| = |i| \cdot |v| = |v|$.

Then we started talking about series, which will allow us to define many useful functions and, later, express many profound ideas in precise language. See p. 56 for definitions. Here is my formal definition of a series, which is slightly different.

6.1 Definition. *Let (z_n) be a sequence of complex numbers with $n \ge 0$. The* **SERIES** $\sum_{k=0}^{\infty} z_k$ *is the sequence of* n**TH PARTIAL SUMS**, *which are* $\sum_{k=0}^{n} z_k$. *That is,*

$$\sum_{k=0}^{\infty} z_k := \left(\sum_{k=0}^{n} z_k\right).$$

Additionally, if the sequence of nth partial sums converges, then $\sum_{k=0}^{\infty} z_k$ also denotes that *limit and is called the* SUM *of the series. That is, if* $\lim_{n\to\infty}\sum_{k=0}^{n} z_k$ *exists, then*

$$\sum_{k=0}^{\infty} z_k := \lim_{n\to\infty}\sum_{k=0}^{n} z_k.$$

Thus the symbol $\sum_{k=0}^{\infty} z_k$ *may have two very different meanings; it is always a sequence, and it may be the limit of that sequence, if that limit exists. Context will make clear the intended meaning of* $\sum_{k=0}^{\infty} z_k$.

See Example 1.5.13 for the geometric series. I discussed telescoping:

$$\sum_{k=0}^{n}(z^k - z^{k+1}) = 1 - z^{n+1}$$

from the more general telescoping identity

$$\sum_{k=0}^{n}(w_k - w_{k+1}) = w_0 - w_{n+1},$$

valid for any sequence (w_n). We did not prove divergence in the case $|z| = 1$, which the book does using Example 1.5.9. That's a great example to read, but it's more detail than I wanted to show in class.

We briefly discussed some convergence tests, which we'll pick up next time. Broadly, it is good to be familiar with the results on pp. 58–63 (or at least know what page to turn to when you need something), but I will single out three particular tools.

Day 7: Wednesday, January 25

I say these are the three most important convergence tests to know and memorize; you can always look up everything else.

1. Test for divergence: if $\lim_{n\to\infty} z_n \neq 0$, then $\sum_{k=0}^{\infty} z_k$ diverges. The contrapositive is helpful, too: if $\sum_{k=0}^{\infty} z_k$ converges, then $\lim_{n\to\infty} z_n = 0$.

2. Absolute convergence: if $\sum_{k=0}^{\infty}|z_k|$ converges, then $\sum_{k=0}^{\infty} z_k$ also converges. The convergence of $\sum_{k=0}^{\infty} z_k$ does not imply the convergence of $\sum_{k=0}^{\infty}|z_k|$; think about alternating series.

3. Comparison test: if $0 \le v_k \le w_k$ and $\sum_{k=0}^{\infty} w_k$ converges, then $\sum_{k=0}^{\infty} v_k$ also converges.

We frequently combine absolute convergence and the comparison test as follows. To show that $\sum_{k=0}^{\infty} z_k$ converges, first develop an estimate of the form $0 \le |z_k| \le w_k$. The "dominating" terms w_k should be somehow "nice" in that you can show that $\sum_{k=0}^{\infty} w_k$ converges. The comparison test forces convergence of $\sum_{k=0}^{\infty}|z_k|$, and then absolute convergence forces convergence of $\sum_{k=0}^{\infty} z_k$. To develop the convergence of $\sum_{k=0}^{\infty} w_k$, you may need to use the comparison test again or some other test. The important thing is that $w_k \ge 0$, and there are many tests for convergence that apply to series with nonnegative terms. Again, you may need to check pp. 58–63 from time to time.

Now we finally have the tools to do some stuff that's really unique to complex analysis and that doesn't just follow by writing z where previously we wrote x. A fundamental problem is that of *extensions*. Say that $f \colon I \subseteq \mathbb{R} \to \mathbb{R}$ is a function. (Recall notation: this means that f is a function defined on I, and I is a subset of $\mathbb{R}$, and the range of f is a subset of $\mathbb{R}$.) Say that $\mathcal{D} \subseteq \mathbb{C}$ is a set with $I \subseteq \mathcal{D}$. Can we define a function $g \colon \mathcal{D} \to \mathbb{C}$ such that $g(z) = f(z)$ for all $z \in I$? In this case we call g an **EXTENSION** of f to $\mathcal{D}$. Equivalently, if $g\big|_I \colon I \to \mathbb{C} \colon z \mapsto g(z)$ is the **RESTRICTION** of g to I, do we have $g\big|_I = f$?

The answer is "of course," if we are lazy. Just put

$$g(z) := \begin{cases} f(z), & z \in I \\ 0, & z \in \mathcal{D} \setminus I. \end{cases}$$

A better question is if we can find an extension g that shares "meaningful properties" with f.

We started with the exponential and extended it to $\mathbb{C}$ via the power series definition. See pp. 64–66. In particular, Theorem 1.6.2 gives a detailed proof of the **FUNCTIONAL EQUATION** $e^{z+w} = e^z e^w$. I partially showed in class how the functional equation proves that $e^x > 0$ for $x \in \mathbb{R}$. Here's the full proof. First,

$$e^x = e^{x/2+x/2} = e^{x/2}e^{x/2} = (e^{x/2})^2 \geq 0.$$

This inequality is, of course, only true because $e^{x/2} \in \mathbb{R}$; in general, z^2 need not be real (and therefore not positive, negative, or zero) for $z \in \mathbb{C}$. Next, the power series formula for e^0 gives $e^0 = 1$. Then

$$1 = e^0 = e^{x-x} = e^x e^{-x},$$

and so $e^x \neq 0$. Thus $e^x \geq 0$ and $e^x \neq 0$, so $e^x > 0$ for all $x \in \mathbb{R}$.

What is really new is Proposition 1.6.3: Euler's formula, which says

$$e^{iy} = \cos(y) + i\sin(y)$$

for $y \in \mathbb{R}$ (because, as of yet, cosine and sine aren't defined for $y \in \mathbb{C}$), and thus from the functional equation,

$$e^{x+iy} = e^x e^{iy} = e^x\big(\cos(y) + i\sin(y)\big).$$

The consequences of this identity are pretty vast. Here's one: $e^{z+2\pi ik} = e^z$ for all $k \in \mathbb{Z}$, and so the complex exponential is not one-to-one like the real exponential.

Day 8: Friday, January 27

We used the functional equation $e^{z+w} = e^z e^w$ and the direct calculation of $e^0 = 1$ to show that $e^z e^{-z} = 1$, and therefore $e^z \neq 0$ for all $z \in \mathbb{C}$. We knew $e^x \neq 0$ for all $x \in \mathbb{R}$, but this is new. This identity also gives

$$\frac{1}{e^z} = e^{-z}.$$

By definition, we only knew

$$\frac{1}{e^z} = \frac{\overline{e^z}}{|e^z|^2},$$

so this formula for $1/e^z$ is better, and probably not apparent from the power series definition. This was Theorem 1.6.2.

Next we talked about the polar form of e^z; see Corollary 1.6.4 and p. 68. I encourage you to work through the calculations in Examples 1.6.5 and 1.6.6. We saw that $|e^{it}| = 1$ for all $t \in \mathbb{R}$, but can you find $z \in \mathbb{C}$ such that $|e^{iz}| \neq 1$?

Some calculations revealed the (possibly surprising) facts that e^z can be negative ($e^{i\pi} = -1$ and, more generally, $e^{ik\pi} = (-1)^k$ for $k \in \mathbb{Z}$) and that the exponential is not one-to-one on $\mathbb{C}$: $e^0 = 2e^{2\pi i} = 1$, and more generally $e^{z+2\pi ik} = e^z$ for all $k \in \mathbb{Z}$. Then we found all $z \in \mathbb{C}$ such that $e^z = 1$: these are $z = 2\pi ik$ for $k \in \mathbb{Z}$ (you might say $z \in 2\pi i\mathbb{Z}$). This is Proposition 1.6.7. In particular, this proposition tells us that the exponential *is* periodic with period $2\pi ik$ for any $k \in \mathbb{Z}$ but *not* P-periodic for any other P. Indeed, if $e^{z+P} = e^z$ for any z, $P \in \mathbb{C}$, then the functional equation forces $e^P = 1$, and therefore $P = 2\pi ik$ for some $k \in \mathbb{Z}$.

Finally, we exploited the appearance of the complex exponential in polar representations. For $z \in \mathbb{C} \setminus \{0\}$, we can write

$$z = |z|\big(\cos(\theta) + i\sin(\theta)\big),$$

where $\theta \in \mathbb{R}$ is an argument of z. But this is just

$$z = |z|e^{i\theta}.$$

See Proposition 1.6.8 (and note that equation (1.6.19) makes the last problem of Problem Set 2 much easier!).

We can use polar representations to solve algebraic problems. Go all the way back to p. 2 to recall that $\mathbb{C}$ exists because we want to solve $x^2 + 1 = 0$. More generally, let's try solving $z^n = w$ given an integer $n \in \mathbb{Z}$ and $w \in \mathbb{C} \setminus \{0\}$. Note that $w = 0$ is not too interesting, since $z^n = 0$ if and only if $z = 0$. Note that n has to be an integer, since we have not treated noninteger complex powers (they're weird!).

Let's work backward. Suppose $z^n = w$ and write z and w in polar form:

$$z = |z|e^{i\theta} \quad \text{and} \quad w = |w|e^{i\phi}.$$

You should think of $|w| \in [0, \infty)$ and $\phi \in \mathbb{R}$ as given information (feel free to specify $\phi = \text{Arg}(w)$) and $|z|$, θ as unknowns. Then we need

$$|w|e^{i\phi} = w = z^n = [|z|e^{i\theta}]^n = |z|^n[e^{i\theta}]^n = |z|e^{in\theta}.$$

Here is the key identity: $[e^{i\theta}]^n = e^{in\theta}$. This is what we expect from real numbers, e.g., $(x^p)^q = x^{pq}$. While we haven't defined noninteger complex powers yet, we do have the identity $(e^v)^n = e^{nv}$ for $v \in \mathbb{C}$ and $n \in \mathbb{Z}$ thanks to the functional equation.

So, we have

$$z^n = w \Longrightarrow |z|^n e^{in\theta} = |w|e^{i\phi}.$$

Take the modulus of both sides to get

$$\left||z|^n e^{in\theta}\right| = \left||w|e^{i\phi}\right| \Longrightarrow |z|^n = |w|,$$

since $|e^{it}| = 1$ for all $t \in \mathbb{R}$. Thus $|z| = |w|^{1/n}$, where $|w|^{1/n}$ is the usual nth root of $|w|$. (We probably should make that more precise. We will.)

We'll finish this next time. This is morally the same as the material on p. 34.

Day 9: Monday, January 30

Picking up from Friday, we figured out that $e^{in\theta} = e^{i\phi}$, thus $e^{i(n\theta-\phi)} = 1$, and so $i(n\theta - \phi) = 2\pi ik$ for some integer $k \in \mathbb{Z}$. We solve for θ as $\theta = (\phi + 2\pi k)/n$, and therefore every solution to $z^n = w$ has the form

$$z = |w|^{1/n} \exp\left(\frac{i(\phi + 2\pi k)}{n}\right) = |w|^{1/n} \exp\left(\frac{i(\mathrm{Arg}(w) + 2\pi k)}{n}\right) =: z_k.$$

You should check that we get n distinct solutions in the following sense: $z_{k+n} = z_k$ for all $k \in \mathbb{Z}$, but if $1 \le j < k \le n$, then $z_j \ne z_k$. You should also compare this result to Proposition 1.3.10.

Then we solved $z^3 = 1$ and talked about nth roots of unity. See Example 1.3.11 and also Example 1.3.12. Look at the circles in Figures 1.20 and 1.21. Last, use the formula for z_k above to convince yourself that the only solutions to $z^2 = -1$ are $z = \pm i$.

Last, we discussed logarithms. The point of a logarithm is to invert the exponential, i.e., to solve $e^z = w$ for z given w. We know that we have to take $w \ne 0$ since $e^z \ne 0$ for all z. We also don't expect unique solutions, since $e^z = 1$ has the infinitely many solutions $z = 2\pi ik$, $k \in \mathbb{Z}$. We got down to (1.8.3) on p. 86 and then discussed the principal log (Definition 1.8.2) and the αth branch of the argument and the log (Definition 1.8.4). Next time: lots of filthy, filthy calculations with logs.

Day 10: Wednesday, February 1

We started by calculating the thing you've always wanted to do: the logarithm of -1. Specifically, we found $\mathrm{Log}(-1) = i\pi$, $\log_\pi(-1) = 3\pi i$, and $\log(-1) = (2k+1)\pi i$ with $k \in \mathbb{Z}$. That last equality really means

$$\log(-1) = \{(2k+1)\pi i \mid k \in \mathbb{Z}\},$$

but usually we don't write logs in this set notation. Then we checked

$$e^{\log(z)} = z \quad \text{and} \quad \log(e^z) = z + 2\pi ik, \ k \in \mathbb{Z}.$$

Both of these equalities are really set equalities in disguise. The first says

$$\{e^w \mid w \in \log(z)\} = \{z\},$$

or, if you prefer,

$$w \in \log(z) \Longrightarrow e^w = z,$$

and the second says

$$\log(e^z) = \{z + 2\pi ik \mid k \in \mathbb{Z}\}.$$

The takeaway for me is that $e^{\log(z)}$ is what you think it is, but $\log(e^z)$ is always multi-valued.

Calculating $e^{\log(z)} = z$ is not much more than using the definition of $\log(\cdot)$, the functional equation for the exponential, and the identity $e^{2\pi ik} = 1$ for $k \in \mathbb{Z}$. Calculating $\log(e^z) = z + 2\pi ik$ takes more work.

Here's some of that work. Start with

$$\log(e^z) = \ln(|e^z|) + i\operatorname{Arg}(e^z) + 2\pi ik, \quad \text{where} \quad \ln(|e^z|) = \ln(e^{\operatorname{Re}(z)}) = \operatorname{Re}(z).$$

Now recall the polar form

$$e^z = e^{\operatorname{Re}(z)+i\operatorname{Im}(z)} = e^{\operatorname{Re}(z)}e^{i\operatorname{Im}(z)} = |e^z|e^{i\operatorname{Im}(z)},$$

which reminds us that $\operatorname{Im}(z)$ is *an* argument of e^z, but maybe not the *principal* argument. But we can always find a (unique) integer $K(z) \in \mathbb{Z}$ such that $\operatorname{Im}(z) + 2K(z)\pi \in (-\pi, \pi]$, since we can write $\mathbb{R}$ as the disjoint union $\mathbb{R} = \cup_{k=-\infty}^{\infty}(k\pi, (k+2)\pi]$, and therefore $\operatorname{Arg}(z) = \operatorname{Im}(z) + 2K(z)\pi$. Thus

$$\log(e^z) = \operatorname{Re}(z) + i\operatorname{Im}(z) + 2\pi i(k + K(z)) = z + 2\pi i(k + K(z)).$$

Since $k + K(z) \in \mathbb{Z}$, we have the equality above.

You should read Examples 1.8.1, 1.8.3, and 1.8.5 and contemplate deeply the four bullet points at the top of p. 88. They're weird! Here is another good exercise. Let $r > 0$ and $z = x + iy$. Calculate $\log(re^z)$ in terms of r, x and y. What do you do if $y \neq \operatorname{Arg}(z)$?

Last, we talked about powers. We know what z^2 means, but what about $z^{1/2}$? 2^i? To figure out a good definition for the complex power z^a, we thought about the real situation of x^a, where $x > 0$ and $a \in \mathbb{R}$. With $y = x^a$, we calculated $\ln(y) = a\ln(x)$, and so $y = e^{a\ln(x)}$. That is, whatever x^a is, it should satisfy $x^a = e^{a\ln(x)}$. We know what $\ln(\cdot)$ is, because it's the integral $\ln(x) = \int_1^x t^{-1}\,dt$. We know what $e^{(\cdot)}$ is, because it's a power series. And so x^a really is

$$x^a = e^{a\ln(x)} = \sum_{k=0}^{\infty} \frac{[a\ln(x)]^k}{k!}.$$

I love this: we went from what x^a *does* to what x^a *is*. That's how math goes!

From this, we defined, for $z \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{C}$,

$$z^a := e^{a\log(z)}.$$

We exclude $z = 0$ because 0^{anything} should be 0. This definition of z^a is inherently multi-valued:

$$z^a = \{e^{aw} \mid w \in \log(z)\} = \left\{e^{a[\ln(|z|)+i\operatorname{Arg}(z)+2\pi ik} \;\middle|\; k \in \mathbb{Z}\right\} = \left\{e^{a\operatorname{Log}(z)}e^{2\pi ika} \;\middle|\; k \in \mathbb{Z}\right\}.$$

As before, we usually suppress the set notation, and so we will often think of z^a as

$$z^a = e^{a\operatorname{Log}(z)}e^{2\pi ika}, \; k \in \mathbb{Z}.$$

We did the quick calculation

$$(-1)^{1/2} = e^{\mathrm{Log}(-1)/2}e^{2\pi ik/2} = e^{i\pi/2}e^{i\pi k} = i(-1)^k = \pm i,$$

which is exactly what it should be, and so we left class happy. You can find more happiness with powers on pp. 90–91 (which you should read carefully, but you can skip Example 1.8.8).

Day 11: Friday, February 3

We started by calculating $1^i = e^{-2\pi k}$ for $k \in \mathbb{Z}$. What this is really saying is that 1^i is the set $\{e^{-2\pi k} \mid k \in \mathbb{Z}\}$. We probably used to expect $1^x = 1$ for all $x \in \mathbb{R}$, but that may no longer be true. You might want to calculate 1^z for an arbitrary $z \in \mathbb{C}$.

We talked about ambiguity: is e^z the power series that we know and love, or is it $e^z = e^{z\,\mathrm{Log}(e)}e^{2\pi izk}$? For consistency, we will always interpret the symbol e^z as the power series, never the set.

Then we worked through Case (i) on p. 90 and assured ourselves that z^n has only one possible meaning for $n \in \mathbb{Z}$. You should read the other two cases on pp. 90–91, and you might find Project Problems 1.8.53 and 1.8.54 on p. 94 interesting. The book doesn't quite spell this out, but we can specify, for $\alpha \in \mathbb{R}$, the αth branch of the power z^a to be the single number $e^{a\log_\alpha(z)}$.

After that, we thought about identities like $(x^p)^q = x^{pq}$ and $x^px^q = x^{p+q}$, which hold when $x > 0$ and p, $q \in \mathbb{R}$. It is very difficult to make sense of these for $z \in \mathbb{C} \setminus \{0\}$ with complex exponents. For example, since $z^a = \{e^{a\,\mathrm{Log}(z)}e^{2\pi ika} \mid k \in \mathbb{Z}\}$, we might expect that

$$(z^a)^b = \{e^{b\,\mathrm{Log}(w)}e^{2\pi ikb} \mid w \in z^a\}.$$

How on earth can this compare to $z^{ab} = \{e^{ab\,\mathrm{Log}(z)}e^{2\pi ikab} \mid k \in \mathbb{Z}\}$? You could spend weeks of your life puzzling out when, and to what extent, power identities still hold for complex numbers, but you shouldn't.

Here's my takeaway: we tried to defined z^a in as reasonable a manner as possible to be in accord with what we expect from real numbers. Opening powers to complex inputs opens a multiverse of dangerous possibilities! You should be able to compute z^a as a set-valued expression, and from time to time we may use a branch of this power as a useful example of a function. Otherwise, don't worry too much about powers.

Last, we went back to Section 1.7 and talked about trig functions. For me, the sine is a power series:

$$\sin(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}, \ x \in \mathbb{R}.$$

You can show that this converges absolutely using the ratio test. One way to define the sine for complex inputs is to replace x with z:

$$\sin(z) = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}, \ z \in \mathbb{C}.$$

You can, and should, check the absolute convergence of this series, too.

This is an unambiguous definition of the complex sine, but it's not always the most useful definition. Here's a more flexible one for real x:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

See p. 76. This suggests that we should have

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

for $z \in \mathbb{C}$, and indeed we do; grind it out using the power series definition of $e^{\pm iz}$ and the power series definition of $\sin(z)$.

We finished by doing Example 1.7.7 (a) and asking the following question: if we define a function on $\mathbb{R}$ in two ways, and if we extend each of those definitions to $\mathbb{C}$, do we still have the same function? Conversely, if f_1 and f_2 are functions on (the same subset of) $\mathbb{C}$, and $f_1(x) = f_2(x)$ for all $x \in \mathbb{R}$, should we have $f_1(z) = f_2(z)$ for all z?

Day 12: Monday, February 6

We continued talking about trig functions (really, the sine). We showed that the sine doesn't pick up any new periods on $\mathbb{C}$. I don't think this is in the book, so here's a sketch. Suppose $\sin(z + P) = \sin(z)$ for all $z \in \mathbb{C}$. What do we know about P? Take $z = 0$ to see $\sin(P) = \sin(0) = 0$ and therefore $P = k\pi$ for some $k \in \mathbb{Z}$. We can do better, as we expect the periods of the sine to be *even* integer multiples of π. So what if $k = 2j + 1$ for some $j \in \mathbb{Z}$? Then $\sin(z + (2j + 1)\pi) = \sin(z)$ for all $z \in \mathbb{C}$. If we're clever, we can find some value of z that results in a contradiction. We can't use an integer multiple of π (as that would just give $0 = 0$), so try $z = \pi/2$. Use the fact that the sine is already 2π-periodic to get down to $\sin(\pi/2 + \pi) = \sin(\pi/2)$, or $-1 = 1$.

Then we showed that the sine is not bounded on $\mathbb{C}$, whereas $|\sin(x)| \le 1$ for all $x \in \mathbb{R}$. This is Example 1.7.5; see the two equations right above for the hyperbolic trig definitions.

You should read Example 1.7.2 and Propositions 1.7.3 and 1.7.4 for extra practice with the complex sine and cosine. We won't do anything else from Section 1.7. Last, you should use the definition

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

to find all zeros and periods of the cosine on $\mathbb{C}$ and to see that the cosine is unbounded on $\mathbb{C}$. You could use the identity $\cos(z) = \sin(z + \pi/2)$, but I think it's good practice to work directly with this definition.

This brings us to the end of the precalculus phase of our course: we've learned arithmetic, geometry, a bit of algebra, what functions are, and the definitions and properties of the essential transcendental functions (exponentials, logs, powers, and trig). Now it's time for calculus! We began with the pretty cut-and-dry definition of the limit as in Definition 2.2.1: for a function $f \colon \mathcal{D} \to \mathbb{C}$ and numbers a, $L \in \mathbb{C}$, we say $\lim_{z \to a} f(z) = L$ if we can make $f(z)$ arbitrarily close to L by taking z sufficiently close to a. Of course, we measure "close" by studying the moduli $|z-a|$ and $|f(z)-L|$, and we measure the adverbs "arbitrarily" and "sufficiently" via ϵ, δ, and quantifiers. Specifically, $\lim_{z\to a} f(z) = L$ if and only if for all $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |z-a| < \delta$ and $z \in \mathcal{D}$, then $|f(z)-L| < \epsilon$. We require $z \in \mathcal{D}$ so that $f(z)$ is defined; maybe there are $z \in \mathbb{C}$ such that $0 < |z-a| < \delta$ but $z \notin \mathcal{D}$. Also, we specify $0 < |z-a|$ because we don't want to require anything about $z = a$. Maybe $a \notin \mathcal{D}$, in which case $f(a)$ is not defined. Or maybe $f(a) \neq L$; this happens a lot with limits. In any case, best to avoid saying anything about $z = a$ right now. One last thing from the precalculus phase. It absolutely, positively *does not* make sense to try to compare complex numbers using inequalities. For example, the sentence $i < 2i$ has no meaning, although $1 < 2$ and $|i| < |2i|$. This begs the question of $<$. Here is a rigorous answer: it is a fact that there exists a set $\mathcal{P} \subseteq \mathbb{R}$ with the following two properties. First, for each $x \in \mathbb{R}$, exactly one of the following holds: either $x \in \mathcal{P}$ or $-x \in \mathcal{P}$ or $x = 0$. This property is called *trichotomy*. Second, for each x, $y \in \mathcal{P}$, it is the case that $x + y \in \mathcal{P}$ and $xy \in \mathcal{P}$. If $x \in \mathcal{P}$, then we write $0 < x$, and for x, $y \in \mathbb{R}$, we write $x < y$ if $0 < y - x$, equivalently, if $y - x \in \mathcal{P}$. Of course, $\mathcal{P} = (0, \infty)$, but note that we can characterize $\mathcal{P}$ just via trichotomy and algebra. We might say that $\mathcal{P}$ defines an *order* on $\mathbb{R}$ via the *relation* $<$.

Can we define an order on $\mathbb{C}$ in a similar way? Is there a set $\mathcal{P}_{\mathbb{C}} \subseteq \mathbb{C}$ such that for all $z \in \mathbb{C}$, exactly one of $z \in \mathcal{P}_{\mathbb{C}}$, $-z \in \mathcal{P}_{\mathbb{C}}$, or $z = 0$ holds, and such that if z, $w \in \mathcal{P}_{\mathbb{C}}$, then $z + w$, $zw \in \mathcal{P}_{\mathbb{C}}$? I say no. For if there is such a $\mathcal{P}_{\mathbb{C}}$, then either $i \in \mathcal{P}_{\mathbb{C}}$ or $-i \in \mathcal{P}_{\mathbb{C}}$, since $i \neq 0$ (note that if $i = 0$, then $0 = i^2 = -1$). If $i \in \mathcal{P}_{\mathbb{C}}$, then $-1 = i^2 \in \mathcal{P}_{\mathbb{C}}$, and then $-i = (-1)i \in \mathcal{P}_{\mathbb{C}}$. A similar contradiction results if $-i \in \mathcal{P}_{\mathbb{C}}$.

Bottom line: if complex numbers appear in an inequality, they better be plugged into a modulus! Saying $|z| \leq |w|$ is fine; saying $z \leq w$ is not.

Day 13: Wednesday, February 8

We talked about properties of limits. The good news is that the algebra that you know from calculus in $\mathbb{R}$ carries over to calculus in $\mathbb{C}$. This is because $\mathbb{C}$ has basically the same algebraic properties as $\mathbb{R}$ does (except for that little business of $i^2 = -1$), and the proofs of many limit properties care less about whether the numbers involved are real or complex and more about the underlying algebra. See, for example, Theorem 2.2.7 (ignore the language about "accumulation points" for right now) and look at Example 2.2.8.

We discussed a "sequential characterization" of limits that's not in the book. This is an instance of how sequences turn "continuous" concepts (think about the ϵ-δ definition of a limit) into "discrete" ones (sequences are indexed by discrete integers). Here is a formal statement.

13.1 Theorem. *Let $f\colon \mathcal{D} \to \mathbb{C}$ be a function and a, $L \in \mathbb{C}$. Then $\lim_{z\to a} f(z) = L$ if and only if for all sequences (z_n) in $\mathcal{D} \setminus \{a\}$ with $z_n \to a$, it is also the case that $f(z_n) \to L$.*

Here are some comments on the proposition. The sentence $\lim_{z\to a} f(z) = L$ encodes the idea that we can make $f(z)$ arbitrarily close to L by taking z sufficiently close to (but maybe not equal to) a. Saying that the sequence (z_n) converges to a encodes this idea of having z sufficiently close to a. Requiring $z_n \in \mathcal{D} \setminus \{a\}$ removes the possibility of setting $z_n = a$

for all n, for in that case we would have $z_n \to a$ but $f(z_n) = f(a)$, and we do not want to require $f(a) = L$ in the sentence $\lim_{z\to a} f(z) = L$. Last, saying $f(z_n) \to L$ encodes the idea of making $f(z)$ arbitrarily close to L.

We looked at part of Example 2.2.17 and showed that $\lim_{z\to -1} \operatorname{Arg}(z)$ does not exist. We did this by applying the sequential characterization of limits, but really we got the idea by drawing a picture; see Figure 2.10. Define sequences (z_n) and (w_n) by $z_n = e^{i(\pi-1/n)}$ and $w_n = e^{i(-\pi+1/n)}$. Then $z_n \to -1$ and $w_n \to -1$ (break z_n and w_n into real and imaginary parts and use the continuity of the real sine and cosine) but $\operatorname{Arg}(z_n) = \pi - 1/n \to \pi$ while $\operatorname{Arg}(w_n) = -\pi + 1/n \to -\pi$. (To calculate the principal argument, recall $\operatorname{Arg}(e^{i\theta}) = \theta$ if $-\pi < \theta \le \pi$.)

Then we talked about the squeeze theorem for complex-valued functions. See Theorem 2.2.5. As always when using inequalities with complex numbers, there must be moduli involved! We showed $\lim_{z\to 0} z\sin(1/\operatorname{Re}(z)) = 0$, which proceeded in a manner similar to Example 2.2.6.

Last, we defined continuity; see Definition 2.2.12. The work above shows that $\operatorname{Arg}(\cdot)$ is discontinuous on $(-\infty, 0)$. Since $\operatorname{Arg}(\cdot)$ is not defined at 0, it is also discontinuous at 0. Thus $\operatorname{Arg}(\cdot)$ is discontinuous on $(-\infty, 0]$. If you look back at the general formula for $\operatorname{Arg}(\cdot)$ on p. 30, it is possible (though probably tedious) to show that $\operatorname{Arg}(\cdot)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$. This is most of the rest of Example 2.2.17.

Day 14: Friday, February 10

We discussed removable and nonremovable discontinuities. The book defines these implicitly on p. 110; here is a formal definition.

14.1 Definition. *Let $f: \mathcal{D} \to \mathbb{C}$ be a function and let $a \in \mathbb{C}$ be such that $L := \lim_{z\to a} f(z)$. Suppose that either $a \notin \mathcal{D}$ or, if $a \in \mathcal{D}$, then $f(a) \ne L$. Then f has a* **REMOVABLE DISCONTINUITY** *at a. Otherwise, if $\lim_{z\to a} f(z)$ does not exist, then f has a* **NONREMOVABLE DISCONTINUITY** *at a.*

Then we proved the following.

14.2 Theorem. *Suppose that the function $f: \mathcal{D} \to \mathbb{C}$ has a removable discontinuity at $a \in \mathbb{C}$ with $L := \lim_{z\to a} f(z)$ and define*

$$\widetilde{f}(z) := \begin{cases} f(z), & z \in \mathcal{D} \setminus \{a\} \\ L, & z = a. \end{cases}$$

Then $\widetilde{f}$ is continuous at a.

Here is the proof: since $f(z) = \widetilde{f}(z)$ for $z \in \mathcal{D} \setminus \{a\}$, and since $\lim_{z\to a} f(z)$ exists, the limit $\lim_{z\to a} \widetilde{f}(z)$ also exists and equals $\lim_{z\to a} f(z)$. That is,

$$\lim_{z\to a} \widetilde{f}(z) = \lim_{z\to a} f(z) = L = \widetilde{f}(a),$$

and so $\widetilde{f}$ is continuous at a.

Examples 2.2.15, 2.2.16, and 2.2.17 all feature removable and nonremovable discontinuities and are worth reading.

Then we discussed how the existence of limits boils down to studying the limits of real and imaginary parts; see Theorem 2.2.9. Note that this could also be proved using the sequential characterization of limits and Theorem 1.5.8. Likewise, continuity respects real and imaginary parts; see Theorem 2.2.13 and also the juicy algebra therein. Thus problems of limits and continuity in $\mathbb{C}$ often can be resolved by thinking about limits and continuity for functions from $\mathbb{R}^2$ to $\mathbb{R}$ as in multivariable calculus. See Proposition 2.2.18 (to which I would add that $h(x,y) = \phi(x)\psi(y)$ and $k(x,y) = \phi(x)+\psi(y)$ are continuous on $\mathbb{R}^2$ whenever ϕ and ψ are continuous on $\mathbb{R}$) and Example 2.2.19 and Theorem 2.2.21. Note that we could get Example 2.2.20 by thinking about the continuity of the function $(x,y) \mapsto \sqrt{x^2+y^2}$.

Day 15: Monday, February 13

We introduced some topological and set-theoretical concepts from Section 2.1 that we've so far avoided.

15.1 Definition.

(i) *The* **open ball** *of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ is*

$$\mathcal{B}(z_0; r) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

(ii) *The* **closed ball** *of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ is*

$$\overline{\mathcal{B}}(z_0; r) := \{z \in \mathbb{C} \mid |z - z_0| \le r\}.$$

(iii) *The* **punctured (open) ball** *of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ is*

$$\mathcal{B}^*(z_0; r) := \mathcal{B}(z_0; r) \setminus \{z_0\} = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}.$$

See pp. 96–97 and also p. 98 and note the book's slightly different notation.

A good exercise is to check that

$$\mathcal{B}(z_0; r) = \left\{z_0 + se^{it} \;\middle|\; 0 \le s < r,\ 0 \le t \le 2\pi\right\}$$

and to "parametrize" the balls $\overline{\mathcal{B}}(z_0; r)$ and $\mathcal{B}^*(z_0; r)$ in similar ways. Another good exercise, which we did in class, is to rewrite the definition of a limit in terms of open balls. For $f \colon \mathcal{D} \to \mathbb{C}$ and a, $L \in \mathbb{C}$, we have

$$\lim_{z \to a} f(z) = L \iff \text{for all } \epsilon > 0 \text{ there is } \delta > 0 \text{ such that } z \in \mathcal{B}^*(a; \delta) \cap \mathcal{D} \Longrightarrow f(z) \in \mathcal{B}(L; \epsilon).$$

You should revisit the geometry of Figure 2.8 (p. 103) in terms of balls. See pp. 98–99 to review intersections ($\cap$) and unions ($\cup$) of sets.

Then we talked about accumulation points, see p. 98 once again. The book's definition of a limit (Definition 2.2.1) and most of the good results on limits assumes that for

$\lim_{z \to a} f(z) = L$ to exist, the point a has to be an "accumulation point" of the domain of f. Why? Consider the situation of $\mathcal{D} = \mathcal{B}(0;1) \cup \{2i\}$ and f really any function on $\mathcal{D}$. Check that $\mathcal{B}^*(2i;\delta) \cap \mathcal{D}$ is empty for $0 < \delta < 1$. Conclude that the statement $z \in \mathcal{B}^*(2i;\delta) \cap \mathcal{D}$ is false whenever $0 < \delta < 1$, and therefore the if-then statement

$$z \in \mathcal{B}^*(2i;\delta) \cap \mathcal{D} \Longrightarrow f(z) \in \mathcal{B}(L;\epsilon)$$

is true for *any* choice of $L \in \mathbb{C}$ and $\epsilon > 0$. Conclude, finally, that $\lim_{z \to 2i} f(z) = L$ for any function f on $\mathcal{D}$ and any $L \in \mathbb{C}$. In particular, the limit can have infinitely many values.

This is absurd, and excessive, and this is why we really have to restrict the point a in the definition of a limit to be an accumulation point of the domain $\mathcal{D}$. (I intentionally didn't stress this before, but now we are wiser.) As an example, you should check that $2i$ is not an accumulation point of $\mathcal{B}(0;1) \cup \{2i\}$ and, more generally, that if $z \in \mathcal{B}(z_0;r)$, then z is an accumulation point of $\mathcal{B}(z_0;r)$, while if $w \notin \overline{\mathcal{B}}(z_0;r)$, then w is not an accumulation point of $\mathcal{B}(z_0;r)$. Finally, look at Proposition 2.2.2, which proves that limits are unique. I claim that the book is tacitly using the fact that z_0 is an accumulation point of S when it uses the point z such that $0 < |z - z_0| < \delta$.

Finally, we talked about derivatives. I will use the word **holomorphic** as a synonym for "(complex) differentiable" and in preference, for now, to "analytic." All of the material on pp. 114–118 up to and including Proposition 2.3.8 should feel very familiar. Because the algebraic properties of $\mathbb{C}$ (as, strictly speaking, a field) are the same as those of $\mathbb{R}$ (notwithstanding the role of $i \ldots$), *all* of the proofs of differentiation theorems go through *exactly* the same in $\mathbb{C}$ as they do in $\mathbb{R}$. Here are some more interesting questions that we will consider in detail.

1. What can we say about the differentiability of our "new" functions on $\mathbb{C}$, e.g., exponentials, logs, trig, powers? We know they are (more or less) differentiable on $\mathbb{R}$, but what happens when we extend them to $\mathbb{C}$? Already we've seen some weird behavior with limits and continuity.

2. How does two-dimensional geometry/two-dimensional limits affect the existence and properties of the derivative? This has definitely been an issue for continuity.

3. What does the differentiability of f say about its real and imaginary parts? They're functions from $\mathbb{R}^2$ to $\mathbb{R}$, so presumably we could study them with multivariable calculus (i.e., partial derivatives).

Our very last activity was working on part (a) of Example 2.3.9.

Day 16: Wednesday, February 15

We revisited the nondifferentiability of $f(z) = \overline{z}$ using the idea of paths. A **path** in $\mathbb{C}$ is just a function $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$. So, a path is a parametric curve $\gamma(t) = \big(\mathrm{Re}[\gamma(t)], \mathrm{Im}[\gamma(t)]\big)$. We can recast limits not just in terms of sequences but also paths. Here is the "path characterization" of limits.

16.1 Theorem. *Let $f\colon \mathcal{D} \to \mathbb{C}$ be a function and w, $L \in \mathbb{C}$. Then $\lim_{z\to w} f(z) = L$ if and only if for all paths $\gamma\colon [a,b] \to \mathcal{D}$, whenever $\lim_{t\to t_0} \gamma(t) = w$ for some $t_0 \in [a,b]$, it is also the case that $\lim_{t\to t_0} f(\gamma(t)) = L$. (If $t_0 = a$ or $t_0 = b$, only the right or left limit as $t \to t_0$, respectively, is required to hold.)*

Unlike the sequential characterization of limits, which defines limits of *functions* in terms of limits of *sequences*, the path characterization of limits defines limits of functions *also* in terms of limits of functions. However, the path characterization reduces knowledge of limits of functions of a *complex* variable to limits of functions strictly of a *real* variable.

We can use the path characterization of limits to show that $\lim_{z\to x} \operatorname{Arg}(z)$ does not exist for any $x < 0$. Take $\gamma_1(t) = |x|e^{it}$ on $[0,\pi]$, so $\lim_{t\to\pi^-} \gamma_1(t) = |x|e^{i\pi} = -|x| = x$ and, for $0 \le t \le \pi$, $\operatorname{Arg}(\gamma_1(t)) = t$, so $\lim_{t\to\pi^-} \operatorname{Arg}(\gamma_1(t)) = \pi$. Then take $\gamma_2(t) = |x|e^{it}$ on $[-\pi,0]$, so $\lim_{t\to-\pi^+} \gamma_2(t) = |x|e^{-i\pi} = -|x| = x$ and, for $-\pi < t \le 0$, $\operatorname{Arg}(\gamma_2(t)) = t$, so $\lim_{t\to-\pi^+} \operatorname{Arg}(\gamma_2(t)) = -\pi$. Thus $\operatorname{Arg}(\cdot)$ approaches two different limits along the paths γ_1 and γ_2, even though both of those paths approach x, so $\lim_{z\to x} \operatorname{Arg}(z)$ does not exist.

I think that neither the sequential characterization nor the path characterization of limits is as immediately convincing as a good picture, which is how we originally studied $\operatorname{Arg}(\cdot)$. Rather, these two characterizations are methods of encoding and formalizing our mathematical intuition, which allows us to check our work and communicate effectively with each other.

It will sometimes be more convenient to use the path characterization of limits in connection with derivatives than the sequential characterization. This is really how we showed that $f(z) = \overline{z}$ is nowhere differentiable. In the limit of the difference quotients $[f(z+h)-f(z)]/h$ as $h \to 0$, we approached 0 along the path of the real axis, and got a limit of 1, and the imaginary axis, and got a limit of -1. See Figure 2.13 on p. 119.

To test the differentiability of a function f at a point z (or exploit the differentiability if it's known), we might want to approach 0 along multiple paths. This requires us to evaluate $f(z+h)$ for h approaching 0 in arbitrary directions. Equivalently, if we think about the difference quotients $[f(w)-f(z)]/(w-z)$ as $w \to z$, we want to allow w to approach z from arbitrary directions. Depending on the domain $\mathcal{D}$ of f, this may not always be possible; perhaps some directions of approach don't lie in $\mathcal{D}$.

This, however, *is* possible if $\mathcal{D}$ is **open**: for each $z \in \mathcal{D}$, there is $r > 0$ such that $\mathcal{B}(z;r) \subseteq \mathcal{D}$. See Definition 2.1.4 and some slightly different language on pp. 96–98. I claim that if $\mathcal{D}$ is open and if $\gamma\colon [a,b] \to \mathbb{C}$ is a path such that $\lim_{t\to t_0} \gamma(t) = 0$, then for t sufficiently small we have $z + \gamma(t) \in \mathcal{D}$.

16.2 Theorem. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open, and let $f\colon \mathcal{D} \to \mathbb{C}$ be a function and $z \in \mathcal{D}$. If f is differentiable at z, and if $\gamma\colon [a,b] \to \mathbb{C}$ is a path such that, for some $t_0 \in [a,b]$, $\lim_{t\to t_0} \gamma(t) = 0$, then it is also the case that*

$$f'(z) = \lim_{t\to t_0} \frac{f(z+\gamma(t)) - f(z)}{\gamma(t)}.$$

Conversely, suppose that $\gamma_1\colon [a_1,b_1] \to \mathbb{C}$ and $\gamma_2\colon [a_2,b_2] \to \mathbb{C}$ are paths such that for some

$t_1 \in [a_1, b_1]$ and $t_2 \in [a_2, b_2]$,

$$\lim_{t \to t_1} \gamma_1(t) = \lim_{t \to t_2} \gamma_2(t) = 0.$$

If

$$\lim_{t \to t_1} \frac{f(z + \gamma_1(t)) - f(z)}{\gamma_1(t)} \neq \lim_{t \to t_2} \frac{f(z + \gamma_2(t)) - f(z)}{\gamma_2(t)},$$

or if the two limits above do not exist, then f is not differentiable at z.

With this proposition, we revisited $f(z) = \overline{z}$ by taking $\gamma_1(t) = t$ and $\gamma_2(t) = it$, both defined on, say, $[0, 1]$. Then we have

$$\lim_{t \to 0} \gamma_1(t) = \lim_{t \to 0} \gamma_2(t) = 0 \quad \text{but} \quad \lim_{t \to 0} f(\gamma_1(t)) = 1 \quad \text{while} \quad \lim_{t \to 0} f(\gamma_2(t)) = -1.$$

Day 17: Friday, February 17

You took an exam.

