ORDINARY DIFFERENTIAL EQUATIONS

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CONTENTS

OVERVIEW OF NOTES

These are lecture notes for a first course in ordinary differential equations. The prerequisite is Calculus II, not multivariable calculus. The notes largely follow *Differential Equations* (Fourth Edition) by Blanchard, Devaney, and Hall, with some nontrivial departures, augmentations, and alterations here and there.

The notes contain three classes of problems.

(!) Problems marked (!) are meant to be attempted *immediately*. They will directly address or reinforce something that we covered in class. It will be to your great benefit to pause and work (!)-problems as you encounter them.

($\star$) Problems marked ($\star$) are intentionally more challenging and deeper than (!)-problems. The ($\star$)-problems will summarize and generalize ideas that we have discussed in class and give you broader, possibly more abstract perspectives. You should attempt the ($\star$)-problems on a second rereading of the lecture notes, after you have completed the (!)-problems and required problems from the textbook. As you prepare for an exam, you should definitely attempt all ($\star$)-problems in sections that will appear on that exam.

(+) Problems marked (+) are optional but encouraged. These will provide more background and insight than we can cover in class (and, frankly, more than is absolutely necessary for your success in the class). If you feel confident in your mastery of the material and have completed the (!)- and ($\star$)-problems, then you should attempt the (+)-problems. (It is not necessary to do (+)-problems in preparation for an exam, but it may be helpful.) In particular, if you feel that something is "missing" from the story that we are telling, perhaps some "Deeper Magic" underlying the course, you should think about the (+)-problems.

1. INTRODUCTION

1.1. Predicting the future.

The goal of this course is to learn how to predict the future. We will know the future if we can answer the **KEY QUESTION**

$$\text{“\textit{What are things like at a given moment in time?}”} \tag{KQ}$$

Throughout the course, whatever we are doing, a good way to gain perspective is to ask ourselves (KQ) in the context of the particular problem that we are studying.

One plausible answer to the vague question (KQ) is the following **KEY STATEMENT**:

$$\text{“\textit{How things are depends on how things were and how things changed.}} \tag{KS1}$$

Hopefully a moment's thought and reflection on your own personal life experiences will indicate the plausibility of (KS1).

This is a mathematics course, so we will pose our key question (KQ) and (first) key statement (KS1) in mathematical language and notation, specifically in the language and notation of calculus. First, by “things” we will mean the values of a function—maybe the number of rabbits in a certain geographic region, the percentage of a population that has a disease, the volume of a raindrop some time after its formation and before it hits the ground, or the position of an object relative to some point of origin. (There are plenty of “things” that are not so nicely described by a single quantity, and we will deal with them later.) Call this function x and let t be its independent variable. Then “how things are” at time t is the value $x(t)$, and “how things were” is the value $x(t_0)$, where $t_0 < t$.

Since we will work with functions every single day in this course, it might help to think briefly about what they really are.

1.1.1 Undefinition. *Let A and B be sets. A* **FUNCTION FROM** *A* **TO** *B is a rule that pairs each element of A with exactly one element of B.*

This is exactly what we have encountered every time we have worked with a function in our lives, right? For example, if A is a set of real numbers, we have often worked with the function x from A to the real numbers that pairs the number t with its square t^2. Of course, this is the familiar parabola, and we compress things via the notation $x(t) = t^2$. Nonetheless, as diligent learners journeying from mathematical innocence to experience, we should question what the words “rule” and “pairs” actually mean.

1.1.2 Definition. *Let A and B be sets. A* **FUNCTION** *x* **FROM** *A* **TO** *B is a set of ordered pairs such that if t is an element of A, then there is a unique element τ of B such that the pair (t, τ) belongs to the set x. In this case, we write $\tau = x(t)$.*

For example, if A is a set of real numbers, then function x from A to the real numbers that pairs the number t in A with its square t^2 is really the set of all ordered pairs (t, t^2), where t is restricted to belong to A.

1.1.3 Remark. **(i)** *Definition 1.1.2 is the true definition of a function, and it is healthy to think about it once, and then promptly forget it. Undefinition 1.1.1 is all that we need on a daily basis in this course, but if we ever go down the rabbit hole of wondering what it is that we are finding as solutions to our differential equations, the true answer is Definition 1.1.2.*

(ii) *Common to both the undefinition and the definition is the notion that functions have domains, and merely stating a formula for a function without also discussing its domain will never be adequate.*

(iii) *We should never conflate the symbol $x(t)$, which denotes the value of the function x at t, with the function x itself: x is a set of ordered pairs of numbers, while $x(t)$ is a single number. An exception to this is the case of constant functions; for example, if $x(t) = 0$ for all t, then we will just say $x = 0$, and use the symbols x and 0 interchangeably.*

Returning to our original key question and key statement, we have said that "how things are" at time t is the value of some function $x(t)$, and we know that "how things are" depends on "how things were" and "how things changed." "How things were" is just some value $x(t_0)$ with $t_0 < t$. As soon as we hear the word "change" in a mathematical context, we probably should think of the derivative.

1.1.4 Definition. *Let x be a function defined on an interval I, and let t be a point in I. Then x is* DIFFERENTIABLE AT t *if the limit*

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \tag{1.1.1}$$

exists, in which case we call this limit the DERIVATIVE OF x AT t*. If x is differentiable at all t in I, then x is* DIFFERENTIABLE ON I*, and we denote by $\dot{x}$ the function whose value is the limit (1.1.1) at each t in I.*

1.1.5 Remark. **(i)** *It can be shown that the limit (1.1.1) exists if and only if the limit*

$$\lim_{\tau \to t} \frac{x(\tau) - x(t)}{\tau - t} \tag{1.1.2}$$

exists, in which case both limits are equal. That is, if x is differentiable at t, then we may calculate $\dot{x}(t)$ using either (1.1.1) or (1.1.2). By the way, the symbol τ is the Greek letter "tau," and we will use it often when we want to write something that looks almost like t.

(ii) *We will sometimes use the notation*

$$\frac{d}{dt}$$

to indicate the "process" of taking the derivative symbolically. For example, if x is the function defined by $x(t) := t^2$, then we might express the calculation of the derivative of x

as

$$\dot{x}(t) = \frac{d}{dt}[t^2] = 2t.$$

Likewise, if x and y are differentiable functions, then we may express the calculation of the derivative of their product as

$$\frac{d}{dt}[xy] = \dot{x}y + x\dot{y},$$

as the product rule teaches us. In other words, the notation d/dt is helpful bookkeeping for signifying symbolic computations.

(iii) *We will not, however, use the notation*

$$\frac{dx}{dt}$$

for the derivative. We will almost never use the notation x', except to distinguish variables; for example, if $x(t) = t^2$ and $f(x) = \cos(x)$, and if $y(t) := f(x(t))$, then the chain rule says

$$\dot{y}(t) = \frac{d}{dt}[f(x(t)] = f'(x(t))\dot{x}(t) = -\sin(t^2)(2t) = -2t\sin(t^2).$$

Now we can put together "how things were" and "how things changed" to figure out "how things are." This will involve the derivative, but, unfortunately, not the derivative at any one moment in the past. Namely, calculus tells us that the net change in "things" from time t_0 to time t is the integral

$$\int_{t_0}^{t} \dot{x}(\tau)\ d\tau.$$

And, in particular, the fundamental theorem of calculus tells us

$$\int_{t_0}^{t} \dot{x}(\tau)\ d\tau = x(t) - x(t_0).$$

Rearranging, we have

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\tau)\ d\tau. \tag{FTC}$$

The statement (FTC) is the exact mathematical formulation of our first key statement (KS1).

$$\text{how things are} \longrightarrow x(t) = \underbrace{x(t_0)}_{\text{how things were}} + \overbrace{\int_{t_0}^{t} \dot{x}(\tau)\ d\tau}^{\text{how things changed}}$$

And so it looks like our key question (KQ) is easy to resolve: figure out one value in the past and all the values of the derivative between the past and the present, then integrate and add. All too easy! But how do we figure out that derivative? We have gone from needing the value of x at one time t to needing the values of $\dot{x}$ at all times τ between t_0 and t. This seems like a worse problem.

Perhaps it will help to think about a concrete situation. Consider the growth of a population of rabbits. (We will be doing this *a lot*.) The growth rate probably depends on the amount of food available to the rabbits, and that amount should vary over the course of the year. So, the growth rate should depend on the time at which we are measuring the growth. And if there are more rabbits, it is likely that there are more mating pairs available, and thus even more rabbits to come. But if there are too many rabbits, maybe they will eat all the food, and the population will decline, and then there will be fewer pairs to produce new rabbits. So, the growth rate of the rabbit population should depend on both time and the number of rabbits.

This suggests that "how things change" depends on two variables: the time at which we are measuring the change and the state of things at that time. We formalize this as a *second* **KEY STATEMENT**:

"How things change depends on the time of change and how things are then." (KS2)

We write this symbolically as

$$\dot{x}(t) = f(t, x(t)). \tag{ODE}$$

Here f is a function of the *two* variables t and x, and so we will often consider values of the form $f(t, x)$. Using x as both the *dependent* variable of a function that depends on t and as the *independent* variable of a function that also depends on x can lead to no end of confusion, but we will simply have to live with it.

We have now met the principal object of study in this course: the ordinary differential equation. More precisely, the equation (ODE) above is a **FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION (ODE)**, since only the first derivative $\dot{x}$ of x appears in the equation. We will spend most of the course solving equations like (ODE), as well as contemplating what "solving" actually means.