Day 18: Monday, February 20

We proved the Cauchy–Riemann equations. See pp. 130–132. The crux of the proof is the geometry in Figures 2.21 and 2.22 and the algebra of (1) equating real and imaginary parts of limits and (2) using the identity $1/i = -i$. Then we used the Cauchy–Riemann equations to give a shorter proof that $f(z) = \overline{z}$ is not differentiable at any $z \in \mathbb{C}$. Last, we stated the converse to the Cauchy–Riemann equations. The book packages the equations as Theorem 2.5.1; here is the version that we used in class. We proved part (i) but not part (ii).

18.1 Theorem. *Suppose that $\mathcal{D} \subseteq \mathbb{C}$ is open and let $f \colon \mathcal{D} \to \mathbb{C}$ be a function. Write $f(x + iy) = u(x, y) + iv(x, y)$.*

(i) *Suppose that f is differentiable at a point $z = x_0 + iy_0 \in \mathcal{D}$. Then the partial derivatives u_x, u_y, v_x, and v_y exist at (x_0, y_0) and satisfy the* **Cauchy–Riemann equations**

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0). \end{cases} \qquad (*)$$

Moreover,

$$f'(x_0 + iy_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \qquad (**)$$

(ii) *Let $x_0 + iy_0 \in \mathcal{D}$ and let $r > 0$ be such that $\mathcal{B}(x_0 + iy_0; r) \subseteq \mathcal{D}$. Suppose that the four partial derivatives u_x, u_y, v_x, and v_y exist and are continuous on $\mathcal{B}(x_0 + iy_0; r)$. Moreover, suppose that the partials satisfy the Cauchy–Riemann equations $(*)$ at $x_0 + iy_0$. Then f is differentiable at $x_0 + iy_0$ and $(**)$ holds.*

Day 19: Wednesday, February 22

First we used the Cauchy–Riemann equations to prove the differentiability of the exponential, and then we used the chain rule and the linearity of the derivative to prove the differentiability of the sine. These are Examples 2.5.3 and 2.5.4.

Then we inquired about the differentiability of the principal logarithm. We have

$$\mathrm{Log}(x+iy) = \ln(\sqrt{x^2+y^2}) + i\,\mathrm{Arg}(x+iy),$$

so we would want to study $u(x,y) = \ln(\sqrt{x^2+y^2})$ and $v(x,y) = \mathrm{Arg}(x+iy)$. Working with u is not bad, but how do we differentiate v? The piecewise definition on p. 30 is true but tough.

The right idea was to use a variant of the reverse chain rule. The book does this in Theorem 2.3.12 and Example 2.5.5. Here is our approach from class, which used somewhat different tools. First we proved a lemma, then a variant of Theorem 2.3.12, and then we did Example 2.5.5.

19.1 Lemma (Difference quotient). *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f\colon \mathcal{D} \to \mathbb{C}$ be differentiable. Fix $a \in \mathcal{D}$ and define*

$$\phi\colon \mathcal{D} \to \mathbb{C}\colon z \mapsto \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \in \mathcal{D} \setminus \{a\} \\ f'(a), & z = a. \end{cases}$$

Then ϕ is differentiable on $\mathcal{D} \setminus \{a\}$ and continuous on $\mathcal{D}$.

Proof. Continuity on $\mathcal{D} \setminus \{a\}$ will follow from differentiability on $\mathcal{D} \setminus \{a\}$. Continuity at a follows from the calculation

$$\lim_{z \to a} \phi(z) = \lim_{z \to a} \frac{f(z)-f(a)}{z-a} = f'(a) = \phi(a).$$

Here we are using the "other" definition of the derivative.

As for the differentiability on $\mathcal{D} \setminus \{a\}$, this is essentially the quotient rule. The map $z \mapsto f(z) - f(a)$ is differentiable on $\mathcal{D}$ as f is differentiable on $\mathcal{D}$ and $f(a)$ is constant; the map $z \mapsto z - a$ is differentiable on $\mathbb{C}$, so the quotient $z \mapsto \big(f(z) - f(a)\big)/(z-a)$ is differentiable as long as the denominator is not zero, i.e., on $\mathcal{D} \setminus \{a\}$. ■

19.2 Theorem (Reverse chain rule). *Let $\mathcal{D}_1$, $\mathcal{D}_2 \subseteq \mathbb{C}$ with $\mathcal{D}_1 \subseteq \mathcal{D}_2$. Let $f\colon \mathcal{D}_1 \to \mathcal{D}_2$ be continuous and let $g\colon \mathcal{D}_2 \to \mathbb{C}$ be differentiable. Suppose that $g(f(z)) = z$ for all $z \in \mathcal{D}_1$ and $g'(f(z)) \neq 0$ for all $z \in \mathcal{D}_1$. Then f is differentiable on $\mathcal{D}_1$ and*

$$f'(z) = \frac{1}{g'(f(z))}.$$

First, a remark: if we also know that f is differentiable, then the formula for f' follows from the chain rule as usual. Indeed, since $g(f(z)) = z$, we differentiate both sides to find $g'(f(z))f'(z) = 1$, and then we solve for $f'(z)$. But here we do not know that f is differentiable, so we have work to do.

Proof. Fix $a \in \mathcal{D}_1$. We need to show that

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \frac{1}{g'(f(z))}.$$

The hypothesis $g(f(z)) = z$ for all z lets us rewrite the difference quotient as

$$\frac{f(z) - f(a)}{z - a} = \frac{f(z) - f(a)}{g(f(z)) - g(f(a))}. \tag{19.1}$$

Now put

$$\phi \colon \mathcal{D}_2 \to \mathbb{C} \colon w \mapsto \begin{cases} \dfrac{g(w) - g(f(a))}{w - f(a)}, \ w \in \mathcal{D}_2 \setminus \{f(a)\} \\ \\ g'(f(a)), \ w = f(a). \end{cases}$$

The difference quotient lemma tells us that ϕ is continuous on $\mathcal{D}_2$, so in particular

$$\lim_{w \to f(a)} \phi(w) = \phi(f(a)) = g'(f(a)) \neq 0.$$

Properties of continuity then tell us that for w close to $f(a)$, we have $\phi(w) \neq 0$, and so the quotient $1/\phi$ is also continuous at $f(a)$. That is,

$$\lim_{w \to f(a)} \frac{1}{\phi(w)} = \frac{1}{\phi(f(a))} = \frac{1}{g'(f(a))}.$$

Since f is continuous at a, we have $\lim_{z \to a} f(z) = f(a)$. Then properties of limits and function composition give

$$\lim_{z \to a} \frac{1}{\phi(f(z))} = \frac{1}{\phi(f(a))} = \frac{1}{g'(f(a))}.$$

We claim that

$$\frac{1}{\phi(f(z))} = \frac{f(z) - f(a)}{g(f(z)) - g(f(a))}.$$

If this is true, then (19.1) shows

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{z \to a} \frac{1}{\phi(f(z))} = \frac{1}{g'(f(a))},$$

as desired. To prove the claim, we just need to check that $f(z) \neq f(a)$ for z close to a. Then the claim will follow from the piecewise definition of ϕ. If $f(z) = f(a)$ for some $z \neq a$, then the hypotheses imply $a = g(f(a)) = g(f(z)) = z$, a contradiction. Thus the claim is true. ■

We applied these results to the principal logarithm by setting $f(z) = \text{Log}(z)$, $\mathcal{D}_1 = \mathbb{C} \setminus (-\infty, 0]$, $g(z) = e^z$, and $\mathcal{D}_2 = \mathbb{C}$. Then $g(f(z)) = e^{\text{Log}(z)} = z$, $g'(z) = e^z$, and $g'(f(z)) = e^{\text{Log}(z)} = z \neq 0$, since $z \neq 0$ here.

Day 20: Friday, February 22

Recommended reading

Today's material departed from the text, so the following notes are more detailed. We are working our way up to proving Theorem 2.5.7, but I am trying to motivate the need for the topological machinery of connectedness (pp. 99–101). We ended by proving a "weak" version of Theorem 2.5.7 in the case that the domain is only open, not a "region." Then we discussed piecewise continuous differentiability, see Definition 3.2.1, and some other issues of calculus for functions $\gamma \colon [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ as on pp. 143–145.

Motivated by the powerful relationship between the real and imaginary parts of a differentiable function, we discussed the situation where $f \colon \mathcal{D} \to \mathbb{C}$ is differentiable, $\mathcal{D}$ is open, and $\text{Im}(f(z)) = 0$ for all $z \in \mathcal{D}$. Then $f(x + iy) = u(x, y) + iv(x, y)$ with $v(x, y) = 0$, so the Cauchy–Riemann equations gave $u_x = u_y = 0$. Since $v_x = 0$ as well, we have $f'(x + iy) = u_x(x, y) + iv(x, y) = 0$. Instinct tells us that f should be constant if its derivative is 0, but instinct is wrong: consider

$$f \colon \mathbb{C} \setminus i\mathbb{R} \to \mathbb{C} \colon z \mapsto \begin{cases} -1, \text{Re}(z) < 0 \\ 1, \text{Re}(z) > 0. \end{cases}$$

This f is only locally constant.

20.1 Definition. *A function $f \colon \mathcal{D} \to \mathbb{C}$ is* LOCALLY CONSTANT *if for each $z \in \mathcal{D}$, whenever $r > 0$ is such that $\mathcal{B}(z; r) \subseteq \mathcal{D}$, then f is constant on $\mathcal{B}(z; r)$.*

To prove the next theorem, we will use (among other options) the fundamental theorem of calculus: if $f \colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable and f' is continuous, then

$$\int_a^b f'(t)\, dt = f(b) - f(a).$$

We will eventually prove the FTC, but we will just accept it as true for now. This generalizes nicely to partial derivatives. If $u \colon \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$ has continuous partial derivatives u_x and u_y on $\mathcal{D}$, and if $x + iy \in \mathcal{D}$ for $a \le x \le b$, then

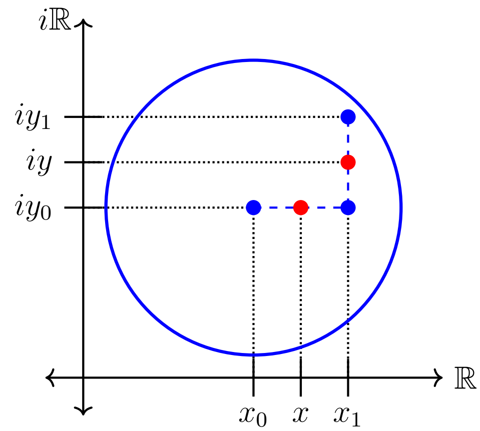
$$\int_a^b u_x(x, y)\, dx = u(b, y) - u(a, y).$$

20.2 Theorem. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open. Suppose that $f \colon \mathcal{D} \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in \mathcal{D}$. Then f is locally constant on $\mathcal{D}$.*

Proof. Let $z_0 = x_0 + iy_0 \in \mathcal{D}$ and $r > 0$ such that $\mathcal{B}(z_0; r) \subseteq \mathcal{D}$. We want to show that $f(z_0) = f(z)$ for all $z \in \mathcal{B}(z_0; r)$.

Write $f(x + iy) = u(x, y) + iv(x, y)$. Since $\mathcal{D}$ is open and f is differentiable, the Cauchy–Riemann equations tell us that the partial derivatives u_x, u_y, v_x, and v_y exist on $\mathcal{D}$. Moreover, since $f'(x+iy) = 0$ for all $x+iy \in \mathcal{D}$, we have $u_x = u_y = v_x = v_y = 0$ on $\mathcal{D}$. It therefore suffices to show that for $x_1 + iy_1 \in \mathcal{B}(z_0; r)$, we have $u(x_1, y_1) = u(x_0, y_0)$ and $v(x_1, y_1) = v(x_0, y_0)$. We will only do this for u, as the proof for v is analogous.

We claim that since $x_0 + iy_0 = z_0 \in \mathcal{B}(z_0; r)$ and $x_1 + iy_1 \in \mathcal{B}(z_0; r)$, we have $x + iy \in \mathcal{B}(z_0; r)$ for all x between x_0 and x_1 and all y between y_0 and $_1$. (By "between," we mean that if $x_0 \le x_1$, then $x_0 \le x \le x_1$, and if $x_1 \le x_0$, then $x_1 \le x \le x_0$.) We leave the proof of this claim as an exercise. Here is the picture of one arrangement of $x_0 + iy_0$ and $x_1 + iy_1$.



Now we "add zero" and rewrite

$$u(x_1, y_1) - u(x_0, y_0) = \big[u(x_1, y_1) - u(x_1, y_0)\big] + \big[u(x_1, y_0) - u(x_0, y_0)\big].$$

Since $u_x(x, y_0) = 0$ for all $x \in [x_0, x_1]$ and $u_y(x_1, y) = 0$ for all $y \in [y_0, y_1]$, we have

$$u(x_1, y_1) - u(x_1, y_0) = \int_{y_0}^{y_1} u_y(x_1, y)\ dy = \int_{y_0}^{y_1} 0\ dy = 0$$

and

$$u(x_1, y_0) - u(x_0, y_0) = \int_{x_0}^{x_1} u_x(x, y_0)\ dx = \int_{x_0}^{x_1} 0\ dx = 0.$$

This proves that u is constant. Exactly the same arguments work for v, and otherwise we switch the limits of integration around in the event, say, that $x_1 \le x_0$. ■

20.3 Corollary. *Suppose that $\mathcal{D} \subseteq \mathbb{C}$ is open and $f \colon \mathcal{D} \to \mathbb{C}$ is differentiable with either* $\mathrm{Re}(f)$ *or* $\mathrm{Im}(f)$ *locally constant on $\mathcal{D}$. Then f is locally constant on $\mathcal{D}$.*

Proof. Write $f(x+iy) = u(x,y) + iv(x,y)$. Say that $u = \mathrm{Re}(f)$ is locally constant on $\mathcal{D}$. Then $u_x = u_y = 0$, so $v_x = v_y = 0$ as well, and therefore $f' = u_x + iv_x = 0$. ■

This suggests that the values of a complex differentiable function defined on an open subset of $\mathbb{C}$ must exhibit a certain "diversity." It is no problem for a $f \colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ to be differentiable, strictly real-valued, and not locally constant; this is the meat and potatoes of real-valued calculus, of course. But as soon as we expand the domain of f to be an open subset of $\mathbb{C}$ (now is a good time to check that no subset of $\mathbb{R}$ is open in $\mathbb{C}$), then f cannot take just real (or just imaginary) values without being very "dull."

We can strengthen the preceding results significantly if we introduce some new topological machinery. First, we need to recall some technical facts about limits, continuity, and differentiability for functions $\gamma\colon \mathcal{D} \to \mathbb{C}$ where $\mathcal{D} \subseteq \mathbb{R}$. Note that the original definition of the limit of a function $f\colon \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$ at a point $a \in \mathbb{C}$ (Definition 2.2.1) permitted, or at least did not prohibit, $\mathcal{D} \subseteq \mathbb{R}$. Likewise, although Definition 2.3.1 for the derivative required $\mathcal{D} \subseteq \mathbb{C}$ to be open, which explicitly excludes $\mathcal{D} \subseteq \mathbb{R}$ (why?), nothing in the definition of the derivative *intrinsically required* $\mathcal{D}$ to be open, and none of the proofs of the major consequences of differentiability (Theorems 2.3.4 and 2.3.5) required $\mathcal{D}$ to be open, either. The bottom line is that *the calculus of limits, continuity, and derivatives works the way it should* for functions $\gamma\colon \mathcal{D} \subseteq \mathbb{R} \to \mathbb{C}$, and results like Proposition 3.1.6 and Theorem 3.1.8 do not need special singling out. (Note, though, that the mean value theorem need not hold if γ is not strictly real-valued; see pp. 145.) Nonetheless, for clarity, we single out some important properties of functions from subintervals of $\mathbb{R}$ to $\mathbb{C}$.

20.4 Definition. *Let $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ be a function.*

(i) *γ is* **continuous** *on $[a,b]$ if $\lim_{\tau \to t} \gamma(\tau) = \gamma(t)$ for all $t \in [a,b]$. A close reading of the formal definition of the limit (Definition 2.2.1) actually requires*

$$\lim_{\tau \to a^+} \gamma(\tau) = \gamma(a), \qquad \lim_{\tau \to t} \gamma(\tau) = \gamma(t),\ a < t < b, \qquad \text{and} \qquad \lim_{\tau \to b^-} \gamma(\tau) = \gamma(b).$$

(ii) *$\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ is* **differentiable** *on $[a,b]$ if the limits*

$$\gamma'(t) := \begin{cases} \displaystyle\lim_{h \to 0^+} \frac{\gamma(a+h) - \gamma(a)}{h}, & t = a \\[2ex] \displaystyle\lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}, & a < t < b \\[2ex] \displaystyle\lim_{h \to 0^-} \frac{\gamma(b+h) - \gamma(b)}{h}, & t = b \end{cases}$$

all exist.

(iii) *$\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ is* **continuously differentiable** *on $[a,b]$ if γ is differentiable on $[a,b]$ in the sense of part (ii) and if γ' is continuous on $[a,b]$ in the sense of part (i).*

(iv) *$\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ is* **piecewise continuously differentiable** *on $[a,b]$ if there are numbers $t_0, \ldots, t_n \in [a,b]$ such that $a = t_0 < \cdots < t_n = b$ and such that the restrictions $\gamma\big|_{[t_{k-1},t_k]}$ are continuously differentiable on $[t_{k-1}, t_k]$ in the sense of part (iii).*

Here, if $I \subseteq \mathbb{R}$ is an interval and $J \subseteq I$, then the **restriction to** J of a function $f\colon I \to \mathbb{C}$

is the function

$$f\big|_J \colon J \to \mathbb{C} \colon t \mapsto f(t).$$

20.5 Example. The map $\gamma \colon [-1,1] \to \mathbb{C} \colon t \mapsto |t|$ is piecewise continuously differentiable but not continuously differentiable. If we put $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$, then $\gamma(t) = -t$ for $-1 \le t \le 0$ and $\gamma(t) = t$ for $0 \le t \le 1$. Then $\gamma\big|_{[-1,0]}$ and $\gamma\big|_{[0,1]}$ are differentiable with $(\gamma\big|_{[-1,0]})'(t) = -1$ and $(\gamma\big|_{[0,1]})'(t) = 1$. Nonetheless,

$$\lim_{h \to 0} \frac{\gamma(h) - \gamma(0)}{h}$$

does not exist, so γ is not differentiable at 0.

20.6 Remark. A function $\gamma \colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ is continuous/differentiable/continuously differentiable/piecewise continuously differentiable if and only if $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are. In particular, if $\gamma = \mu + i\nu$, where μ, $\nu \colon [a,b] \to \mathbb{R}$, and if γ is differentiable, then $\gamma' = \mu' + i\nu'$.

It follows at once from the definition that if $\gamma \colon [a,b] \to \mathbb{C}$ is continuously differentiable, then γ is also piecewise continuously differentiable on $[a,b]$. Likewise, since differentiable functions are continuous, continuously differentiable functions are continuous. It also turns out that piecewise continuously differentiable functions are continuous, although this takes a bit of work. We did not prove the following result in class, and you should feel free to skip the proof.

20.7 Lemma. *If $\gamma \colon [a,b] \to \mathbb{C}$ is piecewise continuously differentiable, then γ is also continuous in the sense of (i).*

Proof. If $n = 1$ in part (iv) above, then γ is continuously differentiable on all of $[a,b]$, and there is nothing to prove. Otherwise, suppose $n \ge 2$.

We first show continuity at a. Since $\gamma\big|_{[t_0,t_1]}$ is continuously differentiable and therefore continuous in the sense of part (i), and since $t_0 = a$, we have

$$\lim_{\tau \to a^+} \gamma(\tau) = \lim_{\tau \to t_0^+} \gamma\big|_{[t_0,t_1]}(\tau) = \gamma\big|_{[t_0,t_1]}(t_0) = \gamma(t_0) = \gamma(a).$$

A similar argument shows $\lim_{\tau \to b^-} \gamma(\tau) = \gamma(b)$.

Now suppose $t \in (a,b)$. Then there is k such that $t \in [t_{k-1}, t_k]$. If $t_{k-1} < t < t_k$, then since $\gamma\big|_{[t_{k-1},t_k]}$ is continuously differentiable and therefore continuous in the sense of part ??, we have

$$\lim_{\tau \to t} \gamma(\tau) = \lim_{\tau \to t} \gamma\big|_{[t_{k-1},t_k]}(\tau) = \gamma\big|_{[t_{k-1},t_k]}(t) = \gamma(t).$$

Otherwise, if t is an endpoint of $[t_{k-1}, t_k]$, we need to do more work. If $t = t_{k-1}$ and $k = 1$, then $t = t_0 = a$, so we already know continuity at t. So, suppose $t = t_{k-1}$ and $k \ge 2$.

(This is allowed since $k \le n$ and $n \ge 2$.) Then the continuity of $\gamma\big|_{[t_{k-1},t_k]}$ gives

$$\lim_{\tau \to t_{k-1}^+} \gamma(\tau) = \lim_{\tau \to t_{k-1}^+} \gamma\big|_{[t_{k-1},t_k]}(\tau) = \gamma\big|_{[t_{k-1},t_k]}(t_{k-1}) = \gamma(t_{k-1}).$$

But since $k \ge 2$, we can also use the continuity of $\gamma\big|_{[t_{k-2},t_{k-1}]}$ to find

$$\lim_{\tau \to t_{k-1}^-} \gamma(\tau) = \lim_{\tau \to t_{k-1}^-} \gamma\big|_{[t_{k-2},t_{k-1}]}(\tau) = \gamma\big|_{[t_{k-2},t_{k-1}]}(t_{k-1}) = \gamma(t_{k-1}).$$

Similar arguments prove continuity at t_k if we consider the case $k = n$ (which implies $t_n = b$) separately from the case $k < n$. ■

Nonetheless, a piecewise continuously differentiable function $\gamma\colon [a,b] \to \mathbb{C}$ need not be differentiable on all of $[a,b]$. The left and right derivatives

$$\lim_{h \to 0^+} \frac{\gamma(t+h) - \gamma(t)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{\gamma(t+h) - \gamma(t)}{h}$$

are indeed defined for all $t \in (a,b)$ and, moreover, equal for all but finitely many $t \in (a,b)$. However, at a $t_k \in (a,b)$, we may have

$$\lim_{h \to 0^+} \frac{\gamma(t_k+h) - \gamma(t_k)}{h} \ne \lim_{h \to 0^-} \frac{\gamma(t_k+h) - \gamma(t_k)}{h},$$

in which case $\gamma'(t_k)$ will not be defined. And so we should not hope that γ' will be continuous on all of $[a,b]$ if γ is only piecewise continuously differentiable.

Day 21: Monday, February 27

Recommended reading

We discussed antiderivatives and integrals. Today's discussion corresponded, broadly, to Definition 3.1.1, Definition 3.1.10, Example 3.1.12, Example 3.1.14, Definition 3.1.4, and Example 3.1.5.

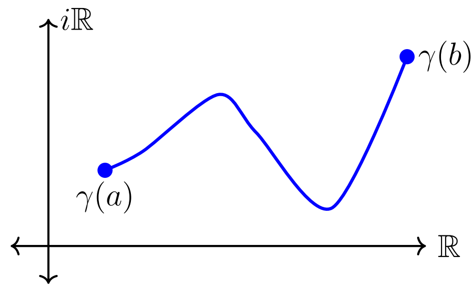
This diversion into continuity and differentiability helps us define a new tool that we will regularly deploy for the rest of the course.

21.1 Definition. *A* PATH *in $\mathbb{C}$ is a piecewise continuously differentiable map $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$. A* PATH IN $\mathcal{D} \subseteq \mathbb{C}$ *is a path $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathcal{D}$. The* IMAGE *of a path $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ is the set*

$$\gamma([a,b]) := \{\gamma(t) \mid a \le t \le b\}$$

i.e., the image of γ is the range of γ. (Below, the image of γ is the blue curve.) The INITIAL POINT *of γ is $\gamma(a)$ and the terminal point of γ is $\gamma(b)$; if $\gamma(a) = \gamma(b)$, then γ is*

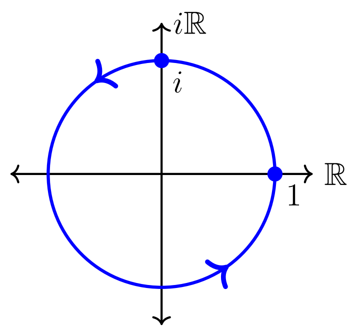
CLOSED.



Here is a gentle example of one of the most important paths in all of complex analysis.

21.2 Example. The map $\gamma\colon [0,2\pi] \to \mathbb{C}\colon t \mapsto e^{it}$ is a path, and the image of this path is the unit circle. That is,

$$\left\{e^{it} \;\middle|\; 0 \le t \le 2\pi\right\} = \{z \in \mathbb{C} \mid |z| = 1\}.$$



The image of this path has an inherent orientation: it starts at 1 and "moves counterclockwise" to i, then to -1, then to $-i$ and, last, back to 1. Thus γ is closed.

Note carefully that a path is a function, while the image of a path is a set. A given set in $\mathbb{C}$ may be the image of many paths; for example, the unit circle is also the image of $\gamma_k\colon [0,2\pi] \to \mathbb{C}\colon t \mapsto e^{ikt}$ for any $k \in \mathbb{Z}$.

21.3 Problem. Prove this. That is, show that if $k \in \mathbb{Z}$, then

$$\{z \in \mathbb{C} \mid |z| = 1\} = \left\{e^{ikt} \;\middle|\; 0 \le t \le 2\pi\right\}.$$

Here is the precise differentiation among "set," "path," and "image."

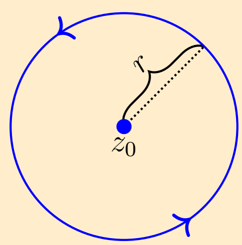
21.4 Definition. *A set $\Gamma \subseteq \mathbb{C}$ is* **PARAMETRIZED** *by the path $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ if the image of γ is Γ, i.e., if $\Gamma = \gamma([a,b])$. In this case we say that γ is a* **PARAMETRIZATION** *of Γ.*

21.5 Example. Here are four different parametrizations of the the unit circle, which is the set $\{z \in \mathbb{C} \mid |z| = 1\}$:

$$\gamma_1 \colon [0, 2\pi] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto e^{it}$$
$$\gamma_2 \colon [0, 2\pi] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto e^{-it}$$
$$\gamma_3 \colon [0, \pi] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto e^{2it}$$
$$\gamma_4 \colon [0, 2\pi] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto e^{4it}.$$

The path γ_1 is probably what we think of as the "usual" parametrization, which "traces out" the unit circle "counterclockwise." (Hopefully the overabundance of quotation marks emphasizes that none of these words or phrases has been given a rigorous mathematical definition yet.) The path γ_2 traces out the unit circle clockwise, e.g., $\gamma_2(\pi/2) = -i$, whereas $\gamma_2(\pi/2) = i$. The path γ_3 traces out the unit circle in "half the time" as γ_1 and γ_2, e.g., $\gamma_3(\pi/4) = i$. And the path γ_4 traces out the unit circle a whopping four times, e.g., $\gamma_4(t) = 1$ for $k = 0$, $\pi/2$, π, $3\pi/2$, and 2π. It turns out that the paths γ_1, γ_2, and γ_3 are closely related and can be "obtained" from each other by various operations that we will define on paths in general.

21.6 Problem. Define the circle of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ to be the set of all points z that lie at a distance r from z_0, i.e., this circle is the set $\{z \in \mathbb{C} \mid |z - z_0| = r\}$. Show that $\gamma \colon [0, 2\pi] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto z_0 + re^{it}$ is a parametrization of this circle. Can you describe γ "dynamically" as encoding a translation and a dilation of the unit circle?



$$\gamma(t) = z_0 + re^{it},\ 0 \le t \le 2\pi$$

21.7 Remark. Do not confuse the image of a path $\gamma \colon [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ with the graph of that path. In calculus, typically we think of the graph of a function $f \colon I \subseteq \mathbb{R} \to \mathbb{R}$ as the set $\{(t, f(t)) \mid t \in I\} \subseteq \mathbb{R}^2$. From our earlier formal definition of a function as a set of ordered pairs, we should see that a function's graph *is* that function, i.e., $f = \{(t, f(t)) \mid t \in I\}$.

But then the corresponding "graph" of $\gamma \colon [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ would be the set $\{(t, \gamma(t)) \mid a \le t \le b\}$. Since $\gamma(t) = \mathrm{Re}(\gamma(t)) + i\,\mathrm{Im}(\gamma(t)) = \big(\mathrm{Re}(\gamma(t)), \mathrm{Im}(\gamma(t))\big)$, per our underlying definition of complex numbers as ordered pairs, we might identify the graph of γ as the set of ordered *triples* $\big\{\big(t, \mathrm{Re}(\gamma(t)), \mathrm{Im}(\gamma(t))\big) \;\big|\; a \le t \le b\big\} \subseteq \mathbb{R}^3$. However, the image of γ is just a subset of $\mathbb{C} = \mathbb{R}^2$. In general, we will never "graph" a path as we did in prior calculus, but we will frequently study a path's image.

21.8 Problem. Do not judge the smoothness (i.e., the differentiability) of a path by the smoothness of its image.

(i) What is the image of

$$\gamma_1 \colon [-1,1] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto |t|?$$

Is this image the same as that of

$$\gamma_2 \colon [-1,1] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto t + i|t|?$$

(ii) Show that the image of γ_2 is the same as the image of

$$\gamma_3 \colon [-1,1] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto t^3 + i|t|^3.$$

Conclude that a set $\Gamma \subseteq \mathbb{C}$ may be both the graph of a function from a subset of $\mathbb{R}$ to $\mathbb{R}$ that is not differentiable and the image of a continuously differentiable map from a subset of $\mathbb{R}$ to $\mathbb{C}$.

In addition to circles, line segments are also among the most important paths that we will study.

21.9 Definition. *Let z_1, $z_2 \in \mathbb{C}$. The* **LINE SEGMENT FROM** z_1 **TO** z_2 *is the path*

$$\gamma \colon [0,1] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto (1-t)z_1 + tz_2.$$

We will often refer to both this path (as a function) and its image (as a set) as $[z_1, z_2]$. Context will make our intentions clear.

If z_1, $z_2 \in \mathbb{R}$, the notation $[z_1, z_2]$ may permit uncomfortable expressions like $[1, 0]$. Such is life. We will continue to emphasize that the domain of a path is a subset of $\mathbb{R}$ by writing, as we have always done, $\gamma \colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$, not $\gamma \colon [a,b] \to \mathbb{C}$, when γ is a path. A path is always defined on a closed, bounded subset of $\mathbb{R}$.

21.10 Problem.

(i) Sometimes it is convenient to represent the same line segment in multiple ways. Let z_1, $z_2 \in \mathbb{C}$. Show that

$$\begin{aligned}\{(1-t)z_1 + tz_2 \mid 0 \le t \le 1\} &= \{\tau z_1 + (1-\tau)z_2 \mid 0 \le \tau \le 1\} \\ &= \{tz_1 + \tau z_2 \mid 0 \le t,\ \tau \le 1 \text{ and } t + \tau = 1\}\end{aligned}$$

In particular, conclude that as sets $[z_1, z_2] = [z_2, z_1]$.

(ii) If a, $b \in \mathbb{R}$ with $a \le b$, then of course we want to think of the line segment $[a, b]$ as the set $\{x \in \mathbb{R} \mid a \le x \le b\}$. Show that this is still the case per Definition 21.9. That is, show

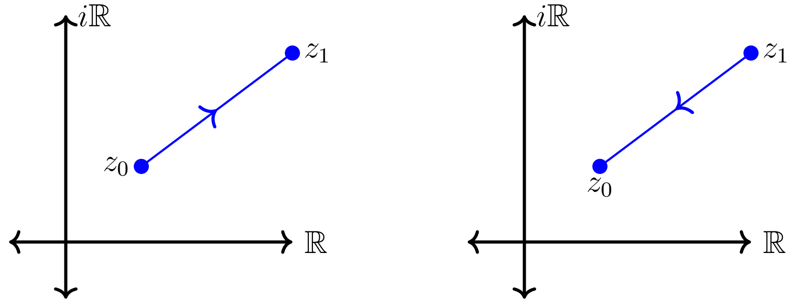
$$\{x \in \mathbb{R} \mid a \le x \le b\} = \{(1-t)a + tb \mid 0 \le t \le 1\}.$$

21.11 Example. Let z_1, $z_2 \in \mathbb{C}$ be distinct points. What is the difference between the paths

$$\gamma_1(t)\colon [0,1] \subseteq \mathbb{R} \to \mathbb{C}\colon t \mapsto (1-t)z_1+tz_2 \quad \text{and} \quad \gamma_2(t)\colon [0,1] \subseteq \mathbb{R} \to \mathbb{C}\colon t \mapsto (1-t)z_2+tz_1?$$

How does this situation resemble the paths γ_1 and γ_2 from Example 21.5?

Solution. The path γ_1 parametrizes $[z_1, z_2]$, while the path γ_2 parametrizes $[z_2, z_1]$. The images of these paths are the same, since as sets $[z_1, z_2] = [z_2, z_1]$. However, $\gamma_1(0) = z_1 \neq z_2 = \gamma_2(0)$, so γ_1 and γ_2 are distinct functions. (In fact, one can check that $\gamma_1(t) = \gamma_2(t)$ if and only if $t = 1/2$, so these functions certainly are not equal.) This calculation also tells us that the initial point of γ_1 is the terminal point of γ_2, and vice-versa. It appears, then, that γ_1 and γ_2 both "trace out" the same image but in the "reverse direction."



In fact, a little algebra shows

$$\gamma_1(t) = \gamma_2(1-t), \ 0 \le t \le 1. \tag{21.2}$$

This is the same situation as with γ_1 and γ_2 in Example 21.5. There, we had $\gamma_1(t) = e^{it}$ and $\gamma_2(t) = e^{-it}$, both defined on $[0, 2\pi]$. The images of both paths were closed, so the initial and terminal points were all the same, but intuitively we saw that γ_2 proceeded in the "reverse orientation" from γ_1. We also had the equality $\gamma_2(t) = \gamma_1(2\pi - t)$, which resembles (21.2). ▲

We formally define this notion of "reverse."

21.12 Definition. *Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a path. The* **REVERSE** *of γ is the path*

$$\gamma^-(t) := \gamma(a+b-t), \ a \le t \le b.$$

Some sources denote this path by $-\gamma$ or γ^* instead.

This definition shows that the pairs of paths γ_1 and γ_2 from Examples 21.5 and 21.11 are the reverse paths of each other.

21.13 Problem. Let $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ be a path. Check that $\gamma^-(a) = \gamma(b)$ and $\gamma^-(b) = \gamma(a)$, so γ^- does indeed "reverse" the initial and terminal points of γ. Check moreover that if $a \le t \le b$, then $a \le a + b - t \le b$, and so $\gamma(a + b - t)$ is indeed defined if γ is defined on $[a, b]$.

Day 22: Wednesday, March 1

We talked about some issues with the exam and the most recent problem set and renewed our understanding of domains of functions, limits, and (non)removable discontinuities. Then we finished Example 21.11.

Now we consider the relationship of the paths

$$\gamma_1 \colon [0, 2\pi] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto e^{it} \quad \text{and} \quad \gamma_3 \colon [0, \pi] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto e^{2it}$$

from Example 21.5. We said that both paths appeared to trace out the unit circle in the same orientation (starting from $z = 1$ and moving counterclockwise), and both paths traced out the unit circle only once.

Observe that $\gamma_3(t) = \gamma_1(2t)$, and the map $\varphi_{13} \colon [0, \pi] \to [0, 2\pi] \colon t \mapsto 2t$ is continuously differentiable with $\varphi_{13}'(t) = 2 > 0$ for all t. Observe also that $\gamma_1(t) = \gamma_3(t/2)$, and the map $\varphi_{31} \colon [0, 2\pi] \to [0, \pi] \colon t \mapsto t/2$ is continuously differentiable with $\varphi_{31}'(t) = 1/2 > 0$ for all t. This dual way of viewing γ_3 as the composition $\gamma_3 = \gamma_1 \circ \varphi_{13}$ and of viewing γ_1 as the composition $\gamma_1 = \gamma_3 \circ \varphi_{31}$ is an illustration of a more general phenomenon.

Day 23: Friday, March 3

Recommended reading

We continued studying paths. See Definition 3.1.11, Examples 3.1.12 and 3.1.13, and Definition 3.2.16. We also talked about connectedness, see pp. 99–101. I would call the book's definition of connectedness (Definition 2.1.5) *polygonal* connectedness; our definition and the book's our equivalent, but one direction of that equivalence takes some work. You should read the statement of Proposition 2.1.7 but you are not responsible for its proof.

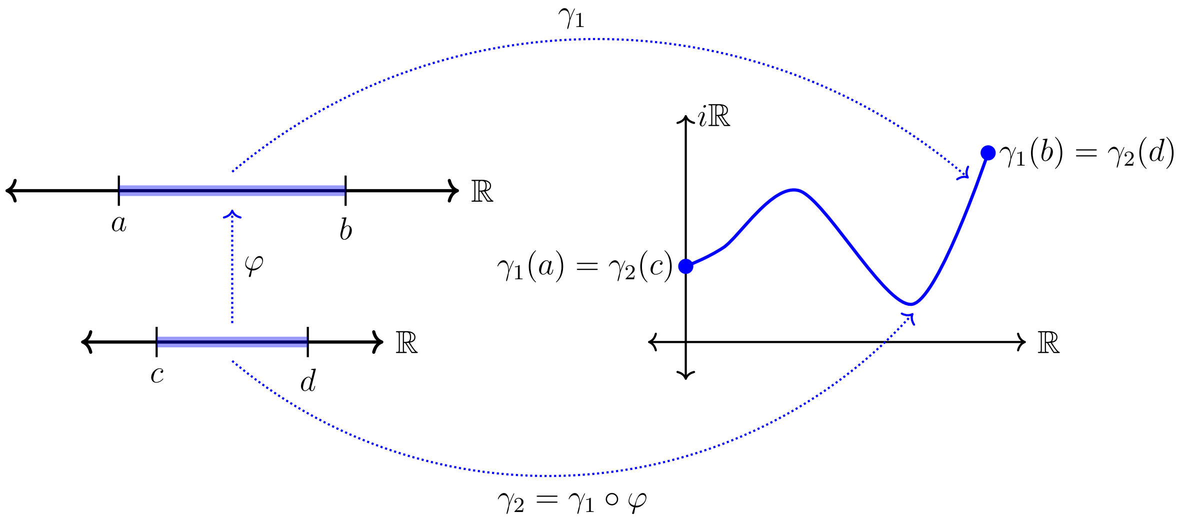
23.1 Definition. *Let $\gamma_1 \colon [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ and $\gamma_2 \colon [c, d] \subseteq \mathbb{R} \to \mathbb{C}$ be paths. Then γ_1 and γ_2 are* **EQUIVALENT** *if there is a continuously differentiable map $\varphi \colon [c, d] \to [a, b]$ such that*

(i) $\varphi'(t) > 0$ *for all* $t \in [c, d]$.

(ii) $\varphi(c) = a$ *and* $\varphi(d) = a$.

(iii) $\gamma_2(t) = \gamma_1(\varphi(t))$ *for all* $t \in [c, d]$.

We say that γ_1 and γ_2 are **REPARAMETRIZATIONS** *of each other.*



The condition that $\varphi'(t) > 0$ in this definition ensures that φ is strictly increasing on $[a, b]$ and therefore one-to-one. This, morally, encodes the idea that $\gamma_2 = \gamma_1 \circ \varphi$ "traces out the same image" as γ_1 does in the "same orientation."

23.2 Problem. It is sometimes convenient to assume that the domain of a path is the interval $[0, 1]$. Show that it is always possible to reparametrize a path $\gamma\colon [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ by finding a continuously differentiable map $\psi\colon [a, b] \to [0, 1]$ that satisfies the conditions of Definition 23.1.

23.3 Problem. Show that if the path γ_1 is a reparametrization of the path γ_2, then γ_1 and γ_2 have the same image.

23.4 Problem. Is the reverse of a path ever a reparametrization of that path?

We now have a way of constructing a new path from an old one (the reverse) and a way of relating two paths that may be different formulaically but really are the same (the reparametrization). If the reverse is morally the equivalent of "multiplying by -1" (which is what multiplying by -1 usually does), then it is only natural that we have an analogue of "adding" paths.

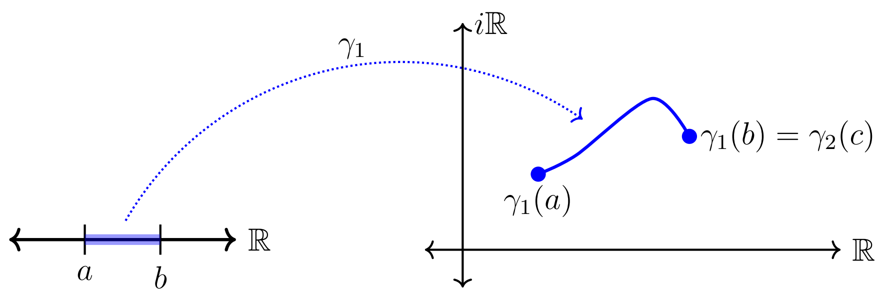
23.5 Definition. *Suppose $\gamma_1\colon [a, b] \to \mathbb{C}$ and $\gamma_2\colon [c, d] \to \mathbb{C}$ are two paths with $\gamma_1(b) = \gamma_2(c)$. Then the* COMPOSITION *of γ_1 and γ_2 is the path*

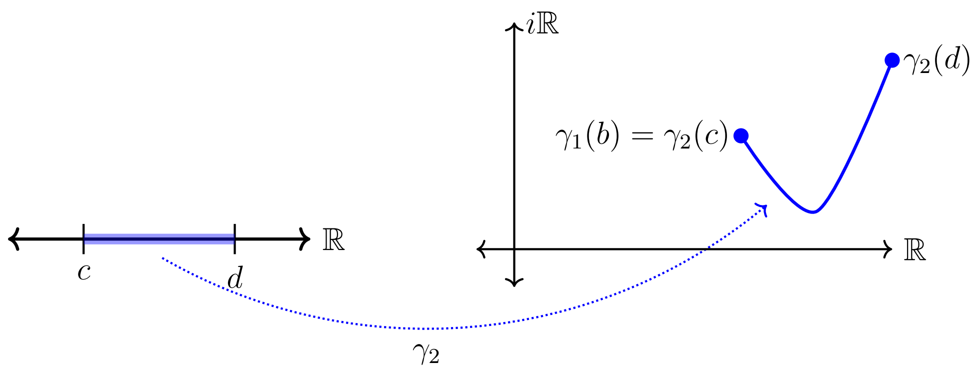
$$\gamma_1 \oplus \gamma_2\colon [a, b + (d - c)] \to \mathbb{C}\colon t \mapsto \begin{cases} \gamma_1(t), & a \le t \le b \\ \gamma_2(t - b + c), & b \le t \le b + d - c. \end{cases}$$

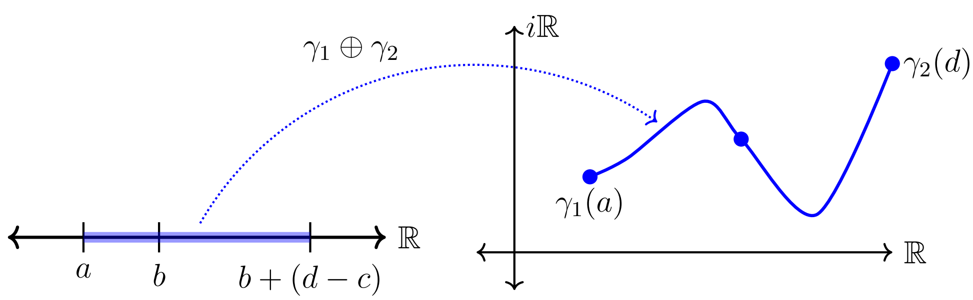
Sometimes this path is denoted by $\gamma_1 + \gamma_2$ or $[\gamma_1, \gamma_2]$ instead.