For now, here is why we still have a problem, and why we have a whole course left to complete together. The equation (FTC) expresses the values of x in terms of one past value $x(t_0)$ and the derivative $\dot{x}$. But (ODE) says that $\dot{x}$ depends on x! If we combine (FTC) and (ODE), then we get

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau. \tag{IE}$$

The equation (IE) is an **INTEGRAL EQUATION** for x.

This seems to be useless. To know the value $x(t)$, we have to be able to calculate that integral over the interval $[t_0, t]$, and so we will need to know the value $f(t, x(t))$ — which means we need to know $x(t)$. But that is exactly what we are trying to find.

The goal of a course in ordinary differential equations is not only to predict the future but also to resolve the tension among the statements (FTC), (ODE), and (IE). While the study of integral equations like (IE) is a highly worthwhile and fruitful activity, it turns out that the differential equation (ODE) will be more tractable to study. Specifically, we will learn how to write a variety of worthwhile problems in the form (ODE), or a form more or less[1] like that. Then we will learn techniques for *solving* (ODE), which will depend greatly on what f is.

Setting up a differential equation to model a particular phenomenon is itself a nontrivial task. Most of our models and phenomena will involve many simplifying assumptions; the goal here is to get to the differential equations, not model the whole world. In fact, even when we have reduced a phenomenon into a reasonable mathematical model, we will often consider special, borderline silly, cases of that model as "toy problems" to illustrate mathematical techniques without the burden of reality.

"Solving" an equation will involve three related approaches.

1. In the *analytic* approach, we find explicit formulas for solutions to problems; this is probably what we think "solving" an equation means (and for good reason), but only very special equations have explicit solution formulas.

2. In the the *qualitative* approach, we use certain features of problems to guarantee the existence of solutions and then predict their behavior; often knowing how a solution behaves over long times is more useful than knowing its precise formula.

3. In the *numerical* approach, we convert our "continuous" problems to "discrete" ones that a computer can be taught to solve with results that a human brain can be taught to interpret.

We should not expect any one approach to work all the time, and even within one of these three camps there may be different variations on the same theme that we might want to consider. In particular, you, as the individual student, scientist, and human being that you are, will sometimes need to decide which approach works best for *you* and your particular problem—and that approach may be different from one a colleague selects. Additionally, we should not view these three approaches as immutably separate and distinct; very often we will tackle the same problem with more than one approach and learn different things in the process.

In olden times, when we did not have the advantage of computers that we now do today, the study of differential equations was largely restricted to the analytic approach, and a course in differential equations was sometimes viewed as a "cookbook" class to learn "recipes" for analytic approaches only. *But no more.* Now, this is not to disparage "formulaic" or "symbolic" techniques, and indeed there are a handful (at least four, but probably no more than seven) that you simply *must* know to be successful in this course and whatever requires differential equations afterward. We will certainly strive to develop a robust understanding of those analytic techniques. A computer with its symbolic toolbox can do most of those techniques, but a computer probably cannot *interpret* the results of those techniques, or even set up the problem for you in the first place so that it's amenable to those techniques. Furthermore, if you have to go to a computer for every little thing, like calculating

$$\frac{d}{dt}[t^2] = 2t \quad \text{or} \quad \int \cos(3t)\ dt = \frac{\sin(3t)}{3} + C,$$

[1] Since "force = mass × acceleration" involves the second derivative $\ddot{x}$, not all interesting problems have quite the form (ODE), but nonetheless (ODE) is the paradigm for most problems to come.

then something has gone very wrong with your education. (Harsh but true.) On the other hand, if you do not have a computer to help you with a problem that actually matters to your life and the larger world, then you probably have much bigger problems!

We will therefore approach analytic techniques with something of a dual mindset. We will celebrate them when we find them but neither limit our studies strictly to them nor despair if we cannot find them. The following **ANALYST'S CREED** thus summarizes our relationship with formulas in this course:

"Having a formula for something is not the same as understanding that thing." (AC)

We shall profess the creed frequently.

This is where we finished on Monday, August 18, 2023.

1.2. Exponential growth.

Our first model of population growth is *exponential growth*. This is a very simple model—too simple, in the end, to be realistic. But the mathematical structure of this model, our methods of solving it, and its similarities to and differences from physical reality all set the tone for how we will proceed in the rest of the course. We begin by thinking a bit about population models in general.

1.2.1. Population models.

First, a disclaimer: the value of these models in our course is that they offer fairly straightforward translations from "physical" principles into the language of ODE, and the resulting ODE are all essential problems for you to understand. Our hypotheses may seem silly, or restrictive; they almost surely are. A perennial challenge in mathematical modeling is the balance between maintaining a *physically realistic* model and a *mathematically tractable* problem.

Second, a calculus caveat: typically when we count populations, we do so with integers. But calculus is inherently continuous, and taking the derivative of an integer-valued function should make us uneasy. We'll always assume that either the population is so large, or our units of measurement are so skewed, that taking noninteger, fractional counts of this population makes sense — like saying that 8.8 million people live in New York City right now.

With the fine print out of the way, how do we get our hands on models? Frequently models arise because of *proportional relationships* between quantities. Intuitively, two quantities are proportional if one is always a multiple of another; for example, the circumference of every circle is proportional to its radius (equivalently, its diameter). Here is a formalization of this concept for future use.

1.2.1 Definition. *Two time-dependent quantities A and B are* **PROPORTIONAL** *if there exists a real number r such that*

$$A(t) = rB(t)$$

for all t at which A and B are defined.

For this definition to make sense, we probably should take $r \neq 0$. Otherwise, if $r = 0$, then since $0 = 0 \cdot B(t)$, any time-dependent quantity B could be seen as proportional to 0, which is absurd.

In constructing population models, the time-dependent quantity A will be the rate of change of the population. That is, if there are $x(t)$ members of the population at time t, then $A(t) = \dot{x}(t)$. The choice of B will vary from model to model; indeed, the construction of B is the heart of our modeling process.

1.2.2. The exponential growth model.

How fast a population is growing depends on many factors. As we noted earlier, a higher population allows for more interactions among members and thus more mating opportunities and thus more offspring; a lower population could do the opposite. One very simple model of population growth, then, is to assume that a population's rate of change is directly proportional to the current population. Then we are assuming that x and $\dot{x}$ are proportional; in the notation of Definition 1.2.1, we have $A(t) = \dot{x}(t)$ and $B(t) = x(t)$, and so we are assuming there is a constant r such that

$$\dot{x}(t) = rx(t) \tag{1.2.1}$$

for all times t.

With r fixed, define a function f of the two real numbers t and x by

$$f(t, x) := rx.$$

Note that f is really independent of t, and so writing f as a function of both t and x is redundant at best and artificial at worst. Then the equation (1.2.1) has the form

$$\dot{x}(t) = f(t, x(t)), \tag{1.2.2}$$

which is the general form of the first-order ODE, as we saw in (ODE).

For simplicity, we often suppress some of the t-dependence in our notation. For example, we write

$$\dot{x} = rx \quad \text{instead of} \quad \dot{x}(t) = rx(t) \quad \text{or} \quad \dot{x} = f(t, x) \quad \text{instead of} \quad \dot{x}(t) = f(t, x(t)).$$

Context will often make clear whether we are referring to x as the independent variable of f or the dependent variable of t.

The number r is a **PARAMETER** of the problem (1.2.1)—a number that is constant in a given incarnation of the problem but whose value could change to allow the problem to model different scenarios. Depending on the type of population that we are trying to model with exponential growth, we will probably need different values of r.

We almost done with the set-up of our model, but we are missing one key piece of data. Populations typically do not arise *ex nihilo*. Say that we are tracking the growth of this population from time $t = 0$. (There will be plenty of circumstances when we want to track growth starting at some time $t_0 \neq 0$, but starting with $t_0 = 0$ is just simpler here.) Assume that we know the initial population: $x(0) = x_0$ for some number x_0. Then we want to solve a more specific problem than (1.2.1): the *pair* of equations

$$\begin{cases} \dot{x}(t) = rx(t) \\ x(0) = x_0. \end{cases} \tag{1.2.3}$$

This pair of equations is not merely an ordinary differential equation but rather an **INITIAL VALUE PROBLEM (IVP)**, since it asks for the solution x of the ODE $\dot{x} = rx$ with the "initial value" $x(0) = x_0$.

1.2.3. The analytic solution for exponential growth.

The good news, but also the least important news, is that we already know how to solve (1.2.3) analytically. Calculus teaches us the following.

1.2.2 Theorem. *There is a function E defined on $(-\infty, \infty)$ such that*

$$\begin{cases} \dot{E}(t) = E(t), \ -\infty < t < \infty \\ E(0) = 1. \end{cases} \tag{1.2.4}$$

Specifically, E can be defined via the power series

$$E(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} := \lim_{n \to \infty} \sum_{k=0}^{n} \frac{t^k}{k!}. \tag{1.2.5}$$

That is, E solves the IVP (1.2.3) with $r = 1$ and $x_0 = 1$. From this IVP, it is possible to deduce a host of wonderful properties of E, including the following.

1.2.3 Corollary. *The function E that solves (1.2.4) also enjoys the following properties.*

(exp1) $E(t) > 0$ *for all* t.

(exp2) $E(t + \tau) = E(t)E(\tau)$ *for all numbers* t *and* τ.