23.6 Problem. Check that if $b \le t \le b+d-c$, then $c \le t-b+c \le d$, and so $\gamma_2(t-b+c)$ is defined if γ_2 is defined on $[c,d]$.

Here is a visualization of composition.







23.7 Example. Let z_1, z_2, $z_3 \in \mathbb{C}$. Then the line segment from z_1 to z_2 is parametrized by

$$\gamma_1\colon [0,1] \to \mathbb{C}\colon t \mapsto (1-t)z_1 + tz_2$$

and the line segment from z_2 to z_3 is parametrized by

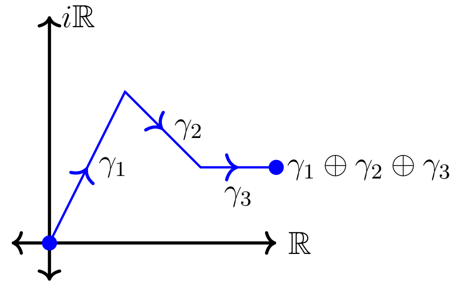
$$\gamma_2\colon [0,1] \to \mathbb{C}\colon t \mapsto (1-t)z_2 + tz_3.$$

Since $\gamma_1(1) = z_1 = \gamma_2(0)$, we can compose γ_1 and γ_2 as

$$\gamma_1 \oplus \gamma_2 \colon [0,2] \to \mathbb{C} \colon t \mapsto \begin{cases} \gamma_1(t), \ 0 \le t \le 1 \\ \gamma_2(t-1), \ 1 \le t \le 2. \end{cases}$$

Then we can calculate

$$(\gamma_1 \oplus \gamma_2)(t) = \begin{cases} (1-t)z_1 + tz_2, \ 0 \le t \le 1 \\ (1-(1-t))z_2 + (t-1)z_3, \ 1 \le t \le 2 \end{cases} = \begin{cases} (1-t)z_1 + tz_2, \ 0 \le t \le 1 \\ tz_2 + (t-1)z_3, \ 1 \le t \le 2. \end{cases}$$



23.8 Problem. Let $\gamma \colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ be a path.

(i) Show that γ and γ^- have the same image.

(ii) Show that if $\phi = \gamma^-$, then $\phi^- = \gamma$. That is, show $(\gamma^-)^- = \gamma$.

(iii) Let $\mu \colon [c,d] \subseteq \mathbb{R} \to \mathbb{C}$ also be a path such that the initial point of μ is the terminal point of γ. By considering the domains of $(\gamma \oplus \mu)^-$ and $\mu^- \oplus \gamma^-$, explain why we should not expect $(\gamma \oplus \mu)^- = \mu^- \oplus \gamma^-$ in general. However, if $a = c = 0$ and $b = d = 1$, show that the equality $(\gamma \oplus \mu)^- = \mu^- \oplus \gamma^-$ is true. (In practice, we could always reparametrize γ and μ so that both are defined on $[0,1]$, and any question that we have about $(\gamma \oplus \mu)^-$ and $\mu^- \oplus \gamma^-$ would likely be invariant under this parametrization.)

We will frequently need to compose more than two paths at once, and so we extend the previous definition accordingly.

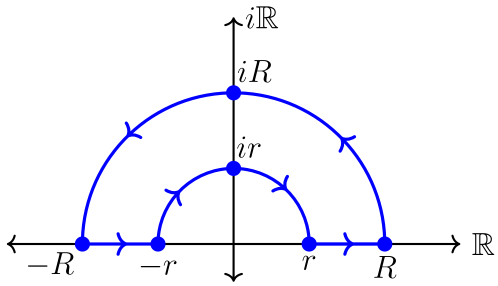
23.9 Definition. *More generally, if, for $k = 1, \ldots, n$ and $n \ge 2$ there are paths $\gamma_k \colon [a_k, b_k] \to \mathbb{C}$ with $\gamma_k(b_k) = \gamma_{k+1}(a_{k+1})$ for $k = 1, \ldots, n-1$, then we define their composition $\oplus_{k=1}^n \gamma_k$ recursively via*

$$\oplus_{k=1}^n \gamma_k = \gamma_1 \oplus \cdots \oplus \gamma_n := \begin{cases} \gamma_1 \oplus \gamma_2, \ n = 2 \\ \\ \left(\oplus_{k=1}^{n-1} \gamma_k \right) \oplus \gamma_n, \ n \ge 2. \end{cases} \tag{23.3}$$

Sometimes this composition is denoted by $\gamma_1 + \cdots + \gamma_n$ or $[\gamma_1, \ldots, \gamma_n]$.

When we consider a "large" composition like $\oplus_{k=1}^n \gamma_k$ in (23.3) above, we will rarely need to know what the domain of $\oplus_{k=1}^n \gamma_k$ actually is; it usually suffices to keep track of the individual domains of the components.

23.10 Example. Let $0 < r < R$. Find four paths γ_1, γ_2, γ_3, γ_4 such that the image of $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ is the path below



Solution. The line segment from $z = r$ to $z = R$ is parametrized by

$$\gamma_1(t) := (1-t)r + tR = (R-r)t + r, \ 0 \le t \le 1.$$

The upper half of the circle of radius R with "counterclockwise" orientation is parametrized by

$$\gamma_2(t) := Re^{it}, \ 0 \le t \le \pi.$$

The line segment from $z = -R$ to $z = -r$ is parametrized by

$$\gamma_3(t) := (1-t)(-R) + t(-r) = (t-1)R - rt = (R-r)t - R, \ 0 \le t \le 1.$$

And the upper half of the circle of radius r with "clockwise" orientation is parametrized by

$$\gamma_4(t) := -re^{i(\pi - t)}, \ 0 \le t \le \pi.$$

Note that the path γ_4 needs to be the reverse of the path $t \mapsto re^{it}$ on $[0, \pi]$. ▲

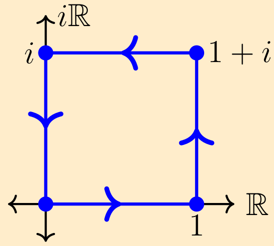
In the preceding example, we could write a piecewise formula for $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ over some domain $[0, b]$ for some $b > 0$. However, we will actually never use such a formula when we work with compositions of paths later, and such a formula would only obscure the four individual domains above. Indeed, although a path need not be continuously differentiable, it can always be expressed as the composition of continuously differentiable.

23.11 Lemma. *Suppose that $\gamma \colon [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ is a path. Then there exist a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ and continuously differentiable paths $\gamma_k \colon [t_{k-1}, t_k] \subseteq \mathbb{R} \to \mathbb{C}$ such that $\gamma = \oplus_{k=1}^n \gamma_k$.*

23.12 Problem. Prove this lemma as follows, referring to Definition 20.4 as needed. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b]$ such that $\gamma_k := \gamma\big|_{[t_{k-1}, t_k]}$ is continuously differentiable. Check that $\gamma_k(t_k) = \gamma_{k+1}(t_k)$ for $k = 1, \ldots, n-1$, and so the composition

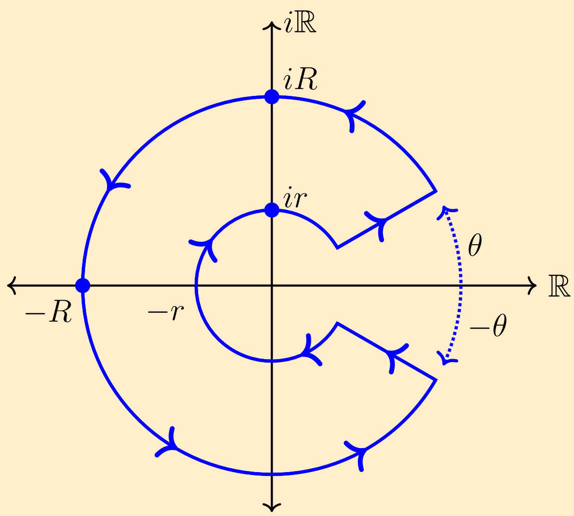
$\oplus_{k=1}^{n}\gamma_k$ is defined. Check that the domain of $\oplus_{k=1}^{n}\gamma_k$ is $[a,b]$ and that $\big(\oplus_{k=1}^{n}\gamma_k\big)(t) = \gamma(t)$ for all t. You may assume that $n = 3$; this will be just complicated enough to illustrate the definition of $\oplus_{k=1}^{n}\gamma_k$.

23.13 Problem. Find formulas for four paths γ_k, $k = 1, 2, 3, 4$, such that the image of $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ is the unit square sketched below.



You do not need to find a "common domain" for the composition but instead can just give formulas for the four paths as in Example 23.10. Note that the curve as drawn is oriented roughly "counterclockwise" in the sense that as you traverse the curve in the direction indicated the "inside" square stays on your left. This is the same phenomenon that happens when we orient a circle counterclockwise.

23.14 Problem. Let $0 < r < R$ and $0 < \theta < \pi/2$. Parametrize the "keyhole contour" below by finding formulas for four paths γ_1, γ_2, γ_3, and γ_4 such that the image of $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ is the curve below. The angle of the "opening" is 2θ radians. Again, you do not need to find a "common domain" for the composition but instead can just give formulas for the four paths as in Example 23.10.

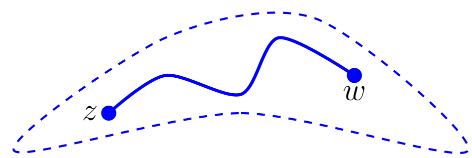


Do you see a "counterclockwise" orientation to this curve?

We conclude with an application of paths that strengthens our prior results about functions with identically zero derivatives. First, we need to augment the geometry of our underlying domains.

23.15 Definition. *A set $\mathcal{D} \subseteq \mathbb{C}$ is* CONNECTED *if for any z, $w \in \mathcal{D}$, there is a path $\gamma \colon [a, b] \subseteq \mathbb{R} \to \mathcal{D}$ such that $\gamma(a) = z$ and $\gamma(b) = w$. Sometimes such a set is called* PATH-CONNECTED, *not just connected.*

Informally, any points z, $w \in \mathcal{D}$ can be "connected" by a path that lies entirely in $\mathcal{D}$.



23.16 Example.

(i) Any open ball $\mathcal{B}(z_0; r)$ is connected. Given z, $w \in \mathcal{B}(z_0; r)$, it is intuitively plausible that we could connect them by the line segment from z to z_0 and then from z_0 to w, or just by the line segment from z to w. This turns out to be true.

(ii) The set $\{z \in \mathbb{C} \mid \text{Re}(z) < -1 \text{ or } \text{Re}(z) > 1\}$ is not connected. Intuitively, any curve connecting z with $\text{Re}(z) < -1$ to w with $\text{Re}(w) > 1$ must pass through the strip $\{z \in \mathbb{C} \mid -1 \leq \text{Re}(z) \leq 1\}$. Proving this requires some thought, possibly involving the intermediate value theorem.

23.17 Problem. Prove both of the claims in the previous example.

23.18 Theorem. *Suppose that $\mathcal{D} \subseteq \mathbb{C}$ is open and connected. If $f : \mathcal{D} \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in \mathcal{D}$, then f is constant on $\mathcal{D}$: there is $c \in \mathbb{C}$ such that $f(z) = c$ for all $z \in \mathcal{D}$.*

Proof. Fix z, $w \in \mathcal{D}$; we will show that $f(z) = f(w)$. Since $\mathcal{D}$ is path-connected, there is a path $\gamma \colon [a, b] \subseteq \mathbb{R} \to \mathcal{D}$ such that $\gamma(a) = z$ and $\gamma(b) = w$.

First suppose that γ is continuously differentiable on all of $[a, b]$. Set $g(t) := f(\gamma(t))$, so g is also differentiable on $[a, b]$ by the chain rule, and $g'(t) = f'(\gamma(t))\gamma'(t) = 0$. By the mean value theorem, $\text{Re}(g)$ and $\text{Im}(g)$ are constant, so g is constant. Thus $f(z) = g(a) = g(b) = f(w)$.

Now suppose that γ is piecewise continuously differentiable on $[a, b]$. For simplicity, take $n = 2$ in part (iv) of Definition 20.4 and suppose there is $t_1 \in (a, b)$ such that $\gamma\big|_{[a,t_1]}$ and $\gamma\big|_{[t_1,b]}$ are continuously differentiable. Define

$$g_1 \colon [a, t_1] \to \mathcal{D} \colon t \mapsto f(\gamma\big|_{[a,t_1]}(t)) \quad \text{and} \quad g_2 \colon [t_1, b] \to \mathcal{D} \colon f(\gamma\big|_{[t_1,b]}(t)).$$

Of course, $g_1(t) = f(\gamma(t))$ and $g_2(t) = f(\gamma_2(t))$, but the domains of g_1 and g_2 are different intervals, so g_1 and g_2 are different functions. However, the utility of taking different domains is that g_1 and g_2 are now differentiable, with $g_1' = 0$ and $g_2' = 0$, so g_1 is constant on $[a, t_1]$ and g_2 is constant on $[t_1, b]$. Then

$$f(z) = g_1(a) = g_1(t_1) = f(\gamma(t_1)) = g_2(t_1) = g_2(b) = f(w).$$

If γ is piecewise continuously differentiable with an arbitrary n from part (iv) of Definition 20.4, we can generalize this argument to include n functions $g_k := f \circ \gamma\big|_{[t_{k-1}, t_k]}$, and we obtain $g_k(t_{k-1}) = g_k(t_k) = f(\gamma(t_k)) = g_{k+1}(t_k) = g_{k+1}(t_{k+1}) = f(\gamma(t_{k+1}))$ for $k = 1, \ldots, n-1$. ■

23.19 Problem. Reread the proof of Theorem 20.2. Recall that we fixed $z_0 \in \mathcal{D}$ and took $r_0 > 0$ such that $\mathcal{B}(z_0; r_0) \subseteq \mathcal{D}$. By Problem 23.17, the ball $\mathcal{B}(z_0; r_0)$ is open and connected, and so f is constant on $\mathcal{B}(z_0; r_0)$. Do you see where in the proof of Theorem 20.2 we used the connectedness of $\mathcal{B}(z_0; r_0)$? [Hint: *consider* $[x_0 + iy_0, x_1 + iy_0] \oplus [x_1 + iy_0, x_1 + iy_1]$.]

Day 24: Monday, March 13

Recommended reading

We discussed antiderivatives and definite integrals. This broadly corresponds to Definition 3.2.2, Proposition 3.2.3, Definition 3.2.5, and Theorem 3.2.7, with a number of deviations and alterations. See also Definition 3.3.1 and Example 3.3.2 for antiderivatives of functions of a complex variable.

We have now completed the first two phases of the course. We are adept in the precalculus of complex numbers—arithmetic, geometry, algebra, and elementary functions—and the differential calculus—limits, continuity, derivatives, and the surprising relationships among the real and imaginary parts of a holomorphic function. Now we will integrate. Our initial goal in this third phase will be the construction of antiderivatives.

24.1 Definition. *Let $\mathcal{D} \subseteq \mathbb{C}$. A holomorphic function $F\colon \mathcal{D} \to \mathbb{C}$ is an* ANTIDERIVATIVE *of $f\colon \mathcal{D} \to \mathbb{C}$ if $F'(z) = f(z)$ for all $z \in \mathcal{D}$.*

24.2 Example.

(i) The function $F(z) = z$ is an antiderivative of the function $f(z) = 1$ on $\mathcal{D} = \mathbb{C}$.

(ii) By the chain rule, the function $F(t) = -ie^{it}$ satisfies $F'(t) = -i^2 e^{it} = e^{it}$, and so F is an antiderivative of $f(t) = e^{-it}$.

When $\mathcal{D} = [a, b] \subseteq \mathbb{R}$ is a real interval, it turns out that there is not much new about antidifferentiation on $\mathcal{D}$; one simply antidifferentiates the real and imaginary parts of $f\colon [a, b] \to \mathbb{C}$. But when $\mathcal{D} \subseteq \mathbb{R}$ is open and therefore genuinely two-dimensional, the antiderivative problem becomes much more surprising, rather like the question of differentiating. We need two tools to resolve the antiderivative problem. We have already mastered the first: paths will play a key role, as we will "integrate over" paths, not just intervals in $\mathbb{R}$. That is, we will study *line* *integrals*, first for their role as antiderivatives and subsequently for their tremendous value as *instruments* that reveal key features of functions.

But to construct these line integrals, we need a second tool: a definite integral for functions defined on a closed bounded interval $[a, b] \subseteq \mathbb{R}$ but now taking values in $\mathbb{C}$. We will build this integral out of the ordinary (Riemann) integral.

What is an integral? We will separate the question of what an integral *is* from the question of what an integral *does*. The former can be quite technical to define precisely, but the latter is actually quite simple. Here are four fundamental "behaviors" that a "good" integral should exhibit.

($\int$1) First, the integral of a function $f\colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ should somehow measure the net area of the region between the graph of f and the interval $[a, b]$. Since the most fundamental area is the area of a rectangle, we should expect

$$\int_a^b 1 \; dt = b - a.$$

($\int$2) If f is nonnegative, the net area of the region between the graph of f and the interval $[a, b]$ should be the genuine area of the region between the graph of f and the interval $[a, b]$, and this should be a positive quantity. So, we expect that if $0 \le f(t)$ on $[a, b]$, then

$$0 \le \int_a^b f(t) \; dt.$$

($\int$3) If we divide the region between the graph of f and the interval $[a, b]$ into multiple components, measure the net area of those components, and add those net areas together, we should get the total net area of the region between the graph of f and the interval $[a, b]$. There are many such ways to accomplish this division, but perhaps one of the most straightforward is to split $[a, b]$ up into two or more subintervals and consider the net areas of the regions between the graph of f and those subintervals. So, we expect that if $a \le c \le b$, then

$$\int_a^c f(t) \; dt + \int_c^b f(t) \; dt = \int_a^b f(t) \; dt.$$

($\int$4) Adding two functions f, $g\colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ should "stack" the graphs of f and g on top of each other. Then the region between the graph of f and the interval $[a, b]$ gets "stacked" on top of region between the graph of g and the interval $[a, b]$. Consequently, the net area of the region between the graph of $f + g$ and the interval $[a, b]$ should just be the sum of these two areas:

$$\int_a^b f(t) \; dt + \int_a^b g(t) \; dt = \int_a^b \big[f(t) + g(t)\big] \; dt.$$

Next, multiplying a function $f\colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ by a constant $\alpha \in \mathbb{R}$ should somehow "scale" the net area of the region between the graph of f and the interval $[a, b]$ by that factor α. For example, the area under the graph of $2f$ over $[a, b]$ should be double the area under the graph. Consequently, the net area of the region between the graph of αf and the interval $[a, b]$ should be the product

$$\int_a^b \alpha f(t) \; dt = \alpha \int_a^b f(t) \; dt.$$

These four properties are exactly the properties of a "good" integral that we will need—no more, no less. Below, we will assert that we can always integrate continuous functions in a manner consistent with the properties above. Before that, we give in to temptation and drop one aspect of integral notation.

24.3 Remark. Contrary to everything that we are taught in calculus, we will typically not write a variable of integration unless we actually need one for clarification (say, to write out the formula for the integrand explicitly, or when changing variables). That is, we write

$$\int_a^b f, \quad \text{not} \quad \int_a^b f(t)\,dt \quad \text{or} \quad \int_a^b f(\tau)\,d\tau.$$

However, when we do include the variable of integration, we follow the custom that any variable may be used, thus

$$\int_a^b f = \int_a^b f(t)\,dt = \int_a^b f(\tau)\,d\tau = \int_a^b f(s)\,ds = \cdots.$$

Also, we will use the words "integral" and "definite integral" more or less interchangeably. (Eventually we will meet a "line integral," and we will sometimes call that just an "integral"—we will add the adjectives "definite" or "line" as needed for emphasis.) But we will never use the words "indefinite integral."

Our view of the definite integral will be "dynamic": the integral is characterized by what it does. And integrals act on both integrands and limits of integration.

24.4 Theorem. *Let $I \subseteq \mathbb{R}$ be an interval and denote by $\mathcal{C}(I)$ the set of all continuous functions from I to $\mathbb{R}$. There exists a map*

$$\int : \{(f,a,b) \mid f \in \mathcal{C}(I),\ a,b \in I\} \to \mathbb{R} \colon (f,a,b) \mapsto \int_a^b f$$

with the following properties.

($\int$1) [Constants] *If a, $b \in I$, then*

$$\int_a^b 1 = b - a.$$

($\int$2) [Monotonicity] *If $f \in \mathcal{C}(I)$ and a, $b \in I$ with $a \le b$ and $0 \le f(t)$ for all $t \in [a,b]$, then*

$$0 \le \int_a^b f.$$

($\int$3) [Additivity of the domain] *If $f \in \mathcal{C}(I)$ and a, b, $c \in I$, then*

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

($\int$4) [Linearity in the integrand] *If f, $g \in \mathcal{C}(I)$, a, $b \in I$, and $\alpha \in \mathbb{R}$, then*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b \alpha f = \alpha \int_a^b f.$$

The number $\int_a^b f$ is the **DEFINITE INTEGRAL OF** f **FROM** a **TO** b.

Properties ($\int$4) encodes the linearity of the integral as an operator on the integrand with the limits of integration fixed, while property ($\int$3) is its **ADDITIVITY** over subintervals with the integrand fixed. Property ($\int$2) encodes the idea that a nonnegative function should have a nonnegative integral, while property ($\int$1) defines the one value of the integral that it most certainly should have from the point of view of area.

While these properties tell us what an integral does, they do not really tell us what an integral is. That is, except for constant functions, the properties above do not give us a formula for computing an integral. This is the challenge of integration; the integral is a limit of sorts, but it is not nearly as transparent a limit as, say, the derivative. It turns out that if f is continuous on the interval $[a, b]$, then

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right). \tag{24.4}$$

The terms of the sequence on the right are the right-endpoint Riemann sums for f over $[a, b]$. Taking this limit as the definition of the integral—and tacitly assuming that the sequence of Riemann sums converges if f is continuous—we can prove properties ($\int$1), ($\int$2), and ($\int$4) quite easily. Property ($\int$3) is not so obvious from (24.4), and in fact this property hinges on expressing $\int_a^b f$ as a "limit" of several kinds of Riemann sums, not just the right-endpoint sum. And then there is still the challenge of ensuring that limits of all sorts of "well-behaved" Riemann sums for f (including, but not limited to, left and right endpoint and midpoint sums) all converge to the same number. Moreover, it is plausible that one might want to integrate functions that are not continuous.

These deeper questions of integration, while tremendously worthwhile, will have no bearing on our further study of complex analysis. We will only need to integrate continuous functions, and we will only need properties ($\int$1), ($\int$2), ($\int$3), and ($\int$4). So, equipped with the integral for real-valued functions, we turn to the complex-valued case.

24.5 Definition. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{C}$ be continuous and let a, $b \in I$. The* **INTEGRAL** *of f from a to b is*

$$\int_a^b f := \int_a^b \operatorname{Re}(f) + i \int_a^b \operatorname{Im}(f). \tag{24.5}$$

24.6 Remark. The terms $(b-a)n^{-1}\sum_{k=1}^n f(a + k(b-a)/n)$ in the Riemann sum limit (24.4) are perfectly well-defined for a function $f\colon I \subseteq \mathbb{R} \to \mathbb{C}$, if a, $b \in I$. Thus one could in principle prove Theorem 24.4 not assuming that f is only real-valued. There are, however, certain advantages to assuming that f is indeed real-valued—namely, the ability to manipulate inequalities involving Riemann sums.

The complex-valued integral inherits many properties from the real-valued version.

24.7 Problem. Let $I \subseteq \mathbb{R}$ be an interval and f, $g \colon I \to \mathbb{C}$ be continuous. Let a, b, $c \in I$ and $\alpha \in \mathbb{C}$. Using only Definition 24.5 and Theorem 24.4, prove the following.

(i) $\operatorname{Re}\left(\int_a^b f\right) = \int_a^b \operatorname{Re}(f)$ and $\operatorname{Im}\left(\int_a^b f\right) = \int_a^b \operatorname{Im}(f)$

(ii) $\overline{\int_a^b f} = \int_a^b \overline{f}$, where $\overline{f}(t) := \overline{f(t)}$

(iii) [Generalization of ($\int$1)] $\int_a^b \alpha = \alpha(b-a)$

(iv) [Generalization of ($\int$3)] $\int_a^c f + \int_c^b f = \int_a^b f$

(v) [Generalization of ($\int$4)] $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ and $\int_a^b \alpha f = \alpha \int_a^b f$

(vi) $\int_a^a f = 0$

(vii) $\int_a^b f = -\int_b^a f$

24.8 Problem. Use induction to generalize additivity as follows. Let $I \subseteq \mathbb{R}$ be an interval and $f \colon I \to \mathbb{C}$ be continuous. If $t_0, \ldots, t_n \in I$, then

$$\int_{t_0}^{t_n} f = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f.$$

Note that Problem 24.7 does not discuss the monotonicity of the integral, as inequalities do not make sense for functions that are complex-and-not-real-valued. If we return to real-valued functions, then we can extend monotonicity in a useful way.

24.9 Problem. Let $I \subseteq \mathbb{R}$ be an interval.

(i) Suppose that f, $g \colon I \to \mathbb{R}$ are continuous and a, $b \in \mathbb{R}$ with $a \le b$. If $f(t) \le g(t)$ for all $t \in [a, b]$, show that

$$\int_a^b f \le \int_a^b g. \tag{24.6}$$

(ii) Suppose that $f\colon I \to \mathbb{R}$ is continuous and there are m, $M \in \mathbb{R}$ such that $m \le f(t) \le M$ for all $t \in [a,b]$. Show that

$$m(b-a) \le \int_a^b f \le M(b-a). \tag{24.7}$$

A double application of (24.6) yields one of the most important estimates on integrals possible.

24.10 Theorem (Real triangle inequality). *Let $I \subseteq \mathbb{R}$ be an interval, let $f\colon I \to \mathbb{R}$ be continuous, and let a, $b \in I$ with $a \le b$. Then*

$$\left|\int_a^b f\right| \le \int_a^b |f|. \tag{24.8}$$

Proof. Use the inequalities $-|f(t)| \le f(t) \le |f(t)|$ and (24.6) to find

$$\int_a^b -|f| \le \int_a^b f \le \int_a^b |f|.$$

By linearity, this is

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|,$$

and by properties of absolute value, this is equivalent to (24.8). ■

24.11 Problem. Show that if we remove the hypothesis $a \le b$, then the estimate (24.8) becomes

$$\left|\int_a^b f\right| \le \left|\int_a^b |f|\right|.$$

Why is the extra absolute value on the right necessary here?

The triangle inequality is also true in the complex-valued setting, but it needs a new proof, since the proof of Theorem 24.10 used monotonicity.

24.12 Theorem (Complex triangle inequality). *Let $I \subseteq \mathbb{R}$ be an interval, let $f\colon I \to \mathbb{C}$ be continuous, and let a, $b \in I$ with $a \le b$. Then*

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Proof. The proof employs polar coordinates. If $\int_a^b f = 0$, then there is nothing to prove, as the inequality simply reads $0 \le \int_a^b |f|$. Otherwise, if $z := \int_a^b f \ne 0$, we can write z in its polar form: $z = |z|e^{i\theta}$. Then $|z| = e^{-i\theta}z$. Note that $e^{-i\theta}z \in \mathbb{R}$, since $|z| \in \mathbb{R}$. Thus

$e^{-i\theta}z = \mathrm{Re}(e^{-i\theta}z)$. Since $\mathrm{Re}(e^{-i\theta}z) \in \mathbb{R}$, we have $\mathrm{Re}(e^{-i\theta}z) \le |\mathrm{Re}(e^{-i\theta}z)|$. We conclude

$$|z| = e^{-i\theta}z = \mathrm{Re}(e^{-i\theta}z) \le |\mathrm{Re}(e^{-i\theta}z)|. \tag{24.9}$$

Returning to the definition $z = \int_a^b f$, (24.9) reads

$$\left|\int_a^b f\right| \le \left|\mathrm{Re}\left(e^{-i\theta}\int_a^b f\right)\right|. \tag{24.10}$$

The linearity of the integral gives

$$e^{-i\theta}\int_a^b f = \int_a^b e^{-i\theta} f,$$

and then part (i) of Problem 24.7 gives

$$\mathrm{Re}\left(\int_a^b e^{-i\theta} f\right) = \int_a^b \mathrm{Re}(e^{-i\theta} f). \tag{24.11}$$

Then the triangle inequality for *real*-valued integrands gives

$$\left|\int_a^b \mathrm{Re}(e^{-i\theta} f)\right| \le \int_a^b |\mathrm{Re}(e^{-i\theta} f)|. \tag{24.12}$$

Since

$$|\mathrm{Re}(e^{-i\theta} f(t))| \le |e^{-i\theta} f(t)| = |f(t)|$$

for any $t \in [a, b]$, monotonicity for the *real*-valued integrand (part (i) of Problem 24.9) gives

$$\int_a^b |\mathrm{Re}(e^{-i\theta} f)| \le \int_a^b |f|. \tag{24.13}$$

We put (24.10), (24.11), (24.12), and (24.13) together to read

$$\left|\int_a^b f\right| \le \left|\mathrm{Re}\left(e^{-i\theta}\int_a^b e^{-i\theta} f\right)\right| = \left|\int_a^b \mathrm{Re}(e^{-i\theta} f)\right| \le \int_a^b |\mathrm{Re}(e^{-i\theta} f)| \le \int_a^b |e^{-i\theta} f| = \int_a^b |f|.$$

This is the triangle inequality. ■

We now have only a handful of results about the definite integral, and yet they are enough to prove the fundamental theorem of calculus.

24.13 Theorem (FTC1). *Let $f \colon I \to \mathbb{C}$ be continuous and fix $a \in I$. Define*

$$F \colon I \to \mathbb{C} \colon t \mapsto \int_a^t f$$

Then F is an antiderivative of f on I.

Proof. Fix $t \in I$. We need to show that F is differentiable at t with $F'(t) = f(t)$. That is, we want

$$\lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = f(t),$$

equivalently,

$$\lim_{h \to 0} \frac{F(t+h) - F(t) - hf(t)}{h} = 0.$$

We first compute

$$\begin{aligned} F(t+h) - F(t) &= \int_a^{t+h} f(\tau) \, d\tau - \int_a^t f(\tau) \, d\tau \\ &= \int_a^{t+h} f(\tau) \, d\tau + \int_t^a f(\tau) \, d\tau \\ &= \int_t^{t+h} f(\tau) \, d\tau. \end{aligned}$$

Next,

$$hf(t) = f(t)[(t+h) - t] = f(t) \int_t^{t+h} 1 \, d\tau = \int_t^{t+h} f(t) \, d\tau.$$

We then have

$$F(t+h) - F(t) - hf(t) = \int_t^{t+h} f(\tau) \, d\tau - \int_t^{t+h} f(t) \, d\tau = \int_t^{t+h} \big[f(\tau) - f(t)\big] \, d\tau.$$

Note that this is one instance in which using the variable of integration τ clarifies the fact that t is constant here. It therefore suffices to show that

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \big[f(\tau) - f(t)\big] \, d\tau = 0,$$

and we do that in the following lemma. ∎

24.14 Lemma. *Let $I \subseteq \mathbb{R}$ be an interval and let $f \colon I \to \mathbb{C}$ be continuous. Then*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \big[f(\tau) - f(t)\big] \, d\tau = 0$$

for any $t \in I$.

Proof. Fix $t \in I$. It suffices to show that the left and right limits

$$\lim_{h \to 0^{\pm}} \frac{1}{h} \int_t^{t+h} \big[f(\tau) - f(t)\big] \, d\tau = 0 \tag{24.14}$$

hold separately. We do this only for the right limit and leave the left limit as an exercise.

We want to show that given $\epsilon > 0$, there is $\delta > 0$ such that if $0 < h < \delta$, then

$$\left|\int_t^{t+h} \frac{1}{h}\big[f(\tau) - f(t)\big]\ d\tau\right| < \epsilon. \tag{24.15}$$

The continuity of f at t provides $\delta > 0$ such that if $|\tau - t| < \delta$, then $|f(\tau) - f(t)| < \epsilon$. Take $0 < h < \delta$. Then if $t \le \tau \le t + h$, we have $0 \le \tau - t \le h < \delta$, and so $|\tau - t| < \delta$. Thus $|f(\tau) - f(t)| < \epsilon$ for all $\tau \in [t, t + h]$, and so the triangle inequality gives

$$\left|\int_t^{t+h} \big[f(\tau) - f(t)\big]\ d\tau\right| \le \int_t^{t+h} \big|f(\tau) - f(t)\big|\ d\tau \le \int_t^{t+h} \epsilon\ d\tau = \epsilon(t + h - t) = \epsilon h.$$

Dividing by h, we obtain (24.15). ■

24.15 Problem. Prove that the left limit (24.14) holds. What specific changes are needed when $h < 0$?

Day 25: Wednesday, March 15

With FTC1, we can prove a second version that facilitates the computation of integrals via antiderivatives.

25.1 Corollary (FTC2). *Let $I \subseteq \mathbb{R}$ be an interval and let $f \colon I \to \mathbb{C}$ be continuous. If F is any antiderivative of f on I, then*

$$\int_a^b f = F(b) - F(a)$$

for all a, $b \in I$.

Proof. Let $F_\star(t) = \int_a^t f$, so $F_\star$ is also an antiderivative of f. Put $h = F_\star - F$, so $h' = 0$ on I. If we also write $h(t) = h_1(t) + ih_2(t)$ for real-valued functions h_1 and h_2, we have $h_1' = h_2' = 0$ on I. Since I is an interval, by the mean value theorem (for real-valued functions of a real variable), h_1 and h_2 are constant, so h is constant, say, $h(t) = h(a)$ for all t. Then $F_\star(t) = F(t) + h(a)$ for all t, so

$$\int_a^b f = F_\star(b) = h(b) + F(b) = h(a) + F(b) = F_\star(a) - F(a) + F(b) = F(b) - F(a)$$

since $F_\star(a) = 0$. ■

25.2 Example. Since $F(t) = e^{it}/i$ is an antiderivative of $f(t) = e^{it}$, we have

$$\int_0^{2\pi} e^{it}\ dt = \frac{e^{it}}{i}\bigg|_{t=0}^{t=2\pi} = \frac{e^{2\pi i} - e^0}{i} = \frac{1-1}{i} = 0.$$

The fundamental theorems of calculus are, of course, the keys to both substitution and integration by parts, two of the most general techniques for evaluating integrals in terms of simpler functions.

25.3 Theorem (Substitution). *Let I, $J \subseteq \mathbb{R}$ be intervals with a, $b \in J$. Let $\varphi \colon J \to I$ be continuously differentiable and let $f \colon I \to \mathbb{C}$ be continuous. Then*

$$\int_a^b (f \circ \varphi)\varphi' = \int_{\varphi(a)}^{\varphi(b)} f.$$

Proof. Put

$$F(t) := \int_{\varphi(a)}^t f,$$

so $F'(t) = f(t)$ for all $t \in I$ by FTC1. Next, put

$$G(t) := F(\varphi(t)),$$

so

$$G'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t).$$

That is, G is an antiderivative of $(f \circ \varphi)\varphi'$, and so

$$\int_a^b (f \circ \varphi)\varphi' = G(b) - G(a)$$

by FTC2. But

$$G(b) - G(a) = F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f - \int_{\varphi(a)}^{\varphi(a)} f = \int_{\varphi(a)}^{\varphi(b)} f. \qquad \blacksquare$$

25.4 Example. Let $k \in \mathbb{Z} \setminus \{0\}$ and put $\varphi(\tau) = k\tau$. Then

$$\int_0^{2\pi} e^{ik\tau} \, d\tau = \frac{1}{k}\int_0^{2\pi} e^{ik\tau}k \, d\tau = \frac{1}{k}\int_0^{2\pi} e^{i\varphi(\tau)}\varphi'(\tau) \, d\tau = \frac{1}{k}\int_{\varphi(0)}^{\varphi(2\pi)} e^{it} \, dt = \frac{1}{k}\int_0^{2k\pi} e^{it} \, dt$$
$$= \frac{1}{ik}e^{it}\bigg|_{t=0}^{t=2\pi} = \frac{e^{2\pi i} - e^0}{ik} = \frac{1-1}{ik} = 0.$$

A recurring theme of our subsequent applications of integrals will be that we are trying to estimate or control some kind of difference (this is roughly 90% of analysis), and it turns out to be possible to rewrite that difference in a tractable way by introducing an integral. It may be possible to manipulate further terms under consideration by rewriting them as integrals, too. The fundamental identity that we will use in the future is (25.16) below.

25.5 Example. FTC2 allows us to rewrite a functional difference as an integral. When we incorporate substitution, we can get a very simple formula for that difference. Suppose that $I \subseteq \mathbb{R}$ is an interval, $f: I \to \mathbb{C}$ is continuously differentiable, and t, $t+h \in I$. Then

$$f(t+h) - f(t) = \int_t^{t+h} f'.$$

With

$$\varphi(\tau) = (1-\tau)t + \tau(t+h) = t + h\tau,$$

we have $\varphi'(\tau) = h$, $\varphi(0) = t$, and $\varphi(1) = t+h$, so substitution implies

$$\int_t^{t+h} f' = \int_0^1 f'(t+h\tau)h \, d\tau.$$

Thus if f is differentiable and f' continuous on an interval containing t and $t+h$, then

$$f(t+h) - f(t) = h\int_0^1 f'(t+h\tau) \, d\tau. \tag{25.16}$$

This equality allows us to control the distance between $f(t+h)$ and $f(t)$ using the explicit factor of h on the right above and the triangle inequality on the integral with the constant limits of 0 and 1. In particular, knowing the size of f' controls the difference. We could obtain a similar result from the mean value theorem (at least, if f is real-valued), but the explicit formula (25.16) eliminates a possibly vague "existential" result from the MVT.

This identity can be generalized to partial derivatives, e.g., if $f = f(t,s)$ is differentiable with respect to t and f_t is continuous, then

$$f(t+h,s) - f(t,s) = h\int_0^1 f_t(t+\tau h, s) \, d\tau.$$

25.6 Problem. Prove the following variant of Example 25.5: if $I \subseteq \mathbb{R}$ is an interval, $f: I \to \mathbb{C}$ is continuously differentiable, and a, $b \in I$, then

$$f(b) - f(a) = (b-a)\int_0^1 f'(a + t(b-a)) \, dt.$$

Integration by parts works nicely for complex-valued functions of a real variable, because the product rule, the FTC, and antiderivatives work as we think they should in this setting.

25.7 Theorem (Integration by parts). *Suppose that f, $g: [a,b] \to \mathbb{C}$ are differentiable with f', g' continuous. Then*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g. \tag{25.17}$$

Proof. Put $H(t) = f(t)g(t)$, so the product rule (for complex-valued functions of a real variable) gives

$$H'(t) = f'(t)g(t) + f(t)g'(t).$$

Then FTC2 gives

$$\int_a^b H' = H(b) - H(a) = f(b)g(b) - f(a)g(a). \tag{25.18}$$

But by linearity

$$\int_a^b H' = \int_a^b f'g - \int_a^b fg', \tag{25.19}$$

and so (25.17) follows by equating (25.18) and (25.19). ■

Day 26: Friday, March 17

Recommended reading

We discussed line integrals (which the book calls path or contour integrals). See Definition 3.2.9. We did a version of Example 3.2.10. Examples 3.2.11, 3.2.13, and 3.2.14 provide valuable detail; you should read and work through them carefully. The book treats our version of the FTC in the proof of Theorem 3.3.4 (see p. 169 and the discussion on how part (a) implies (b) there). Proposition 3.2.12 contains useful properties of the line integral and proofs for some of the problems below.

We extend the definite integral to functions of a complex variable as a line integral. While there is some reasonable motivation for the following definition as a "limit" of certain Riemann sums, we do not consider that (although see Problem 26.6 to view a definite integral as a specific line integral). Instead, we take the position that the line integral is simply the best instrument for extracting critical information about functions, although its full utility will not be apparent for some time.

26.1 Definition. *Let $\mathcal{D} \subseteq \mathbb{C}$ and let $f \colon \mathcal{D} \to \mathbb{C}$ be continuous. Let $\gamma \colon [a, b] \subseteq \mathbb{R} \to \mathcal{D}$ be continuously differentiable. Then the* **LINE INTEGRAL OF** f **OVER** γ *is*

$$\int_\gamma f = \int_\gamma f(z)\ dz := \int_a^b f(\gamma(t))\gamma'(t)\ dt = \int_a^b (f \circ \gamma)\gamma'. \tag{26.20}$$

26.2 Remark. The integrand in (26.20) is the product $(f \circ \gamma)\gamma'$. This is a continuous function since γ is continuously differentiable. Thus Definition 24.5 applies.

As with definite integrals, we will often omit the variable of integration in line integrals and include it for clarity when necessary. When we do include it, we continue the custom that we can change the symbol at will:

$$\int_\gamma f = \int_\gamma f(z)\ dz = \int_\gamma f(w)\ dw = \int_\gamma f(\xi)\ d\xi = \cdots .$$

We will frequently integrate over lines and circles, and so the following two examples contain extremely important calculations.