(exp3) E *is strictly increasing in the sense that if* $t_1 < t_2$*, then* $E(t_1) < E(t_2)$.

(exp4) $\lim_{t \to \infty} E(t) = \infty$ *and* $\lim_{t \to -\infty} E(t) = 0$.

Of course, we usually do not write E for this function.

1.2.4 Definition. *We define $e := E(1)$ and define the symbol e^t by $e^t := E(t)$. We also write* $\exp(t) = E(t)$.

Now, if we follow our calculus intuition and define

$$x(t) := x_0 e^{rt},$$

then we can check that x does solve (1.2.3). This situation will often arise throughout the course: we have a differential equation and we have a function that we *think* solves it. Maybe we we have a rigorous procedure for generating a formula for that function, or maybe we have a dodgy, speculative, fly-by-the-seat-of-our-pants method[2], or maybe we have neither, and we have just made a very lucky guess. Regardless of *how* we got the putative solution, we can always *check* whether or not it is a solution by the venerable method of plugging and chugging.

Remember that our problem is

$$\dot{x} = rx \quad \text{and} \quad x(0) = x_0.$$

The initial condition does not require calculus to check, so we do that first:

$$x(t) = x_0 e^{rt} \Longrightarrow x(0) = x_0 e^{r \cdot 0} = x_0 e^0 = x_0 \cdot 1 = x_0.$$

So, the initial condition works.

Now we check the actual differential equation. It is best to do this working out each side separately. The left side of the differential equation requires us to compute $\dot{x}$, and we do so by the chain rule:

$$x(t) = x_0 e^{rt} \Longrightarrow \dot{x}(t) = \frac{d}{dt}[x_0 e^{rt}] = x_0 \frac{d}{dt}[e^{rt}] = x_0 r e^{rt} = r(x_0 e^{rt}). \tag{1.2.6}$$

The right side of the differential equation is just multiplication (later in the course the right side will be more complicated!):

$$x(t) = x_0 e^{rt} \Longrightarrow rx(t) = r(x_0 e^{rt}). \tag{1.2.7}$$

We compare (1.2.6) and (1.2.7) and see that

$$x(t) = x_0 e^{rt} \Longrightarrow \dot{x}(t) = r(x_0 e^{rt}) = rx(t).$$

Thus $x(t) = x_0 e^{rt}$ does indeed solve the IVP (1.2.3).

1.2.5 Remark. *Here is the point: to check that a function solves an IVP, see if the initial condition is true, and then see if the function solves the differential equation by computing each side of the differential equation separately. We can check if a function solves a problem even if we have no idea how to construct that function in the first place.*

But this also illustrates the limits of the analytic approach. Say that we knew all about derivatives but we never met an exponential before. (Such calculus classes do happen, sometimes under the umbrella of "later transcendentals.") We would know what the equation $\dot{x} = rx$ is *asking*, but we would have no idea of the correct formula for its solution.

Or say that we did know that the exponential was the solution, but we had no idea of the limit behavior of the exponential at $\pm\infty$, or how to evaluate/approximate the exponential

[2] Such a dodgy, speculative, or fly-by-the-seat-of-our-pants procedure is often called **FORMAL**—in the sense that we only care about the "form" of how things appear and nothing more rigorous.

at particular finite moments in time. Then we would not be able to say anything about what the solution is really *doing*, and so we would not be meeting the goal of the course and predicting the future. This is our first invocation of the analyst's creed (AC) in practice: having a formula is not the same as understanding.

We will now take up a qualitative analysis of the exponential growth model (1.2.3) that would enable us to predict the end behavior of the model assuming that we know calculus but not that we know calculus of exponentials, or that exponentials are involved in the solution to the IVP (1.2.3). If this seems silly now, bear in mind that shortly we will meet a more complicated model of population growth for which our existing calculus background will offer absolutely no help in constructing analytic solutions.

One last remark—we have found *an* analytic solution to the IVP (1.2.3). Is that it? Are there any other solutions? This is the question of *uniqueness* of solutions—if we are going to predict the future, we should be predicting at most one future. This is a bit tricky, and it will require some subsequent cleverness and new tools to address.

1.2.6 Problem (!). Explain why just the ODE $\dot{x} = rx$, not the full IVP (1.2.3), definitely does not have unique solutions. In general, to get uniqueness we need to impose an initial condition.

1.2.4. Qualitative analysis of exponential growth.

We will study the **EXPONENTIAL GROWTH** model

$$\dot{x} = rx \quad \text{and} \quad x(0) = x_0$$

qualitatively in the following steps. Throughout, we will pretend that we do not know that the solution is $x(t) = x_0 e^{rt}$, although we should be looking out for the similarities in our results here to the exponential that will inevitably, naturally arise.

1. We have to start somewhere, so assume that $r = 0$. Then the differential equation $\dot{x} = rx$ just becomes

$$\dot{x}(t) = 0$$

for all t at which x is defined. This means that x is not changing, and so x is constant. Then for all t at which x is defined,

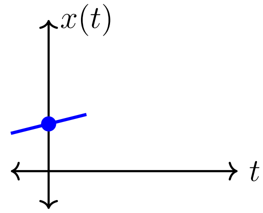
$$x(t) = x(0) = x_0$$

And since constant functions are defined at all real numbers, it looks like the solution is just $x(t) = x_0$ for all t.

2. Now suppose $r > 0$. It is reasonable to assume that for any "real" population, the initial number of members is positive: $x_0 > 0$. Then since $r > 0$ and $x_0 > 0$, we have

$$\dot{x}(0) = rx(0) = rx_0 > 0.$$

Thus x is increasing at time $t = 0$; equivalently, the slope of x at $t = 0$ is positive. So, if we look at the graph of x "close" to $t = 0$, it looks like this.



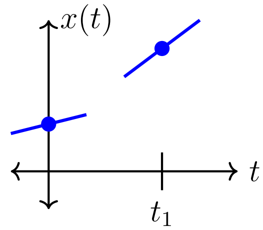
We know more: since x is increasing at time $t = 0$, for a time t_1 "close to" but greater than $t = 0$, we should have $x(0) < x(t_1)$. In particular, $x(t_1) > 0$. Thus

$$\dot{x}(t_1) = rx(t_1) > 0,$$

and so x is again increasing at time $t = t_1$. Moreover, x is increasing *faster* at time $t = t_1$ than at $t = 0$, since

$$\dot{x}(t_1) = rx(t_1) > rx(0) = \dot{x}(0).$$

Then if we sketch the slopes of x at both times $t = 0$ and $t = t_1$, we get a picture like this.

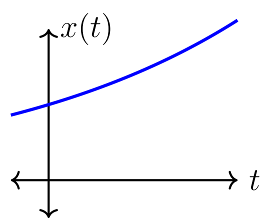


In particular, since $\dot{x}(t_1) > \dot{x}(0)$, the slope of the tangent at time $t = t_1$ should be steeper than the tangent at time $t = 0$.

We can then iterate this analysis starting at time $t = t_1$ to suggest that x is strictly increasing on its domain. Moreover, we can study the concavity of x by calculating its second derivative, $\ddot{x}$. We have

$$\ddot{x}(t) = \frac{d}{dt}[\dot{x}(t)] = \frac{d}{dt}[rx(t)] = r\dot{x}(t) = r(rx(t)) = r^2x(t). \tag{1.2.8}$$

Thus $\ddot{x}(t) > 0$ whenever $x(t) > 0$. Since $x(0) > 0$ and x is strictly increasing, we expect $x(t) > 0$ for all times t at which x is defined. And so $\ddot{x}(t) > 0$ for all t, which means that x is concave up. Moreover, since $\ddot{x}(t) > 0$ for all t, the derivative $\dot{x}$ is strictly increasing, and so the graph of x keeps getting steeper as time goes on. Here, then, is a candidate for the graph of x.



In particular, because x is increasing and always getting steeper, we expect the values of x to blow up to ∞ over long times. That is, if x is defined for all times t, we expect

$$\lim_{t\to\infty} x(t) = \infty.$$

3. But this raises a question. Is the solution x really defined for all times $t > 0$, or all times t in $(-\infty, \infty)$? Plenty of functions are not defined for all real numbers; maybe there is some "end" time $t = T_\omega$ at which x fails to be defined.

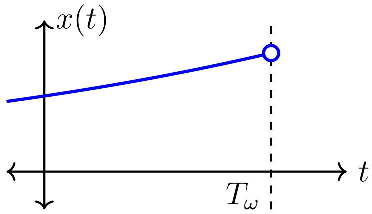
The assumptions and work above tell us that x is strictly increasing and concave up on $[0, T_\omega)$ with $x(0) > 0$, and so calculus tells us that

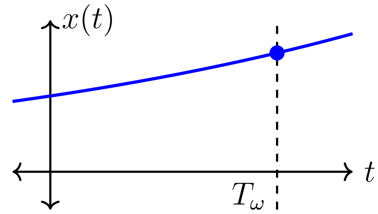
$$L := \lim_{t\to T_\omega^-} x(t)$$

should exist, either as a finite (and positive) real number. If $L < \infty$, then

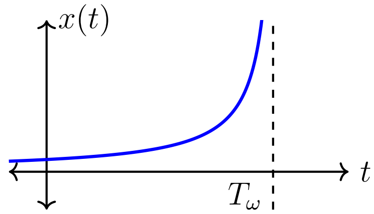
$$\lim_{t\to T_\omega^-} \dot{x}(t) = \lim_{t\to T_\omega^-} rx(t) = rL.$$

Thus x would have finite slope at $t = T_\omega$, and so it looks like we could just continue drawing the graph of x past $t = T_\omega$. But then x could be defined for values of t larger than T_ω, which goes against our assumption above that x is not defined past T_ω.





So, we expect that if $T_\omega < \infty$, then $\lim_{t\to T_\omega^-} x(t) = \infty$. Since x is concave up, we conclude that the graph of x might have a vertical asymptote at $t = T_\omega$ like the picture below.



4. We did all the analysis above assuming $r \geq 0$ and $x_0 > 0$. If $x_0 = 0$, then

$$\dot{x}(0) = rx(0) = rx_0 = 0,$$

and so the graph of x has a horizontal tangent at $t = 0$. This does not tell us if x is increasing, or decreasing, near $t = 0$.

However, if we stare at the differential equation $\dot{x} = rx$ long enough (and it is always a good idea to stare long and hard at differential equations), we might see that plugging in $x = 0$ on both sides makes for a true equality. That is, suppose $x(t) = 0$ for all t. Then

$$\dot{x}(t) = \frac{d}{dt}[0] = 0 \quad \text{and} \quad rx(t) = r \cdot 0 = 0.$$

Thus taking $x(t) = 0$ for all t solves $\dot{x} = rx$, and consequently the initial value problem with $x_0 = 0$. In other words, a nonexistent population that grows exponentially. . .remains nonexistent. In particular,

$$\lim_{t\to\infty} x(t) = 0,$$

a pretty sharp contrast to the case $x_0 > 0$.

5. Suppose $x_0 < 0$, but keep $r > 0$. From the modeling point of view, a negative population is probably useless, but mathematically it merits examination. We redo the analysis above more succinctly.

We have

$$\dot{x}(0) = rx_0 < 0,$$

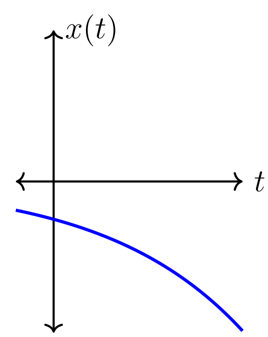
so x is decreasing at time $t = 0$. Then for times t close but greater than 0, we have

$$x(t) < x(0) < 0,$$

so x is decreasing and becoming "more negative." Also, by (1.2.8), we have

$$\ddot{x}(t) = r^2 x(t) < 0,$$

so x is concave down. Then the graph of x might be

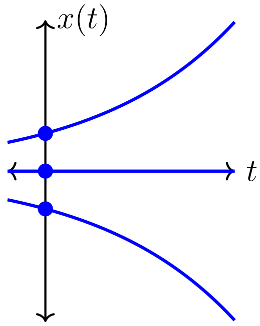


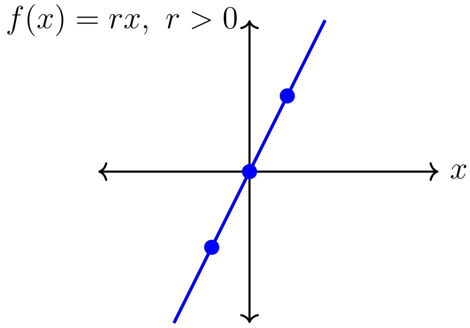
This suggests that, if x is defined for all times up to and not beyond some T_ω, which may be finite or infinite, then

$$\lim_{t\to\infty} x(t) = -\infty.$$

This is a remarkable change in the end behavior of x from the cases $x_0 > 0$ and $x_0 = 0$!

6. We can summarize all of the work above by graphing three putative solutions (as functions of time t) to $\dot{x} = rx$ with $r > 0$ and different signs on $x(0) = x_0$. We see that as soon as the initial value $x(0)$ "passes through" the constant solution $x = 0$, the end behavior changes radically. We also graph $f(x) = rx$ as a function of x.





To be clear, in the left graph, the independent variable is t and the dependent variable is x, while on the right x is independent and f is dependent. (Notation is a nightmare!) It is no accident that the graphs of increasing solutions x have initial conditions $x(0) = x_0$ where $f(x_0) > 0$, while the graphs of decreasing solutions have initial conditions $x(0) = x_0$ with $f(x_0) < 0$. This suggests a strategy going forward: to gain intuition about the behavior of solutions to a problem $\dot{x} = f(x)$, study f.

1.2.7 Problem ($\star$). Repeat all of the analysis and arguments above for the IVP

$$\dot{x} = rx \quad \text{and} \quad x(0) = x_0$$

assuming $r < 0$. Where specifically does the condition $r < 0$ change the work above?

1.2.8 Problem (+). Parameters and initial conditions are ubiquitous and essential in models: they allow us to "tune" the model to reflect different relevant physical scenarios. However, from the point of view of analytically solving a problem, parameters and initial conditions are often superfluous. One way to set them to convenient values (often 0 or 1) is the broad technique of **RESCALING**.

(i) Suppose that we want to solve analytically the exponential growth IVP

$$\begin{cases} \dot{x} = rx \\ x(0) = x_0 \end{cases} \quad \text{with} \quad r \neq 0 \quad \text{and} \quad x_0 \neq 0, \tag{1.2.9}$$

but we have no idea where to start. (We are assuming $r \neq 0$ and $x_0 \neq 0$ to prevent the problem from being totally trivial.) The problem might simpler with $r = 1$ and $x_0 = 1$.

Here is how we rescale to achieve this: let α and β be fixed real numbers and define

$$y(\tau) := \alpha x(\beta\tau). \tag{1.2.10}$$

Show that if x solves (1.2.9), then y solves

$$\begin{cases} \dot{y} = \beta r y \\ y(0) = \alpha x_0 \end{cases} \tag{1.2.11}$$

(ii) Since $r \neq 0$ and $x_0 \neq 0$, the form of (1.2.11) suggests that we take $\alpha = 1/x_0$ and $\beta = 1/r$. Then y solves

$$\begin{cases} \dot{y} = y \\ y(0) = 1. \end{cases} \tag{1.2.12}$$

This is, comparatively, simpler than (1.2.9). Suppose that y solves (1.2.12) and define, via (1.2.10),

$$x(t) := \frac{1}{\alpha} y\left(\frac{t}{\beta}\right) = x_0 y(rt).$$

Using *only* this definition of x and the fact that y solves (1.2.12), show that x solves (1.2.9). *Do not use the fact that $y(\tau) = e^{\tau}$, even though you know it. The point is to show that if we can solve the simpler rescaled problem* (1.2.12)*, then we can solve the original problem* (1.2.9)*. This allows us to focus our attention solely on the simpler rescaled problem.*

Our qualitative analysis of exponential growth revealed a number of gaps in our understanding of the model. These gaps motivate the following four fundamental questions, which we will later state more precisely. Answering these questions will be a central part of our analysis of any differential equation.

1. Do solutions exist?

2. Are solutions unique?

3. Where are solutions defined?

4. What do solutions do over long times?

1.3. More complicated population models.

Exponential growth models, as the name indicates, allow for only two kinds of end behavior for a population (assuming a positive initial condition): either the population explodes to possess infinitely many members ($x(t) \to \infty$), or it dies off ($x(t) \to 0$). This is wholly unrealistic for many populations, which often exhibit neither kind of extreme behavior. But "real" populations do not just grow in a manner dependent on the current population. Many other factors affect (negatively) the rate of a population's growth: internal conflict/interaction within the population or with another population; limited food, space, or necessary resources; the spread of a disease through the population; harvesting of the population by some outside source; birth control. By diversifying the notion of proportionality that gave us the exponential growth model, we can develop other population models whose end behaviors are more realistic. These models will help us motivate and test many of our forthcoming techniques.

1.3.1. Growth with a time-dependent rate.

Recall that the exponential growth model reads $\dot{x} = r_0 x$, where r_0 is a fixed real number (we are writing r_0 now, not r, so we can use r for another object soon), and we obtained this model by assuming that the growth rate of a population was directly proportional to its current size. If we relax our definition of "proportional" (Definition 1.2.1), we can obtain some more nuanced models.

Suppose that the "constant" of proportionality is time-dependent; instead of a real number r_0, use a function of time, so the model now reads

$$\dot{x} = r(t)x. \tag{1.3.1}$$

This allows us to tune the growth rate to be more sensitive to the time at which we are measuring growth. At any moment in time, the growth rate is still proportional to the population at that time, but the rate of proportionality can vary. Specifically, this could incorporate the periodic, nonconstant effects of seasons or fertility cycles on the population.

The ODE (1.3.1) generalizes exponential growth (take $r(t) = r_0$ for all t) and will turn out to be one of the most important equations that we solve analytically later. Right now, our calculus knowledge probably does not help us conjure up solutions to (1.3.1) from scratch, but we can always check a solution candidate.

1.3.1 Example. The ODE

$$\dot{x} = \sin(t)x$$

takes the "constant" of proportionality to be periodic. In particular, since the sine oscillates between 1 and -1, sometimes the growth rate is positive, and sometimes negative; thus the population oscillates between increasing and decreasing.

We can check that $x(t) := e^{-\cos(t)}$ is a solution for all t:

$$\dot{x}(t) = \frac{d}{dt}[e^{-\cos(t)}] = e^{-\cos(t)}(-1)(-\sin(t)) = \sin(t)e^{-\cos(t)} = \sin(t)x(t).$$

This solution is itself periodic: $x(t + 2\pi) = x(t)$. This might model a periodic, recurring pattern of growth and death in a population.