26.3 Example. Parametrize the line segment $[0, i]$ by

$$\gamma \colon [0,1] \subseteq \mathbb{R} \to \mathbb{C} \colon t \mapsto (1-t)0 + ti = it.$$

Then $\gamma'(t) = i$ for all t. The function $f(z) := \overline{z}$ is continuous on $\mathbb{C}$, and so we may compute

$$\int_\gamma \overline{z}\ dz = \int_0^1 \overline{\gamma(t)}\gamma'(t)\ dt = \int_0^1 \overline{it}(i)\ dt = \int_0^1 -it(i)\ dt = \int_0^1 t\ dt = \frac{1}{2}.$$

26.4 Example. Let $z_0 \in \mathbb{C}$, $r > 0$, and $n \in \mathbb{Z}$ and parametrize the circle of radius r centered at z_0 by $\gamma(t) := z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then $\gamma'(t) = ire^{it}$, and so

$$\int_\gamma (z-z_0)^n\ dz = \int_0^{2\pi} \big((z_0+re^{it})-z_0\big)^n (ire^{it})\ dt = ir\int_0^{2\pi} (re^{it})^n e^{it}\ dt = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\ dt.$$

If $n = -1$, then

$$\int_\gamma \frac{dz}{z-z_0} = i\int_0^{2\pi} 1\ dt = 2\pi i.$$

If $n \neq -1$, then since $F(t) := e^{i(n+1)t}/(i(n+1))$ is an antiderivative of $f(t) := e^{i(n+1)t}$, we have

$$\int_\gamma (z-z_0)^n\ dz = ir^{n+1}\frac{e^{i(n+1)t}}{i(n+1)}\bigg|_{t=0}^{t=2\pi} = ir^{n+1}\left(\frac{1-1}{i(n+1)}\right) = 0.$$

Here it is essential that $n \in \mathbb{Z}$.

26.5 Remark. Since we will integrate over line segments and circles so often, we will use a special, suggestive notation for their line integrals that will relieve us from writing out their parameterizations each time. Assume below that f is continuous at least on the given line segment and circle.

(i) For z_1, $z_2 \in \mathbb{C}$, define

$$\int_{[z_1,z_2]} f := (z_2 - z_1)\int_0^1 f((1-t)z_1 + tz_2)\ dt.$$

This line integral is oriented "from z_1 to z_2."

(ii) For $z_0 \in \mathbb{C}$ and $r > 0$, define

$$\int_{|z-z_0|=r} f := ir\int_0^{2\pi} f(z_0 + re^{it})e^{it}\ dt.$$

This line integral is oriented with the circle traversed "counterclockwise."

26.6 Problem. Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be continuous. Show that for any a, $b \in I$, we have

$$\int_{[a,b]} f = \int_a^b f,$$

where the integral on the left is the line integral over the path $[a, b]$, and the integral on the right is the ordinary Riemann integral.

Now we extend our definition of the line integral to paths that are only piecewise continuously differentiable, not just continuously differentiable.

26.7 Definition. *Let $\mathcal{D} \subseteq \mathbb{C}$ and let $f: \mathcal{D} \to \mathbb{C}$ be continuous. Let $\gamma: [a, b] \subseteq \mathbb{R} \to \mathcal{D}$ be a path in $\mathcal{D}$ and write $\gamma = \oplus_{k=1}^n \gamma_k$, where $\gamma_k: [a_k, b_k] \subseteq \mathbb{R} \to \mathcal{D}$ is continuously differentiable. Then the* **LINE INTEGRAL OF f OVER γ** *is*

$$\int_\gamma f = \int_\gamma f(z)\ dz := \sum_{k=1}^n \int_{\gamma_k} f = \sum_{k=1}^n \int_{a_k}^{b_k} f(\gamma_k(t))\gamma_k'(t)\ dt.$$

Since the preceding definition depends on the representation chosen for γ as a composition of particular paths, we need to be sure that different choices of representations actually do not give different line integrals.

26.8 Problem. Check that the line integral from Definition 26.7 is well-defined in the following sense. Let $\mathcal{D} \subseteq \mathbb{C}$ and let $f: \mathcal{D} \to \mathbb{C}$ be continuous. Suppose that $\gamma: [a, b] \subseteq \mathbb{R} \to \mathcal{D}$ be a path in $\mathcal{D}$ with

$$\gamma = \oplus_{k=1}^n \gamma_k \quad \text{and} \quad \gamma = \oplus_{k=1}^m \mu_k,$$

where $\gamma_k: [a_k, b_k] \subseteq \mathbb{R} \to \mathcal{D}$ and $\mu_k: [c_k, d_k] \subseteq \mathbb{R} \to \mathcal{D}$ are all continuously differentiable. Show that

$$\sum_{k=1}^n \int_{\gamma_k} f = \sum_{k=1}^m \int_{\mu_k} f.$$

The line integral enjoys mostly obvious properties that generalize those of the definite integral.

26.9 Problem. Let $\mathcal{D} \subseteq \mathbb{C}$ and let $f: \mathcal{D} \to \mathbb{C}$ be continuous. The following results hold for all paths, but in your work you may assume that the paths are continuously differentiable, not merely piecewise continuously differentiable. In the context of Problem 26.6, how do parts (i), (ii), and (iii) below generalize results from Problem 24.7?

(i) Let γ_1 and γ_2 be paths in $\mathcal{D}$ and suppose that the terminal point of γ_1 is the initial point of γ_2. Show that

$$\int_{\gamma_1 \oplus \gamma_2} f = \int_{\gamma_1} f \oplus \int_{\gamma_2} f.$$

(ii) Let γ be a path in $\mathcal{D}$. Show that

$$\int_{\gamma^-} f = -\int_\gamma f.$$

(iii) Let γ be a path in $\mathcal{D}$, let $g \colon \mathcal{D} \to \mathbb{C}$ also be continuous, and let $\alpha \in \mathbb{C}$. Show that

$$\int_\gamma (f+g) = \int_\gamma f + \int_\gamma g \quad \text{and} \quad \int_\gamma \alpha f = \alpha \int_\gamma f.$$

(iv) Show that if γ_1 and γ_2 are equivalent paths in $\mathcal{D}$, i.e., γ_1 is a reparametrization of γ_2, then

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

26.10 Problem. Let $\mathcal{D} \subseteq \mathbb{C}$ and let $f \colon \mathcal{D} \to \mathbb{C}$ be continuous.

(i) Let γ be a path in $\mathcal{D}$. What is the value of

$$\int_{\gamma \oplus \gamma^-} f?$$

Does this remind you of a result from Problem 24.7?

(ii) Fix $z_0 \in \mathcal{D}$ and let γ be the "constant" path $\gamma \colon [a,b] \subseteq \mathbb{R} \to \mathcal{D} \colon t \mapsto z_0$. What is the value of

$$\int_\gamma f?$$

Does this remind you of a result from Problem 24.7?

(iii) Explain why we should expect, in general, that

$$\overline{\int_\gamma f} \neq \int_\gamma \overline{f},$$

and give a specific example of f and γ for which the equality does not hold.

The fundamental theorem of calculus nicely extends to line integrals and thereby generalizes the FTC for definite integrals.

26.11 Theorem (FTC for line integrals). *Let $\mathcal{D} \subseteq \mathbb{C}$ and $f \colon \mathcal{D} \to \mathbb{C}$ be continuous.*

Suppose that $F: \mathcal{D} \to \mathbb{C}$ is an antiderivative of f. Then if $\gamma: [a,b] \subseteq \mathbb{R} \to \mathcal{D}$ is a path,

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

Proof. We only give the proof in the special case that γ is continuously differentiable. Then

$$\int_\gamma f = \int_a^b (f \circ \gamma)\gamma' = \int_a^b (F \circ \gamma)' = F(\gamma(b)) - F(\gamma(a)).$$

If γ is only piecewise continuously differentiable, express γ as the composition of continuously differentiable paths, apply the result just proved to each of those paths, add, and simplify using telescoping. ■

26.12 Problem.

(i) Let $\mathcal{D} \subseteq \mathbb{C}$ and $f: \mathcal{D} \to \mathbb{C}$ be continuous. Show that if f has an antiderivative on $\mathcal{D}$ and γ is a closed path in $\mathcal{D}$, then

$$\int_\gamma f = 0.$$

(ii) Compute

$$\int_{|z|=1} \overline{z}\, dz.$$

Does $f(z) = \overline{z}$ have an antiderivative on $\mathcal{D} = \mathbb{C}$?

(iii) Compute

$$\int_{|z|=1} \frac{dz}{z}.$$

Does $f(z) = 1/z$ have an antiderivative on $\mathcal{D} = \mathbb{C} \setminus \{0\}$?

Day 27: Monday, March 20

Recommended reading

We discussed arc length for paths and the ML-inequality for line integrals, see pp. 161–164. Then we discussed challenges in extending the FTC for functions on subintervals of $\mathbb{R}$ to functions on subsets of $\mathbb{C}$. We concluded with the need for independence of path, which the book discusses in Theorem 3.3.4.

In calculus we learned that if $f: [a,b] \to \mathbb{R}$ is continuously differentiable, then, by a limiting argument with Riemann sums, the integral

$$\int_a^b |f'(t)|\, dt$$

captures the natural notion of the "length" of the graph of f. We import this concept to paths.

27.1 Definition. *The* **ARC LENGTH** *of a continuously differentiable path* $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ *is*

$$\ell(\gamma) := \int_a^b |\gamma'(t)|\,dt.$$

If $\gamma = \oplus_{k=1}^n \gamma_k$ *is piecewise continuously differentiable with each* γ_k *continuously differentiable, then the arc length of* γ *is*

$$\ell(\gamma) := \sum_{k=1}^n \ell(\gamma_k).$$

27.2 Problem.

(i) Let $k \in \mathbb{Z}$ and define $\gamma_k\colon [0,2\pi] \subseteq \mathbb{R} \to \mathbb{C}\colon t \mapsto e^{ikt}$. What is $\ell(\gamma_k)$? Is this what you expected?

(ii) What is the arc length of a line segment? Is it what you expected?

27.3 Problem. Check that arc length is well-defined in the sense that if γ is piecewise continuously differentiable with both $\gamma = \oplus_{k=1}^n \gamma_k$ and $\gamma = \oplus_{k=1}^m \mu_k$, then

$$\sum_{k=1}^n \ell(\gamma_k) = \sum_{k=1}^m \ell(\mu_k).$$

We have not yet stated a triangle inequality for line integrals; in fact, the natural (but, alas, naive) estimate

$$\left|\int_\gamma f\right| \le \int_\gamma |f|$$

does not even make sense.

27.4 Problem. Why not? Explain why we should not expect the quantity $\int_\gamma |f|$ to be real-valued, and therefore it has no place in an inequality.

Instead, the concept of arc length permits the correct adaptation of the triangle inequality for line integrals. The following estimate is sometimes called the "ML-inequality" or "ML-estimate" because the right side is the product of a *m*aximum and an arc *l*ength. In particular, it is an extension of part (ii) of Problem 24.9.

27.5 Theorem (ML-inequality). *Let* $\mathcal{D} \subseteq \mathbb{C}$ *and suppose that* $f\colon \mathcal{D} \to \mathbb{C}$ *is continuous.*

Let $\gamma\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ be a path in $\mathcal{D}$. Then

$$\left|\int_\gamma f\right| \le \left(\max_{a \le t \le b} |f(\gamma(t))|\right)\ell(\gamma).$$

Proof. The definition of the line integral and the triangle inequality for definite integrals yield the following estimate:

$$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\ dt\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\ dt = \int_a^b |f(\gamma(t))||\gamma'(t)|\ dt.$$

The function

$$g\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}\colon t \mapsto |f(\gamma(t))|$$

is continuous (because f, γ, and the modulus are all continuous), and so g has a maximum on the closed, bounded interval $[a,b]$ by the extreme value theorem. (Here it is important that g is real-valued, as otherwise the notion of maximum does not make sense.) Then for all $t \in [a,b]$, we have

$$|f(\gamma(t))||\gamma'(t)| \le M|\gamma'(t)|,$$

and so monotonicity for the definite integral of a real-valued function provides

$$\int_a^b |f(\gamma(t))\gamma'(t)|\ dt \le M\int_a^b |\gamma'(t)|\ dt = M\ell(\gamma).$$

If γ is only piecewise continuously differentiable, then we express $\gamma = \oplus_{k=1}^n \gamma_k$ with γ_k continuously differentiable and use the argument above, the triangle inequality, and the definition of arc length. ■

27.6 Problem. How does the ML-inequality generalize the estimate (24.7) for definite integrals?

27.7 Example. Let $\mathcal{D} = \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| \le 1\}$, i.e., $\mathcal{D}$ is an infinite horizontal strip of width 2 containing the real line. Suppose that $f\colon \mathcal{D} \to \mathbb{C}$ is holomorphic and satisfies $|f(z)| \le |\operatorname{Re}(z)|^{-1}$ when $z \in \mathcal{D}$ with $|z| \ge 1$. We can use the ML-inequality to show

$$\lim_{R\to\infty}\left|\int_{[R,R+i]} f\right| = 0.$$

The length of the line segment $[R, R+i]$ is $|(R+i) - R| = |i| = 1$, so we estimate

$$\left|\int_{[R,R+i]} f\right| \le \max_{z\in[R,R+i]} |f(z)| \le \max_{z\in[R,R+i]} \frac{1}{|\operatorname{Re}(z)|}.$$

Parametrize this line segment, as usual, by $t \mapsto (1-t)R + t(R+i) = R + it$, so if $z \in [R, R+i]$, then $\operatorname{Re}(z) = R$. Thus

$$\left|\int_{[R,R+i]} f\right| \le \frac{1}{R} \to 0 \text{ as } R \to \infty.$$

At last we tackle the problem of antiderivatives on a general subset of $\mathbb{C}$. FTC1 tells us that if $I \subseteq \mathbb{R}$ is an interval and $f \colon I \to \mathbb{C}$ is continuous, then f has an antiderivative, and specifically FTC1 *constructs* an antiderivative for f. For any fixed $a \in I$, an antiderivative is $F(t) := \int_a^t f$; from the point of view of the line integral (recall Problem 26.6), we have integrated f over the line segment $[a, t]$. This approach to antiderivatives will not quite succeed if we broaden the domain beyond real intervals.

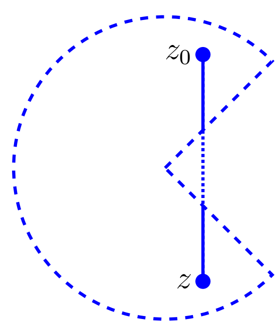
First, continuity alone does not guarantee antiderivatives; the functions in parts (ii) and (iii) of Problem 26.12 are continuous on their domains but do not have antiderivatives. Rather, part (i) gives a necessary condition for the existence of an antiderivative: the integral around a closed path must be zero.

Second, even if we knew that a function under consideration integrated to zero around closed paths, how might we try to construct its antiderivative? Could we replicate the technique of FTC1? We could fix some $z_0 \in \mathcal{D}$ and try to "base" our antiderivative there. We might then try to define an antiderivative as

$$F(z) := \int_{[z_0,z]} f,$$

where $[z_0, z]$ is the line segment from z_0 to z.

This attempted definition has two underlying assumptions. First, this presumes that f is continuous on $\mathcal{D}$, as we have only defined line integrals for continuous functions, although we know from Problem 26.12 that continuity alone will ultimately not be enough to get an antiderivative. Second, this presumes that $[z_0, z] \subseteq \mathcal{D}$ for any $z \in \mathcal{D}$, as f needs to be defined over $[z_0, z]$ for $\int_{[z_0,z]} f$ to be defined. However, depending on the geometry of $\mathcal{D}$, we have no guarantee that $[z_0, z] \subseteq \mathcal{D}$ for all $z \in \mathcal{D}$.

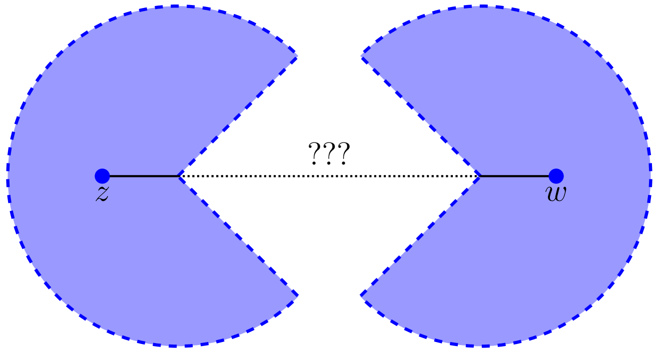


The next option would be not to restrict ourselves to line segments. Suppose we take an arbitrary path γ_z in $\mathcal{D}$ whose initial point is z_0 and whose terminal point is z. Then we could define

$$F(z) := \int_{\gamma_z} f \tag{27.21}$$

and perhaps that would be an antiderivative of f.

There are, again, problems with this approach. First, we have no guarantee that there is a point $z_0 \in \mathcal{D}$ such that for any $z \in \mathcal{D}$, there is also a path in $\mathcal{D}$ connecting z_0 and z.



This is not so hard to remediate; just take $\mathcal{D}$ to be path connected. For technical reasons that are, perhaps, less obvious than the value of path connectedness, we will also want our functions to be defined on open sets, and so we pause for a definition.

27.8 Definition. *A set $\mathcal{D} \subseteq \mathbb{C}$ that is both open and connected will be called, hereafter, a* **DOMAIN**. *Some books use the term* **REGION** *instead of domain.*

Neither term in the preceding definition is ideal: every function has a domain, but not every function has a domain that is a domain! Every subset of $\mathbb{C}$ could reasonably be called a region, but not every region is a region!

Now, even if $\mathcal{D}$ is a domain, how do we know that the function F in (27.21) is *well-defined*? That is, perhaps there are paths γ_z and ϕ_z in $\mathcal{D}$ whose initial points are both z_0 and whose terminal points are both z, but for which

$$\int_{\gamma_z} f \neq \int_{\phi_z} f.$$

In that case, would the antiderivative depend on which path we pick? How would we know which one to choose? Or could the integral of f over a path connecting z_0 and z be "independent of path" in the sense that the integral is the same no matter what the path is (provided those endpoints z_0 and z remain the same)?

This turns out to be a tremendously significant issue, so we first formalize it in a definition and then state and prove a theorem.

27.9 Definition. *Let $\mathcal{D} \subseteq \mathbb{C}$. A continuous function $f \colon \mathcal{D} \to \mathbb{C}$ is* **PATH INDEPENDENT ON** $\mathcal{D}$ *or* **INDEPENDENT OF PATH ON** $\mathcal{D}$ *if whenever γ_1 and γ_2 are paths in $\mathcal{D}$ with the same initial and terminal points, then*

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

It is important to specify the set on which a function is path independent. We will see examples of functions $f \colon \mathcal{D} \to \mathbb{C}$ that are path independent on some smaller $\mathcal{D}_0 \subseteq \mathcal{D}$ but not on all of $\mathcal{D}$.

Day 28: Wednesday, March 22

Recommended reading

We studied independence of path. This is covered in Section 3.3, which you should read in its entirety. We proved Theorem 3.3.4 in two steps and cited but did not prove Lemma 3.3.3. Examples 3.3.5 through 3.3.9 are diverse, useful, and interesting, and you may find them helpful for future work. Last, we introduced the new geometric tool of star-shaped sets, see Definition 3.5.3 and Figure 3.39 on pp. 184–185.

28.1 Theorem (Path independence). *Let $\mathcal{D} \subseteq \mathbb{C}$ be a domain and suppose that $f \colon \mathcal{D} \to \mathbb{C}$ is continuous and path independent. Then f has an antiderivative on $\mathcal{D}$.*

Proof. Fix $z_\star \in \mathcal{D}$. For $z \in \mathcal{D}$, define

$$F(z) := \int_{\gamma_z} f,$$

where γ_z is a path in $\mathcal{D}$ whose initial point is $z_\star$ and whose terminal point is z. Such a path exists because $\mathcal{D}$ is a domain. The function F above is well-defined because f is independent of path: if μ_z is another path in $\mathcal{D}$ whose initial point is $z_\star$ and whose terminal point is z, then

$$\int_{\gamma_z} f = \int_{\mu_z} f.$$

We will show that F is differentiable on $\mathcal{D}$ with $F'(z) = f(z)$ for all $z \in \mathcal{D}$. The proof is very similar to that of FTC1 (Theorem 24.13). Fix $z \in \mathcal{D}$. We need to show that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

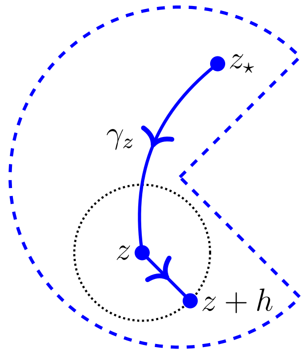
equivalently,

$$\lim_{h \to 0} \frac{F(z+h) - F(z) - hf(z)}{h} = 0. \tag{28.22}$$

Let $h \in \mathbb{C} \setminus \{0\}$ with $|h|$ small enough that $[z, z+h] \subseteq \mathcal{D}$. This is possible since $\mathcal{D}$ is open, and so there is $r > 0$ such that $\mathcal{B}(z; r) \subseteq \mathcal{D}$. Let γ_z be any path in $\mathcal{D}$ with initial point z_0 and terminal point z. Then $\gamma_z \oplus [z, z+h]$ is a path in $\mathcal{D}$ with initial point z_0 and terminal point $z+h$, so

$$F(z+h) = \int_{\gamma_z \oplus [z,z+h]} f = \int_{\gamma_z} f + \int_{[z,z+h]} f.$$

Here is a sketch of this situation.



We therefore may calculate

$$F(z+h) - F(z) = \left(\int_{\gamma_z} f + \int_{[z,z+h]} f\right) - \int_{\gamma_z} f = \int_{[z,z+h]} f. \tag{28.23}$$

Parametrize the line segment $[z, z+h]$ by $t \mapsto (1-t)z + t(z+h) = z + th$, $0 \le t \le 1$, as usual, so $\gamma'(t) = h$ and

$$\int_{[z,z+h]} f = \int_0^1 f(z+th)h\ dt = h\int_0^1 f(z+th)\ dt.$$

We combine this with (28.23) to find

$$\begin{aligned} F(z+h) - F(z) - hf(z) = h\int_0^1 f(z+th)\ dt - hf(z) &= h\int_0^1 f(z+th)\ dt - h\int_0^1 f(z)\ dt \\ &= h\int_0^1 \big[f(z+th) - f(z)\big]\ dt. \end{aligned} \tag{28.24}$$

Then

$$\frac{F(z+h) - F(z) - hf(z)}{h} = \int_0^1 \big[f(z+th) - f(z)\big]\ dt. \tag{28.25}$$

To prove the desired limit (28.22), it therefore suffices to show

$$\lim_{h \to 0} \int_0^1 \big[f(z+th) - f(z)\big]\ dt = 0.$$

This is true by the continuity of f at z, as we show in the following lemma. ■

28.2 Lemma. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f \colon \mathcal{D} \to \mathbb{C}$ be continuous. Then*

$$\lim_{h \to 0} \int_0^1 \big[f(z+th) - f(z)\big]\ dt = 0$$

for each $z \in \mathcal{D}$.

Proof. This proof, too, is very similar to that of FTC1 (Theorem 24.13). Fix $z \in \mathcal{D}$. We need to show that given $\epsilon > 0$, there is $\delta > 0$ such that if $h \in \mathbb{C} \setminus \{0\}$ with $|h| < \delta$, then

$$\left|\int_0^1 \big[f(z+th) - f(z)\big]\ dt\right| < \epsilon. \tag{28.26}$$

Since f is continuous at z, there is $\delta > 0$ such that if $w \in \mathcal{D}$ with $|w - z| < \delta$, then $|f(w) - f(z)| < \epsilon$. Now suppose $0 < |h| < \delta$. If $0 \le t \le 1$, then

$$|(z+th) - z| = |th| = |t||h| \le |h| < \delta,$$

and therefore

$$\max_{0 \le t \le 1} |f(z+th) - f(z)| < \epsilon.$$

Then the triangle inequality shows

$$\left|\int_0^1 \big[f(z+th) - f(z)\big]\ dt\right| \le \int_0^1 \big|f(z+th) - f(z)\big|\ dt < \int_0^1 \epsilon\ dt = \epsilon.$$ ■

28.3 Problem. Compare the proofs for definite integrals of Theorem 24.13 (FTC1) and Lemma 24.14 to the proofs for line integrals of Theorem 28.1 and Lemma 28.2. Identify explicitly where the proofs are identical and how, if at all, they are different.

We have now obtained a *sufficient* condition for a continuous function (whose domain is a. . .domain. . .) to have an antiderivative: the function should be path independent on that domain. We might ask if we could weaken or change this condition and still guarantee an antiderivative's existence.

We cannot.

In fact, earlier, in Problem 26.12 we saw a *necessary* condition for an antiderivative's existence: if a function has an antiderivative, then that function integrates to zero over closed paths. This condition turns out to be sufficient in that it implies path independence and thus the existence of an antiderivative. We collect these seemingly disparate results into one theorem.

28.4 Theorem. *Let $\mathcal{D} \subseteq \mathbb{C}$ be a domain and let $f\colon \mathcal{D} \to \mathbb{C}$ be continuous. The following are equivalent:*

(i) *f has an antiderivative on $\mathcal{D}$.*

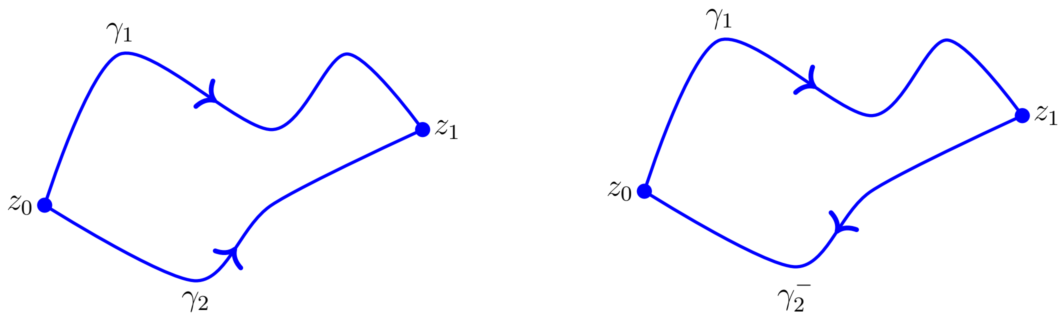
(ii) *If γ is a closed path in $\mathcal{D}$, then*

$$\int_\gamma f = 0.$$

(iii) *f is independent of path in $\mathcal{D}$.*

Proof. (i) $\Longrightarrow$ (ii) This is part (i) of Problem 26.12.

(ii) $\implies$ (iii) Suppose that γ_1 and γ_2 are paths in $\mathcal{D}$ with the same initial point z_0 and the same terminal point z_1, as in the sketch below.



Then the path $\gamma_1 \oplus \gamma_2^-$ is closed, so part (ii) implies

$$0 = \int_{\gamma_1 \oplus \gamma_2^-} f = \int_{\gamma_1} f - \int_{\gamma_2} f$$

and so

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

(iii) $\implies$ (i) This is Theorem 28.1. ∎

We now have new tools available in our quest for antiderivatives: we could check independence of path, or we could check that integrals over closed paths vanish. For a given function, both conditions are arguably somewhat difficult to check, as they require an *infinite* number of conditions to be met. The integral over *every* closed path must vanish—and any domain contains infinitely many closed paths (just consider all the circular ones). Or the integrals over *any* pair of paths with the same initial and terminal points must be the same—and we can probably find infinitely many paths connecting any two points in a domain.

A repeated theme in our course has been the starring roles of algebra ($i^2 = -1$), analysis (limits), and geometry ($\mathbb{C} = \mathbb{R}^2$). We will find a surprising resurgence of these roles in the antiderivative problem. For a function $f\colon \mathcal{D} \to \mathbb{C}$ to be guaranteed to have an antiderivative, it turns out that both the differentiability of f and a sufficiently nice geometric structure of $\mathcal{D}$ must be ensured.

Recall again Problem 26.12, which presented two continuous functions that did not integrate to 0 over certain closed paths in their domains. Consequently, those functions do not have antiderivatives on those domains. Here are some details.

- The function

$$f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}\colon z \mapsto \frac{1}{z}$$

is differentiable on $\mathbb{C} \setminus \{0\}$ but has no antiderivative there. The geometry of $\mathbb{C} \setminus \{0\}$ turns out to be problematic—even though $\mathbb{C} \setminus \{0\}$ is a domain, i.e., open and connected.

- The function

$$g\colon \mathbb{C} \to \mathbb{C}\colon z \mapsto \overline{z}$$

is continuous but nowhere holomorphic on $\mathbb{C}$ and has no antiderivative on $\mathbb{C}$. The set $\mathbb{C}$ is also a domain, like $\mathbb{C} \setminus \{0\}$, and turns out to have some nicer structural properties ($\mathbb{C}$ is star-shaped—see Definition 28.5; $\mathbb{C} \setminus \{0\}$ is not.)

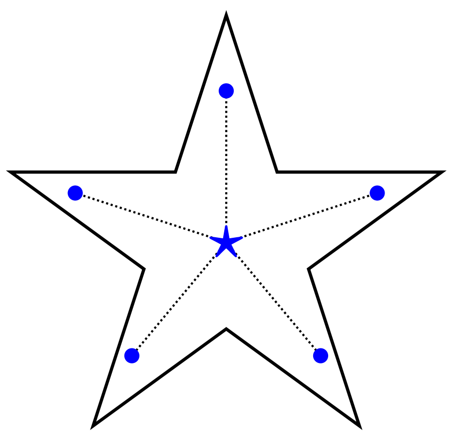
These examples reinforce the comment above that f must be differentiable (an analytic property) and the domain of f must be sufficiently nice (a geometric property). This continues our theme of successively demanding more refined geometric properties of the sets on which our functions are defined. First we worked on any $\mathcal{D} \subseteq \mathbb{C}$, which permitted $\mathcal{D} \subseteq \mathbb{R}$. To achieve the Cauchy–Riemann equations, we specialized to open $\mathcal{D}$, which excluded subsets of $\mathbb{R}$. To prove that differentiable functions f on $\mathcal{D}$ with $f' = 0$ were genuinely constant, not locally constant, we introduced the domain (i.e., the open and connected set—here "domain" is not just the set on which a function is defined!), and that served us quite well with independence of path.

Now we want a special kind of domain. In Theorem 28.1, we worked on a domain $\mathcal{D}$, fixed a point $z_\star \in \mathcal{D}$, and integrated over paths connecting $z_\star$ to other $z \in \mathcal{D}$. We will consider those domains $\mathcal{D}$ for which the path connecting $z_\star$ to z is always the line segment $[z_\star, z]$.

28.5 Definition. *A set $\mathcal{D} \subseteq \mathbb{C}$ is* **STAR-SHAPED** *if there is a point $z_\star \in \mathcal{D}$ such that $[z_\star, z] \subseteq \mathcal{D}$ for all $z \in \mathcal{D}$. The point $z_\star$ is called a* **STAR-CENTER** *for $\mathcal{D}$. A* **STAR-SHAPED DOMAIN** *or a* **STAR DOMAIN** *is a domain that is also star-shaped.*

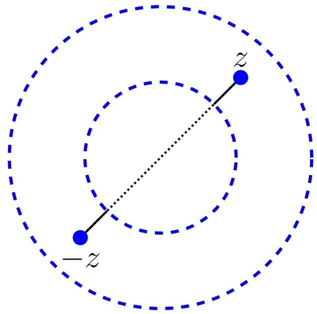
28.6 Example.

(i) We should (unsurprisingly!) expect that the set below is star-shaped, and its star-center should be the point indicated by the symbol $\star$.

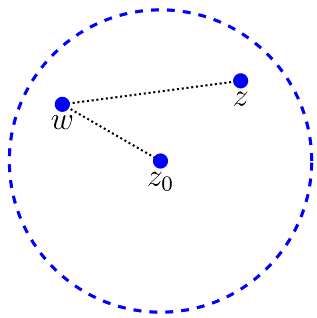


(ii) Let $0 \le r < R \le \infty$. Any **ANNULUS** of the form $\mathcal{A} := \{z \in \mathbb{C} \mid r < |z| < R\}$ is not star-shaped: if $z \in \mathcal{A}$, then $-z \in \mathcal{A}$. However, $0 \in [z, -z]$ and $0 \notin \mathcal{A}$, so $[z, -z] \not\subseteq \mathcal{A}$. That is, no matter what point z we try to pick for the star-center, we cannot connect z to $-z$

by a line segment that is wholly contained in $\mathcal{A}$.



(iii) For any $z_0 \in \mathbb{C}$ and $r > 0$, the open ball $\mathcal{B}(z_0; r)$ is star-shaped, and any point in $\mathcal{B}(z_0; r)$ is a star-center. Below we see that the line segments from both the center of the ball z_0 and an arbitrary point z in the ball can reach any other point w in the ball.



28.7 Problem. Fill in the following technical details from Example 28.6.

(i) For any $z \in \mathbb{C}$, show that $0 \in [z, -z]$.

(ii) Fix $z_0 \in \mathbb{C}$ and $r > 0$. Show that if z, $w \in \mathcal{B}(z_0; r)$, then $[z, w] \subseteq \mathcal{B}(z_0; r)$. [Hint: *show that* $z_0 - ((1-t)z + tw) = (1-t)(z_0 - z) + t(z_0 - w)$.]

28.8 Problem. Let $z =\in \mathbb{C} \setminus \{0\}$. Show that $0 \in [z, -z]$. Why does this tell you that no point in $\mathbb{C} \setminus \{0\}$ can be a star center for $\mathbb{C} \setminus \{0\}$, and therefore that $\mathbb{C} \setminus \{0\}$ is not star-shaped?

28.9 Problem.

(i) Prove that any star-shaped set is connected.

(ii) A set $\mathcal{D} \subseteq \mathbb{C}$ is **CONVEX** if $[z, w] \subseteq \mathbb{C}$ for any z, $w \in \mathbb{C}$. Prove that every convex set is connected and every star-shaped set is convex.

(iii) Is every connected set star-shaped? Is every convex set star-shaped?

Day 29: Friday, March 24

Recommended reading

We discussed differentiating under the (definite) integral and proved a "weak" version of Cauchy's integral theorem. If you're interested, the book presents a version of differentiating under the integral for line integrals in Theorem 3.8.5, although we will not use that. The book also presents a version of Cauchy's integral theorem proved using Green's theorem from vector calculus; see Section 3.4. This requires a version of the Jordan curve theorem (p. 177), which I wholeheartedly disdain. You are not responsible for the material in Section 3.4, but you can read Examples 3.4.7 and 3.4.8 with our version of the Cauchy integral theorem from class. Also, read Remark 3.4.6 and think about why $\mathbb{C} \setminus \{0\}$ is not a star domain.

We can now show that with the additional geometric structure of the star domain, we can give a simple condition under which a function has an antiderivative. Namely, we show that if f is holomorphic on the star domain $\mathcal{D}$ and f' is continuous on $\mathcal{D}$, then $\int_\gamma f = 0$ for all closed paths γ in $\mathcal{D}$. By Theorem 28.4, this implies that f has an antiderivative on $\mathcal{D}$.

We will employ an important auxiliary technique called "differentiating under the integral." Suppose that $I \subseteq \mathbb{R}$ is an interval and $f \colon I \times [a,b] \to \mathbb{R}$ is a function, where

$$I \times [a,b] = \{t + is \in \mathbb{C} \mid t \in I, \ a \le s \le b\}.$$

Denote by $f(t,\cdot)$ the map $f(t,\cdot) \colon [a,b] \to \mathbb{C} \colon s \mapsto f(t,s)$. If for each $t \in I$, the map $f(t,\cdot)$ is integrable on $[a,b]$, then we can define a new function via

$$\phi(t) := \int_a^b f(t,s)\, ds.$$

It is natural to ask if ϕ is differentiable, and since the integral has many properties in common with sums, and since *finite* sums and integrals can readily be interchanged, we might expect that

$$\phi'(t) = \frac{d}{dt}\int_a^b f(t,s)\, ds = \int_a^b \frac{\partial}{\partial t}[f(t,s)]\, ds,$$

at least if f is differentiable with respect to t and if the partial derivative $f_t(t,\cdot)$ is integrable.

Happily, this turns out to be the case, although the proof requires some nuance to make rigorous this interchange of derivative and integral. In particular, we need the tool of uniform continuity, as developed in real analysis or, more generally, metric space topology.

29.1 Lemma (Uniform continuity). *Suppose that $f \colon \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$ is continuous, where $\mathcal{D}$ is a set of the form*

$$\mathcal{D} = \{x + iy \mid x_0 \le x \le x_1, \ y_0 \le y \le y_1\} \quad \text{or} \quad \mathcal{D} = \overline{\mathcal{B}}(x_0 + iy_0; r_0). \tag{29.27}$$

Then for all $\epsilon > 0$, there exists $\delta > 0$ such that if w, $z \in \mathcal{D}$ with $|w - z| < \delta$, then

$$|f(w) - f(z)| < \epsilon.$$

We will not prove this lemma, but we contrast its "uniformity" with "ordinary continuity," which would say that for all $\epsilon > 0$ and $z \in \mathcal{D}$, there is $\delta > 0$ such that if $|w - z| < \delta$, then $|f(w) - f(z)| < \epsilon$. In "ordinary" continuity, the threshold δ can depend on both ϵ and z; in "uniform" continuity, the same δ works for the whole set $\mathcal{D}$. The key is that the two varieties of $\mathcal{D}$ in (29.27) are closed and bounded sets (and so Lemma 29.1 turns out to hold for much more general $\mathcal{D}$ than these varieties, though we will not need them).

29.2 Theorem (Leibniz's rule for differentiating under the integral). *Suppose that $I \subseteq \mathbb{R}$ is an interval and $a, b \in \mathbb{R}$ with $a \le b$. Let $f \colon I \times [a,b] \to \mathbb{C} \colon (t,s) \mapsto f(t,s)$ be a continuous function such that f_t exists and is continuous on $I \times [a,b]$. Then the map $\phi(t) := \int_a^b f(t,s)\ ds$ is defined and differentiable on I and*

$$\phi'(t) = \int_a^b f_t(t,s)\ ds.$$

Proof. Fix $t \in I$. We want to show that

$$\lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h} = \int_a^b f_t(t,s)\ ds,$$

equivalently,

$$\lim_{h \to 0} \frac{1}{h} \left(\phi(t+h) - \phi(t) - h \int_a^b f_t(t,s)\ ds \right) = 0.$$

That is, we want to show that for all $\epsilon > 0$, there is $\delta > 0$ such that if $|h| < \delta$, then

$$\left| \frac{1}{h} \left(\phi(t+h) - \phi(t) - h \int_a^b f_t(t,s)\ ds \right) \right| < \epsilon. \tag{29.28}$$

We compute

$$\begin{aligned} \phi(t+h) - \phi(t) - h \int_a^b f_t(t,s)\ ds &= \int_a^b f(t+h,s)\ ds - \int_a^b f(t,s)\ ds - h \int_a^b f_t(t,s)\ ds \\ &= \int_a^b \big[f(t+h,s) - f(t,s) - h f_t(t,s) \big]\ ds. \end{aligned} \tag{29.29}$$

It therefore suffices to show

$$\lim_{h \to 0} \int_a^b \frac{f(t+h,s) - f(t,s) - h f_t(t,s)}{h}\ ds = 0. \tag{29.30}$$

By definition of the partial derivative, we know that

$$\lim_{h \to 0} \frac{f(t+h,s) - f(t,s) - h f_t(t,s)}{h} = 0$$

for any fixed t and s. Our challenge is now to make this limit hold "uniformly" over all $s \in [0,1]$ so that we can "pass the limit through the integral" in (29.30).

Example 25.5 allows us to rewrite

$$f(t+h,s)-f(t,s)=h\int_0^1 f_t(t+h\tau,s)\ d\tau,$$

and so

$$\int_a^b \big[f(t+h,s)-f(t,s)-hf_t(t,s)\big]\ ds=\int_a^b\left[h\int_0^1 f_t(t+h\tau,s)\ d\tau-hf_t(t,s)\right]\ ds. \quad (29.31)$$

Now rewrite

$$f_t(t,s)=f_t(t,s)\int_0^1 1\ d\tau=\int_0^1 f_t(t,s)\ d\tau,$$

so that

$$\int_a^b\left[h\int_0^1 f_t(t+h\tau,s)\ d\tau-hf_t(t,s)\right]\ ds=h\int_a^b\int_0^1\big[f_t(t+h\tau,s)-f_t(t,s)\big]\ d\tau\ ds. \quad (29.32)$$

We combine (29.29), (29.31), and (29.32) to conclude that

$$\frac{1}{h}\left(\phi(t+h)-\phi(t)-h\int_a^b f_t(t,s)\ ds\right)=\int_a^b\int_0^1\big[f_t(t+h\tau,s)-f_t(t,s)\big]\ d\tau\ ds,$$

and so we estimate with *two* applications of the triangle inequality that

$$\begin{aligned}\left|\frac{1}{h}\left(\phi(t+h)-\phi(t)-h\int_a^b f_t(t,s)\ ds\right)\right| &\le (b-a)\max_{a\le s\le b}\left|\int_0^1\big[f_t(t+h\tau,s)-f_t(t,s)\big]\ d\tau\right| \\ &\le (b-a)\max_{a\le s\le b}\left(\max_{0\le\tau\le 1}\big|f_t(t+h\tau,s)-f_t(t,s)\big|\right).\end{aligned}$$

Now we will use uniform continuity. Since I is an interval and $t\in I$, there are t_0, $t_1\in I$ such that $t_0<t<t_1$ and $[t_0,t_1]\subseteq I$. Then f_t is continuous on a set $\mathcal{D}$ of the first form in (29.27), and so given $\epsilon>0$, there is $\delta>0$ such that both $[t-\delta,t+\delta]\subseteq I$ and, if $|\xi-t|<\delta$, then

$$\big|f_t(\xi,s)-f_t(t,s)\big|<\frac{\epsilon}{b-a}$$

for all $s\in[a,b]$. What is critical here is that we can make the difference above uniformly small over all $s\in[a,b]$.

Take $0<|h|<\delta$, so that $|(t+h\tau)-t|=|h||\tau<|h|<\delta$, since $0\le\tau\le 1$. This guarantees

$$\big|f_t(t+h\tau,s)-f_t(t,s)\big|<\frac{\epsilon}{b-a},$$

and thus

$$(b-a)\max_{a\le s\le b}\left(\max_{0\le\tau\le 1}\big|f_t(t+h\tau,s)-f_t(t,s)\big|\right)<\epsilon$$

when $0<|h|<\delta$. This proves the desired estimate (29.28). ■

29.3 Problem. Let

$$\phi(t) := \left[\int_0^1 s\cos(s^2+t)\ ds\right].$$

Calculate ϕ' in two ways in two ways: first by evaluating the integral with FTC2 and differentiating the result and second by differentiating under the integral and then simplifying the result with FTC2. (The point is to convince you that differentiating under the integral works.)

Now we are ready to prove our first pass[1] at the Cauchy integral theorem.

29.4 Theorem ("Weak" Cauchy integral theorem). *Let $\mathcal{D} \subseteq \mathbb{C}$ be a star domain and let $f\colon \mathcal{D} \to \mathbb{C}$ be holomorphic with f' continuous on $\mathcal{D}$. Then*

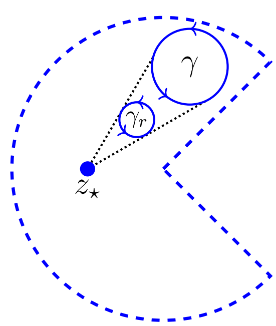
$$\int_\gamma f = 0$$

for any closed path γ in $\mathcal{D}$.