1.3.2 Problem ($\star$). While we do not yet know how to solve $\dot{x} = r(t)x$ for a general function r, we might be able to guess. We have seen solutions for exponential growth $\dot{x} = r_0 x$ in the form $x(t) = Ce^{r_0 t}$, where C is a constant, and the function $R_0(t) := r_0 t$ satisfies $\dot{R}_0(t) = r_0$. We might generalize this as follows. Check that $x(t) = e^{R(t)}$ solves $\dot{x} = r(t)x$, where R is an antiderivative of R—that is, $\dot{R}(t) = r(t)$.

This is where we finished on Wednesday, August 16, 2023.

1.3.2. Logistic growth.

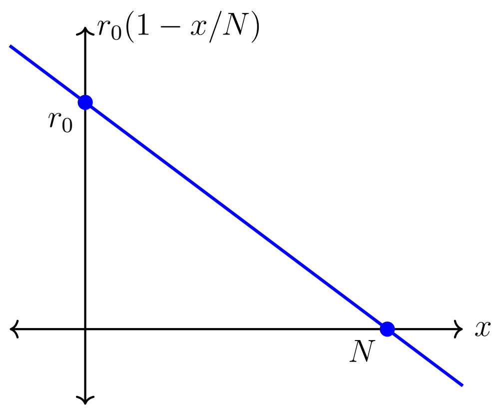
Here is another way of relaxing the definition of "proportional." Suppose that the growth rate of a population is proportional to its current population in the sense that the "constant" of proportionality depends on the current population. Then the population model would read

$$\dot{x} = r(x(t))x, \tag{1.3.2}$$

where r is a function not of time but of the population itself. Annoyingly, we will write x both for the function determining the population and for the independent variable of the function r.

This model (1.3.2) allows us to tune the growth rate to increase or decrease as the population varies. Perhaps we want the growth rate to decrease if the population passes a certain "sustainability" threshold: beyond this level of population, the population is too large to continue to grow (perhaps due to competition among its members for resources like space or food). That is, we might want there to be a population level N such that $r(x) > 0$ for $x < N$ but $r(x) < 0$ for $x > N$. Additionally, to keep the growth model as simple as possible when the population is indeed growing—that is, to keep the growth at an exponential level for suitably small populations—we might want $r(x) \approx r_0$ for $x \approx 0$ and some constant r_0. There are many such functions r that do this; one of the simplest is

$$r(x) = r_0 \left(1 - \frac{x}{N}\right), \quad r_0,\ N > 0.$$



Then (1.3.2) reads

$$\dot{x} = r_0 \left(1 - \frac{x}{N}\right) x.$$

This ODE is called the **LOGISTIC EQUATION**. Unlike exponential growth, it has two parameters, both r_0 and N. And unlike exponential growth, it is **NONLINEAR** in the sense that it reads

$$\dot{x} = f(x),$$

where

$$f(x) := r_0 \left(1 - \frac{x}{N}\right) x = r_0 x - \frac{r_0 x^2}{N}$$

is not a linear function of the state variable x (rather, it is quadratic).

It is possible to find analytic solutions to the logistic equation, but the calculus and algebraic steps are much more burdensome than for exponential growth, and the final formula is not very enlightening without further algebraic manipulation (here it helps to know what you are looking for before you start looking for it). We will see how a combination of qualitative[3], numerical, and theoretical techniques can yield a rich amount of information about the logistic equation without having a formula for it—and then we will see how those techniques leave unanswered one particular question for which a formula will be very much appreciated.

1.3.3 Example. Previously we saw that the constant function $x(t) = 0$ solved the exponential growth equation $\dot{x} = r_0 x$. More generally, for an ODE $\dot{x} = f(x)$, we might find numbers x such that $f(x) = 0$. Then since the derivative of a constant function is 0, any such number will solve $\dot{x} = f(x)$.

For the logistic equation $\dot{x} = r_0(1 - x/N)x$, if we put $f(x) = r_0(1 - x/N)x$, then the roots of f are $x = 0$ and $x = N$. (This is assuming $r_0 \neq 0$; otherwise, f has many more roots!). If we put, say, $x(t) = N$ for all t, then

$$\dot{x}(t) = \frac{d}{dt}[N] = 0 \quad \text{and} \quad r_0\left(1 - \frac{x(t)}{N}\right)x(t) = r_0\left(1 - \frac{N}{N}\right)N = 0,$$

thus $x(t) = N$ solves the logistic equation. Such constant solutions to ODE are called **EQUILIBRIUM SOLUTIONS**, and we should always look for them first; finding them is a root-finding task, not a calculus one.

1.3.4 Problem ($\star$). The logistic equation appears to build in the feature that the population will start to decrease if it gets too large for its environment. (Whether or not the population actually does this—whether or not the logistic equation actually has solutions, and what their long-time behavior is—has yet to be established.) We can modify the logistic equation further by demanding that the population decreases if it is too *small*—perhaps if there are too few members to support a viable number of mating pairs. By studying where the function

$$f(x) := r_0(\frac{x}{M} - 1)\left(1 - \frac{x}{N}\right)x$$

is positive or negative, check that the ODE

$$\dot{x} = r_0(\frac{x}{M} - 1)\left(1 - \frac{x}{N}\right)x$$

has these features. What are the roles of the parameters M and N in this model?

1.3.5 Problem (+). In Problem 1.2.8, we rescaled the exponential growth model to simplify the parameter and initial condition. Like much of our treatment of exponential growth in Section 1.2, that work was not really necessary, as we had very explicit solutions for the model. Rather, the work illustrated what can be complicated techniques in a familiar

[3] These qualitative techniques will be similar to, but more precise and nuanced than, our previous explorations with exponential growth. We could repeat some of that qualitative analysis on the logistic equation using just our background in calculus, but we would still have to contend with the same unanswered questions that remained from exponential growth. Instead, we will use logistic growth as a regular "toy" problem when meeting new techniques.

setting.

Here is a less familiar setting. Suppose that x solves the logistic IVP

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{N}\right) \\ x(0) = x_0, \end{cases} \qquad \text{where} \quad r \neq 0 \quad \text{and} \quad N \neq 0. \tag{1.3.3}$$

(i) Let α and β be fixed real numbers. Show that if

$$y(\tau) = \alpha x(\beta\tau), \tag{1.3.4}$$

where x solves (1.3.3), then y solves

$$\begin{cases} \dot{y}(\tau) = (\beta r)y\left(1 - \dfrac{y}{\alpha N}\right) \\ y(0) = \alpha x_0. \end{cases}$$

(ii) Conclude that if we choose $\alpha = 1/N$, $\beta = 1/r$, and $y_0 = x_0/N$, then y solves the much simpler IVP

$$\begin{cases} \dot{y} = y(1 - y) \\ y(0) = y_0. \end{cases} \tag{1.3.5}$$

Note that unlike in Problem 1.2.8, we do not have enough degrees of freedom to set $y_0 = 1$ here.

(iii) Following (1.3.4), put

$$x(t) := \frac{1}{\alpha}y\left(\frac{t}{\beta}\right) = Ny(rt),$$

where y solves (1.3.5). Using *only* this definition of x and the fact that y solves (1.3.5), conclude that x solves the original, more complicated logistic equation (1.3.3). Thus, if all we care about are formulas for solutions, we just have to solve (1.3.5).

This is where we finished on Friday, August 18, 2023.

1.4. Fundamental terminology and guiding questions.

If we are going to talk sensibly and successfully about a mathematical concept, we better be sure that we understand all the words involved in that concept. So, we will now formalize the notion of ODE that we have used in the previous two sections. Finally, we will crystalize some guiding questions for the future.

Here is our primary object of study for the foreseeable future.

1.4.1 Definition. *An* **FIRST-ODER ORDINARY DIFFERENTIAL EQUATION (ODE)** *is an equation of the form*

$$\dot{x} = f(t, x), \tag{1.4.1}$$

where f is a function defined for t in some interval (a, b) and x in some interval (c, d). The values $a = -\infty$, $c = -\infty$, $b = \infty$, and $d = \infty$ are all allowed. We will sometimes call t the **TIME** *or* **TEMPORAL** *variable and x the* **STATE** *or* **PHASE** *variable.*

A **SOLUTION** *to the equation* (1.4.1) *is a differentiable function x defined on an interval I such that*

$$\dot{x}(t) = f(t, x(t)) \tag{1.4.2}$$

for all x in I and that $\dot{x}$ is continuous on I. For (1.4.2) *to make sense, tacitly we require I to be a subinterval of (a, b) and $x(t)$ to belong to (c, d).*

Why *this* definition? Definitions are not handed down to us from on high, even though it often looks like that; definitions exist because, over time and after thought, people come to realize that those definitions are the best way to capture a concept.

1. Hopefully the necessity of the "pointwise" condition (1.4.2) is obvious. After all, (1.4.1) is really an equality of functions, and functions are equal when they are equal at every point in their domains.

2. Next, why does a solution have to be defined on an interval? Remember that we should be thinking of t as time. If our model only predicts the future from, say, 9 am to 12:19 pm, and then again from 1:11 pm to 5 pm, that would be a pretty strange model—it just stops working for one hour during the day. Requiring the solution to an ODE to be defined on an interval helps keep the flow of time unbroken. Do not neglect consideration of the domain of a solution to an ODE; often our solutions will end up defined for all time in $(-\infty, \infty)$, but not always. *Whenever you find a formula for a solution to a differential equation, you must be able to state its domain.*

3. Third, why should the solution's derivative be continuous? This is actually a pretty natural condition to demand. Most of the time, things not only change continuously; their rate of change evolves continuously. (Not always: flip a light switch.) Requiring $\dot{x}$ to be continuous affords our model extra control over reality.

Definition 1.4.1 is actually rather easy to check in practice, once you have a candidate function for a solution. In general, when someone asks you to "check" that a certain function solves an ODE, you do not have to come up with the solution from scratch and get what they got. Rather, plug and chug.

1.4.2 Example. The ODE

$$\dot{x} = \sin(t)x + 2te^{-\cos(t)}$$

looks like nothing that we have seen before. Nonetheless, we can check that the function

$$x(t) := t^2 e^{-\cos(t)}$$

solves it. We just compute

$$\dot{x}(t) = \frac{d}{dt}[t^2e^{-\cos(t)}] = 2te^{-\cos(t)} + t^2e^{-\cos(t)}(-1)(-\sin(t)) = \sin(t)t^2e^{-\cos(t)} + 2te^{-\cos(t)}.$$

Then we recognize that the first term on the right really contains x. That is, $\sin(t)t^2e^{-\cos(t)} = \sin(t)x(t)$. And so we have shown

$$\dot{x}(t) = \sin(t)x(t) + 2te^{-\cos(t)},$$

as desired.

1.4.3 Remark. *We will sometimes overwork the letter x—for example, we might start out by saying something like "Let x and y solve the ODE $\dot{x} = f(t,x)$." The first appearance of x in that sentence is to serve as a particular function, while the second is to serve as a placeholder variable. This sentence really means that x and y are both functions that satisfy*

$$\dot{x}(t) = f(t,x(t)) \quad \text{and} \quad \dot{y}(t) = f(t,y(t))$$

for all t in their domains.

As we saw in our population models, most of the time in a "physical" scenario we do not meet just an ODE by itself, but rather one with an initial condition appended.

1.4.4 Definition. *An* **FIRST-ORDER INITIAL VALUE PROBLEM (IVP)** *is a pair of equations of the form*

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0, \end{cases} \tag{1.4.3}$$

where $\dot{x} = f(t,x)$ is an ODE, and t_0 and x_0 are given real numbers with f defined at $t = t_0$. A **SOLUTION** *to the IVP* (1.4.3) *is a function x that solves the ODE $\dot{x} = f(t,x)$ in the sense of Definition 1.4.1, that is defined at $t = t_0$, and that satisfies $x(t_0) = x_0$.*

1.4.5 Problem (!). **(i)** Check that if C is any real number, then the function

$$x(t) := Ce^{-\cos(t)} + t^2e^{-\cos(t)}$$

solves the ODE from Example 1.4.2.

(ii) Select the right constant C to solve the IVP

$$\begin{cases} \dot{x} = \sin(t)x + e^{-\cos(t)} \\ x(0) = 2. \end{cases}$$

There are plenty of interesting and worthwhile problems that are not first-order. A

SECOND-ORDER problem like

$$\ddot{x} = \dot{x} + x + \sin(t),$$

where $\ddot{x}$ denotes the second derivative of x, arises naturally in mechanical problems that are derived from Newton's second law. There are **HIGHER-ORDER** problems with derivatives beyond the second, but they rather rarely show up in practice A **FIRST-ORDER PLANAR SYSTEM** like

$$\begin{cases} \dot{x} = x + xy \\ \dot{y} = -y + 2xy, \end{cases}$$

in which x and y are *both* unknown functions, arises naturally in modeling interacting populations, such as predators and prey. Here **PLANAR** refers to the presence of two, and only two, unknown functions in the problem; we could think of the solution as an ordered pair (x, y) of functions in the two-dimensional plane. Certainly there could be more than two functions involved in a system of differential equations, but most of the key ideas for those larger systems appear without much fuss in the planar setting.

It turns out that second-order ODE and first-order systems consisting of *two* ODE are intimately related, and we can learn a lot about the one from the other. We will do so after a thorough study of first-order problems, which encompass valuable models and yield both useful techniques and useful insights for studying other kinds of problems.

Motivated by some of our evocative, but ultimately imprecise, calculus-based analysis of population models, and spurred by our success with direct integration, here are the fundamental questions of our course. They are all deeper, more nuanced versions of our original key question (KQ), now couched in the framework of ODE.

1. Do solutions to an ODE (IVP) exist? If so, how do we know that they exist? Our profession of the analyst's creed (AC) notwithstanding, is there a procedure for finding formulas for those solutions? Most broadly, *does this ODE (IVP) model allow us to predict the future?*

2. Are solutions to an ODE (IVP) unique? (Problems 1.2.6 and 1.4.5 illustrate that apparently ODE by themselves need not have unique solutions, but maybe imposing an initial condition does force uniqueness.) If we have found a solution, is it the only one? Most broadly, *does this ODE (IVP) model predict only one*[4] *future?*

3. If solutions exist to an ODE (IVP), where are they defined? What is their domain? Are they defined for all real numbers[5] in $(-\infty, \infty)$ or strictly for a subinterval? Most broadly, *for how long does this ODE (IVP) model allow us to predict the future?*

4. What do solutions to an ODE (IVP) do at the limit of their existence? For example, if a solution x is defined for all time $t \geq 0$, does the limit $\lim_{t\to\infty} x(t)$ exist as a finite real number, or as an extended real number ($\pm\infty$)? If the solution does not have a limit at ∞, is it *asymptotic* to some more familiar function x_∞ in the sense that $\lim_{t\to\infty}[x(t) - x_\infty(t)] = 0$? If a numerical or asymptotic limit exists, can we quantify "how fast" x approaches that limit?

[4] Or will we commit crimes against the Sacred Timeline by getting multiple solutions and predicting multiple futures?

[5] For all time. Always.

And if a solution is only defined up to some finite time T_ω, what happens as t gets close to T_ω? Is there some "breakdown" of the model at T_ω? Most broadly, *what happens in the future?!*

1.5. Integration.

Believe it or not, the key step in analytically solving many ODE boils down to a "direct integration" or "antidifferentiation" problem of the form

$$\dot{x} = f(t). \tag{1.5.1}$$

Here x is, as always, the unknown function, while f is some *given* function—in principle, we know everything that we might want to know about f. In particular, f should be defined and continuous on some interval I, and (1.5.1) says that

$$\dot{x}(t) = f(t)$$

for all t in I. That is, x is an **ANTIDERIVATIVE** for f on I. Calculus then teaches us that if F is another antiderivative for f on I, then there is a constant C such that

$$x(t) = F(t) + C \tag{1.5.2}$$

for all t in I.

More precisely, here is a theorem that we will use, explicitly or implicitly, quite often.

1.5.1 Theorem. *Let f be continuous on the interval I. Suppose that F is an antiderivative of F on I (so $\dot{F}(t) = f(t)$ for all t in I) and that x solves $\dot{x} = f(t)$ on I. Then there is a constant C such that*

$$x(t) = F(t) + C$$

for all t in I.

Proof. One approach is to invoke the celebrated mean value theorem. Put $G(t) := x(t) - F(t)$. Then

$$\dot{G}(t) = \dot{x}(t) - \dot{F}(t) = f(t) - f(t) = 0$$

for all t in I. This implies that G is constant on I: there is a constant C such that $G(t) = C$ for all t in I, and therefore $x(t) = F(t) + C$ for all t in I.

Why is G constant? This is the mean value theorem, which, for any distinct t_1 and t_2 in I, lets us write

$$\frac{G(t_2) - G(t_1)}{t_2 - t_1} = \dot{G}(\tau) = 0$$

for some τ between t_1 and t_2, thus $G(t_1) = G(t_2)$ for all t_1 and t_2 in I, thus G is constant on I. ■

And so, to solve (1.5.1), all that we have to do is find *one* antiderivative of f on I; then all solutions to (1.5.1) are given by (1.5.2).

1.5.2 Example. **(i)** To solve the ODE

$$\dot{x} = 2t,$$

we need to find an antiderivative of the function

$$f(t) := 2t.$$

One candidate is

$$F(t) := t^2,$$

and so all solutions to this ODE are

$$x(t) = t^2 + C$$

for some constant C. However, it would be equally correct to use the antiderivative

$$F_1(t) := t^2 + 1,$$

and then all solutions would have the form

$$x(t) = t^2 + 1 + C.$$

(ii) To solve the ODE

$$\dot{x} = 2t(t^2 + 1)^{2023},$$

we need to find an antiderivative of

$$f(t) := 2t(t^2 + 1)^{2023}.$$

If we put

$$u(t) = t^2 + 1,$$

we recognize that f is really the product

$$f(t) = [u(t)]^{2023}\dot{u}(t),$$

and therein we might recognize the chain rule to see that f is the derivative of

$$F(t) := \frac{[u(t)]^{2024}}{2024} = \frac{(t^2 + 1)^{2024}}{2024}.$$

Thus all solutions are

$$x(t) = \frac{(t^2 + 1)^{2024}}{2024} + C.$$

(iii) To solve the ODE

$$\dot{x} = e^{t^2},$$

we want to say that

$$x(t) = F(t) + C,$$

where F is an antiderivative of

$$f(t) := e^{t^2},$$

but experience in calculus might teach us that there is no “elementary” representation for this antiderivative F. (We could use the Taylor series for the exponential, but that is not quite “elementary.”)