Proof. We prove this only in the case that $\gamma\colon [0,1] \subseteq \mathbb{R} \to \mathcal{D}$ is continuously differentiable. For simplicity, assume that γ is (re)parametrized over $[0,1]$. Let $z_\star$ be a star center for $\mathcal{D}$. Then $[z_\star, \gamma(t)] \subseteq \mathcal{D}$ for each $t \in [0,1]$, so $(1-r)z_\star + r\gamma(t) \in \mathcal{D}$ for each $r \in [0,1]$ and $t \in [0,1]$. Define

$$\gamma_r\colon [0,1] \subseteq \mathbb{R} \to \mathcal{D}\colon t \mapsto (1-r)z_\star + r\gamma(t).$$

Then γ_r is a continuously differentiable path in $\mathcal{D}$ with $\gamma_1 = \gamma$ and $\gamma_0(t) = z_\star$. Here is a sketch.



We integrate f over γ_r and define

$$\mathcal{I}\colon [0,1] \subseteq \mathbb{R} \to \mathbb{C}\colon r \mapsto \int_{\gamma_r} f = \int_0^1 f((1-r)z_\star + r\gamma(t))r\gamma'(t)\ dt.$$

Note that

$$\mathcal{I}(0) = 0 \quad \text{and} \quad \mathcal{I}(1) = \int_\gamma f.$$

[1] I learned this proof from my advisor, Doug Wright, at Drexel University. I'm calling it a "weak" version not because the math is wimpy but because of the hypothesis that f' is continuous—a result is "stronger" if we can prove it with fewer hypotheses. Shortly we will drop this hypothesis, at the expense of much more work.

We will show that $\mathcal{I}$ is constant on $[0,1]$ by computing $\mathcal{I}'$ via differentiation under the integral; we will obtain $\mathcal{I}'(r) = 0$ for each r, and thus $\mathcal{I}(1) = \mathcal{I}(0) = 0$.

The integrand here is

$$g(r,t) := f((1-r)z_\star + r\gamma(t))r\gamma'(t) = f(z_\star + (\gamma(t) - z_\star)r)r\gamma'(t).$$

Since γ is continuously differentiable on $[0,1]$ and since f is holomorphic on $\mathcal{D}$ with f' continuous, it follows that g is continuous on $J := [0,1] \times [0,1]$, that g is differentiable with respect to r on J, and that g_r is continuous on J. In particular, the product rule gives

$$g_r(r,t) = f'(z_\star + (\gamma(t) - z_\star)r)(\gamma(t) - z_\star)r\gamma't(t) + f(z_\star + (\gamma(t) - z_\star)r)\gamma'(t).$$

Then

$$\mathcal{I}'(r) = \int_0^1 g_r(r,t)\,dt = \int_0^1 f'(z_\star + (\gamma(t) - z_\star)r)(\gamma(t) - z_\star)r\gamma't(t)\,dt + \int_0^1 f(z_\star + (\gamma(t) - z_\star)r)\gamma'(t)\,dt. \tag{29.33}$$

We evaluate

$$\int_0^1 f'(z_\star + (\gamma(t) - z_\star)r)(\gamma(t) - z_\star)r\gamma't(t)\,dt$$

using integration by parts. Take

$$\begin{aligned} u &= \gamma(t) - z_\star & dv &= f'(z_\star + (\gamma(t) - z_\star)r)r\gamma'(t)\,dt \\ du &= \gamma'(t)\,dt & v &= f'(z_\star + (\gamma(t) - z_\star)r). \end{aligned}$$

Then

$$\int_0^1 f'(z_\star + (\gamma(t) - z_\star)r)(\gamma(t) - z_\star)r\gamma't(t)\,dt = \big(\gamma(t) - z_\star\big)\big(f((1-r)z_\star + r\gamma(t)\big)\Big|_{t=0}^{t=1} - \int_0^1 f'(z_\star + (\gamma(t) - z_\star)r)\,dt. \tag{29.34}$$

Since γ is closed, we have $\gamma(0) = \gamma(1)$, and so it follows that

$$\big(\gamma(t) - z_\star\big)\big(f((1-r)z_\star + r\gamma(t)\big)\Big|_{t=0}^{t=1} = 0. \tag{29.35}$$

Combining (29.33), (29.34), and (29.35) yields $\mathcal{I}'(r) = 0$ for all $r \in [0,1]$. ■

29.5 Problem. Check that (29.35) is true. Remember that $\gamma(0) = \gamma(1)$.

29.6 Problem. Adapt the proof of the "weak" Cauchy integral theorem to the case where γ is only piecewise continuously differentiable. Proceed as follows. First, write $\gamma = \oplus_{k=1}^n \gamma_k$, where each γ_k is continuously differentiable on $[0,1]$ with $\gamma_{k-1}(1) = \gamma_k(0)$ for $k = 1, \ldots, n$. Then put $\gamma_{k,r}(t) := (1-r)z_\star + r\gamma_k(t)$ and $\gamma_r := \oplus_{k=1}^n \gamma_{k,r}$. Set $\mathcal{I}_k(r) := \int_{\gamma_{k,r}} f$, so $\mathcal{I}(r) =$

$\sum_{k=1}^{n} \mathcal{I}_k(r)$. Differentiate under each integral and obtain

$$\mathcal{I}_k'(r) = \big(\gamma_k(t) - z_\star\big)\big(f((1-r)z_\star + r\gamma_k(t)\big)\Big|_{t=0}^{t=1}.$$

Use this to recognize $\sum_{k=1}^{n} \mathcal{I}_k(r)$ as a telescoping sum.

29.7 Example. It is notoriously difficult (impossible) in calculus to find a formula in terms of "elementary functions" for an antiderivative of $f(x) = e^{x^2}$; we know that one exists on $\mathbb{R}$ because f is continuous, and so we can use the fundamental theorem of calculus. When we extend f to $\mathbb{C}$, we observe that $f(z) = e^{z^2}$ is holomorphic with $f'(z) = 2ze^{z^2}$; this is continuous, and so f has an antiderivative on $\mathbb{C}$.

In fact, we could take this antiderivative to be

$$F(z) := \int_{[0,z]} e^{w^2} \, dw,$$

if we repeat the proof of Theorem 28.1 for this particular integrand. We will do just that shortly under different hypotheses and in a broader context.

Day 30: Monday, March 27

Recommended reading

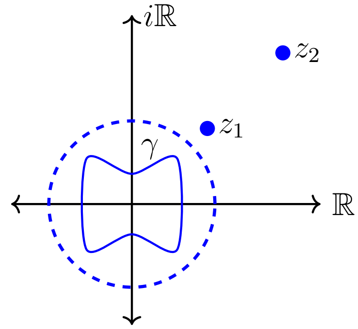
We proved a specialized "deformation" lemma as a consequence of the Cauchy integral theorem. The notion of "deformation of curves" can be made both much more precise and much more general. See Section 3.6; in particular, the three cases on p. 187 and Definition 3.6.1 offer this precision and generality. The integral invariance that we proved is really a special case of Theorem 3.6.5. You are not required to read any of this, but know that there's a big world beyond what we discussed.

30.1 Example. Let z_1, $z_2 \in \mathbb{C}$ with $0 \le |z_1| < |z_2|$. Fix $0 < R < |z_1|$ and let γ be any closed curve in $\mathcal{B}(0;R)$. Then

$$\int_\gamma \frac{dz}{(z-z_1)(z-z_2)^2} = 0,$$

since the function $f(z) := 1/[(z-z_1)(z-z_2)^2]$ is holomorphic on the star domain $\mathcal{B}(0;R)$

with f' continuous there.



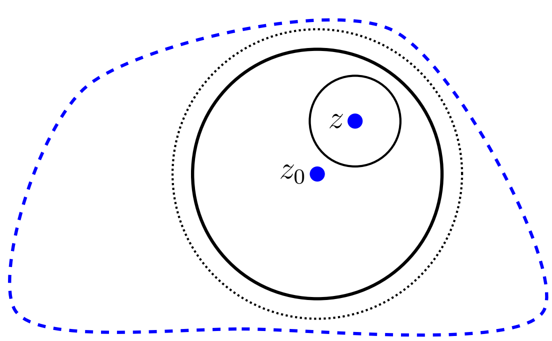
A leitmotif of complex integration theory turns out to be deformation of curves. It may be possible to "deform" one curve onto another in a "continuous" way; if the underlying domain is suitably nice (possibly, but not necessarily, a star domain) and if the integrand is suitably nice (holomorphic), then a line integral of a function over one curve should equal a line integral of that function over the other curve. We saw this in our proof of the weak Cauchy integral theorem; the curve γ was deformed onto the "constant" curve $z_\star$, or, really, the line segment $[z_\star, z_\star]$, and the integral over this line segment was 0.

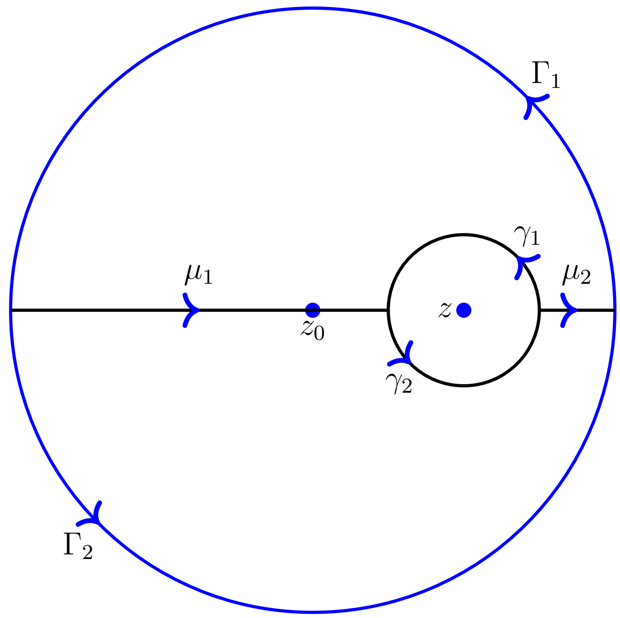
It is possible to make this notion of deformation very precise and to prove a version of the Cauchy integral theorem stating that the line integral of a holomorphic function is invariant under deformation of curves if the domain is geometrically suitable. We will not explore this and will instead be content with one very specific kind of deformation involving circles.

30.2 Lemma (Death Star). *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and suppose that z, $z_0 \in \mathcal{D}$ and R, r, $s > 0$ with $\overline{\mathcal{B}}(z;s) \subseteq \overline{\mathcal{B}}(z_0;r) \subseteq \mathcal{B}(z_0;R) \subseteq \mathcal{D}$. (See the sketch below.) Suppose that $f \colon \mathcal{D} \setminus \{z\} \to \mathbb{C}$ is holomorphic. Then*

$$\int_{|w-z_0|=r} f = \int_{|w-z|=s} f.$$

Proof. First we sketch the set-up of the subset containment $\overline{\mathcal{B}}(z;s) \subseteq \overline{\mathcal{B}}(z_0;r) \subseteq \mathcal{B}(z_0;R) \subseteq \mathcal{D}$. The dotted circle is the circle of radius R centered at z_0; the thick solid circle is the circle of radius r centered at z_0; and the thin solid circle is the circle of radius s centered at z.

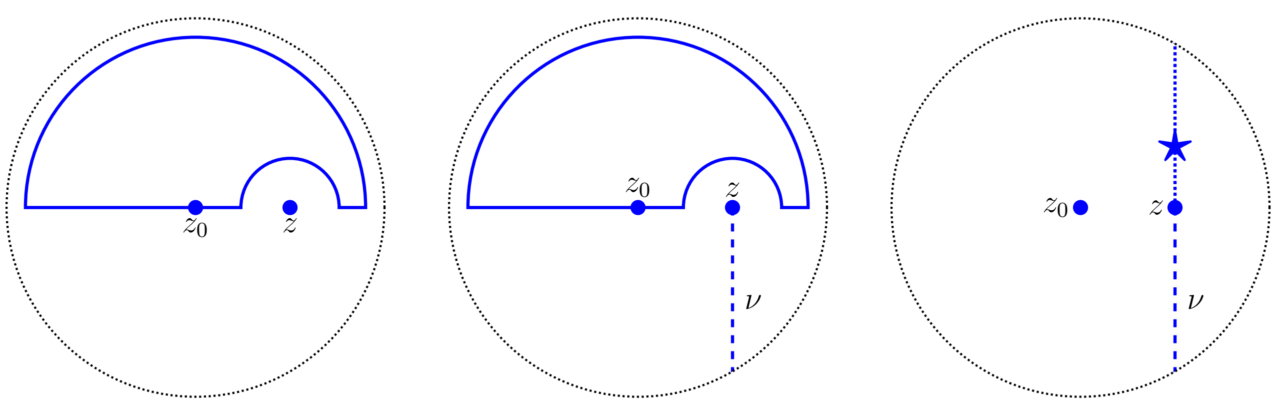




We split the circle of radius r centered at z_0 and the circle of radius s centered at z into a number of auxiliary paths. The paths Γ_1 and γ_1 are the upper halves of their respective circles, and Γ_2 and γ_2 are the lower halves. The paths μ_1 and μ_2 are line segments. In particular, we have

$$\int_{|w-z_0|=r} f = \int_{\Gamma_1 \oplus \Gamma_2} f \quad \text{and} \quad \int_{|w-z|=s} f = \int_{\gamma_1 \oplus \gamma_2} f. \tag{30.36}$$

Consider the path $\mu_1 \oplus \gamma_1^- \oplus \gamma_2 \oplus \Gamma_1$, which we draw in solid blue in the first circle below. This is a closed path contained in $\mathcal{B}(z_0; R)$; we draw the circle of radius R centered at z_0 in dotted black below. Delete from $\mathcal{B}(z_0; R)$ the line segment from z to the circle of radius R centered at z_0 and call the resulting set $\mathcal{V}$; this is the second circle below. Then $\mu_1 \oplus \gamma_1^- \oplus \gamma_2 \oplus \Gamma_1$ is still a path in $\mathcal{V}$. Also, f is holomorphic on $\mathcal{V}$ since $z \notin \mathcal{V}$. Finally, $\mathcal{V}$ is a star domain; this is somewhat technical to prove precisely, but any point $\star$ on the dotted blue line in the third circle below will be a star center for $\mathcal{V}$.



The Cauchy integral theorem[2] then implies that

$$\int_{\mu_1 \oplus \gamma_1^- \oplus \mu_2 \oplus \Gamma_1} f = 0. \tag{30.37}$$

Exactly the same arguments show that

$$\int_{\mu_1 \oplus \gamma_2 \oplus \mu_2 \oplus \Gamma_2^-} f = 0. \tag{30.38}$$

Equating (30.37) and (30.38) and using the algebra and arithmetic of line integrals shows

$$\int_{\Gamma_1 \oplus \Gamma_2} f = \int_{\gamma_1 \oplus \gamma_2} f, \tag{30.39}$$

and, by (30.36), this is the desired conclusion. ■

30.3 Problem. Carry out the algebra and arithmetic of line integrals to prove (30.39), assuming that (30.37) and (30.38) hold.

30.4 Problem. Use the Death Star lemma and Example 26.4 to show that

$$\int_{|w-z_0|=r} \frac{dw}{w-z} = 2\pi i$$

for all $r > 0$ and z, $z_0 \in \mathbb{C}$ such that $|z - z_0| < r$. Be sure to check that all the (very technical) hypotheses of the Death Star lemma are met.

30.5 Remark. Don't be too proud of this technological terror we've constructed in Lemma 30.2. The ability to deform one circle onto another and preserve the line integral is insignificant next to the power of the Cauchy theorems.

[2] Specifically, Theorem 31.4, since we are not assuming that f' is continuous here. This version of the Cauchy integral theorem does not depend on the Death Star lemma; we are proving this lemma first due to my travel plans.

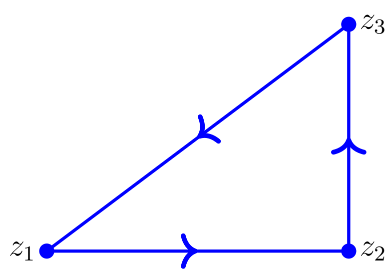
Day 31: Wednesday, March 29

Recommended reading

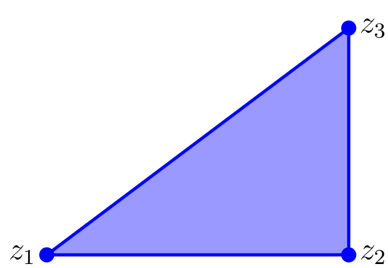
The book divides the proof of the Cauchy–Goursat theorem into two stages. The first is Lemma 3.5.1; this is, broadly, the "analysis" stage, which estimates an integral using the definition of the derivative and the calculation of a polynomial antiderivative. The second is Theorem 3.5.2; this is the "geometry" stage, which estimates an integral by rewriting it as a sum of integrals over related paths. The Cauchy–Goursat theorem illustrates the power of the triangle: if we can get a challenging, technical result only for line integrals over triangles (a fairly benign sort of path), then we can get a powerful result for many other kinds of curves. You are not required to know either Lemma 3.5.1 or Theorem 3.5.2, but you should at least glance at their proofs. The book's proof of Theorem 3.5.4 closely resembles our version.

We can improve our initial version of the Cauchy integral theorem in a subtle, but important, way: the derivative does not have to be continuous. To accomplish this, we need more powerful tools, and so we will call upon the fearsome power of the triangle. This completes our return to kindergarten geometry begun with lines and circles.

What is a triangle? Let z_1, z_2, $z_3 \in \mathbb{C}$. Surely the path below is a triangle.

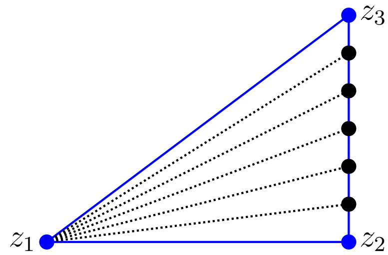


We recognize this path as the composition of three line segments in a particular order, namely $[z_1, z_2] \oplus [z_2, z_3] \oplus [z_3, z_1]$. However, we might also argue that the two-dimensional region below is a triangle as well.



Both "triangular paths" and "triangular regions" will be very useful to us, and so we should give precise definitions of them both, and use notation that distinguishes them. While we recognized the triangular path above as a composition of line segments, how might we tractably describe the triangular *region* above in terms of z_1, z_2, z_2?

One useful approach is to recognize the region as a *union* of line segments—specifically, all line segments whose initial point is z_1 and whose terminal point lies on the line segment $[z_2, z_3]$.



Based on this reasoning, we make the following definition.

31.1 Definition. *Let $z_1,\ z_2,\ z_3 \in \mathbb{C}$.*

(i) *The* **TRIANGLE** *spanned by z_1, z_2, and z_3 is the set*

$$\Delta(z_1, z_2, z_3) := \bigcup_{0 \le s \le 1} [z_1, (1-s)z_2 + sz_3] = \left\{(1-t)z_1 + t\big((1-s)z_2 + sz_3\big)\ \middle|\ 0 \le s,\ t \le 1\right\}. \tag{31.40}$$

(ii) *The* **TRIANGULAR PATH** *spanned by z_1, z_2, and z_3 is the closed path*

$$\partial\Delta(z_1, z_2, z_3) := [z_1, z_2] \oplus [z_2, z_3] \oplus [z_3, z_1]. \tag{31.41}$$

31.2 Problem. Let $z_1,\ z_2,\ z_3 \in \mathbb{C}$.

(i) Prove that the order in which we specify the endpoints of a triangle is irrelevant in the sense that

$$\Delta(z_1, z_2, z_3) = \Delta(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$$

for any function $\sigma\colon \{1, 2, 3\} \to \{1, 2, 3\}$ that is one-to-one and onto (i.e., any permutation). Explain why the order of the points matters very much when we are working with a triangular *path*.

(ii) Suppose that two or more of the points $z_1,\ z_2,\ z_3$ are equal, or that all three points belong to some line segment $[z, w]$. Prove that $\Delta(z_1, z_2, z_3)$ is really a line segment. (Remarkably, this "degenerate" case will not require any special treatment in our subsequent use of triangles!)

The key to a version of the Cauchy integral theorem that drops the hypothesis of continuity on f' is that f should integrate to 0 over triangles. This turns out to be true.

31.3 Theorem (Cauchy–Goursat theorem). *Suppose that f is holomorphic on an open set $\mathcal{D}$ (which need not be star-shaped or even a domain). Let $z_1,\ z_2,\ z_3 \in \mathcal{D}$ such that*

$\Delta(z_1, z_2, z_3) \subseteq \mathcal{D}$. *Then*

$$\int_{\partial\Delta(z_1,z_2,z_3)} f = 0.$$

We will not prove this theorem here; its proof is a wonderful union of analysis (careful estimates using the definition of the derivative and the triangle inequality for integrals) and geometry (breaking a given triangle into an infinite sequence of nested triangles) and more analysis (estimating integrals over those nested triangles and finding a subsequence of triangles whose intersection is nonempty).

At last we are ready to prove that a holomorphic function integrates to 0 around closed paths *without* assuming that the derivative is continuous and *without* assuming that the path is a triangle.

31.4 Theorem (Cauchy integral theorem). *Let $\mathcal{D} \subseteq \mathbb{C}$ be a star domain and let $f \colon \mathcal{D} \to \mathbb{C}$ be holomorphic on $\mathcal{D}$. Then*

$$\int_{\gamma} f = 0$$

for any closed path γ in $\mathcal{D}$.

Proof. Let $z_\star$ be a star center for $\mathcal{D}$. We show that

$$F(z) := \int_{[z_\star, z]} f$$

is an antiderivative of f on $\mathcal{D}$. The proof is very similar to that of Theorem 28.1, except we have replaced the general path connecting $z_\star$ and z with the line segment $[z_\star, z]$.

Fix $z \in \mathcal{D}$. As always, we want to show that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

equivalently,

$$\lim_{h \to 0} \frac{F(z+h) - F(z) - hf(z)}{h} = 0.$$

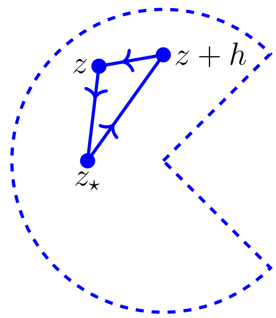
By Problem 31.5 below, there is $r > 0$ such that if $h \in \mathbb{C}$ with $|h| < r$, then $\Delta(z_\star, z, z+h) \subseteq \mathcal{D}$. Assume that $h \in \mathbb{C}$ satisfies $|h| < r$ from now on.

We calculate

$$F(z+h) - F(z) = \int_{[z_\star, z+h]} f - \int_{[z_\star, z]} f = \int_{[z_\star, z+h]} f + \int_{[z, z_\star]} f.$$

If we add and subtract the integral of f over $[z+h, z]$, then we will have integrated f over the triangle $\partial\Delta(z_\star, z+h, z)$, and this integral is 0 by the Cauchy–Goursat theorem. That is,

$$\int_{\partial\Delta(z_\star, z+h, z)} f = 0.$$



So, we do just that:

$$\begin{aligned} F(z+h) - F(z) &= \int_{[z_\star, z+h]} f + \int_{[z+h, z]} f + \int_{[z, z_\star]} f - \int_{[z+h, z]} f \\ &= \int_{[z_\star, z+h] \oplus [z+h, z] \oplus [z, z_\star]} f + \int_{[z, z+h]} f \\ &= \int_{\partial \Delta(z_\star, z+h, z)} f + \int_{[z, z+h]} f \\ &= \int_{[z, z+h]} f. \end{aligned}$$

Then

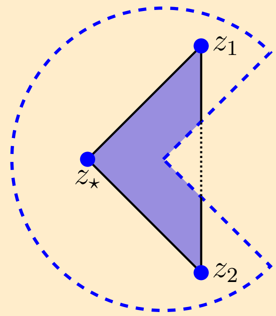
$$F(z+h) - F(z) - hf(z) = \int_{[z, z+h]} f - hf(z) = h \int_0^1 \big[f(z+th) - f(z)\big]\, dt,$$

as we previously calculated in (28.24). Lemma 28.2 then implies

$$\lim_{h \to 0} \frac{F(z+h) - F(z) - hf(z)}{h} = \lim_{h \to 0} \int_0^1 \big[f(z+th) - f(z)\big]\, dt = 0,$$

as desired. ■

31.5 Problem. Let $\mathcal{D} \subseteq \mathbb{C}$ be a star domain with star center $z_\star$ and let $z \in \mathcal{D}$. Since $\mathcal{D}$ is open, there is $r > 0$ such that $\mathcal{B}(z; r) \subseteq \mathcal{D}$. Prove that if $h \in \mathbb{C}$ with $|h| < r$, then $\Delta(z_\star, z, z+h) \subseteq \mathcal{D}$. [Hint: *use the definition of a triangle as a union of line segments, the definition of an open ball, and a lot of estimates.*] Note that for *arbitrary* z_1, $z_2 \in \mathcal{D}$, the triangle $\Delta(z_\star, z_1, z_2)$ need not be wholly contained in $\mathcal{D}$.



In the context of our quest for antiderivatives, the Cauchy integral theorem was a welcome result. In lieu of checking independence of path, it gave us a simple sufficient condition for the existence of an antiderivative: differentiability itself. That is, for a function defined on a star domain to have an antiderivative on that star domain, it suffices for the function to be differentiable. However, one might rightly quibble that the star domain is a very special geometric form. Are there more "relaxed" geometries that guarantee the existence of antiderivatives for suitably nice functions?

We answer our question with a (somewhat circular) definition.

31.6 Definition. *A domain (i.e., open and connected) $\mathcal{D} \subseteq \mathbb{C}$ is an* **ELEMENTARY DOMAIN** *if every holomorphic function on $\mathcal{D}$ has an antiderivative on $\mathcal{D}$.*

31.7 Problem. Is $\mathbb{C} \setminus \{0\}$ an elementary domain?

Certainly star domains are elementary domains, thanks to Cauchy's integral theorem, but are there others? It turns out that we can easily build new elementary domains out of given ones, and so in particular we can build elementary domains out of star domains. To do this, we need to be able to "glue" certain holomorphic functions together to produce a new holomorphic function that agrees, under certain restrictions, with the old ones.

31.8 Lemma (Merging). *Let $\mathcal{D}_1$, $\mathcal{D}_2 \subseteq \mathbb{C}$ be open with $\mathcal{D}_1 \cap \mathcal{D}_2$ open and nonempty. Let $f_1 \colon \mathcal{D}_1 \to \mathbb{C}$ and $f_2 \colon \mathcal{D}_2 \to \mathbb{C}$ be holomorphic, and suppose $f_1(z) = f_2(z)$ for all $z \in \mathcal{D}_1 \cap \mathcal{D}_2$. Then there is a unique holomorphic function $f \colon \mathcal{D}_1 \cup \mathcal{D}_2 \to \mathbb{C}$ such that $f\big|_{\mathcal{D}_1} = f_1$ and $f\big|_{\mathcal{D}_2} = \mathcal{D}_2$. In particular,*

$$f'(z) = \begin{cases} f_1'(z), & z \in \mathcal{D}_1 \\ f_2'(z), & z \in \mathcal{D}_2. \end{cases}$$

Proof. Define

$$f \colon \mathcal{D}_1 \cup \mathcal{D}_2 \to \mathbb{C} \colon z \mapsto \begin{cases} f_1(z), & z \in \mathcal{D}_1 \setminus \mathcal{D}_2 \\ f_1(z), & z \in \mathcal{D}_1 \cap \mathcal{D}_2 \\ f_2(z), & z \in \mathcal{D}_2 \setminus \mathcal{D}_1. \end{cases}$$

Since the three sets $\mathcal{D}_1 \setminus \mathcal{D}_2$, $\mathcal{D}_1 \cap \mathcal{D}_2$, and $\mathcal{D}_2 \setminus \mathcal{D}_1$ are disjoint, the function f is well-defined. We now show that f is holomorphic. Consider the following three cases on $z \in \mathcal{D}_1 \cup \mathcal{D}_2$.

1. $z \in \mathcal{D}_1 \setminus \mathcal{D}_2$. Since $\mathcal{D}_1$ is open, there is $r > 0$ such that $\mathcal{B}(z;r) \subseteq \mathcal{D}_1$. We claim that $f\big|_{\mathcal{B}(z;r)} = f_1\big|_{\mathcal{B}(z;r)}$. In that case, for $w \in \mathcal{D}_1 \cup \mathcal{D}_2$ sufficiently close to z, we have $w \in \mathcal{B}(z;r)$, and then

$$\frac{f(w) - f(z)}{w - z} = \frac{f_1(w) - f_1(z)}{w - z},$$

from which we have

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} = f_1'(z).$$

Now we prove the claim. Take $w \in \mathcal{B}(z;r)$, so $w \in \mathcal{D}_1$. If $w \in \mathcal{D}_1 \setminus \mathcal{D}_2$, then $f(w) = f_1(w)$ by definition. And if $w \in \mathcal{D}_1 \cap \mathcal{D}_2$, then, again, $f(w) = f_1(w)$ by definition.

2. $z \in \mathcal{D}_1 \cap \mathcal{D}_2$. Again, take $r > 0$ such that $B(z;r) \subseteq \mathcal{D}_1$. If $w \in \mathcal{B}(z;r)$, then $f(w) = f_1(w)$ by the argument in the preceding paragraph, from which it follows as before that f is differentiable at z with $f'(z) = f_1'(z)$.

3. $z \in \mathcal{D}_2 \setminus \mathcal{D}_1$. Take $r > 0$ such that $\mathcal{B}(z;r) \subseteq \mathcal{D}_2$ and let $w \in \mathcal{B}(z;r)$. If $w \in \mathcal{D}_2 \setminus \mathcal{D}_1$, then $f(w) = f_2(w)$ by definition, and it $w \in \mathcal{D}_1 \cap \mathcal{D}_2$, then $f(w) = f_1(w)$ by definition, but then $f(w) = f_2(w)$ by hypothesis. Then $f\big|_{\mathcal{B}(z;r)} = f_2\big|_{\mathcal{B}(z;r)}$, and so f is differentiable at z with $f'(z) = f_2'(z)$.

Last, for uniqueness, if $g\colon \mathcal{D}_1 \cup \mathcal{D}_2 \to \mathbb{C}$ satisfies $g\big|_{\mathcal{D}_1} = f_1$ and $g\big|_{\mathcal{D}_2} = f_2$, then $g(z) = f_1(z)$ for all $z \in \mathcal{D}_1 \setminus \mathcal{D}_2$, $g(z) = f_1(z)$ for all $z \in \mathcal{D}_1 \cap \mathcal{D}_2$, and $g(z) = f_2(z)$ for all $z \in \mathcal{D}_2 \setminus \mathcal{D}_1$, thus $g = f$, and so the extension f is unique. ■

31.9 Theorem. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be elementary domains such that their intersection*

$$\mathcal{D}_1 \cap \mathcal{D}_2 := \{z \in \mathbb{C} \mid z \in \mathcal{D}_1 \text{ and } z \in \mathcal{D}_2\}$$

is nonempty and connected. Then their union

$$\mathcal{D}_1 \cup \mathcal{D}_2 := \{z \in \mathbb{C} \mid z \in \mathcal{D}_1 \text{ or } z \in \mathcal{D}_2\}$$

is also an elementary domain.

Proof. Let $f\colon \mathcal{D}_1 \cup \mathcal{D}_2 \to \mathbb{C}$ be holomorphic; we want to show that f has an antiderivative on all of $\mathcal{D}_1 \cup \mathcal{D}_2$. The restrictions $f\big|_{\mathcal{D}_1}$ and $f\big|_{\mathcal{D}_2}$ are also holomorphic; since $\mathcal{D}_1$ and $\mathcal{D}_2$ are elementary domains, there are holomorphic maps $F_1\colon \mathcal{D}_1 \to \mathbb{C}$ and $F_2\colon \mathcal{D}_2 \to \mathbb{C}$ such that $F_1'(z) = f(z)$ for $z \in \mathcal{D}_1$ and $F_2'(z) = f(z)$ for $z \in \mathcal{D}_2$.

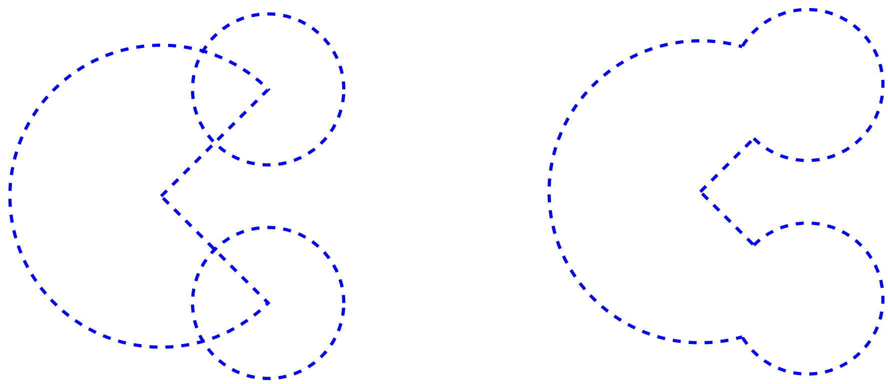
Now define

$$g\colon \mathcal{D}_1 \cap \mathcal{D}_2 \to \mathbb{C}\colon z \mapsto F_1(z) - F_2(z).$$

Then $g'(z) = 0$ for all $z \in \mathcal{D}_1 \cap \mathcal{D}_2$. Since $\mathcal{D}_1 \cap \mathcal{D}_2$ is a domain, Theorem 23.18 implies that g is constant on $\mathcal{D}_1 \cap \mathcal{D}_2$; take $C \in \mathbb{C}$ such that $g(z) = C$ for all $z \in \mathcal{D}_1 \cap \mathcal{D}_2$. Thus $F_1(z) = F_2(z) + C$ for all $z \in \mathcal{D}_1 \cap \mathcal{D}_2$.

The functions F_1 on $\mathcal{D}_1$ and $F_2 + C$ on $\mathcal{D}_2$ therefore satisfy the hypotheses of the merging lemma, and so there is a (unique) holomorphic function $F \colon \mathcal{D}_1 \cup \mathcal{D}_2 \to \mathbb{C}$ such that $F'\big|_{\mathcal{D}_1} = F_1' = f_1$ and $F'\big|_{\mathcal{D}_2} = F_2' = f_2$. Thus $F' = f$ on $\mathcal{D}_1 \cup \mathcal{D}_2$, so F is an antiderivative of f. ■

31.10 Example. Since open balls are star domains, we can "glue" overlapping balls onto an existing star domain and get an elementary domain that is (probably) not a star domain.



Day 32: Friday, March 31

You took an exam. Good times!

Day 33: Monday, April 3

Recommended reading

We proved the Cauchy integral formula. Our proof closely resembled that of Theorem 3.8.1 in the book with two major exceptions. First, we only proved the Cauchy integral formula over circles, not arbitrary curves; we had no need of "positive orientation" or "interior." Second, we did not use Theorem 3.7.2 but rather the Death Star lemma. You should read Example 3.8.2. Next, try rewriting the integral in Example 3.8.3 using partial fractions and then use the Cauchy integral formula; don't apply Theorem 3.7.2 as in the book. Finally, we started to discuss the generalized Cauchy integral formula, which appears in Theorem 3.8.6.

We will now prove one of the most important results in complex analysis, a formula that relates the values of a function in the interior of a ball to its values on the (circular) boundary of that ball. The full utility of this result will probably not be apparent right now, but it will serve us for the rest of the course.

33.1 Theorem (Cauchy integral formula). *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f \colon \mathcal{D} \to \mathbb{C}$ be holomorphic. Let $z_0 \in \mathcal{D}$ and $R > 0$ such that $\overline{\mathcal{B}}(z_0; R) \subseteq \mathcal{D}$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z}\, dw \tag{33.42}$$

for $0 < r < R$ *and all* $z \in \mathcal{B}(z_0; r)$.

Proof. The key to the proof, and indeed the motivation for the identity (33.42), is Problem 30.4, which told us

$$\int_{|w-z_0|=r} \frac{dw}{w-z} = 2\pi i.$$

Then we have

$$f(z) = \frac{1}{2\pi i}\big(f(z)2\pi i\big) = \frac{1}{2\pi i} f(z) \int_{|w-z_0|=r} \frac{dw}{w-z} = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(z)}{w-z}\, dw.$$

Then (33.42) is equivalent to

$$\frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)-f(z)}{w-z}\, dw = 0. \tag{33.43}$$

We will achieve (33.43) by showing that

$$\left| \int_{|w-z_0|=r} \frac{f(w)-f(z)}{w-z}\, dw \right| < \epsilon \tag{33.44}$$

for all $\epsilon > 0$. Indeed, if (33.44) is true and (33.43) is false, then we have

$$\epsilon_* := \left| \int_{|w-z_0|=r} \frac{f(w)-f(z)}{w-z}\, dw \right| > 0,$$

in which case (33.44) is false with $\epsilon = \epsilon_*/2$.

Here is how we prove (33.44). We have seen a function like the integrand before, in the difference quotient lemma (Lemma 19.1). In particular, this integrand is holomorphic on $\mathcal{D} \setminus \{z\}$, and so we may appeal to the Death Star lemma. Specifically, since $\mathcal{B}(z_0; r)$ is open, we may take $s_0 > 0$ such that $\mathcal{B}(z; s_0) \subseteq \mathcal{B}(z_0; r)$. Then if $0 < s < s_0/2$, we have $\overline{\mathcal{B}}(z; s) \subseteq \overline{\mathcal{B}}(z_0; r) \subseteq \mathcal{B}(z_0; R)$. The Death Star lemma therefore implies

$$\int_{|w-z_0|=r} \frac{f(w)-f(z)}{w-z}\, dw = \int_{|w-z|=s} \frac{f(w)-f(z)}{w-z}\, dw. \tag{33.45}$$

We will estimate the integral on the right with the ML-inequality. Given $\epsilon > 0$, the continuity of f at z allows us to choose $s > 0$ so that if $|w-z| < s$, then $|f(w)-f(z)| < \epsilon/2\pi$. Then

$$\max_{|w-s|=s} \left| \frac{f(w)-f(z)}{w-z} \right| \le \frac{\epsilon}{2\pi s},$$

and so the ML-inequality implies

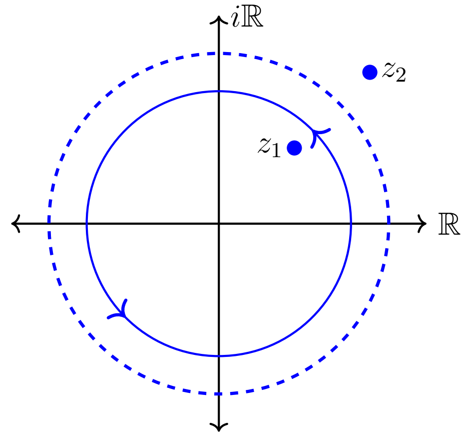
$$\left| \int_{|w-z|=s} \frac{f(w)-f(z)}{w-z}\, dw \right| \le 2\pi s \left(\frac{\epsilon}{2\pi s} \right) = \epsilon.$$

By (33.45), this proves the desired inequality (33.44) for any $\epsilon > 0$. ■

33.2 Example. Let z_1, $z_2 \in \mathbb{C}$ with $0 \le |z_1| < |z_2|$. Fix $|z_1| < r < \rho < |z_2|$. Then

$$\int_{|z|=r} \frac{dz}{(z-z_1)(z-z_2)^2} = \int_{|z|=r} \frac{1/(z-z_2)^2}{z-z_1}\,dz = \frac{2\pi i}{(z_1-z_2)^2},$$

since the function $f(z) := 1/(z-z_2)^2$ is holomorphic on the open set $\mathcal{D} := \mathcal{B}(0;\rho)$ and, with $R := (r+\rho)/2$, we have $r < R$ and $\overline{\mathcal{B}}(0;R) \subseteq \mathcal{D}$.



33.3 Problem. Contrast the result (and the drawing) above with Example 30.1.

33.4 Problem. Explain why the Cauchy integral formula does not (apparently) allow us to evaluate

$$\int_{|z|=2} \frac{dz}{z^2-1}.$$

Then rewrite the integrand using partial fractions and realize that the Cauchy integral formula (or maybe just the Death Star lemma!) does, in fact, apply.

The true value of the Cauchy integral formula (CIF) is not that it enables us to compute certain line integrals that would otherwise be difficult or impossible (although it does). Rather, the CIF provides an *integral representation* of a function, and integrals are *the* key instrument for extract information about functions.

Specifically, the CIF uses one-dimensional information about a function f—the values of $w \mapsto f(w)/(w-z)$ on the circle of radius r centered at z_0—to compute two-dimensional information about f—its values on the ball of radius r centered at z_0. This may feel similar to the fundamental theorem of calculus, which reads

$$\int_a^b f' = f(b) - f(a)$$

when f is differentiable on $[a,b]$ and f' is continuous on $[a,b]$. Both the CCIF and the FTC give information about a function from an integral whose integrand is related to that function.

The CIF, however, might have at least two advantages over our beloved FTC. First, the FTC requires information about the derivative on the whole interval $[a, b]$ to produce information about f at the endpoints; we need one-dimensional data (values on an interval) to get zero-dimensional data (the difference of values at the endpoints). Second, the FTC requires information about a function other than f (namely, the derivative of f), whereas the integrand in the CIF is really just f gussied up via division by a linear polynomial.

We only proved the Cauchy integral formula for line integrals over circles, whereas the Cauchy integral theorem holds for line integrals over arbitrary closed paths. We will eventually generalize the integral formula to permit more arbitrary closed paths, but that will also require us to account for a notion of "orientation" on the paths. As it stands, our version of the integral formula above is perfectly suited to give us a rich amount of information about functions.

Here is the first of many deep consequences of the Cauchy integral formula. Suppose that the hypotheses of the Cauchy integral formula are met. That is, we have an open set $\mathcal{D}$ and a holomorphic function $f\colon \mathcal{D} \to \mathbb{C}$, and we have fixed $z_0 \in \mathcal{D}$ and $R > 0$ such that $\overline{\mathcal{B}}(z_0; R) \subseteq \mathcal{D}$. Then for any $r \in (0, R)$ and $z \in \mathcal{B}(z_0; r)$, we can write

$$f(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r} \mathcal{K}(w, z)\ dw, \qquad \text{where} \qquad \mathcal{K}(z, w) := \frac{f(w)}{w - z}.$$

The map $\mathcal{K}$ is defined on the set

$$\mathcal{D}_0 := \left\{(z, w) \in \mathbb{C}^2 \ \middle|\ |z - z_0| < r,\ |w - z_0| = r\right\}.$$

In particular, for $(z, w) \in \mathcal{D}_0$, we have $z \neq w$. It should follow, then, that $\mathcal{K}$ is continuous on $\mathcal{D}_0$ (this needs some development, since we have not discussed continuity for functions defined on subsets of $\mathbb{C}^2$) and that $\mathcal{K}$ is differentiable with respect to z (this too needs development, since we have not discussed partial derivatives for functions of several complex variables), and that

$$\mathcal{K}_z(z, w) = \frac{f(w)}{(w - z)^2}.$$

If this is all indeed true (it is), then we might expect that we could differentiate under the (line) integral as in Leibniz's rule (Theorem 29.2) and find

$$f'(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r} \mathcal{K}_z(z, w)\ dw = \frac{1}{2\pi i}\int_{|w-z_0|=r} \frac{f(w)}{(w - z)^2}\ dw.$$

Now look at this integrand. Exactly the same reasoning as above suggests that we can differentiate under the integral *again* to conclude that f' is differentiable and

$$f''(z) = 2\left(\frac{1}{2\pi i}\int_{|w-z_0|=r} \frac{f(w)}{(w - z)^3}\ dw\right).$$

Turn the crank and be convinced that f'' is differentiable. . .