1.5.3 Problem (!). **(i)** Suppose that $F(t) = t^3$ is an antiderivative of the function f and that G is an antiderivative of the function g. Find all solutions to the ODE

$$\dot{x} = f(t) + g(t) + 1.$$

[Hint: *your answer will involve* G.]

(ii) Suppose we also know that the function G above satisfies $G(0) = 0$. Find all solutions to the IVP

$$\begin{cases} \dot{x} = f(t) + g(t) + 1 \\ x(0) = 3. \end{cases}$$

[Hint: *there is only one solution.*]

The last example above reminds us that we may not always be able to find a "nice" formula for an antiderivative—but we know that, per the analyst's creed (AC), having a formula is not the same as understanding. Can we guarantee that antiderivatives always exist for reasonably well-behaved (i.e., continuous) functions? We certainly can, thanks to the tool of the definite integral.

A rigorous definition of the definite integral involves Riemann sums and limits of sequences—two valuable tools rather removed from the tamer limits that define derivatives, like (1.1.1). Adopting the perspective that "what things do defines what things are," we will not spend any length of time working with a general formula for the definite integral of a continuous function; instead, we will define it via its most fundamental properties. Intuitively, the definite integral $\int_a^b f(t)\ dt$ should encode the "net signed area" between the graph of f and the t-axis, where the graph is considered between the points $t = a$ and $t = b$.

Furthermore, we do not prove that any continuous function defined on an interval containing the points a and b has a definite integral $\int_a^b f(t)\ dt$. Instead, we only state the following theorem which asserts its existence and details its most fundamental and useful properties. For completeness, we do define the integral in (1.5.3) as a limit of right-endpoint Riemann sums. It turns out that if one assumes that this limit exists, then it is possible to prove all of the properties below except ($\int$3) directly from this limit definition. Perhaps the greatest conceptual and technical challenge in working rigorously with the definite integral is that there is not just one limit definition of it (unlike, essentially, the derivative); the definite integral can be expressed as virtually infinitely many different kinds of limits of Riemann sums. Far more important for us in differential equations than these limits are the properties below, the fact that all properties correspond to an intuitive notion of area, and the fact that these properties give us the fundamental theorem of calculus, which in turn gives us antiderivatives.

1.5.4 Theorem. *Let I be an interval. For each continuous function f defined on I and all points a and b in I, the limit*

$$\int_a^b f(t)\ dt := \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) \tag{1.5.3}$$

exists. This number $\int_a^b f(t)\ dt$ *is called the* **DEFINITE INTEGRAL OF** f **FROM** a **TO** b, *and it has the following properties.*

($\int$1) [Linearity in the integrand] *If* f *and* g *are continuous functions on* I *and if* α *is a real number, then*

$$\int_a^b \big(f(t)+g(t)\big)\ dt = \int_a^b f(t)\ dt + \int_a^b g(t)\ dt \quad \text{and} \quad \int_a^b \alpha f(t)\ dt = \alpha \int_a^b f(t)\ dt.$$

($\int$2) [Constants] *If* a *and* b *are any points in* I, *then*

$$\int_a^b 1\ dt = b - a.$$

($\int$3) [Additivity of the domain] *If* f *is continuous on* I *and* a, b, *and* c *are any points in* I, *then*

$$\int_a^c f(t)\ dt + \int_c^b f(t)\ dt = \int_a^b f(t)\ dt.$$

($\int$4) [Monotonicity] *If* f *is continuous on* I *and if* a *and* b *are points in* I *with* $a \le b$ *and* $f(t) \ge 0$ *for all* t *in* $[a,b]$, *then*

$$\int_a^b f(t)\ dt > 0.$$

In the number $\int_a^b f(t)\ dt$, the number a is the **LOWER LIMIT OF INTEGRATION**, the number b is the **UPPER LIMIT OF INTEGRATION**, and the function f is the **INTEGRAND**. The properties above, except ($\int$4), do not require the lower limit to be less than or equal to the upper limit. Property ($\int$1) covers the algebra of integrands; property ($\int$2) encodes what we expect the area of a rectangle to be; property ($\int$3) encodes arithmetic on the limits of integration; and property ($\int$4) reflects the notion that the graph of a nonnegative function should lie above/overlapping the t-axis and therefore have a "nonnegative" area underneath it. We also follow the convention that the definite integral is independent of the variable of integration, thus

$$\int_a^b f(t)\ dt = \int_a^b f(\tau)\ d\tau = \int_a^b f(s), \tag{1.5.4}$$

and so on. However, we never allow the variable of integration to be the same symbol that we use for a limit of integration, and so we would write

$$\int_a^t f(\tau)\ d\tau, \quad \text{not} \quad \int_a^t f(t)\ dt.$$

1.5.5 Problem ($\star$). Suppose that f and g are continuous functions defined on the interval

$I = (1, \infty)$. Suppose also that

$$\int_2^4 f(t)\ dt = 1, \qquad \int_2^3 f(t)\ dt = 2, \qquad \text{and} \qquad \int_3^4 g(t)\ dt = 3.$$

Compute

$$\int_3^4 \big(4f(t) + 5g(t)\big)\ dt.$$

Which properties from Theorem 1.5.4 did you use in this calculation?

1.5.6 Problem ($\star$). Let f be continuous on the interval I.

(i) Let a be a point in I. Take $b = a$ and $c = a$ in part ($\int$3) of Theorem 1.5.4 to conclude that

$$2\int_a^a f(t)\ dt = \int_a^a f(t)\ dt,$$

and therefore

$$\int_a^a f(t)\ dt = 0.$$

Why can we paraphrase this as "the area under a point is zero"?

(ii) Let a and b be points in I. Use ($\int$3) of Theorem 1.5.4 and the work above to conclude that

$$\int_a^b f(t)\ dt + \int_b^a f(t)\ dt = 0,$$

and therefore

$$\int_b^a f(t)\ dt = -\int_a^b f(t)\ dt.$$

Why can we paraphrase this as "flipping the limits of integration flips the sign of the integral"?

This is where we finished on Monday, August 21, 2023.

Remarkably, the six properties of integrals in Theorem 1.5.4 are enough to give us the fundamental theorem of calculus and from that, antiderivatives.

1.5.7 Theorem (FTC1). *Let f be continuous on an interval I and let t_0 be a point in I. Then the function*

$$F(t) := \int_{t_0}^t f(\tau)\ d\tau \tag{1.5.5}$$

is an antiderivative of f, i.e., $\dot{F}(t) = f(t)$ for all t in I.

1.5.8 Problem (+). Think about how you would want to prove this and explain why it suffices to show

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\left[f(\tau)-f(t)\right]\,d\tau=0.$$

You do not have to explain why this limit is true, but instead use the definition of F in (1.5.5), the definition of the derivative (1.1.1), and the properties of definite integrals from Theorem 1.5.4 to explain why this limit is the key thing to prove. [Hint: *show that for any fixed* t*, we can write the number* $f(t)$ *as* $f(t)=\left(\int_t^{t+h}f(t)\,d\tau\right)/h$*. Be careful with notation:* τ *is the variable of integration, while* t *is a fixed number.*]

1.5.9 Example. We return to part (iii) of Example 1.5.2. The function $f(t)=e^{t^2}$ is continuous on $(-\infty,\infty)$, and so $F(t):=\int_0^t e^{\tau^2}\,d\tau$ is an antiderivative of f on $(-\infty,\infty)$. Then all solutions to $\dot{x}=e^{t^2}$ have the form

$$x(t)=\int_0^t e^{\tau^2}\,d\tau+C$$

for some constant C. There was no need to use 0 as the lower limit of integration in F; any point t_0 in $(-\infty,\infty)$ would do. While we cannot simplify $\int_0^t e^{\tau^2}\,d\tau$ any further symbolically, we could use numerical methods to approximate this definite integral for any choice of t.

1.5.10 Problem (!). All three functions below have the same derivative. Why? (And what is that derivative?)

$$x_1(t):=\int_0^t\cos(\tau^2)\,d\tau,\qquad x_2(t):=-\int_t^1\cos(\tau^2)\,d\tau,\qquad x_3(t):=-\int_2^{-t}\cos(\tau^2)\,d\tau.$$

We can evaluate definite integrals if we know an antiderivative of the integrand. If x is a function defined at the points a and b, we frequently use the notation

$$x(t)\Big|_{t=a}^{t=b}:=x(b)-x(a).$$

1.5.11 Theorem (FTC2). *Let* x *be differentiable on the interval* I*, and suppose that* $\dot{x}$ *is continuous on* I*. Then*

$$\int_a^b\dot{x}(t)\,dt=x(t)\Big|_{t=a}^{t=b}=x(b)-x(a).$$

for any points a *and* b *in* I*.*

1.5.12 Problem (+). Prove this theorem as follows. Define $F(t):=\int_a^t\dot{x}(\tau)\,d\tau$. Then F is an antiderivative of $\dot{x}$, and so there is a constant C such that $x(t)=F(t)+C$ for all t. Conclude $x(t)\big|_{t=a}^{t=b}=F(b)-F(a)$, and use Problem 1.5.6 to find $F(a)=0$.