If this reasoning holds, then we have discovered something remarkble. A holomorphic function is not just once differentiable but *infinitely* many times differentiable, and we can represent all derivatives as a line integral of the quotient of the *original* function and a polynomial. This result is called the generalized Cauchy integral formula, and it has many proofs.

Day 34: Wednesday, April 5

Recommended reading

We completed our discussion of the generalized Cauchy integral formula and some of its consequences. Our proof of the generalized formula followed that of Theorem 3.8.6, but we did not use the more general differentiation under the integral technique that the book did. As with the original formula, we got the generalized Cauchy integral formula only for line integrals over circles, not arbitrary curves. Read Example 3.8.7 (a). After that, we discussed Liouville's theorem (Theorem 3.9.2).

The first proof of the generalized Cauchy integral formula that we will give hinges on the venerable mathematical technique known as brute force.

34.1 Remark. Brute force is the best force.

Here is the brute force part of the proof.

34.2 Lemma. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f \colon \mathcal{D} \to \mathbb{C}$ be continuous. Fix $z_0 \in \mathcal{D}$ and take $R > 0$ such that $\overline{\mathcal{B}}(z_0; R) \subseteq \mathcal{D}$. Let $r \in (0, R)$ and let $m \geq 1$ be an integer. Define*

$$F_m \colon \mathcal{B}(z_0; r) \to \mathbb{C} \colon z \mapsto \int_{|w-z_0|=r} \frac{f(w)}{(w-z)^m}\, dw. \tag{34.46}$$

Then F_m is holomorphic with $F_m' = mF_{m+1}$.

Proof. This is essentially a differentiation under the integral argument for a very specific integrand. We need to show that for any $z \in \mathcal{B}(z_0; r)$, we have

$$\lim_{h \to 0} \frac{F_m(z+h) - F_m(z)}{h} = mF_{m+1}(z),$$

equivalently,

$$\lim_{h \to 0} \frac{F_m(z+h) - F_m(z) - hmF_{m+1}(z)}{h}. \tag{34.47}$$

We compute

$$F_m(z+h) - F_m(z) - hmF_{m+1}(z) \\ = \int_{|w-z_0|=r} f(w) \left[\frac{(w-z)^{m+1} - (w-z)((w-z)-h)^m - hm((w-z)-h)^m}{(w-z)^{m+1}((w-z)-h)}\right] dw. \tag{34.48}$$

This calculation just requires finding a common denominator inside the integral.

We claim that for all integers $m \geq 1$, there is a function $P_m \colon \mathbb{C}^2 \to \mathbb{C}$ and a constant $C_m > 0$ such that

$$\xi^{m+1} - \xi(\xi+h)^m + mh(\xi+h)^m = h^2 P_m(\xi, h) \quad \text{and} \quad |P_m(\xi, h)| \leq C_m |h|^2 (|\xi| + |h|)^{m-1} \tag{34.49}$$

for all ξ, $h \in \mathbb{C}$. The proof of this claim is Problem 34.3. With this claim in hand, we can estimate the integral on the right in (34.48) via the ML-inequality.

Put

$$M := \max_{|w-z_0|=r} |f(w)|.$$

Since $h \to 0$, we may as well assume that

$$|h| \le \min\{1, (r - |z-z_0|)/2\}. \tag{34.50}$$

Since $|w - z_0| = r$ and $|z - z_0| < r$, the triangle inequality gives

$$|w - z| = |(w - z_0) + (z_0 - z)| \le |w - z_0| + |z - z_0| = r + |z - z_0| < 2r.$$

Then

$$\begin{aligned}\big|(w-z)^{m+1} - (w-z)((w-z)-h)^m - hm((w-z)-h)^m\big| &= |P_m(w-z,-h)| \\ &\le C_m|h|^2(|w-z| + |-h|)^{m-1} \le C_m|h|^2(2r+1)^{m-1}.\end{aligned}$$

Next, the reverse triangle inequality implies

$$|w - z| = |(w - z_0) - (z - z_0)| \ge |w - z_0| - |z - z_0| = r - |z - z_0|$$

and

$$|(w-z) - h| \ge |w - z| - |h| \ge r - |z - z_0| - \frac{r - |z-z_0|}{2} = \frac{r - |z - z_0|}{2}.$$

Here we have used the estimate (34.50) on $|h|$.

We combine all of these estimates to conclude that if $|w - z_0| = r$, then

$$\big|F_m(z+h) - F_m(z) - hmF_{m+1}(z)\big| \le \frac{2\pi r M C_m |h|^2 (2r+1)^{m-1}}{\big(r - |z-z_0|\big)^{m+1}\left(\dfrac{r - |z-z_0|}{2}\right)}.$$

If we divide both sides by $|h|$, we conclude

$$\left|\frac{F_m(z+h) - F_m(z) - hmF_{m+1}(z)}{h}\right| \le C|h|,$$

where C depends on r, m, z, z_0, and M, but not on h. The squeeze theorem then yields the limit (34.47). ■

34.3 Problem. Prove the claim (34.49) using one of the following options.

(i) Add and subtract $(\xi + h)^{m+1}$ to find

$$\xi^{m+1} - \xi(\xi+h)^m + mh(\xi+h)^m = -\big((\xi+h)^{m+1} - \xi^{m+1}\big) + (m+1)h(\xi+h)^m.$$

Rewrite

$$(\xi+h)^{m+1}-\xi^{m+1}=(m+1)h\int_0^1(\xi+th)^m\,dt$$

using the fundamental theorem of calculus and obtain

$$\xi^{m+1}-\xi(\xi+h)^m+mh(\xi+h)^m=(m+1)h\left(\int_0^1\left[(\xi+h)^m-(\xi+th)^m\right]dt\right).$$

Use the fundamental theorem of calculus again to rewrite

$$\int_0^1\left[(\xi+h)^m-(\xi+th)^m\right]dt=mh\int_0^1\int_0^1(1-t)\big(\xi+th+\tau h(1-t)\big)^{m-1}\,d\tau\,dt.$$

Define

$$P_m(\xi,h):=m(m+1)\int_0^1\int_0^1(1-t)\big(\xi+th+\tau h(1-t)\big)^{m-1}\,d\tau\,dt.$$

Prove the estimate on P_m using multiple applications of the triangle inequality.

(ii) Expand $(\xi+h)^m$ using the binomial theorem:

$$(\xi+h)^m=\sum_{k=0}^m\binom{m}{k}\xi^k h^{m-k}=\xi^m+m\xi^{m-1}h+\sum_{k=0}^{m-2}\binom{m}{k}\xi^k h^{m-k}.$$

Then do arithmetic.

34.4 Theorem (Generalized Cauchy integral formula). *Suppose that $\mathcal{D}\subseteq\mathbb{C}$ is open and $f\colon\mathcal{D}\to\mathbb{C}$ is holomorphic. Then f is infinitely differentiable on $\mathcal{D}$. In particular, if $z_0\in\mathcal{D}$ with $\overline{\mathcal{B}}(z_0;R)\subseteq\mathcal{D}$, then for any $r\in(0,R)$, $z\in\mathcal{B}(z_0;R)$, and $n\geq 0$, the nth derivative of f is*

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{(w-z)^{n+1}}\,dw. \tag{34.51}$$

Proof. We induct on n, starting with $n=0$, i.e., $f^{(0)}=f$. Then (34.51) is just the Cauchy integral formula. Assume that (34.51) holds for some $n\geq 0$; then

$$f^{(n)}=\frac{n!}{2\pi i}F_{n+1},$$

where F_{n+1} was defined in (34.46). Lemma 34.2 implies that F_{n+1} is holomorphic with $F'_{n+1}=(n+1)F_{n+2}$. Consequently, $f^{(n)}$ is differentiable with

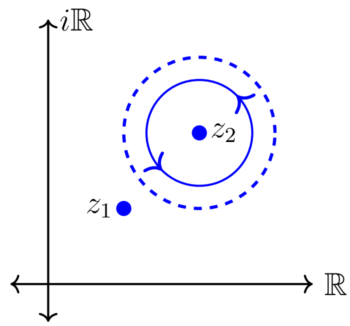
$$f^{(n+1)}(z)=(f^{(n)})'(z)=(n+1)F_{n+2}(z)=\frac{(n+1)n!}{2\pi i}F_{n+2}(z)=\frac{(n+1)!}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{(w-z)^{n+2}}\,dw.$$

This is the desired form of $f^{(n+1)}$ from (34.51). ∎

34.5 Example. Let $z_1, z_2 \in \mathbb{C}$ with $0 < |z_1| < |z_2|$. Let $0 < \rho < |z_2| - |z_1|$ and $0 < r < \rho$. Then

$$\int_{|z-z_2|=r} \frac{dz}{(z-z_1)(z-z_2)^2} = \int_{|z-z_2|=r} \frac{1/(z-z_1)}{(z-z_2)^{1+1}}\, dz = 2\pi i \frac{d}{dz}\left[\frac{1}{z-z_1}\right]\bigg|_{z=z_2} = -\frac{2\pi i}{(z_1-z_2)^2},$$

since the function $f(z) := 1/(z-z_1)$ is holomorphic on the open set $\mathcal{D} := \mathcal{B}(z_2;\rho)$ and, with $R := (r+\rho)/2$, we have $r < R$ and $\overline{\mathcal{B}}(z_2;R) \subseteq \mathcal{D}$.



34.6 Problem. Show that the function

$$f\colon \mathbb{R} \to \mathbb{C}\colon t \mapsto \begin{cases} t^2, & t \ge 0 \\ -t^2, & t < 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and that f' is continuous on $\mathbb{R}$ but not differentiable at 0.

At last, we can fully characterize when a function has an antiderivative. This effectively completes the third phase of our course—the integral calculus phase—and opens the way to a multiverse of complex analytic possibilities.

34.7 Problem.

(i) Let $\mathcal{D} \subseteq \mathbb{C}$ be an elementary domain (such as, but not necessarily, a star domain—see Definition 31.6). Show that a function $f\colon \mathcal{D} \to \mathbb{C}$ is holomorphic if and only if f has an antiderivative on $\mathcal{D}$. [Hint: *one direction is the definition; for the other, if $F' = f$, what do you know about F''?*]

(ii) Suppose that $\mathcal{D} \subseteq \mathbb{C}$ is open (not necessarily a star domain or even connected). Show that $f\colon \mathcal{D} \to \mathbb{C}$ is holomorphic if and only if f is "locally antidifferentiable" in the sense that if $z_0 \in \mathcal{D}$ and $r > 0$ such that $\mathcal{B}(z_0;r) \subseteq \mathcal{D}$, then there is a holomorphic function $F\colon \mathcal{B}(z_0;r) \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \mathcal{B}(z_0;r)$. [Hint: *an open ball $\mathcal{B}(z_0;r)$ is a star domain.*]

(iii) Use part (iii) of Problem 26.12 to show that the result in part (ii) above is the best that we can expect when the underlying set is not necessarily an elementary domain.

Here is a first result from that multiverse of possibilities. We have not used the following definition all that much, so now is a good time to bring it up.

34.8 Definition. *A holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is called* **ENTIRE**. *That is, $f: \mathbb{C} \to \mathbb{C}$ is entire if f is differentiable at each $z \in \mathbb{C}$.*

If we replace $\mathbb{C}$ by $\mathbb{R}$ in the preceding definition, we are familiar with many functions that are infinitely differentiable on $\mathbb{R}$. And many of those functions are bounded; consider $f(t) = \sin(t)$, which satisfies $|\sin(t)| \le 1$ for all $t \in \mathbb{R}$. It turns out that only the most trivial of bounded functions can be entire.

34.9 Theorem (Liouville). *Suppose that $f: \mathbb{C} \to \mathbb{C}$ is entire and bounded, i.e., there is $M > 0$ such that $|f(z)| \le M$ for all z. Then f is constant.*

Proof. We show that $f'(z) = 0$ for all z; since $\mathbb{C}$ is a domain, it follows that f is constant. Fix $z \in \mathbb{C}$ and $r > 0$. The generalized Cauchy integral formula (with $R = 2r$) gives

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{(w-z)^2} \, dw,$$

and if $|w - z| = r$, then we can estimate the integrand as

$$\left| \frac{f(w)}{(w-z)^2} \right| = \frac{|f(w)|}{|w-z|^2} = \frac{|f(w)|}{r^2} \le \frac{M}{r^2}.$$

Then the ML-inequality implies

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{(w-z)^2} \, dw \right| \le \frac{2\pi r M}{2\pi r^2} = \frac{M}{r^2}.$$

Since this is true for an arbitrary $r > 0$, we can use the squeeze theorem and send $r \to \infty$ to conclude $|f'(z)| = 0$, thus $f'(z) = 0$. ■

Day 35: Friday, April 7

Recommended reading

We continued discussing Liouville's theorem and its useful consequences. Specifically, we proved a version of the fundamental theorem of algebra (Theorem 3.9.4). There are many, many proofs of this theorem, and we will see a stronger version of it later. Then we returned to the Cauchy integral formula and exposed the power series lurking within it. That is, we proved Theorem 4.3.1. You should read Definition 4.2.1 and Example 4.2.2. Theorem 4.2.5 is a more precise version of our statement about the radius of convergence, but it requires the tool of the limsup, which you don't need to know for this class.

35.1 Example. Previously we have seen that $\sin(\cdot)$ is unbounded on $\mathbb{C}$, e.g., by considering

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i},$$

thus

$$|\sin(iy)| = \frac{|e^{-y} - e^{y}|}{2} \to \infty \text{ as } y \to \pm\infty.$$

But even without this estimate, since we know that $\sin(\cdot)$ is entire and not constant (e.g., $\sin(0) = 0$ and $\sin(\pi/2) = 1$), we are guaranteed that $\sin(\cdot)$ is unbounded on $\mathbb{C}$. This is, of course, a marked contrast to the familiar estimate $|\sin(x)| \le 1$ for $x \in \mathbb{R}$.

As an application of Liouville's theorem, we derive a first (somewhat rough) version of the fundamental theorem of algebra, which states that every polynomial with complex coefficients has a root in $\mathbb{C}$. Note that not every polynomial with real coefficients has a root in $\mathbb{R}$ (think of the most famous quadratic in the world, which is, from one point of view, the reason this course exists).

35.2 Theorem. *Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n \ge 1$, i.e., $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \ne 0$. Then f has a root in $\mathbb{C}$: there is $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.*

Proof. Suppose not. Then $f(z) \ne 0$ for all $z \in \mathbb{C}$, and so the function $g := 1/f$ is defined on $\mathbb{C}$; moreover, g is holomorphic on $\mathbb{C}$. If we can show that g is also bounded on $\mathbb{C}$, i.e., there is $M > 0$ such that $|g(z)| \le M$ for all $z \in \mathbb{C}$, then Liouville's theorem will tell us that g is constant. That is, there is $c \in \mathbb{C}$ such that $g(z) = c$ for all $z \in \mathbb{C}$, and so $f(z) = 1/c$ for all $z \in \mathbb{C}$. But then f is not a polynomial of degree at least 1.

The proof that g is bounded is somewhat technical, so here is a sketch of how it would proceed for a polynomial with real coefficients on $\mathbb{R}$. That is, suppose that $a_k \in \mathbb{R}$ for each k. We know from calculus that $\lim_{x\to\pm\infty} |f(x)| = \infty$, so $\lim_{x\to\pm\infty} |g(x)| = 0$. Then for some $r > 0$ we have $|g(x)| \le 1$ for all $|x| \ge r$. Since g is continuous on $[-r, r]$, there is $M_r > 0$ such that $|g(x)| \le M_r$ for $x \in [-r, r]$. Put $M = \max\{1, M_r\}$ to see that $|g(x)| \le M$ for all $x \in \mathbb{R}$. Of course, the boundedness of the infinitely differentiable function $g = 1/f$ on $\mathbb{R}$ alone is not enough to imply that g is constant; there are many bounded infinitely differentiable functions on $\mathbb{R}$. ■

The fact that a once-differentiable function is really infinitely many times differentiable should be surprising, if not shocking. We will now develop a result that is nothing short of *staggering*. Suppose that $\mathcal{D} \subseteq \mathbb{C}$ is open and $z_0 \in \mathcal{D}$ with $R > 0$ such that $\overline{B}(z_0; R) \subseteq \mathcal{D}$. Fix $z \in \mathcal{B}(z_0; r)$ with $0 < r < R$. Then the Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w - z}\, dw. \tag{35.52}$$

It has been some time, but we once expanded $1/(w - z)$ using the geometric series.

Specifically, since $|w - z_0| > 0$ and $|z - z_0| < |w - z_0|$, we have

$$\frac{1}{w - z} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(w - z_0)^{k+1}}. \tag{35.53}$$

Then

$$\int_{|w-z_0|=r} \frac{f(w)}{w - z}\,dw = \int_{|w-z_0|=r} f(w) \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(w - z_0)^{k+1}}\,dw$$

Suppose for the moment that we can "interchange" the line integral and the series, i.e.,

$$\int_{|w-z_0|=r} f(w) \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(w - z_0)^{k+1}}\,dw = \sum_{k=0}^{\infty} \int_{|w-z_0|=r} f(w) \frac{(z - z_0)^k}{(w - z_0)^{k+1}}\,dw. \tag{35.54}$$

This is certainly true if the series is just a finite sum, and morally it should smack of differentiating under the integral; both there and here we are swapping an integral and a limiting procedure. Now,

$$\sum_{k=0}^{\infty} \int_{|w-z_0|=r} f(w) \frac{(z - z_0)^k}{(w - z_0)^{k+1}}\,dw = \sum_{k=0}^{\infty} \left(\int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}}\,dw \right) (z - z_0)^k,$$

and so if (35.54) is indeed permitted, then we have shown

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}}\,dw \right) (z - z_0)^k.$$

If we put

$$a_k := \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}}\,dw, \tag{35.55}$$

then this just compresses to

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

and we might remember that

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

In other words, if (35.54) is true, then f is really a *power series*—at least locally, around a given point—and the coefficients in this power series expansion effectively arise from the generalized Cauchy integral formula.

We will explore the consequences of this calculation in detail, and later we will justify the interchange (35.54). For now, we collect (and slightly rephrase) the result above as a formal theorem.

35.3 Theorem (Taylor). *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f\colon \mathcal{D} \to \mathbb{C}$ be holomorphic. Let*

$z_0 \in \mathcal{D}$ and $R > 0$ such that $\mathcal{B}(z_0; R) \subseteq \mathcal{D}$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad a_k := \frac{f^{(k)}(z_0)}{k!} \tag{35.56}$$

for each $z \in \mathcal{B}(z_0; R)$. The series (35.56) is the **TAYLOR SERIES OF f AT z_0**.

Proof. Up to justifying (35.54), all that we need to change from the work above is the former hypothesis $\overline{\mathcal{B}}(z_0; r) \subseteq \mathcal{D}$ from the Cauchy integral formula. Now we are just assuming $\mathcal{B}(z_0; R) \subseteq \mathcal{D}$. So, fix $z \in \mathcal{B}(z_0; R)$ and take r, $\rho > 0$ such that $|z - z_0| < r < \rho < R$. Then $\overline{\mathcal{B}}(z_0; \rho) \subseteq \mathcal{D}$, so we can apply the Cauchy integral formula on $\overline{\mathcal{B}}(z_0; \rho)$ and get (35.52). From there the work above proceeds to give the power series expansion. ■

We will now step away from (ostensibly) studying holomorphic functions to review some essential features of power series. We will return to discuss Taylor series extensively. Incidentally, calculus textbooks usually call the Taylor series at $z_0 = 0$ (if f is defined there and holomorphic on a ball centered at 0) the Maclaurin series. Outside of calculus classes, virtually no one uses this terminology.

35.4 Definition. *Let (a_k) be a sequence in $\mathbb{C}$ and $z_0 \in \mathbb{C}$. The* **POWER SERIES CENTERED AT z_0 WITH COEFFICIENTS (a_k)** *is the series*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Recall that the symbol $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ plays the dual role of denoting the sequence of partial sums $\left(\sum_{k=0}^{n} a_k (z - z_0)^k\right)$ and the limit of this sequence, if this limit exists. A power series carries z as an extra parameter, and so the convergence of a power series will depend on the value of z. In particular, a power series centered at z_0 always converges at $z = z_0$.

35.5 Problem. To what? Recall the convention of denoting $z^0 = 1$, even when $z = 0$.

We will now state a general convergence theorem for power series which we also likely saw for real power series in calculus. We will not prove it here, as the proof will not teach us anything new specifically about complex analysis.

35.6 Theorem. *Let (a_k) be a sequence in $\mathbb{C}$ and $z_0 \in \mathbb{C}$. There exists a unique (extended) real number $R \geq 0$ such that the power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. This number R is the* **RADIUS OF CONVERGENCE** *of the power series.*

While there is a formula for R in terms of the coefficients (a_k), and while this formula *always* works, it is both complicated and unwieldy. Often it is best to use the ratio or root tests or to recognize the power series as the Taylor series for a holomorphic function. Indeed, we can paraphrase Theorem 35.3 in the following useful way.

35.7 Corollary. *The radius of convergence of the Taylor series centered at z_0 for a holomorphic function f on an open set is at least as large as the radius of any open ball centered at z_0 and contained in that open set. Unlike the Taylor series of a function of a real variable, we do not have to check any estimates on the remainder in the series; we just squeeze the largest open ball possible into the domain of our holomorphic function.*

As in real calculus, a power series may converge or diverge for $z \in \mathbb{C}$ with $|z - z_0| = R$. The behavior varies from series to series. It is even possible for a series to converge at some z with $|z - z_0| = R$ and diverge at others.

35.8 Example. The coefficients and center of the series

$$\sum_{k=0}^{\infty} \frac{z^k}{k^2+1}$$

are $a_k = 1/(k^2+1)$ and 0, respectively. For a fixed $z \in \mathbb{C}$, the terms of this series are $A_k := z^k/(k^2+1)$. To study the convergence of this series, we use the ratio test and compute

$$\left|\frac{A_{k+1}}{A_k}\right| = \left|\frac{z^{k+1}}{(k+1)^2+1} \cdot \frac{k^2+1}{z^k}\right| = |z|\frac{k^2+1}{(k+1)^2+1} \to |z| \text{ as } k \to \infty.$$

Per the ratio test, the series therefore converges for $|z| < 1$ and diverges for $|z| > 1$.

When $|z| = 1$, we need to study the series differently; the ratio test will be inconclusive. In this case, we attempt to discern absolute convergence and compute

$$\left|\frac{z^k}{k^2+1}\right| = \frac{|z|^k}{k^2+1} = \frac{1}{k^2+1} \le \frac{1}{k^2}$$

for $k \ge 1$. By comparison with the p-series $\sum_{k=1}^{\infty} k^{-2}$, we obtain absolute convergence for all $z \in \mathbb{C}$ with $|z| = 1$, and therefore convergence.

Day 36: Monday, April 10

Recommended reading

We did a number of examples of Taylor series. The book has many more: see Examples 4.3.4, 4.3.6, 4.3.7, 4.3.8, and 4.3.9.

36.1 Example. The familiar geometric series

$$\sum_{k=0}^{\infty} z^k$$

is certainly a power series with coefficients $a_k = 1$ for all k and center 0. We have previously established its convergence for $|z| < 1$ by computing its partial sums and its divergence for $|z| \geq 1$ via the test for divergence (which we reiterate below in the case $|z| = 1$). For extra practice, we could use the root test and check

$$|z^k|^{1/k} = (|z|^k)^{1/k} = |z| \to |z| \text{ as } k \to \infty,$$

and thus the geometric series converges for $|z| < 1$ and diverges for $|z| > 1$. When $|z| = 1$, we have $|z^k| = |z|^k = 1$ for all k, and so

$$\lim_{k\to\infty} |z^k| = 1 \neq 0.$$

Thus $\lim_{k\to\infty} z^k \neq 0$ either, for $|z| = 1$, and so the test for divergence shows that $\sum_{k=0}^{\infty} z^k$ diverges for $|z| = 1$.

36.2 Example. Consider the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^k.$$

We use the ratio test and study

$$\left| \frac{(-1)^{k+1}}{(k+1)+1} z^{k+1} \cdot \frac{k+1}{(-1)^k z^k} \right| = |z| \frac{k+1}{k+2} \to |z| \text{ as } k \to \infty.$$

Thus (like our previous two examples) the series converges for $|z| < 1$ and diverges for $|z| > 1$.

When $|z| = 1$, we may have convergence or divergence: take $z = 1$ to see that the series is the alternating harmonic series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (1)^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} = -\sum_{j=1}^{\infty} \frac{(-1)^j}{j},$$

which converges. Take $z = -1$ to see that the series is the harmonic series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (-1)^k = \sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{j=1}^{\infty} \frac{1}{j},$$

which diverges.

To determine the Taylor series for a function at a given point, we often have three options, which we list below from least to most preferred.

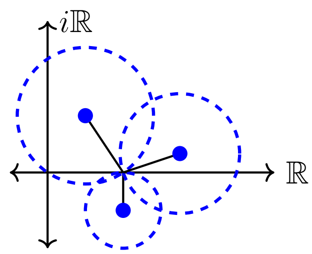
1. Calculate the coefficients using the generalized Cauchy integral formula, e.g., (35.55).

2. Calculate lots of derivatives of f.

3. Recognize f as some modification of a function whose Taylor series is known, and manipulate that known Taylor series.

36.3 Example. Let $f(z) = 1/(1-z)$. We know that f is holomorphic on $\mathbb{C} \setminus \{1\}$ and that $f(z) = \sum_{k=0}^{\infty} z^k$ for $|z| < 1$, but what is the Taylor series expansion for f centered at an arbitrary $z_0 \in \mathbb{C} \setminus \{1\}$, and what is the largest ball on which that series converges? There are several ways of proceeding here.

(i) We could draw pictures and just figure out what is the largest ball $\mathcal{B}(z_0; R)$ contained in $\mathbb{C} \setminus \{1\}$. Then we could use Theorem 35.3 or Corollary 35.7 to ensure convergence of the Taylor series on $\mathcal{B}(z_0; R)$. Note, though, that these results do not imply the *divergence* of the Taylor series outside $\mathcal{B}(z_0; R)$. Pretty quickly the pictures will convince us that $R = |1 - z_0|$, and then we would have to do some further analysis to ensure that R cannot be larger.



(ii) Here is one approach to that further analysis. We could differentiate f repeatedly and observe patterns:

$$f(z) = (1-z)^{-1}, \qquad f'(z) = -(1-z)^{-2}(-1) = (1-z)^{-2},$$
$$f''(z) = -2(1-z)^{-3}(-1) = 2(1-z)^{-3}, \qquad f^{(3)}(z) = -6(1-z)^{-4}(-1) = 6(1-z)^{-4}, \ldots$$

A formal induction argument establishes

$$f^{(k)}(z) = k!(1-z)^{-(k+1)},$$

and so the Taylor series for f centered at z_0 is

$$\sum_{k=0}^{\infty} \frac{k!(1-z_0)^{-(k+1)}}{k!}(z-z_0)^k = \sum_{k=0}^{\infty} \frac{1}{(1-z_0)^{k+1}}(z-z_0)^k.$$

Since f is holomorphic on the open set $\mathbb{C} \setminus \{1\}$, this Taylor series converges on any ball $\mathcal{B}(z_0; R)$ such that $\mathcal{B}(z_0; R) \subseteq \mathbb{C} \setminus \{1\}$. How can we find R just from the coefficients of this series? We could use the ratio test and calculate

$$\left|\frac{(z-z_0)^{k+1}}{(1-z_0)^{(k+1)+1}} \cdot \frac{(1-z_0)^{k+1}}{(z-z_0)^k}\right| = \frac{|z-z_0|}{|1-z_0|} \to \frac{|z-z_0|}{|1-z_0|} \text{ as } k \to \infty.$$

Thus the series converges for $|z-z_0| < |1-z_0|$ and diverges for $|z-z_0| > |1-z_0|$. (We will not investigate convergence at $|z-z_0| = |1-z_0|$ except to note divergence at $z = 1$ via the test for divergence.)

(iii) In lieu of the differentiation above, we could try to use a known Taylor series. Specifically, we would write, for z, $z_0 \in \mathbb{C} \setminus \{1\}$,

$$f(z) = \frac{1}{1-z} = \frac{1}{1-z_0+z_0-z} = \frac{1}{1-z_0-(z-z_0)} = \frac{1}{(1-z_0)\left[1-\left(\dfrac{z-z_0}{1-z_0}\right)\right]}$$

$$= \left(\frac{1}{1-z_0}\right)\left(\frac{1}{1-\left(\dfrac{z-z_0}{1-z_0}\right)}\right) = \frac{1}{1-z_0} f\left(\frac{z-z_0}{1-z_0}\right).$$

Since $f(w) = \sum_{k=0}^{\infty} w^k$ for $|w| < 1$, we therefore have

$$f(z) = \frac{1}{1-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{1-z_0}\right)^k \qquad \text{for} \qquad \left|\frac{z-z_0}{1-z_0}\right| < 1,$$

and this gives the same Taylor series as above.

Day 37: Wednesday, April 12

Recommended reading

We did an example that illustrated Remark 4.3.3 in the book. See the paragraph after Example 4.3.6 as well for another curious situation, and (re)read the first paragraph on p. 227. This will shed further light on the distinction between real analytic functions on $\mathbb{R}$ and holomorphic/analytic functions on $\mathbb{C}$. Then we talked about differentiating power series, which is Corollary 4.2.9 in the book. See Examples 4.2.12 and 4.2.13 to practice recognizing certain power series as derivatives of known functions. Proposition 4.3.5 partially answers our question of the existence and uniqueness of analytic continuations, which we will resolve in more detail later.

37.1 Example. The function $f(z) = \operatorname{Log}(z)$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and, for z in this set, its derivative is (as we expect) $f'(z) = 1/z = z^{-1}$. Taking more derivatives, we have

$$f''(z) = -z^{-2}, \qquad f'''(z) = 2z^{-3}, \qquad f^{(4)}(z) = -6z^{-4}, \qquad f^{(5)}(z) = 24z^{-5}, \ldots,$$

and so, observing this pattern and/or inducting, we find

$$f^{(k)}(z) = (-1)^{k+1}(k-1)!z^{-k}.$$

Then the Taylor series for $\mathrm{Log}(\cdot)$ centered at any $z_0 \in \mathbb{C} \setminus (-\infty, 0]$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \mathrm{Log}(z_0) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{k!} z_0^{-k}(z-z_0)^k$$
$$= \mathrm{Log}(z_0) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{kz_0^k}(z-z_0)^k.$$

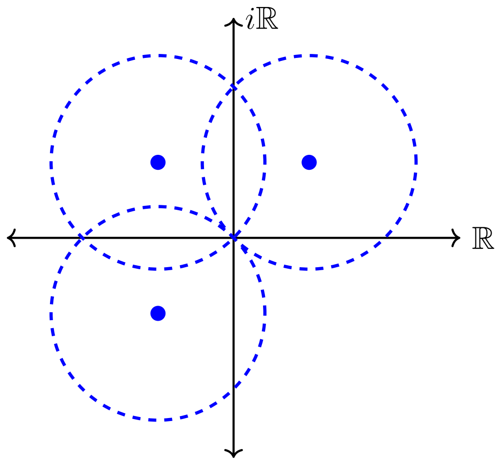
By Theorem 35.3 and Corollary 35.7, this series definitely converges for any ball $\mathcal{B}(z_0; R) \subseteq \mathbb{C} \setminus (-\infty, 0]$.

We can also test the convergence of the series (starting with $k = 1$, since we can ignore finitely many terms in the series without affecting convergence) with the ratio test:

$$\left| \frac{(-1)^{(k+1)+1}}{(k+1)z_0^{k+1}}(z-z_0)^{k+1} \cdot \frac{kz_0^k}{(-1)^{k+1}(z-z_0)^k} \right| = \frac{k}{k+1}\left(\frac{|z-z_0|}{|z_0|}\right) \to \frac{|z-z_0|}{|z_0|} \text{ as } k \to \infty.$$

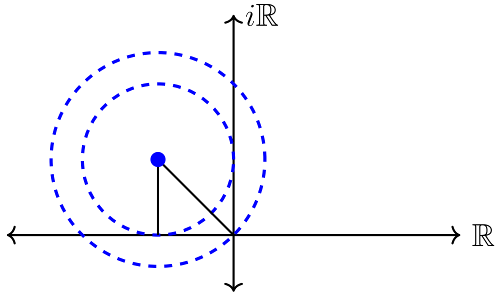
Consequently, the series converges if $|z - z_0| < |z_0|$.

How does this compare to what ? Here are sketches of some balls of the form $\mathcal{B}(z_0; |z_0|)$ for various $z_0 \in \mathbb{C}$.



These pictures suggest that if $\mathrm{Re}(z_0) < 0$, then $\mathcal{B}(z_0; |z_0|)$ may intersect nontrivially with $(-\infty, 0]$, whereas the largest ball centered at z_0 that does not intersect $(-\infty, 0]$ is

$\mathcal{B}(z_0, |\operatorname{Im}(z_0)|)$.



Thus the Taylor series for $\operatorname{Log}(\cdot)$ converges on a larger ball than one that fits in the domain of $\operatorname{Log}(\cdot)$, and so the Taylor series cannot converge to $\operatorname{Log}(\cdot)$ on all of this larger ball!

Notwithstanding the oddities above, power series are some of the nicest functions in existence, because calculus-type computations with them are very easy—and so it is a wonder of nature that holomorphic functions are (locally) power series. Here is another theorem about power series that should be familiar from calculus, and which we will not prove.

37.2 Theorem. *Suppose that the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges on $\mathcal{B}(z_0; R)$. Then the function $f(z) := \sum_{k=0}^{\infty} a_k(z-z_0)^k$ is holomorphic on $\mathcal{B}(z_0; R)$ with*

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \left(\prod_{j=0}^{n-1} (k-j) \right) a_k(z-z_0)^{k-n} = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k(z-z_0)^{k-n} \tag{37.57}$$

for each $z \in \mathcal{B}(z_0; R)$ and each integer $n \geq 0$. In particular, the series in (37.57) converges on $\mathcal{B}(z_0; R)$, and

$$a_k = \frac{f^{(n)}(z_0)}{k!}. \tag{37.58}$$

Note that (37.58) follows directly from (37.57) by substituting $z = z_0$.

37.3 Problem. To motivate the equality (37.57), try differentiating $f(z) = z^k$ some n times, observe patterns, and try to rewrite the coefficients in the derivatives as quotients of factorials. For example, calculate $f', \ldots, f^{(6)}$ for $f(z) = z^5$.

37.4 Problem. The coefficients of a power series are unique in the following sense. Let $z_0 \in \mathbb{C}$ and let (a_k) and (b_k) be sequences in $\mathbb{C}$ such that for some $R > 0$,

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} b_k(z-z_0)^k$$

for all $z \in \mathcal{B}(z_0; R)$. Show that $a_k = b_k$ for all k. [Hint: *let f be the difference of the series and use Theorem 37.2.*]

37.5 Problem. Here is another proof of Liouville's theorem (Theorem 34.9). Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is entire; explain why

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_k := \frac{f^{(k)}(0)}{k!}$$

for all $z \in \mathbb{C}$. If $M > 0$ satisfies $|f(z)| \le M$ for all $z \in \mathbb{C}$, use the generalized Cauchy integral theorem to show

$$|a_k| \le Mr^{-k}$$

for all $r > 0$ and $k \ge 0$. Send $r \to \infty$ and conclude $a_k = 0$ for $k \ge 1$.

37.6 Example. Recognizing a given power series as the derivative of another is a useful skill. For example, at first glance the series

$$\sum_{k=2}^{\infty} k(k-1)z^k$$

looks like a second derivative, since the starting index is 2. For $k \ge 2$, we calculate

$$\frac{k!}{(k-2)!} = \frac{k(k-1)(k-2)!}{(k-2)!} = k(k-1),$$

and so all that is "wrong" in this series is the power of z. We therefore rewrite

$$\sum_{k=2}^{\infty} k(k-1)z^k = z^2 \sum_{k=2}^{\infty} \frac{k!}{(k-2)!} z^{k-2} = z^2 f''(z), \qquad \text{where} \qquad f(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

This works assuming $|z| < 1$, and so, for such z, we obtain

$$\sum_{k=2}^{\infty} k(k-1)z^k = z^2 \left(\frac{2}{(1-z)^3} \right) = \frac{2z^2}{(1-z)^3}.$$

Last, we can use the ratio test to check that the original series converges for $|z| < 1$ and diverges for $|z| > 1$, and so the restriction to $|z| < 1$ above is reasonable.

The identity (37.58) agrees with Theorem 35.3 and shows that we can either start with a holomorphic function and obtain a power (Taylor) series, or we can start with a power series and recognize it as the Taylor series of a function. Morally, the two approaches to complex analysis are the same. For this reason, we now introduce a piece of standard terminology that we have heretofore delayed.

37.7 Definition. *Let $\mathcal{D} \subseteq \mathbb{C}$. A function $f\colon \mathcal{D} \to \mathbb{C}$ is* ANALYTIC *on $\mathcal{D}$ if for each $z_0 \in \mathcal{D}$,* *there is $r > 0$ and a sequence (a_k) in $\mathbb{C}$ such that*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{37.59}$$

for each $z \in \mathcal{B}(z_0; r) \cap \mathcal{D}$.

Theorems 35.3 and 37.2 combine to tell us that analytic functions are precisely the holomorphic functions.

37.8 Theorem. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open. A function $f \colon \mathcal{D} \to \mathbb{C}$ is analytic if and only if f is holomorphic, in which case the series expansion (37.59) of f about a point $z_0 \in \mathcal{D}$ is its Taylor series.*

37.9 Example. Many familiar functions are analytic on $\mathbb{C}$ because of how we chose to define them as power series. This includes the exponential, the sine, and the cosine:

$$e^z = \exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \qquad \sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \qquad \text{and} \qquad \cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}.$$

Although we do not usually employ this terminology in real-variable calculus, it is entirely possible for a function defined on (a subinterval of) $\mathbb{R}$ to be analytic in the sense that for each point in that interval, the function equals its Taylor series around that point. Indeed, that is probably how we first rigorously met the exponential and trigonometric functions in calculus.

37.10 Definition. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f \colon I \to \mathbb{R}$ is* REAL ANALYTIC *on I if for each $t_0 \in I$, there is a sequence of real numbers (a_k) and a real number $\delta > 0$ such that for $t \in (t_0 - \delta, t_0 + \delta) \cap I$,*

$$f(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^k. \tag{37.60}$$

Theorem 37.2 holds for real analytic functions with essentially the same proof as in the complex-variable case: a real analytic function is infinitely differentiable, and in the expansion (37.60), the coefficients satisfy $a_k = f^{(k)}(t_0)/k!$. However, Theorem 37.8 does not. There are plenty of infinitely differentiable functions on $\mathbb{R}$ that are not real analytic; a classical counterexample is

$$f(t) := \begin{cases} e^{-1/t^2}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

One can show that f is infinitely differentiable on $\mathbb{R}$ and $f^{(k)}(0) = 0$ for all k. Thus the Taylor series for f centered at 0 converges to the zero function $\mathbb{R}$, and that is definitely not

f. This is in line with our previous remarks that, as we know well from calculus, a function on $\mathbb{R}$ can be n-times differentiable but not $(n+1)$-times differentiable. Differentiability on $\mathbb{C}$ is much stronger: the existence of one derivative guarantees the existence of all derivatives *and* the convergence of the Taylor series back to the original function to boot.

But in the happy case that we do have a real analytic function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$, can we extend it to an analytic function on some open set $\mathcal{D} \subseteq \mathbb{C}$ with $I \subseteq \mathcal{D}$? After all, we did that quite successfully with the exponential and trigonometric functions. Such an extension has a formal name.

37.11 Definition. *Let $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \mathbb{C}$. A function $f\colon \mathcal{D} \to \mathbb{C}$ is an* **ANALYTIC CONTINUATION** *of a function $f_0\colon \mathcal{D}_0 \to \mathbb{C}$ if f is analytic and if $f\big|_{\mathcal{D}_0} = f_0$, i.e., if $f(z) = f_0(z)$ for all $z \in \mathcal{D}_0$.*

So, when does a real analytic function have an analytic continuation from a real interval to an open subset of the plane? And if a function has an analytic continuation, is that continuation unique? That is, could a function f_0 have two analytic continuations, f_1 and f_2, with $f_1 \neq f_2$? Such a possibility should be frightening, as it might mean that there is more than one way to extend, say, the exponential to the plane—and so perhaps we have been working with the wrong exponential all along!

Of course, this is nonsense. Analytic continuations, if they exist, are unique. Forcing two functions f_1 and f_2 to be the same is really saying that $f_1 - f_2 = 0$. And so we will take up the study of the *zeros* of an analytic function: if f is analytic, what can we say about those z at which $f(z) = 0$? In particular, what is the minimum amount of data about a function that we need to conclude that it is *always* zero? (Not much.)

Day 38: Friday, April 14

Recommended reading

We discussed the zeros of an analytic function, which corresponds to Definition 4.5.1, Theorem 4.5.2, and Example 4.5.3.

Power series are, euphemistically, "just" polynomials of "infinite" degree. A spot of work with the roots of polynomials, then, will motivate some of the broader results on analytic functions that we will develop.

Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n \geq 1$. Note that this formula for f is its Taylor expansion centered at 0, since $f^{(k)}(z) = 0$ for all integers $k \geq n+1$ and all $z \in \mathbb{C}$. By the fundamental theorem of algebra, f has a root $z_1 \in \mathbb{C}$. Since f is entire, we may expand f as a power series centered at z_1: $f(z) = \sum_{k=0}^{\infty} b_k (z - z_1)^k$. Here $b_0 = f(z_1) = 0$, and also $b_k = f^{(k)}(z_1)/k! = 0$ for $k \geq n+1$. Thus $f(z) = \sum_{k=1}^{n} b_k (z - z_1)^k$, and so we may factor

$$f(z) = (z - z_1) \sum_{k=1}^{n} b_k (z - z_1)^{k-1} = (z - z_1) p_1(z), \qquad p_1(z) = \sum_{j=0}^{n-1} b_{j+1} (z - z_1)^j.$$

We now recognize p_1 as a polynomial of degree $n - 1$; if $n = 1$, then p_1 is constant, and in particular $p_1(z_1) \neq 0$. Otherwise, $f = 0$, and then f would not be a polynomial of degree at least 1. If $n \geq 2$, then either $p_1(z_1) \neq 0$, or $p_1(z_1) = 0$, in which case we can repeat the argument above and factor

$$p_1(z) = (z - z_1)p_2(z),$$

where p_2 is a polynomial of degree $n - 2$. In this case, we can rewrite

$$f(z) = (z - z_1)^2 p_2(z).$$

And then the process continues to allow us to conclude that for some integer $m_1 \geq 1$, there is a polynomial p_1 of degree $n - m_1$ such that $p_1(z_1) \neq 0$ and

$$f(z) = (z - z_1)^{m_1} p_1(z). \tag{38.61}$$

We want to call the integer m_1 the **MULTIPLICITY** or **ORDER** of z_1 as a root of f. As with most integer-dependent processes, a rigorous proof of the factorization (38.61) would use induction on n.

We could go further from (38.61) and say that, if $m < n$, then p_1 is a polynomial of degree at least 1, and therefore p_1 has a root z_2. Note that $z_2 \neq z_1$ since $p_1(z_1) \neq 0$. Then we could write $p_1(z) = (z - z_2)^{m_2} p_2(z)$, where $p_2(z_1) \neq 0$. And so on. Eventually we would factor

$$f(z) = a(z - z_1)^{m_1} \cdots (z - z_r)^{m_r},$$

where $z_1, \ldots, z_r \in \mathbb{C}$ are distinct and $m_1, \ldots, m_r \geq 1$ are integers with $m_1 + \cdots + m_r = n$. The coefficient $a \in \mathbb{C} \setminus \{0\}$ is the constant polynomial that arises from the very last factorization of p_r, i.e., $p_r(z) = (z - z_r)^{m_r} a$. This factorization *is* the fundamental theorem of algebra, and a rigorous proof also needs induction.

But we will not do that. Instead, viewing analytic functions as "infinite degree polynomials," we will see just how much the behavior of zeros of analytic functions resembles the results above for polynomials.

38.1 Theorem. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and suppose that $f \colon \mathcal{D} \to \mathbb{C}$ is analytic. Let $z_0 \in \mathcal{D}$ such that $f(z_0) = 0$ and take $r > 0$ such that $\mathcal{B}(z_0; r) \subseteq \mathcal{D}$. Then one, and only one, of the following holds:*

(i) *$f(z) = 0$ for all $z \in \mathcal{B}(z_0; r)$.*

(ii) *There is an analytic function $g \colon \mathcal{B}(z_0; r) \to \mathbb{C}$ and an integer $m \geq 1$ such that $f(z) = (z - z_0)^m g(z)$ for $z \in \mathcal{B}(z_0; r)$ and, additionally, $g(z_0) \neq 0$. The integer m is the smallest integer k such that $f^{(k)}(z_0) \neq 0$, and $g(z_0) = f^{(m)}(z_0)$.*

Proof. Write $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for $z \in \mathcal{B}(z_0; r)$, where $a_k = f^{(k)}(z_0)/k!$. We consider the following two cases on the coefficients.

(i) $a_k = 0$ for all k. Since $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for all $z \in \mathcal{B}(z_0; r)$, we then have $f(z) = 0$ for all $z \in \mathcal{B}(z_0; r)$. This is (i).

(ii) There is $n \geq 1$ such that $a_n \neq 0$. Note that $a_0 = f(z_0) = 0$, so this is only possible for some $n \geq 1$. Now let $m \geq 1$ be the *smallest* integer satisfying $a_m \neq 0$. (That such a smallest integer exists is a consequence of the well-ordering property of the positive integers.) We may then write

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=m}^{\infty} a_k(z-z_0)^k = \sum_{j=0}^{\infty} a_{j+m}(z-z_0)^{j+m} = (z-z_0)^m \sum_{j=0}^{\infty} a_{j+m}(z-z_0)^j. \tag{38.62}$$

These equalities are valid for $z \in \mathcal{B}(z_0; r)$.

Then, for $z \in \mathcal{B}^*(z_0; r)$, we have

$$\sum_{j=0}^{\infty} a_{j+m}(z-z_0)^j = \frac{f(z)}{(z-z_0)^m}.$$

That is, the series on the left converges for $z \in \mathcal{B}^*(z_0; r)$, and certainly the series converges at $z = z_0$. Thus the map

$$g \colon \mathcal{B}(z_0; r) \to \mathbb{C} \colon z \mapsto \sum_{j=0}^{\infty} a_{j+m}(z-z_0)^j$$

is analytic. Moreover, we have the factorization $f(z) = (z-z_0)^m g(z)$ from (38.62), and by definition of g we compute $g(z_0) = a_m \neq 0$. This is (ii). ■

In case (ii) above, since $g(z_0) \neq 0$, continuity allows us to choose $\rho \leq r$ such that $g(z) \neq 0$ on $\mathcal{B}(z_0; \rho)$. We then give this case a special name.

38.2 Definition. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f \colon \mathcal{D} \to \mathbb{C}$ be analytic. Let $z_0 \in \mathcal{D}$ and let $m \geq 1$ be an integer. Then z_0 is an* **ZERO OF f OF ORDER (MULTIPLICITY)** *m if for some $r > 0$ such that $\mathcal{B}(z_0; r) \subseteq \mathcal{D}$, there is an analytic function $g \colon \mathcal{B}(z_0; r) \to \mathbb{C}$ such that $f(z) = (z-z_0)^m g(z)$ for $z \in \mathcal{B}(z_0; r)$ with $g(z) \neq 0$ for $z \in \mathcal{B}(z_0; r)$. In the case $m = 1$, the zero is sometimes called* **SIMPLE**.

38.3 Example. We find the zeros and their orders for several different functions.

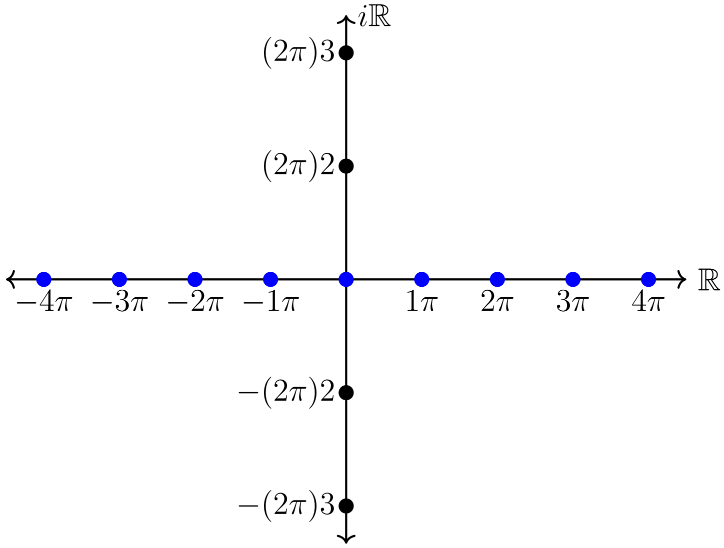
(i) $f_1(z) = z^2$. Here $f_1(z) = 0$ if and only if $z = 0$, and we can basically read off from the definition of f_1 that 0 has order 2. Indeed, with $g(z) = 1$ for all z, we have $f_1(z) = z^2 g(z)$, and certainly $g(0) \neq 0$.

(ii) $f_2(z) = \sin(z)$. Long ago we calculated that the roots of the sine are $z = k\pi$ for $k \in \mathbb{Z}$. Since $f_2'(z) = \cos(z)$ and $\cos(k\pi) = (-1)^k \neq 0$ for all k, these roots are all simple, i.e., they have multiplicity 1. We could factor $\sin(z) = (z - k\pi)g_k(z)$ for a certain analytic function g_k, but here we would need the Taylor series of the sine centered at $k\pi$, as in the proof of Theorem 38.1; constructing g_k is not as transparent as in the case $f(z) = z^2$.

(iii) $f_3(z) = e^{2z} - 2e^z + 1$. Here we use the factorization $w^2 - 2w + 1 = (w-1)^2$ to write

$f_3(z) = (e^z - 1)^2$. Then $f_3(z) = 0$ if and only if $e^z = 1$, so the zeros of f are the numbers $2\pi ik$ for $k \in \mathbb{Z}$. We calculate $f_3'(z) = 2(e^z - 1)$, so $f_3'(2\pi ik) = 0$, and $f_3''(z) = 2e^z$, so $f_3''(2\pi ik) = 2 \neq 0$. Each zero therefore has order 2.

The zeros of the second and third functions in the example above had something in common—not their order, and not whether they were real or purely imaginary. Rather, all of these zeros were *isolated* from each other. We can see this just by plotting points: a blue dot is a zero of $f_2(z) = \sin(z)$ and a black dot is a zero of $f_3(z) = (e^z - 1)^2$. Around each dot we can draw a ball that does not include any other dot.

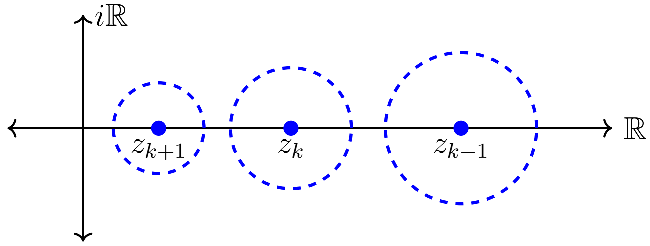


38.4 Definition. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f\colon \mathcal{D} \to \mathbb{C}$ be analytic. A point $z_0 \in \mathcal{D}$ is an* **ISOLATED ZERO** *of f if there is $r > 0$ such that $\mathcal{B}(z_0; r) \subseteq \mathcal{D}$ and $f(z) \neq 0$ for $z \in \mathcal{B}^*(z_0; r)$.*

That is, z_0 is the only zero of f in the ball $\mathcal{B}(z_0; r)$; outside the ball, f certainly may have zeros. Additionally, for different isolated zeros of the same function, there are no guarantees about the relative sizes of the balls surrounding them and excluding other zeros. In Example 38.3, the zeros of the second and third functions were a nice, uniform distance away from each other. This does not always happen.

38.5 Example. Let $\mathcal{D} = \mathbb{C} \setminus \{0\}$ and let $f(z) = \sin(\pi/z)$. Then f is analytic on $\mathcal{D}$ and $f(z) = 0$ if and only if $\pi/z = k\pi$ for some integer k. That is, the zeros of f are the numbers $z_k = 1/k$. These numbers are definitely isolated; after a bit of algebra, we can find $r_k > 0$ such that if $|z - 1/k| < r_k$, then $z \neq 1/j$ for any integer $j \neq k$. But note that $z_k \to 0$ as $k \to \infty$, and in particular the distance between successive zeros z_k and z_{k+1} shrinks as

$k \to \infty$.



Although we cannot guarantee that the zeros of an analytic function are *all* a minimum distance apart, we can be assured that they are isolated, at least for a function that is not always zero. In other words, the only "interesting" zeros—those of a function that is not identically zero—must be isolated. We will actually prove a sort of converse to this statement and, in the process, demonstrate that only a small amount of data must be verified to guarantee that a function is always zero. From this, we will quickly extract a test for determining when two functions really are the same.

Day 39: Monday, April 17

Recommended reading

We proved an augmented version of Theorem 4.5.4, which we used to prove the identity principle (Theorem 4.5.5). Example 4.5.6 offers some different applications of the identity principle.

39.1 Theorem. *Let $\mathcal{D} \subseteq \mathbb{C}$ be a domain and let $f \colon \mathcal{D} \to \mathbb{C}$ be analytic. The following are equivalent.*

(i) *$f(z) = 0$ for all $z \in \mathcal{D}$.*

(ii) *There is $z_0 \in \mathcal{D}$ such that $f^{(k)}(z_0) = 0$ for all $k \geq 0$.*

(iii) *There is a sequence (z_k) in $\mathcal{D}$ of distinct points (i.e., $z_k \neq z_j$ for $j \neq k$) such that $f(z_k) = 0$ for all k and $z_k \to z_0$ for some $z_0 \in \mathcal{D}$.*

(iv) *f has a zero that is not isolated in $\mathcal{D}$.*

Proof. (i) $\Longrightarrow$ (ii) This is essentially a direct calculation: if $f(z) = 0$ for all $z \in \mathcal{D}$, then, fixing z, we have

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Thus $f'(z) = 0$ for all $z \in \mathcal{D}$. Proceeding inductively, we find $f^{(k)}(z) = 0$ for all $z \in \mathcal{D}$ and all integers $k \geq 0$. We can then take any point $z_0 \in \mathcal{D}$ to satisfy the condition in part (ii).

(ii) $\Longrightarrow$ (iii) Fix $r > 0$ such that $\mathcal{B}(z_0; r) \subseteq \mathcal{D}$. Then $f(z) = \sum_{k=0}^{\infty} f^{(k)}(z_0)(z - z_0)^k/k! = 0$

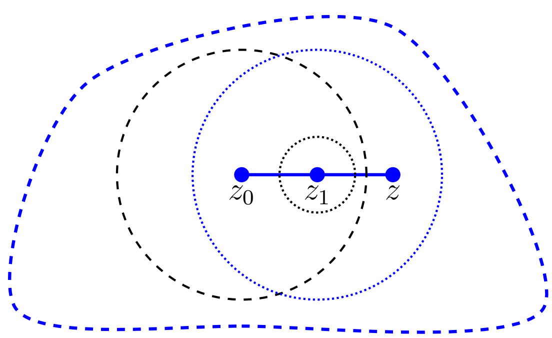
for all $z \in \mathcal{B}(z_0; r)$. Now set $z_k := z_0 + r/(k+1)$. It is straightforward to check that $z_k \neq z_j$ for $j \neq k$, that $z_k \in \mathcal{B}(z_0; r) \subseteq \mathcal{D}$ for each k, and that $z_k \to z_0 \in \mathcal{D}$.

(iii) $\implies$ (iv) We claim that z_0 is this zero that is not isolated, and we prove this by contradiction. If z_0 is isolated, then there is $r > 0$ such that $\mathcal{B}(z_0; r) \subseteq \mathcal{D}$ and $f(z) \neq 0$ for $z \in \mathcal{B}^*(z_0; r)$. Since $z_k \to z_0$, for k sufficiently large we have $z_k \in \mathcal{B}(z_0; r)$. And since the points z_k are all distinct, we have $z_k = z_0$ for at most one $k \geq 1$. Thus for k large, we really have $z_k \in \mathcal{B}^*(z_0; r)$. But $f(z_k) = 0$, which contradicts our prior conclusion that $f(z) \neq 0$ for $z \in \mathcal{B}^*(z_0; r)$.

(iv) $\implies$ (i) Let z_0 be the zero that is not isolated, so for some $r > 0$ with $\mathcal{B}(z_0; r) \subseteq \mathcal{D}$, we have $f(z) = 0$ for all $z \in \mathcal{B}(z_0; r)$. If $\mathcal{D} = \mathcal{B}(z_0; r)$, then we are done. Otherwise, we need to do more work, and it is here that we will use for the first time in the proof the hypothesis that $\mathcal{D}$ is connected, not merely open.

We want to show that $f(z) = 0$ for all $z \in \mathcal{D}$. We give two arguments. The first is geometric and relies on an assertion about subsets of $\mathbb{C}$ that requires more technical tools from analysis than we care to develop here. The second is more rigorous but also possibly more opaque.

Argument #1. Let $z \in \mathcal{D}$ and let γ be a curve in $\mathcal{D}$ from z_0 to z. It is possible to cover the image of γ by a finite sequence of overlapping balls of the same radius $\rho \leq r$ centered at points on the image of γ, starting with $\mathcal{B}(z_0; \rho)$, such that the center of the kth ball is contained in the $(k-1)$st ball, and such that each ball is contained in $\mathcal{D}$. For example, if there are two balls, $\mathcal{B}(z_0; \rho)$ and $\mathcal{B}(z_1; \rho)$ for some z_1 on the image of γ, then the situation looks like this (in which we assume that γ is a line segment for simplicity).



We know that $f(w) = 0$ for all $w \in \mathcal{B}(z_0; \rho)$ and that $z_1 \in \mathcal{B}(z_0; \rho)$. In particular, $f(w) = 0$ for all $w \in \mathcal{B}(z_1; s)$, where $s > 0$ is such that $\mathcal{B}(z_1; s) \subseteq \mathcal{B}(z_0; \rho)$. Then $f^{(k)}(z_1) = 0$ for all k, and so $f(w) = \sum_{k=0}^{\infty} f^{(k)}(z_1)(w - z_1)^k/k! = 0$ for all $w \in \mathcal{B}(z_1; \rho)$. Here we are really using the fact that part (ii) implies (i) when $\mathcal{D}$ is a ball centered at z_0. In particular, then, $f(z) = 0$. If there are more than two balls involved in the covering, then we can "piggyback" this argument to show that f is zero on each successive ball, culminating with the ball that contains (but need not be centered at) z.

The difficulty with this approach is the construction of this special "finite covering" of the image of γ, which needs, among other things, the tools of compactness and uniform continuity. Below we present a less geometrically obvious (but still geometrically motivated) proof that has the advantage of being logically self-contained to the tools that we already possess.

Argument #2. Put

$$\mathcal{D}_1 := \{z \in \mathcal{D} \mid z \text{ is an isolated zero of } f \text{ in } \mathcal{D} \text{ or } f(z) \neq 0\}$$

and

$$\mathcal{D}_2 := \{z \in \mathcal{D} \mid f(z) = 0 \text{ and } z \text{ is not an isolated zero of } f \text{ in } \mathcal{D}\}.$$

Note that $\mathcal{D}_2$ is nonempty, that $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{D}$, and that $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$. We claim that $\mathcal{D}_1$ and $\mathcal{D}_2$ are both open; if this is true, then Problem 39.2 below forces $\mathcal{D}_1 = \varnothing$ since $\mathcal{D}$ is a domain. Then $\mathcal{D} = \mathcal{D}_2$, in which case $f(z) = 0$ for all $z \in \mathcal{D}$.

We first show that $\mathcal{D}_1$ is open. If $z \in \mathcal{D}_1$ is an isolated zero of f in $\mathcal{D}$, let $r > 0$ be such that $\mathcal{B}(z; r) \subseteq \mathcal{D}$ with $f(w) \neq 0$ for $w \in \mathcal{B}^*(z_0; r)$. Thus $\mathcal{B}^*(z; r) \subseteq \mathcal{D}_1$, and since we know $z \in \mathcal{D}_1$ already, we conclude $\mathcal{B}(z; r) \subseteq \mathcal{D}$. If $z \in \mathcal{D}_1$ satisfies $f(z) \neq 0$, then by continuity there is $r > 0$ such that $\mathcal{B}(z; r) \subseteq \mathcal{D}$ and $f(w) \neq 0$ for $w \in \mathcal{B}(z; r)$. This implies that $w \in \mathcal{D}_1$ for all $w \in \mathcal{B}(z; r)$, and so $\mathcal{B}(z; r) \subseteq \mathcal{D}_1$. Either way, we have found $r > 0$ such that $\mathcal{B}(z; r) \subseteq \mathcal{D}_1$.

Now we show that $\mathcal{D}_2$ is open. If $z \in \mathcal{D}_2$, then $f(z) = 0$ and z is not an isolated zero of f in $\mathcal{D}$. So, for some $r > 0$ such that $\mathcal{B}(z; r) \subseteq \mathcal{D}$, we have $f(w) = 0$ for all $w \in \mathcal{B}(z; r)$. That is, each $w \in \mathcal{B}(z; r)$ is a zero of f; now we show that each w is a zero that is not isolated, which will imply $w \in \mathcal{D}_2$ and thus $\mathcal{B}(z; r) \subseteq \mathcal{D}_2$. Given $w \in \mathcal{B}(z; r)$, take $s > 0$ such that $\mathcal{B}(w; s) \subseteq \mathcal{B}(z; r)$. It is still the case that $f(\xi) = 0$ for all $\xi \in \mathcal{B}(w; s)$, so w is a zero of f in $\mathcal{D}$ that is not isolated, as desired. ■

39.2 Problem. Let $\mathcal{D} \subseteq \mathbb{C}$ be nonempty and connected.

(i) Suppose that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where both $\mathcal{D}_1$ and $\mathcal{D}_2$ are open and $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$. Argue by contradiction as follows that if $\mathcal{D}_2 \neq \varnothing$, then $\mathcal{D}_1 = \varnothing$.

Suppose instead that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are nonempty. Explain why the function

$$f \colon \mathcal{D} \to \mathbb{C} \colon z \mapsto \begin{cases} 1, & z \in \mathcal{D}_1 \\ 2, & z \in \mathcal{D}_2 \end{cases}$$

is defined, holomorphic, locally constant, and not constant. Conclude that $\mathcal{D}$ cannot be connected.

(ii) Let $\mathcal{D} = \mathbb{C} \setminus i\mathbb{R}$, $\mathcal{D}_1 = \{z \in \mathbb{C} \mid \text{Im}(z) < 0\}$, and $\mathcal{D}_2 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are open, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, with $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$ but $\mathcal{D}_1 \neq \varnothing$ and $\mathcal{D}_2 \neq \varnothing$. Draw a picture of this situation. Then draw a curve with initial point in $\mathcal{D}_1$ and terminal point in $\mathcal{D}_2$. Point out on your drawing how this curve shows that $\mathcal{D}$ is not connected.

39.3 Problem. In the proof that part (iv) of Theorem 39.1 implies part (i), perhaps a more natural decomposition would be

$$\mathcal{D}_1 := \{z \in \mathcal{D} \mid z \text{ is an isolated zero of } f \text{ in } \mathcal{D}\}$$

and

$$\mathcal{D}_2 := \{z \in \mathcal{D} \mid z \text{ is not an isolated zero of } f \text{ in } \mathcal{D}\}.$$

Explain why $\mathcal{D}_1$ is not open, so this decomposition does not work.

39.4 Problem. Give an example of an open set $\mathcal{D}$ and an analytic function $f\colon \mathcal{D} \to \mathbb{C}$ such that f is not identically zero on but such that f has a zero in $\mathcal{D}$ that is not an isolated zero. That is, f and $\mathcal{D}$ should satisfy the following two conditions.

(i) There exists $z_1 \in \mathcal{D}$ such that $f(z_1) \neq 0$.

(ii) There exist $z_2 \in \mathcal{D}$ and $r > 0$ such that $\mathcal{B}(z_2; r) \subseteq \mathcal{D}$ and $f(z) = 0$ for all $z \in \mathcal{B}(z_2; r)$.

Such an open set $\mathcal{D}$ cannot be connected—why?

39.5 Problem. Does the situation of Example 38.5 contradict the equivalence of parts (i) and (iii) of Theorem 39.1?

Perhaps the most useful "test" to emerge from this theorem is part (iii): f need only be zero on a sequence of distinct points in $\mathcal{D}$ that converges to a point in $\mathcal{D}$ in order for us to conclude that f is always zero on $\mathcal{D}$! For example, if f is zero on a line segment in $\mathcal{D}$ (a one-dimensional subset of an open, and therefore two-dimensional, set), then f is zero on all of $\mathcal{D}$. This is only a very "little" amount of data!

39.6 Problem. Prove this ebullient claim. Specifically, let $\mathcal{D} \subseteq \mathbb{C}$ be a domain with z_1, $z_2 \in \mathcal{D}$ and $z_1 \neq z_2$. Suppose that f_1, $f_2\colon \mathcal{D} \to \mathbb{C}$ are analytic with $f_1(z) = f_2(z)$ for all $z \in \mathcal{D}$. Prove that $f_1 = f_2$ on $\mathcal{D}$.

While Theorem 39.1 is stated for the zeros of a function, this result carries over nicely to comparing two functions: just study where their difference is zero.

39.7 Corollary (Identity principle). *Let $\mathcal{D} \subseteq \mathbb{C}$ be a domain and let f_1, $f_2\colon \mathcal{D} \to \mathbb{C}$ be analytic. Suppose that $f_1(z_k) = f_2(z_k)$ for a sequence (z_k) of distinct points in $\mathcal{D}$ such that $z_k \to z$ for some $z \in \mathcal{D}$. Then $f_1 = f_2$ on $\mathcal{D}$.*

Proof. Put $f = f_1 - f_2$ and use the equivalence of parts (i) and (iii) of Theorem 39.1. ■

39.8 Example. Many "functional identities" that are known on $\mathbb{R}$ remain true for functions extended analytically to $\mathbb{C}$. Often they can be proved brute-force (the best force) from the definitions of these analytic continuations, but we can also use the identity principle.

We know that $\ln(t_1 t_2) = \ln(t_1) + \ln(t_2)$ for $t_1,\ t_2 > 0$. We would like to say that $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ for $z_1,\ z_2 \in \mathbb{C} \setminus \{0\}$, but this probably is not true for the entire plane. Fix $t_0 > 0$ and define

$$f\colon (0,\infty) \to \mathbb{R}\colon t \mapsto \ln(tt_0) - \big[\ln(t) + \ln(t_0)\big].$$

Then, really, $f(t) = 0$ for all $t > 0$, and so certainly f is real analytic. Next, note that if $t_0 > 0$ and $z \in \mathbb{C} \setminus (-\infty, 0]$, then $zt_0 \in \mathbb{C} \setminus (-\infty, 0]$ as well. Thus the function

$$\widetilde{f}\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}\colon z \mapsto \mathrm{Log}(zt_0) - \big[\mathrm{Log}(z) + \mathrm{Log}(t_0)\big]$$

is analytic, since the principal logarithm is analytic except on the branch cut $(-\infty, 0]$. Furthermore, $\widetilde{f}(t) = f(t) = 0$ for all $t \in (0,\infty)$. Thus $\widetilde{f}(z) = 0$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$. Changing our notation slightly, we conclude

$$\mathrm{Log}(zt) = \mathrm{Log}(z) + \mathrm{Log}(t)$$

for all $z \in \mathbb{C} \setminus (-\infty, 0]$ and $t > 0$.

Now let $z,\ w \in \mathbb{C} \setminus \{0\}$. Then $zw = |zw|e^{i[\mathrm{Arg}(z)+\mathrm{Arg}(w)]}$. Changing notation once more, we know $\mathrm{Log}(\xi t) = \mathrm{Log}(\xi) + \mathrm{Log}(t)$ when $t > 0$ and $\xi \in \mathbb{C} \setminus (-\infty, 0]$. Taking $t = |zw|$ and $\xi = e^{i[\mathrm{Arg}(z)+\mathrm{Arg}(w)]}$, we will have

$$\mathrm{Log}(zw) = \mathrm{Log}(|zw|) + \mathrm{Log}(e^{i[\mathrm{Arg}(z)+\mathrm{Arg}(w)]}) \tag{39.63}$$

if $e^{i[\mathrm{Arg}(z)+\mathrm{Arg}(w)]} \notin (-\infty, 0]$. This requires $\mathrm{Arg}(z) + \mathrm{Arg}(w) \neq \pm\pi$. From here, it is an exercise in (hopefully more familiar) properties of the complex logarithm to show $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$.

39.9 Problem. Let $z,\ w \in \mathbb{C} \setminus \{0\}$ such that $\mathrm{Arg}(z) + \mathrm{Arg}(w) \neq \pm\pi$. Use (39.63) to show that $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$.

Day 40: Wednesday, April 19

Recommended reading

We revisited the issue of analytic continuation of real analytic functions raised in Proposition 4.3.5. Then we discussed removable singularities and poles. The book contains a wealth of information on pp. 276–281. All of the examples are worth reading, and the theorems give several additional characterizations of removable singularities and poles. Figure 4.16 is particularly illustrative. The implications (v) $\implies$ (vi) $\implies$ (i) of Theorem 4.5.11 are sometimes called the Riemann removability theorem. You can omit on first reading references to Laurent series, which we will study soon. Example 4.3.10 and Theorem 4.3.11 offer more examples of removable singularities, although that language isn't used there.

Now we can answer a major question that has been driving us since we first extended the exponential to the plane: is there only one way to extend a real analytic function into $\mathbb{C}$? Yes.

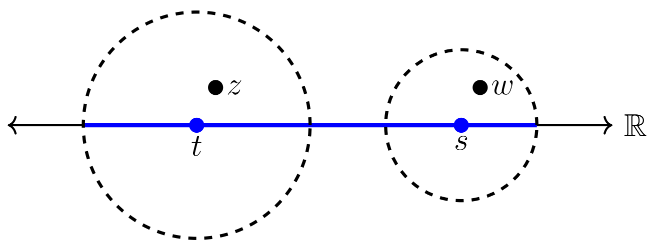
40.1 Theorem (Analytic continuation of real analytic functions). *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be real analytic. Then there exists a domain $\mathcal{D} \subseteq \mathbb{C}$ such that $I \subseteq \mathcal{D}$ and that f has a unique analytic continuation on $\mathcal{D}$.*

Proof. The uniqueness result is the identity theorem; see Problem 40.3.

Now we show existence. First we have to construct the domain $\mathcal{D}$. For each $t \in I$, there is $r_t > 0$ such that the Taylor series for f converges to f on $(t - r_t, t + r_t) \cap I$. We may as well make r_t so small that $(t - r_t, t + r_t) \subseteq I$. Then there is a sequence $(a_{k,t})$ of real numbers such that $f(\tau) = \sum_{k=0}^{\infty} a_{k,t}(\tau - t)^k$ for all $\tau \in (t - r_t, t + r_t)$. Specifically, $a_{k,t} = f^{(k)}(t)/k!$.

Now we set

$$\mathcal{D} := \bigcup_{t \in I} \mathcal{B}(t; r_t) = \{z \in \mathbb{C} \mid |z - t| < r_t \text{ for some } t \in I\}.$$



We claim that $\mathcal{D}$ is open and connected. For openness, fix $z \in \mathcal{D}$ and take $t \in I$ such that $z \in \mathcal{B}(t; r_t)$; since $\mathcal{B}(t; r_t)$ is open, there is $r > 0$ such that $\mathcal{B}(z; r) \subseteq \mathcal{B}(t; r_t)$. For connectedness, fix z, $w \in \mathcal{D}$. Take $z \in \mathcal{B}(t; r_t)$ and $w \in \mathcal{B}(s; r_s)$ for some t, $s \in I$. Let $\gamma = [z, t] \oplus [t, s] \oplus [s, w]$; then γ is a path in $\mathcal{D}$ with initial point z and terminal point w.

Next, we claim that, with the sequence $(a_{k,t})$ and the radius $r_t > 0$ defined above, the series $\sum_{k=0}^{\infty} a_{k,t}(z - t)^k$ converges for each $z \in \mathcal{B}(t; r_t)$. This is because the power series $\sum_{k=0}^{\infty} a_{k,t}(z - t)^k$ has a radius of convergence $R_t \geq 0$ and because we know already that the power series converges for $z \in \mathbb{R}$ with $|z - t| < r_t$. If the power series diverged at some point in the ball $\mathcal{B}(t; r_t)$, then it could not converge on all of the open interval $(t - r_t, t + r_t)$.

Finally, we define the analytic continuation. First, for $t \in I$, define

$$f_t : \mathcal{B}(t; r_t) \to \mathbb{C} : z \mapsto \sum_{k=0}^{\infty} a_{k,t}(z - t)^k.$$

By the work above, f_t is analytic on $\mathcal{B}(t; r_t)$. Next, note that if $\mathcal{B}(t; r_t) \cap \mathcal{B}(s; r_s) \neq \varnothing$ for some t, $s \in I$, then by Problem 40.2 below, there is a sequence of distinct points (w_k) in $\mathcal{B}(t; r_t) \cap \mathcal{B}(s; r_s)$ such that $w_k \to w$ for some $w \in \mathcal{B}(t; r_t) \cap \mathcal{B}(s; r_s)$. Since $f_t(w_k) = f_s(w_k)$ for each k, the identity principle implies that $f_t(z) = f_s(z)$ for each $z \in \mathcal{B}(t; r_t) \cap \mathcal{B}(s; r_s)$.

Consequently, we may define

$$\widetilde{f} : \mathcal{D} \to \mathbb{C} : z \mapsto f_t(z) \text{ if } z \in \mathcal{B}(t; r_t).$$

There is no ambiguity in this definition if $z \in \mathcal{B}(t; r_t) \cap \mathcal{B}(s; r_s)$ for two distinct t, $s \in I$, as the work above shows $f_t(z) = f_s(z)$. Finally, since each f_t is analytic on $\mathcal{B}(t; r_t)$, the function $\widetilde{f}$ is analytic on $\mathcal{D}$. And clearly $\widetilde{f}(t) = f(t)$ for each $t \in I$. ■

40.2 Problem. Let z_1, $z_2 \in \mathbb{C}$ and r_1, $r_2 > 0$ such that $\mathcal{B}(z_1; r_1) \cap \mathcal{B}(z_2; r_2) \neq \varnothing$. Show that there exists a sequence of distinct points (w_k) in $\mathcal{B}(z_1; r_1) \cap \mathcal{B}(z_2; r_2)$ such that $w_k \to w$ for some $w \in \mathcal{B}(z_1; r_1) \cap \mathcal{B}(z_2; r_2)$. [Hint: *as usual when working with balls, start by drawing a picture.*]

40.3 Problem.

(i) Let $\mathcal{D} \subseteq \mathbb{C}$ be a domain and let $I \subseteq \mathbb{R}$ be a nonempty interval such that $I \subseteq \mathcal{D}$. Suppose that f_1, $f_2 \colon \mathcal{D} \to \mathbb{C}$ are analytic with $f_1(t) = f_2(t)$ for all $t \in I$. Prove that $f_1 = f_2$ on $\mathcal{D}$. [Hint: *use Problem 39.6.*]

(ii) Prove that analytic continuations, whether of real analytic functions defined on a real interval or not, are unique. That is, suppose that $\mathcal{D}_0 \subseteq \mathbb{C}$ is a domain and $f \colon \mathcal{D}_0 \to \mathbb{C}$ is analytic. Let $\mathcal{D} \subseteq \mathbb{C}$ also be a domain with $\mathcal{D}_0 \subseteq \mathcal{D}$. Suppose that $\widetilde{f}_1$, $\widetilde{f}_2 \colon \mathcal{D} \to \mathbb{C}$ are both analytic continuations of $\mathcal{D}_0$. Then $\widetilde{f}_1 = \widetilde{f}_2$.

We now know a great deal about analytic functions, especially their power series expansions and their zeros. What happens if a function fails to be analytic, or holomorphic, or differentiable, on some proper subset of its domain? Depending on the geometry of that region of failure, we may still be able to say quite a lot about the function. Studying such failures is not just a natural evolution of our narrative—frequently applications demand consideration of functions that are not analytic in certain controlled ways.

We begin with the simplest failure of analyticity: the isolated singularity.

40.4 Definition. *Let $z_0 \in \mathbb{C}$ and $r > 0$. A function $f \colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ has an* **ISOLATED SINGULARITY** *at z_0 if f is analytic on $\mathcal{B}^*(z_0; r)$.*

It may appear that there are lots of ways for a function to fail to be analytic at a single point in a ball, and lots of possible behaviors on that punctured ball, but the power of analyticity on the punctured ball is such that there are only really three situations to consider. The following three canonical examples, all of which are functions defined and analytic on $\mathbb{C} \setminus \{0\}$, will illustrate those three behaviors:

$$f(z) = \frac{\sin(z)}{z}, \qquad g(z) = \frac{1}{z}, \qquad \text{and} \qquad h(z) = e^{1/z}.$$

The form of these functions illustrates a general truth: most isolated singularities arise in practice via some kind of division by 0.

If $f \colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ has an isolated singularity at z_0, perhaps it is natural to ask about the limit behavior of f at z_0. Either the limit $\lim_{z \to z_0} f(z)$ exists, or it does not. If the limit does exist, our experience with removable discontinuities suggests that we can extend f to z_0 and retain continuity, perhaps analyticity.

We can.

40.5 Theorem. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose that $f\colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ is analytic and $L := \lim_{z \to z_0} f(z)$ exists. Then the function*

$$\widetilde{f}\colon \mathcal{B}(z_0; r) \to \mathbb{C}\colon z \mapsto \begin{cases} f(z), & z \neq z_0 \\ L, & z = z_0 \end{cases}$$

is analytic.

We will not prove this theorem but instead work with one of the canonical examples above.

40.6 Example. The function $f(z) = \sin(z)/z$, defined on $\mathbb{C} \setminus \{0\}$, can be rewritten as follows. First, 0 is a simple root of $g(z) := \sin(z)$, so for some $r > 0$ and $z \in \mathcal{B}(0; r)$, we have $\sin(z) = zq(z)$, where $q\colon \mathcal{B}(0; r) \to \mathbb{C}$ is analytic, $q(z) \neq 0$ for $z \in \mathcal{B}(0; r)$, and

$$q(0) = g'(0) = \cos(0) = 1.$$

Thus, for $z \in \mathcal{B}^*(0; r)$, we have

$$f(z) = \frac{\sin(z)}{z} = \frac{zq(z)}{z} = q(z) \quad \text{and so} \quad \lim_{z \to 0} f(z) = \lim_{z \to 0} q(z) = q(0) = 1.$$

We now name this first kind of isolated singularity.

40.7 Definition. *Let $z_0 \in \mathbb{C}$ and $r > 0$. An analytic function $f\colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ has a* **REMOVABLE SINGULARITY** *at z_0 if the limit $\lim_{z \to z_0} f(z)$ exists.*

Theorem 40.5 says that any analytic function $f\colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ with a removable singularity at z_0 has an analytic continuation to that singularity. Conversely, the existence of an analytic continuation $\widetilde{f}\colon \mathcal{B}(z_0; r) \to \mathbb{C}$ of f implies that f has a removable singularity at z_0, since the limit $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \widetilde{f}(z)$ must exist by the continuity of $\widetilde{f}$ and the equality $f(z) = \widetilde{f}(z)$ on $\mathcal{B}^*(z_0; r)$.

40.8 Problem. Let $\mathcal{D} \subseteq \mathbb{C}$ be open and let $f\colon \mathcal{D} \to \mathbb{C}$ be analytic. Show that f has a removable singularity at every point of $\mathcal{D}$.

40.9 Problem. Let $\mathcal{D} \subseteq \mathbb{C}$ be open, let $f\colon \mathcal{D} \to \mathbb{C}$ be analytic, and let $z_0 \in \mathcal{D}$. Define

$$\phi\colon \mathcal{D} \to \mathbb{C}\colon z \mapsto \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ \\ f'(z_0), & z = z_0. \end{cases}$$

(i) Show that ϕ is analytic on $\mathcal{D}$.

(ii) What is the Taylor series of ϕ centered at z_0?

(iii) Compare these results to the difference quotient lemma (Lemma 19.1).

Suppose next that $f\colon \mathcal{B}^*(z_0;r) \to \mathbb{C}$ is analytic but the limit $\lim_{z\to z_0} f(z)$ does not exist. As we know from calculus, there are different gradations of a limit not existing. An infinite limit (a vertical asymptote) technically does not exist as a real number, but knowing that a limit is infinite surely tells us more information than just saying that the limit does not exist. We have not yet worked with infinite limits of complex-value functions, but the behavior of the canonical example $g(z) = 1/z$ suggests how we might proceed.

40.10 Definition. *Let $z_0 \in \mathbb{C}$ and $r > 0$. For a function $f\colon \mathcal{B}^*(z_0;r) \to \mathbb{C}$, we write $\lim_{z\to z_0} |f(z)| = \infty$ if for all $M > 0$, there is $\delta \in (0,r]$ such that if $0 < |z - z_0| < \delta$, then $M < |f(z)|$.*

It is then definitely the case that $\lim_{z\to 0} |z^{-1}| = \infty$: given $M > 0$, just take $\delta = 1/M$. More generally, suppose that $f\colon \mathcal{B}^*(z_0;r) \to \mathbb{C}$ has an isolated singularity at z_0 with $\lim_{z\to z_0} |f(z)| = \infty$. Take $\delta > 0$ such that if $z \in \mathcal{B}^*(z_0;\delta)$, then $1 < |f(z)|$, so in particular $f(z) \neq 0$ for $z \in \mathcal{B}^*(z_0;\delta)$. Then the function

$$g\colon \mathcal{B}^*(z_0;\delta) \to \mathbb{C}\colon z \mapsto \frac{1}{f(z)}$$

is defined and analytic. Moreover, it is not too much work to check that $\lim_{z\to z_0} g(z) = 0$.

40.11 Problem. Check this.

Then g has a removable singularity at z_0 and therefore an analytic continuation to $\mathcal{B}(z_0;\delta)$ of the form

$$\widetilde{g}\colon \mathcal{B}(z_0;\delta) \to \mathbb{C}\colon z \mapsto \begin{cases} 1/f(z), & z \neq z_0 \\ 0, & z = z_0. \end{cases}$$

Since $1/f(z) \neq 0$ for all $z \in \mathcal{B}^*(z_0;\delta)$, we see that $\widetilde{g}(z) \neq 0$ for $z \in \mathcal{B}^*(z_0;\delta)$, too. Then $\widetilde{g}$ really has an isolated zero at z_0, and so there is an integer $m \geq 1$ and an analytic function $q\colon \mathcal{B}(z_0;\rho) \to \mathbb{C}$ for some $\rho \in (0,\delta]$ such that for $z \in \mathcal{B}(z_0;\rho)$,

$$\widetilde{g}(z) = (z - z_0)^m q(z) \quad \text{and} \quad q(z) \neq 0.$$

Thus for $z \in \mathcal{B}^*(z_0;\rho)$, we have

$$f(z) = \frac{1}{\widetilde{g}(z)} = \frac{1}{(z - z_0)^m q(z)} = \frac{1/q(z)}{(z - z_0)^m}.$$

Put $p(z) := 1/q(z)$ to conclude the following.

40.12 Theorem. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose that $f: \mathcal{B}^*(z_0; r) \to \mathbb{C}$ is analytic and $\lim_{z \to z_0} |f(z)| = \infty$. Then there exist $\rho \in (0, r]$, an integer $m \geq 1$, and an analytic function $p: \mathcal{B}(z_0; \rho) \to \mathbb{C}$ such that $p(z_0) \neq 0$ and*

$$f(z) = \frac{p(z)}{(z - z_0)^m} \quad \text{for} \quad z \in \mathcal{B}^*(z_0; \rho).$$

This gives rise to another kind of named isolated singularity.

40.13 Definition. *Let $z_0 \in \mathbb{C}$ and $r > 0$. An analytic $f: \mathcal{B}^*(z_0; r) \to \mathbb{C}$ has a* **POLE OF ORDER** *m at z_0 if there exist $\rho \in (0, r]$, an integer $m \geq 1$, and an analytic function $p: \mathcal{B}(z_0; \rho) \to \mathbb{C}$ such that $p(z_0) \neq 0$ and*

$$f(z) = \frac{p(z)}{(z - z_0)^m} \quad \text{for} \quad z \in \mathcal{B}^*(z_0; \rho).$$

Day 41: Friday, April 21

Recommended reading

We did an example of an isolated singularity that is a pole and paid attention to series representations centered at the isolated singularity. Then we talked about essential singularities and the Casorati–Weierstrass theorem. Read Examples 4.5.18 and 4.5.21, Theorem 4.5.19, and Remark 4.5.20.

41.1 Example. The function

$$f(z) = \frac{e^{z-1} - 1}{(z - 1)^3}$$

is analytic on $\mathbb{C} \setminus \{1\}$ and has an isolated singularity at 1. The form of f suggests that 1 might be a pole, so we try to rewrite f in the form $f(z) = p(z)/(z - 1)^m$ for some integer $m \geq 1$ and some analytic function p such that $p(1) \neq 0$. Note that the current numerator of f will not work as p, since $g(z) := e^{z-1} - 1$ has a zero at 1. We check $g'(z) = e^{z-1}$, so $g'(1) = 1$, and therefore g has a simple root at 1. Then we can factor $g(z) = (z - 1)p(z)$ for z close to 1 with $p(1) \neq 0$, and so

$$f(z) = \frac{(z - 1)p(z)}{(z - 1)^3} = \frac{p(z)}{(z - 1)^2}.$$

Thus 1 is a pole of order 2.

41.2 Problem. Show that for $z \neq 1$, the function from Example 41.1 can be written in the form

$$\frac{e^{z-1}-1}{(z-1)^3} = \frac{1}{(z-1)^2} + \frac{1}{2(z-1)} + \sum_{k=0}^{\infty} \frac{(z-1)^k}{(k+3)!}.$$

We have now seen two kinds of behaviors at isolated singularities: either $\lim_{z\to z_0} f(z)$ exists, or it does not but $\lim_{z\to z_0} |f(z)| = \infty$. The third possibility, simply, is that neither of these behaviors holds.

41.3 Example. Let $f(z) = e^{1/z}$. Put $z_k = 1/2\pi i k$ to see that $z_k \to 0$ and $f(z_k) = e^{2\pi i k} = 1$. Thus $f(z_k) \to 1$ as well, and so it cannot be the case that $\lim_{z\to z_0} |f(z)| = \infty$.

Now put $w_k = 1/k$ to see that $w_k \to 0$ as well but $f(w_k) = e^k \to \infty$. Then the limit $\lim_{z\to z_0} f(z)$ cannot exist.

We give this situation a name based on the only two characteristics that we see.

41.4 Definition. *Let $z_0 \in \mathbb{C}$ and $r > 0$. An analytic function $f\colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ has an* **ESSENTIAL SINGULARITY** *at z_0 if z_0 is neither a removable singularity nor a pole. That is, the limit $\lim_{z\to z_0} f(z)$ does not exist, but it is also not the case that $\lim_{z\to z_0} |f(z)| = \infty$.*

This is not the most helpful of definitions, as it requires us to check that two conditions do *not* hold. However, the situation of Example 41.3 in fact characterizes essential singularities. Along one "path of approach" to an essential singularity, a function blows up, but along a different, suitably chosen path, the function can become arbitrarily close to *any* $z \in \mathbb{C}$. In Example 41.3, we just saw that with the case of $z = 1$.

41.5 Problem. Fix $z \in \mathbb{C}$. Determine a sequence (z_k) such that $z_k \to 0$ and $e^{1/z_k} \to z$.

41.6 Theorem (Casorati–Weierstrass). *Let $z_0 \in \mathbb{C}$ and $r > 0$. Let $f\colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ be analytic. Then z_0 is an essential singularity of f if and only if both of the following hold.*

(i) *There is a sequence (w_k) in $\mathcal{B}^*(z_0; r)$ such that $w_k \to z_0$ and $|f(w_k)| \to \infty$.*

(ii) *For each $z \in \mathbb{C}$, there is a sequence (z_k) in $\mathcal{B}^*(z_0; r)$ such that $z_k \to z_0$ and $f(z_k) \to z$.*

We introduced removable singularities, poles, and essential singularities via the limit behavior of the function at the singularity. Removable singularities lead to analytic continuations, poles lead to a nice fractional representation, and essential singularities lead to very nervous behaviors. It would, perhaps, be nice if there were one "unified" test that we could apply to singularities to determine their. We will develop such a test by examining the series behavior of functions near isolated singularities.

41.7 Remark. We will not study "non-isolated singularities." We might call a point $z_0 \in \mathbb{C}$ a non-isolated singularity of a function $f\colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ if there is no $\rho \in (0, r)$ such that f

is analytic on $\mathcal{B}^*(z_0;\rho)$. For example, $\text{Log}(\cdot)$ is not analytic on any punctured ball centered at the origin.

Day 42: Monday, April 24

Recommended reading

We talked about the geometry of annuli. See p. 261 and the illustrative Figures 4.10, 4.11, 4.12, 4.13, 4.14, and 4.15. Then we stated a version of Theorem 4.4.1 and did part of an example on finding Laurent decompositions of the same function in different annuli. Examples 4.4.2, 4.4.3, 4.4.4, 4.4.5, 4.4.6, and 4.4.7 provide many illustrations of these techniques. See the geometric series identity in (4.4.8) and also Exercise 30 in Section 4.3 for a very useful technique when handling rational functions.

Here is a summary of the examples of isolated singularities that we have studied and the series behavior of functions at those singularities.

Function	Singularity	Type	Series
$\dfrac{\sin(z)}{z}$	0	Removable	$\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k}$
$\dfrac{1}{z}$	0	Pole of order 1	$\dfrac{1}{z}$
$\dfrac{e^{z-1}-1}{(z-1)^3}$	1	Pole of order 2	$\displaystyle\frac{1}{(z-1)^2}+\frac{1}{2(z-1)}+\sum_{k=0}^{\infty}\frac{1}{(k+3)!}z^k$
$e^{1/z}$	0	Essential singularity	$\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)\frac{1}{z^k}$

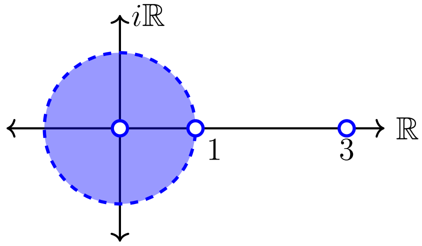
The pattern that might emerge is that removable singularities at z_0 lead to ordinary power series at z_0; poles lead to series with negative powers of $z-z_0$, but only finitely many such negative powers (up to and including the order of the pole); and essential singularities have infinitely many negative powers of $z-z_0$. This pattern is indeed true, but we can prove it in a more general context than the isolated singularity, which requires the function to be analytic on a punctured ball centered at z_0. Punctured balls are not the only reasonable domain for a function with an isolated singularity, especially if the function has more than one singularity.

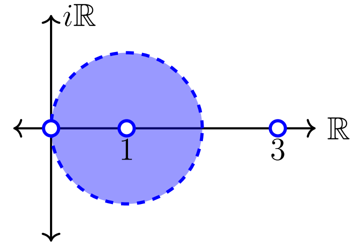
42.1 Example. The function

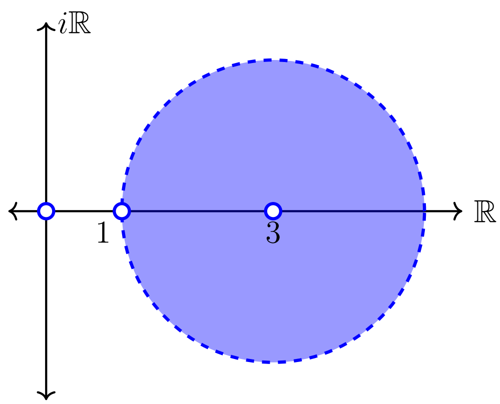
$$f(z)=\frac{1}{z(z-1)(z-3)}$$

is analytic on $\mathbb{C}\setminus\{0,1,3\}$ with simple poles at the points 0, 1, and 3. Much of our prior success hinged on working on open balls on which functions were analytic. Now we might try the next best thing: what are the largest ball-like subsets of $\mathbb{C}$ on which f is analytic? Such subsets would have to exclude the three poles.

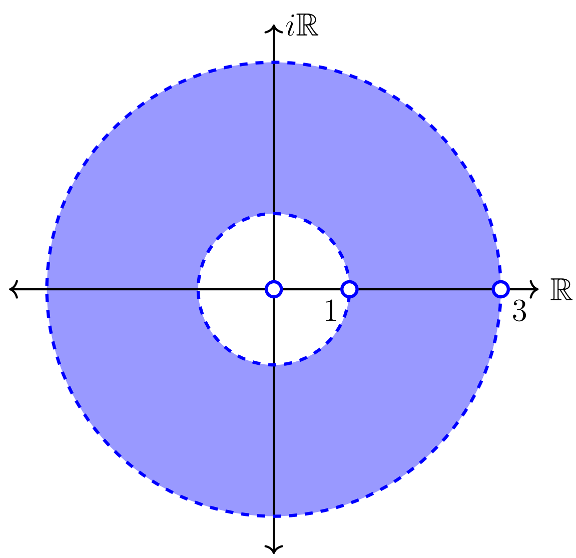
We might start with the largest punctured balls on which f is analytic. These are the sets of points $z \in \mathbb{C}$ such that $0 < |z| < 1$, $0 < |z-1| < 1$, and $0 < |z-3| < 2$.



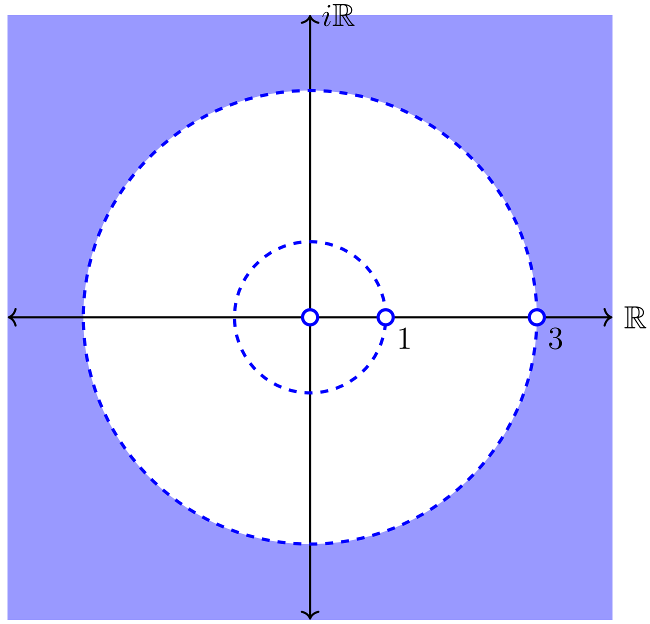




We might also consider regions "between" the singularities. One such region, which is almost a ball, is the "ring" of points z such that $1 < |z| < 3$. This is really the open ball $\mathcal{B}(0;3)$ with the closed ball $\overline{\mathcal{B}}(0;1)$ removed from its center.



Another similar region is the set of z such that $3 < |z|$, which is the whole plane with the ball $\overline{\mathcal{B}}(0;3)$ removed.



We place under one name the different subsets of $\mathbb{C}$ that appeared in the preceding example.

42.2 Definition. *Let $z_0 \in \mathbb{C}$ and $0 \leq r < R \leq \infty$. The* **ANNULUS CENTERED AT** z_0 **OF INNER RADIUS** r **AND OUTER RADIUS** R *is*

$$\mathcal{A}(z_0; r, R) := \{z \in \mathbb{C} \mid r < |z - z_0| < R\}.$$

42.3 Example. The function f from Example 42.1 is analytic on the annuli $\mathcal{A}(0; 0, 1)$, $\mathcal{A}(0; 1, 3)$, $\mathcal{A}(0; 3, \infty)$, $\mathcal{A}(1; 0, 1)$, and $\mathcal{A}(3; 0, 2)$.

42.4 Problem. Let $z_0 \in \mathbb{C}$. Prove the following set equalities for annuli.

(i) If $0 < R < \infty$, then $\mathcal{A}(z_0; 0, R) = \mathcal{B}^*(z_0; R)$.

(ii) $\mathcal{A}(z_0; 0, \infty) = \mathbb{C} \setminus \{z_0\}$.

(iii) If $0 < r < \infty$, then $\mathcal{A}(z_0; r, \infty) = \mathbb{C} \setminus \overline{\mathcal{B}}(z_0; r)$.

We can now state the principal result about the series behavior of an analytic function on an annulus.

42.5 Theorem. *Let $z_0 \in \mathbb{C}$ and $0 \leq r < R \leq \infty$. Suppose that $f \colon \mathcal{A}(z_0; r, R) \to \mathbb{C}$ is* *analytic. Then there exist unique analytic functions*

$$f_R \colon \mathcal{B}(0; R) \to \mathbb{C} \qquad \text{and} \qquad f_r \colon \mathcal{B}(0; 1/r) \to \mathbb{C},$$

where we interpret $\mathcal{B}(0; 1/0) = \mathcal{B}(0; \infty) = \mathbb{C}$*, such that* $f_r(0) = 0$ *and*

$$f(z) = f_R(z - z_0) + f_r\left(\frac{1}{z - z_0}\right)$$

for each $z \in \mathcal{A}(z_0; r, R)$*. We may expand* f_R *and* f_r *as power series centered at 0 to find*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \tag{42.64}$$

where for each $k \in \mathbb{Z}$*, the coefficient* a_k *satisfies*

$$a_k = \frac{1}{2\pi i} \int_{|z - z_0| = s} \frac{f(z)}{(z - z_0)^{k+1}} \, dz \tag{42.65}$$

for any $s \in (r, R)$*.*

The ordered pair (f_R, f_r) *is the* **Laurent decomposition** *of* f *on* $\mathcal{A}(z_0; r, R)$*; the series* (42.64) *is the* **Laurent series** *of* f *on* $\mathcal{A}(z_0; r, R)$*; and the coefficients* (42.65) *are the* **Laurent coefficients** *of* f *on* $\mathcal{A}(z_0; r, R)$*. The function* f_r *is the* **principal part** *of the Laurent decomposition; the mapping* $f_r((\cdot - z_0)^{-1})$ *may also be called the principal part. The doubly infinite series on the right of* (42.64) *is defined to be the sum of the two series on the left.*

The formula (42.65) is useful for estimating the Laurent coefficients in terms of f, but it rarely provides an expedient way of actually calculating the coefficients. As with Taylor series, the strategy is to reduce a new Laurent expansion to an old one (or an old Taylor series).

Laurent decompositions and series meld analysis and geometry. The same function f may be defined on different annuli centered at a point z_0, and it is likely that f will have different Laurent decompositions and series on those different annuli. We saw this with Taylor series: changing the center of the series changes the coefficients of the series. But now the center of the annulus can stay the same, and if the radii change, so may the Laurent decomposition and series.

42.6 Example. The function

$$f(z) = \frac{1}{z(z-1)}$$

is analytic on $\mathbb{C} \setminus \{0, 1\}$. Consequently, f is analytic on the annuli $\mathcal{A}(0; 1, 1) = \mathcal{B}^*(0; 1)$, $\mathcal{A}(1; 0, 1) = \mathcal{B}^*(1; 1)$, and $\mathcal{A}(0; 1, \infty)$. We will find (different) Laurent series for f on each annulus. First, it will be helpful to have the partial fractions decomposition

$$f(z) = -\frac{1}{z} + \frac{1}{z-1}. \tag{42.66}$$

(i) *Decomposition on* $\mathcal{A}(0;1,1)$. We want to write f as a series in the powers z^k. The term $-1/z$ in (42.66) already has this form, so we rewrite

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty}(-1)z^k.$$

Here we used the geometric series, since $|z| < 1$. Then

$$f(z) = -\frac{1}{z} + \sum_{k=1}^{\infty}(-1)z^k, \ 0 < |z| < 1.$$

The principal part is the mapping $z \mapsto -1/z$. This representation of f resembles our prior results for poles: the order of the pole is 1, and the power z^{-1} appears in the sum, but there are no negative powers z^k with $k \le -2$.

(ii) *Decomposition on* $\mathcal{A}(0;1,\infty)$. Again we need a series expansion of the term $1/(z-1)$ in powers of z. We can exploit the geometric series again, if we remember that $1 < |z|$, and therefore $1/|z| < 1$:

$$\frac{1}{z-1} = \frac{1}{z\left(1-\dfrac{1}{z}\right)} = \frac{1}{z}\left(\frac{1}{1-\dfrac{1}{z}}\right) = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}} = \sum_{j=1}^{\infty}\frac{1}{z^j}.$$

Then

$$f(z) = -\frac{1}{z} + \sum_{j=1}^{\infty}\frac{1}{z^j} = \sum_{j=2}^{\infty}\frac{1}{z^j}.$$

Note that now there are infinitely many negative powers of z in the sum!

(iii) *Decomposition on* $\mathcal{A}(1;1,1)$. Now we need to write the term $-1/z$ as a sum of powers of $z-1$. We can make $z-1$ appear by adding zero in the denominator of $1/z$ and then seeing the structure $1/(1-w)$ in $1/z$ for some w:

$$-\frac{1}{z} = -\frac{1}{z-1+1} = -\frac{1}{1-(-1)(z-1)} = -\sum_{k=0}^{\infty}[(-1)(z-1)]^k = \sum_{k=0}^{\infty}(-1)^{k+1}(z-1)^k.$$

Then

$$f(z) = \frac{1}{z-1} + \sum_{k=0}^{\infty}(-1)^{k+1}(z-1)^k, \ 0 < |z-1| < 1.$$

We note that f has a pole of order 1 at 1, and the only negative power of $z-1$ that appears in the sum is $(z-1)^{-1}$.

42.7 Problem. The following identities are often useful when computing the Laurent series of a rational function with simple poles. Let z, $w \in \mathbb{C}$ with $|z| \neq |w|$. Show that

$$\frac{1}{z-w} = \begin{cases} \dfrac{1}{z\left(1-\dfrac{w}{z}\right)} = \displaystyle\sum_{k=0}^{\infty} \frac{w^k}{z^{k+1}}, & |w| < |z| \\ -\dfrac{1}{w\left(1-\dfrac{z}{w}\right)} = -\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}, & |z| < |w|. \end{cases}$$

Day 43: Wednesday, April 26

Recommended reading

We discussed how Laurent coefficients help classify isolated singularities. This is the content of Theorem 4.5.17, whose proof is contained in the proofs of Theorems 4.5.12 and 4.5.15. Then we started integrating functions around closed curves in annuli and saw how one, and only one, of the Laurent coefficients really determines the integral. That is the content of Example 4.4.9. The key takeaway from that example is the last sentence on p. 271. See also the first paragraph in Section 5.1 on pp. 293–294. Finally, we started talking about the winding number. This is not discussed in the book, but it will give us a rigorous way of measuring the orientation of a closed curve relative to a point, and of defining the interior and exterior of a closed curve.

43.1 Example. The function

$$f(z) = \frac{\cos(z)}{z^{2023}}$$

has a pole of order 2023 at 0, and

$$\frac{\cos(z)}{z^3} = \frac{1}{z^{2023}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k!)} z^{2k-2023}.$$

This is the Laurent series for f on $\mathcal{A}(0;0,\infty) = \mathbb{C} \setminus \{0\}$. Of course, we could rewrite the series so that it is given strictly in terms of powers of z, and that series would have the form $\sum_{k=-2023}^{\infty} b_k z^k$ for some coefficients b_k with $b_{-2023} \neq 0$.

The Taylor series for a function analytic on a ball contains all the essential "data" for that function in its coefficients. If we know the countable sequence of coefficients in the Taylor series—a somewhat less than one-dimensional set of data—then we know everything about that function in two dimensions on that ball. What data is contained in the Laurent coefficients of a function? Here we must remember that geometry, not just analysis, plays a role. In the preceding example, we saw that a function could have two very different Laurent series depending on the underlying annuli. If, in the case of an isolated singularity, we choose the annulus to be a punctured ball, we can glean a complete characterization of the singularity from the behavior of the Laurent coefficients.

To ease our passage, we point out that if $f\colon \mathcal{A}(z_0; r, R) \to \mathbb{C}$ is analytic for some $z_0 \in \mathbb{C}$ and $0 \le r < R \le \infty$, and if

$$f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{b_{-k}}{(z - z_0)^k}$$

for $z \in \mathcal{A}(z_0; r, R)$ and some coefficients $b_k \in \mathbb{C}$, then by uniqueness, the coefficients b_k are the Laurent coefficients of f. Specifically, we could define

$$g_R \colon \mathcal{B}(0; R) \to \mathbb{C} \colon w \mapsto \sum_{k=0}^{\infty} b_k w^k \quad \text{and} \quad g_r \colon \mathcal{B}(0; 1/r) \to \mathbb{C} \colon w \mapsto \sum_{k=1}^{\infty} b_{-k} w^k$$

to see that g_R and g_r are analytic and $g_r(0) = 0$. Since $f(z) = g_R(z - z_0) + g_r((z - z_0)^{-1})$ on $\mathcal{A}(z_0; r, R)$, the pair (g_R, g_r) is the Laurent decomposition of f on $\mathcal{A}(z_0; r, R)$.

43.2 Theorem. *Let $z_0 \in \mathbb{C}$ and $R > 0$. Suppose that $f\colon \mathcal{B}^*(z_0; R) \to \mathbb{C}$ is analytic and let $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ be the Laurent expansion of f on the annulus $\mathcal{A}(z_0; 0, R) = \mathcal{B}^*(z_0; R)$. Then*

(i) *f has a removable singularity at z_0 if and only if $a_k = 0$ for $k \le -1$.*

(ii) *f has a pole of order $m \ge 1$ at z_0 if and only if $a_{-m} \ne 0$ and $a_k = 0$ for $k \le -(m+1)$.*

(iii) *f has an essential singularity at z_0 if and only if $a_k \ne 0$ for infinitely many $k \le -1$.*

Proof. **(i)** ($\Longrightarrow$) Suppose that f has a removable singularity at z_0. Then f has an analytic continuation $\widetilde{f}$ to $\mathcal{B}(z_0; R)$. Write $\widetilde{f}(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$; then $f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ for all $z \in \mathcal{B}^*(z_0; R)$. Consequently, this is the Laurent series for f on the annulus $\mathcal{A}(z_0; 0, R) = \mathcal{B}^*(z_0; R)$; by the uniqueness of the decomposition, we have $a_k = 0$ for $k \le -1$.

($\Longleftarrow$) If $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for $z \in \mathcal{B}^*(z_0; R)$, then an analytic continuation of f to $\mathcal{B}(z_0; R)$ is just this series. Consequently, $\lim_{z \to z_0} f(z) = a_0$ exists, and so f has a removable singularity at z_0.

(ii) ($\Longrightarrow$) For some $\rho \in (0, R]$, there is an analytic function $p\colon \mathcal{B}(z_0; \rho) \to \mathbb{C}$ such that

$$f(z) = \frac{p(z)}{(z - z_0)^m}, \ z \in \mathcal{B}^*(z_0; \rho).$$

Write $p(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ with $b_0 = p(z_0) \ne 0$. Then

$$f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^{k-m} = \sum_{j=-m}^{\infty} b_{j+m} (z - z_0)^j, \ z \in \mathcal{B}^*(z_0; \rho).$$

Consequently, this is the Laurent series for f on the annulus $\mathcal{A}(z_0; 0, \rho) = \mathcal{B}^*(z_0; \rho)$, and so by the uniqueness of the decomposition $a_k = 0$ for $k \le -(m+1)$ and $a_{-m} = b_0 \ne 0$.

($\Longleftarrow$) Rewrite, for $z \in \mathcal{A}(z_0; r, R)$,

$$f(z) = \sum_{k=-m}^{\infty} a_k(z-z_0)^k = \sum_{k=-m}^{\infty} a_k(z-z_0)^{k+m}(z-z_0)^{-m} = \frac{1}{(z-z_0)^m}\sum_{k=-m}^{\infty} a_k(z-z_0)^{k+m}$$
$$= \frac{1}{(z-z_0)^m}\sum_{j=0}^{\infty} a_{j-m}(z-z_0)^j.$$

Put $p(z) := \sum_{j=0}^{\infty} a_{j-m}(z-z_0)^j$. Above we factored $f(z) = (z-z_0)^{-m}p(z)$, so the series $p(z)$ does converge for each $z \in \mathcal{A}(z_0; r, R)$. That is, the series converges for $r < |z-z_0| < R$, and so by properties of power series it converges for all $z \in \mathcal{B}(z_0; R)$. Thus p is analytic on $\mathcal{B}(z_0; R)$. Moreover, $p(z_0) = a_{-m} \neq 0$. We conclude $f(z) = (z-z_0)^{-m}p(z)$ with p analytic on a ball centered at z_0 and $p(z_0) \neq 0$; hence f has a pole of order m at z_0.

(iii) ($\Longrightarrow$) Since z_0 is an essential singularity of f, z_0 is not a removable singularity, and so it cannot be the case that $a_k = 0$ for all $k \leq -1$. But z_0 is also not a pole, so it cannot be the case that $a_k = 0$ for all $k \leq -(m+1)$ for some integer $m \geq 1$. Thus, given any integer $m \geq 1$, there must be some integer $k < -m$ such that $a_k \neq 0$. We can therefore construct a sequence of infinitely many distinct points (a_{m_k}) such that $m_{k+1} < m_k < 0$ and $a_{m_k} \neq 0$ for all k.

($\Longleftarrow$) If $a_k \neq 0$ for infinitely many $k \leq -1$, then z_0 cannot be a removable singularity nor a pole, and so z_0 must be an essential singularity. ■

43.3 Problem (Riemann removability criterion). Let $z_0 \in \mathbb{C}$ and $R > 0$. Suppose that $f \colon \mathcal{B}^*(z_0; R) \to \mathbb{C}$ is analytic. Prove that f has a removable singularity at z_0 if and only if there exist $\rho \in (0, R]$ and $M > 0$ such that $|f(z)| \leq M$ on $\mathcal{B}^*(z_0; \rho)$. In other words, f has a removable singularity at z_0 if and only if f is bounded on some ball centered at z_0. [Hint: *first use the fact that if* $\lim_{z\to z_0} f(z)$ *exists, then* f *is bounded near* z_0*. For the converse, let* (a_k) *be the Laurent coefficients of* f*; show that* $a_k = 0$ *for* $k \leq -1$ *by using the integral definition* (42.65) *for* $s \in (0, \rho]$ *and the ML-inequality. What happens in the limit of this integral as* $s \to 0^+$*?*]

43.4 Problem. Let $z_0 \in \mathbb{C}$ and $R > 0$. Suppose that $f \colon \mathcal{B}^*(z_0; R) \to \mathbb{C}$ is analytic. Prove that the following are equivalent.

(i) f has a pole of order $m \geq 1$ at z_0.

(ii) $\lim_{z\to z_0}(z-z_0)^m f(z)$ exists and is nonzero.

(iii) $\lim_{z\to z_0}(z-z_0)^{m+1} f(z) = 0$.

(iv) There exist $\rho \in (0, R]$ and $M > 0$ such that

$$|f(z)| \leq \frac{M}{|z-z_0|^m} \text{ for } z \in \mathcal{B}^*(z_0; \rho).$$

43.5 Problem. Let $z_0 \in \mathbb{C}$ and $R > 0$. Suppose that $f\colon \mathcal{B}^*(z_0; R) \to \mathbb{C}$ is analytic. Prove that f has a removable singularity at z_0 if and only if $\lim_{z \to z_0}(z - z_0)f(z) = 0$. In the context of Problem 43.4, explain why we might euphemistically call a removable singularity a "pole of order 0."

So very many of our labors have involved line integrals. We built and characterized antiderivatives via line integrals, thereby completing one of the major stories of real-variable calculus in the complex setting. Moreover, we learned that the integral is *the* tool for extracting data about functions—specifically via the Cauchy integral formula and Taylor coefficients. That story is more or less complete, and we will not typically succeed in finding antiderivatives for analytic functions on annuli.

43.6 Problem. Explain why by evaluating the line integral

$$\int_{|z-z_0|=s} \frac{dz}{z - z_0}.$$

By taking $r < s < R$, conclude that the annulus $\mathcal{A}(z_0; r, R)$ is not an elementary domain.

Nonetheless, we might ask what we can learn about line integrals of analytic functions over closed curves in annuli. Such integrals appeared so often in our former work that it is natural to pursue them further. So, let $z_0 \in \mathbb{C}$ and $0 \le r < R \le \infty$, and let $f\colon \mathcal{A}(z_0; r, R) \to \mathbb{C}$ be analytic. Let (f_R, f_r) be the Laurent decomposition of f in $\mathcal{A}(z_0; r, R)$, and let γ be a closed curve in $\mathcal{A}(z_0; r, R)$. Then

$$\int_\gamma f = \int_\gamma \left(f_R(z - z_0) + f_r((z - z_0)^{-1})\right) dz = \int_\gamma f_R(z - z_0)\ dz + \int_\gamma f_r((z - z_0)^{-1})\ dz$$
$$= \int_\gamma f_r((z - z_0)^{-1})\ dz. \qquad (43.67)$$

43.7 Problem. Recall that $f_R\colon \mathcal{B}(0; R) \to \mathbb{C}$ is analytic. Use this and the hypothesis that γ is a closed curve in $\mathcal{A}(z_0; r, R)$ to show that

$$\int_\gamma f_R(z - z_0)\ dz = 0.$$

Recall that

$$f_r((z - z_0)^{-1}) = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}.$$

Suppose that we can interchange the sum and integral to find

$$\int_\gamma f_r((z - z_0)^{-1})\ dz = \int_\gamma \sum_{k=1}^{\infty} a_{-k} \frac{a_k}{(z - z_0)^k}\ dz = \sum_{k=1}^{\infty} a_{-k} \int_\gamma \frac{dz}{(z - z_0)^k}. \qquad (43.68)$$

We will justify this interchange eventually. Then, for $k \le -2$,

$$\int_\gamma \frac{dz}{(z-z_0)^k} = 0. \tag{43.69}$$

43.8 Problem. Explain why (43.69) does not follow from any version of the Cauchy integral theorem that we know but does follow from the fundamental theorem of calculus.

We combine (43.67), (43.68), and (43.69) to conclude

$$\int_\gamma f = a_{-1} \int_\gamma \frac{dz}{z-z_0}. \tag{43.70}$$

For the purposes of calculating $\int_\gamma f$, all of the other data from the Laurent series was irrelevant; only the particular coefficient a_{-1} matters. Using the definition of a_{-1} from (42.65), the formula (43.70) reads

$$\int_\gamma f = \left(\frac{1}{2\pi i}\int_{|z-z_0|=s} f(z)\ dz\right)\left(\int_\gamma \frac{dz}{z-z_0}\right). \tag{43.71}$$

The line integral of f over γ is therefore the product of two integrals—one an integral of f over a (more or less) arbitrary circle, and one an integral of a "tame" rational function over the given curve γ. In other words, the data of the line integral—the curve γ and the integrand f—decouple into two integrals, one dependent on f (but not γ), and one dependent on γ (but not f), and both dependent on the center z_0 of the underlying annulus.

Both factors in (43.71) will reappear in our subsequent study of integrals in more general domains. We name and examine the second factor, adjusted slightly, first.

43.9 Definition. *Let γ be a closed curve in $\mathbb{C}$ and let $z \in \mathbb{C}$ be a point that is not in the image of γ. Then the* **WINDING NUMBER OF γ WITH RESPECT TO z** *is*

$$\chi(w;z) := \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}.$$

We can now rewrite (43.71) once again. Here is a summary of our work.

43.10 Theorem. *Let $z_0 \in \mathbb{C}$ and $0 \le r < R \le \infty$. Let $f\colon \mathcal{A}(z_0;r,R) \to \mathbb{C}$ be analytic and let γ be a closed curve in $\mathcal{A}(z_0;r,R)$. Then*

$$\int_\gamma f = \left(\int_{|z-z_0|=s} f(z)\ dz\right)\chi(\gamma;z_0), \ \ r < s < R.$$

We will develop and generalize this formula to the highly useful situation in which f has a finite number of isolated singularities within an elementary domain. First, however, we focus on the geometry of the winding number.

43.11 Example. Although it is not at all obvious at first glance, the winding number does what it promises. For $k \in \mathbb{Z} \setminus \{0\}$, $r > 0$, and $z_0 \in \mathbb{C}$, define

$$\gamma\colon [0, 2\pi] \to \mathbb{C}\colon t \mapsto z_0 + re^{ikt}.$$

Intuitively, we should view γ as "tracing out" the circle of radius r centered at z_0 a total number of $|k|$ times, with the circle oriented counterclockwise if $k > 0$ and clockwise if $k < 0$.

Now let $z \in \mathbb{C}$ with $|z - z_0| \neq r$. We can calculate

$$\int_\gamma \frac{dw}{w - z} = \begin{cases} 2\pi ik, & |z - z_0| < r \\ 0, & |z - z_0| > r, \end{cases} \tag{43.72}$$

and so

$$\chi(\gamma; z) = \begin{cases} k, & |z - z_0| < r \\ 0, & |z - z_0| > r. \end{cases}$$

In other words, $\chi(\gamma; z)$ "counts" the number of times that γ "winds around" z_0: either k times (with the sign of k indicating orientation) if z is "inside" the circle of radius r centered at z_0, or no times at all if z is "outside" this circle.

43.12 Problem. Obtain the first identity in (43.72) by justifying each of the following equalities:

$$\int_0^{2\pi} \frac{rike^{ikt}}{z_0 + re^{ikt} - z}\, dt = \int_0^{2k\pi} \frac{rie^{i\tau}}{z_0 + re^{i\tau} - z}\, d\tau = \sum_{j=1}^k \int_{2(j-1)\pi}^{2j\pi} \frac{rie^{i\tau}}{z_0 + re^{i\tau} - z}\, d\tau$$
$$= k\int_0^{2\pi} \frac{rie^{i\tau}}{z_0 + re^{i\tau} - z}\, d\tau = k\int_{|w - z_0| = r} \frac{dw}{w - z} = 2\pi ik.$$

For the second, use the Cauchy integral theorem. What is the appropriate star domain?

Day 44: Friday, April 28

Recommended reading

We discussed the residue theorem, which is stated and proved rather differently as Theorem 5.1.2. Residues are defined in Definition 5.1.1. Applications of the residue theorem are staggering and manifold. The presentation in the textbook throughout Chapter 5 is excellent and, like the rest of the book, abundant in examples and detail. I strongly encourage you to keep this book for future reference.

The following situation often arises in practice. Let $\mathcal{D} \subseteq \mathbb{C}$ be an elementary domain—so $\mathcal{D}$ is open and connected, and if $h\colon \mathcal{D} \to \mathbb{C}$ is analytic and γ is a closed curve in $\mathcal{D}$, then $\int_\gamma h = 0$. Fix a finite number of distinct points $z_1, \ldots, z_n \in \mathcal{D}$, and let $f\colon \mathcal{D} \setminus \{z_k\}_{k=1}^n \to \mathbb{C}$

be analytic. Choose $r_k > 0$ such that $\mathcal{B}(z_k; r_k) \subseteq \mathcal{D}$; then f is analytic on each $\mathcal{B}^*(z_k; r_k)$, and consequently each z_k is an isolated singularity of f. Let (f_{r_k}, f_{∞_k}) be the Laurent decomposition of f on the annulus $\mathcal{B}^*(z_k; r_k)$. Then each principal part f_{∞_k} is entire.

44.1 Problem. Why?

It turns out that if we "remove" all the principal parts from f, then we are left with a rather nice function.

44.2 Lemma. *Under the hypotheses and notation above, the function*

$$g \colon \mathcal{D} \setminus \{z_k\}_{k=1}^n \to \mathbb{C} \colon z \mapsto f(z) - \sum_{k=1}^n f_{\infty_k}\left(\frac{1}{z - z_k}\right) \tag{44.73}$$

has removable singularities at z_k and consequently has an analytic continuation $\widetilde{g}$ on $\mathcal{D}$.

Proof. Fix an integer j satisfying $1 \le j \le n$. For $1 \le k \le n$ with $k \ne j$, since f_{∞_k} is entire, we have

$$\lim_{z \to z_j} f_{\infty_k}((z - z_k)^{-1}) = f_{\infty_k}((z_j - z_k)^{-1}).$$

Next, since

$$f(z) = f_{r_j}(z - z_j) + f_{\infty_j}\left(\frac{1}{z - z_j}\right)$$

on $\mathcal{B}^*(z_j; r_j)$, and since f_{r_j} is analytic on $\mathcal{B}(0; r_j)$, we also have

$$\lim_{z \to z_j} \big(f(z) - f_{\infty_j}((z - z_j)^{-1})\big) = \lim_{z \to z_j} f_{r_j}(z - z_j) = f_{r_j}(0).$$

Thus

$$\lim_{z \to z_j} g(z) = \lim_{z \to z_j} \left(f(z) - \sum_{k=1}^n f_{\infty_k}((z - z_k)^{-1})\right) = f_{r_j}(0) + \sum_{\substack{k=1 \\ k \ne j}}^n f_{\infty_k}((z_j - z_k)^{-1}).$$

Consequently, g has removable singularities at each z_j and therefore has an analytic continuation to each z_j. Specifically, this analytic continuation is

$$\widetilde{g} \colon \mathcal{D} \to \mathbb{C} \colon z \mapsto \begin{cases} g(z), & z \in \mathcal{D} \setminus \{z_k\}_{k=1}^n \\ \lim\limits_{z \to z_k} g(z), & z = z_k, \end{cases}$$

with the limits given above. ∎

Let $\widetilde{g}$ be as in the preceding lemma and let γ be a closed curve in $\mathcal{D}$. Since $\widetilde{g}$ is analytic and $\mathcal{D}$ is an elementary domain, we have

$$\int_\gamma \widetilde{g} = 0.$$

Now we add the additional hypothesis that none of the points z_k belong to the image of γ. Then $\widetilde{g}(z) = g(z)$ for all z in the image of γ, and so

$$0 = \int_\gamma \widetilde{g} = \int_\gamma g.$$

Using the definition of g in (44.73), we have

$$0 = \int_\gamma f(z)\ dz - \sum_{k=1}^n \int_\gamma f_{\infty_k}\left(\frac{1}{z - z_k}\right)\ dz.$$

Expand f_{∞_k} as the series

$$f_{\infty_k}(w) = \sum_{j=1}^\infty a_{k,-j}w^k,\ w \in \mathbb{C}, \qquad a_{k,-j} = \frac{1}{2\pi i}\int_{|z-z_k|=s} \frac{f(z)}{(z-z_k)^{-j+1}}\ dz,\ 0 < s < r.$$

We claim that

$$\int_\gamma f_{\infty_k}\left(\frac{1}{z - z_k}\right)\ dz = 2\pi i a_{k,-1}\chi(\gamma; z_k)$$

and thus

$$0 = \int_\gamma f - 2\pi i \sum_{k=1}^n a_{k,-1}\chi(\gamma; z_k). \tag{44.74}$$

This is essentially the same reasoning that gave us Theorem 43.10.

Now it is time to name the coefficients $a_{k,-1}$.

44.3 Definition. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose that $f\colon \mathcal{B}^*(z_0; r) \to \mathbb{C}$ is analytic, and let $f(z) = \sum_{k=-\infty}^\infty a_k(z - z_0)^k$ be the Laurent series for f on $\mathcal{B}^*(z_0; r)$. The* **RESIDUE OF** f **AT** z_0 *is the coefficient a_{-1}, and we write*

$$\operatorname{Res}(f; z_0) = a_{-1} = \frac{1}{2\pi i}\int_{|z-z_0|=s} f(z)\ dz,\ 0 < s < r.$$

All of our work up to and including (44.74) can be summarized in one theorem, the mightiest and proudest of the Cauchy theorems.

44.4 Theorem (Cauchy's residue theorem). *Let $\mathcal{D} \subseteq \mathbb{C}$ be an elementary domain and let $z_1, \ldots, z_n \in \mathbb{C}$ be distinct points. Let $f\colon \mathcal{D} \setminus \{z_k\}_{k=1}^n \to \mathbb{C}$ be analytic, and let γ be a closed curve in $\mathcal{D}$. Then*

$$\int_\gamma f = 2\pi i \sum_{k=1}^n \operatorname{Res}(f; z_k)\chi(\gamma; z_k).$$

As with Theorem 43.10, the residue theorem perfectly decouples the problem of computing a line integral into two distinct problems: the analytic problem of finding the residue (which involves the integrand and not the underlying curve) and the geometric problem of computing the winding number (which involves the curve and not the function)—the two problems are connected in that they both involve the isolated singularities of the integrand.

Here is how the residue theorem is often used "in practice," and how the hypothetical situation mentioned above often naturally occurs. Suppose that f is a function analytic on all but finitely many points of $\mathcal{D}$; call those points $z_1, \ldots, z_n$. Suppose that μ is a curve in $\mathcal{D}$ that does not contain these points $z_1, \ldots, z_n$ in its image. And suppose that for some reason we want to compute $\int_\mu f$. We are not assuming that μ is closed, and so we cannot use the residue theorem. However, perhaps we can judiciously choose another path ν in $\mathcal{D} \setminus \{z_k\}_{k=1}^n$ such that the composition $\mu \oplus \nu$ is defined and also closed.

If we are lucky, the line integral $\int_\nu f$ will be "easy" to evaluate—or at least easier than $\int_\mu f$. Then the residue theorem tells us

$$\int_{\mu\oplus\nu} f = 2\pi i \sum_{k=1}^n \operatorname{Res}(f; z_k)\chi(\mu \oplus \nu; z_k),$$

and so we find

$$\int_\mu f = 2\pi i \sum_{k=1}^n \operatorname{Res}(f; z_k)\chi(\mu \oplus \nu; z_k) - \int_\nu f.$$

Success with this sort of "residue calculus" then hinges on two tasks: calculating residues and choosing the auxiliary curve μ. For the former, there are a host of techniques that enable one to avoid the definition; the latter, for better or for worse, is often as much of an art as it is a science.

44.5 Problem. Let $\mathcal{D} \subseteq \mathbb{C}$ be an elementary domain and let $f \colon \mathcal{D} \to \mathbb{C}$ be analytic.

(i) Assume that the residue theorem is true but that we did not know at all the Cauchy integral theorem. (This is absurd, since we used the Cauchy integral theorem in is proof, but, for the nonce, assume it.) Show that the residue theorem implies the Cauchy integral theorem: $\int_\gamma f = 0$ for any closed curve γ in $\mathcal{D}$.

(ii) Show that the residue theorem implies the following more general version of the Cauchy integral formula: if γ is a closed curve in $\mathcal{D}$ and $z \in \mathcal{D}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = f(z)\chi(\gamma; z). \tag{44.75}$$

(iii) Show that (44.75) implies the version of the Cauchy integral formula in Theorem 33.1. [Hint: *use Example 43.11.*]

(iv) Show that, in fact, (44.75) implies the Cauchy integral theorem as stated in (i). [Hint: *it is a fact that there is some $z \in \mathcal{D}$ such that z is not in the image of γ. Set $g(w) = (w-z)f(w)$ and apply* (44.75) *to g in lieu of f.*]