1.5.13 Example. Suppose that x is differentiable on $[1,2]$ with $\dot{x}$ continuous on $[1,2]$. If $x(1)=2$ and $x(2)=4$, then

$$\int_1^2 \frac{\dot{x}(t)}{2}\ dt = \frac{1}{2}\int_1^2 \dot{x}(t)\ dt = \frac{1}{2}x(t)\Big|_{t=1}^{t=2} = \frac{x(2)-x(1)}{2} = \frac{4-2}{2} = 1.$$

1.5.14 Problem (+). Let I and J be intervals, and let a and b be points in I. Suppose that x is a differentiable function from I to J and that $\dot{x}$ is continuous on I. Let f be continuous on J. Show that

$$\int_a^b f(x(t))\dot{x}(t)\ dt = \int_{x(a)}^{x(b)} f(u)\ du.$$

[Hint: *define a function F on J by $F(\tau) := \int_{x(a)}^{\tau} f(u)\ du$ and define a function G on J by $G(\tau) = F(x(\tau))$. Use FTC1 to calculate the derivative $\dot{G}$. Use FTC2 to compute $\int_a^b f(x(t))\dot{x}(t)\ dt = G(b) - G(a)$. What is $G(a)$?*]

1.5.15 Problem (!). Suppose that x is a differentiable function on $[1,2]$ and $\dot{x}$ is continuous on $[1,2]$ with $x(1)=\pi/2$ and $x(2)=\pi$. What is

$$\int_1^2 \sin(x(t))\dot{x}(t)\ dt?$$

[Hint: *use the result of Problem 1.5.14.*]

1.5.16 Problem ($\star$). Suppose that f is a continuous function on $[1,4]$ and $\displaystyle\int_1^2 tf(t)\ dt = 2$. What is

$$\int_1^4 f(\sqrt{t})\ dt?$$

[Hint: *substitute $u=\sqrt{t}$ and note that $dt = 2u\ du$.*]

We can now establish an existence and uniqueness theorem for the direct integration IVP. We will show first that a solution exists and second that this is the only possible solution. Our proof uses both parts of the fundamental theorem of calculus.

1.5.17 Theorem (Direct integration). *Let f be continuous on the interval I, let t_0 be a point in I, and let x_0 be a real number. Then the only solution to the IVP*

$$\begin{cases} \dot{x} = f(t) \\ x(t_0) = x_0 \end{cases} \tag{1.5.6}$$

is

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau)\ d\tau. \tag{1.5.7}$$

This function x is defined on all of I.

Proof. First, since f is continuous on I, the integral $\int_{t_0}^{t} f(\tau)\ d\tau$ is defined for all t in I, and so this formula for x is actually defined. We need to do two things: (1) show that the formula above yields a solution to the IVP and (2) show that this is the only possible solution. (1) is a direct calculation, but (2) requires a little more work. Both (1) and (2) use different parts of the FTC.

1. *The function x as defined actually solves the IVP.* We use FTC1 to calculate

$$\dot{x}(t) = \frac{d}{dt}\left[x_0 + \int_{t_0}^{t} f(\tau)\ d\tau\right] = 0 + \frac{d}{dt}\left[\int_{t_0}^{t} f(\tau)\ d\tau\right] = f(t).$$

Thus x satisfies the ODE part of the IVP. Then we use properties of integrals to calculate

$$x(t_0) = x_0 + \int_{t_0}^{t_0} f(\tau)\ d\tau = x_0 + 0 = x_0.$$

Hence x satisfies the initial condition and so solves the IVP.

2. *The function x above is the only possible solution to the IVP.* Suppose that all we know about the function x is that it satisfies both the ODE $\dot{x} = f(t)$ and the initial condition $x(t_0) = x_0$. The ODE means that x satisfies $\dot{x}(t) = f(t)$ for all t in the domain of x. Consequently, we may integrate both sides of this equality over the same interval:

$$\int_{t_0}^{t} \dot{x}(\tau)\ d\tau = \int_{t_0}^{t} f(\tau)\ d\tau. \tag{1.5.8}$$

Since we are using t as the upper limit of integration, we have changed the variable of integration to τ. We can evaluate the integral on the left using FTC2:

$$\int_{t_0}^{t} \dot{x}(\tau)\ d\tau = x(t) - x(t_0) = x(t) - x_0, \tag{1.5.9}$$

where the second equality uses the initial condition $x(t_0) = x_0$. We combine (1.5.8) and (1.5.9) to conclude

$$x(t) - x_0 = \int_{t_0}^{f} f(\tau)\ d\tau,$$

and so

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau)\ d\tau.$$

Thus the only possible solution to the IVP is given by (1.5.7). ■

1.5.18 Problem ($\star$). Here is a different way to argue uniqueness. Suppose that x and y are both solutions of the same direct integration IVP, i.e.,

$$\begin{cases} \dot{x} = f(t) \\ x(t_0) = x_0 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{y} = f(t) \\ y(t_0) = x_0. \end{cases}$$

We want x and y to be the same, so we want $x(t) - y(t) = 0$ for all t. Define $z(t) := x(t) - y(t)$, and show that z solves the IVP

$$\begin{cases} \dot{z} = 0 \\ z(t_0) = 0. \end{cases}$$

Then use FTC2 to conclude that $z(t) = 0$ for all t, as desired.

1.5.19 Example. **(i)** The only solution to the IVP

$$\begin{cases} \dot{x} = 2t \\ x(1) = 2 \end{cases}$$

is

$$x(t) = 2 + \int_1^t 2\tau \, d\tau = 2 + \tau^2\Big|_{\tau=1}^{\tau=t} = 2 + t^2 - 1 = t^2 + 1.$$

(ii) The only solution to the IVP

$$\begin{cases} \dot{x} = e^{t^2} \\ x(0) = 1 \end{cases}$$

is

$$x(t) = 1 + \int_0^t e^{\tau^2} \, d\tau,$$

which we cannot simplify further.

Direct integration is a "complete success" story: we have a formulaic method for solving any direct integration problem, and we know exactly which functions are solutions. We will have relatively few "complete successes" in this course, so we should cherish them when we find them.

1.5.20 Remark. **(i)** *As in part (ii) of Remark 1.1.5, we will use the indefinite integral notation*

$$\int f(t) \, dt$$

for bookkeeping and euphemistic purposes in the calculation of actual antiderivatives. For *example, if $f(t) = 2t$, then we might say that any antiderivative F of f has the form*

$$F(t) = \int 2t \; dt = t^2 + C \tag{1.5.10}$$

for some constant C.

(ii) *However, we will never write just $F(t) = \int 2t \; dt$ and leave it at that; we will always work through the calculation (1.5.10). Pretending that the symbol $\int f(t) \; dt$ is a function suppresses and obscures the constant of integration. Doing so also conflates the independent variable of an antiderivative F of f with the variable of integration t in the symbol $\int f(t) \; dt$; previously, in (1.5.4), we agreed that the variable of integration is irrelevant, but with indefinite integrals, saying $F(t) = \int f(t) \; dt$ could lead to*

$$F(t) = \int f(t) \; dt = \int f(\tau) \; d\tau = F(\tau),$$

and then we might think that F is constant.

(iii) *In conclusion, if you are overwhelmed by ambiguities when trying to use an indefinite integral to solve a problem, a good idea is to give up and use a definite integral instead.*

1.5.21 Example. We use indefinite integral notation to guide our symbolic computation of an antiderivative of $f(t) := 2t(t^2+1)^{2023}$, which we previously studied with a good guess in part (ii) of Example 1.5.2. We substitute $u = t^2 + 1$ and $du = 2t \; dt$ to obtain

$$\int 2t(t^2+1)^{2023} \; dt = \int u^{2023} \; du = \frac{u^{2024}}{2024} + C = \frac{(t^2+1)^{2024}}{2024} + C.$$

That is, for any constant C, the function F defined by

$$F(t) := \frac{(t^2+1)^{2024}}{2024} + C$$

is an antiderivative of f.

This is where we finished on Wednesday, August 23, 2023.

2. FIRST-ORDER DIFFERENTIAL EQUATIONS

2.1. Separation of variables: a useful toy problem.

It turns out that the time and state dependence of many differential equations can be "separated" in the sense that the equations have the form

$$\dot{x} = g(t)h(x),$$

where g is a function of time t alone, and h is a function of the state x alone. Such equations, unsurprisingly, are called **SEPARABLE**. For example, exponential growth has the form $\dot{x} = rx$, so $g(t) = 1$ and $h(x) = rx$; logistic growth has the form $\dot{x} = rx(1 - x/N)$, so $g(t) = 1$, again, and $h(x) = rx(1 - x/N)$; growth with a time-varying rate has the form $\dot{x} = r(t)x$, so $g(t) = r(t)$ and $h(x) = x$; and direct integration has the form $\dot{x} = f(t)$, so $g(t) = f(t)$ and $h(x) = 1$.

We develop here, in increasing levels of generality, an analytic technique for solving separable ODE. The technique relies on first finding constant solutions (the "equilibrium" solutions that we have previously discussed) and then "separating" the t- and x-dependencies in the ODE into two distinct antidifferentiation problems. Our success will vary greatly from ODE to ODE, and we will spend almost as much time criticizing this method as we will practicing it.

For quite some time we will consider the "toy" problem

$$\dot{x} = x^2. \tag{2.1.1}$$

This has all the essential features of the harder problems (like logistic) that will want to solve without the attendant notational, algebraic, and/or emotional baggage.

The goal is to solve for x. A natural, but bad, idea is direct integration: since $\dot{x} = x^2$, if F is any antiderivative of x, then

$$x(t) = F(t) + C \tag{2.1.2}$$

for some constant C. Equivalently, for all t and t_0 in its domain, the solution x to (2.1.1) also satisfies the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\tau)\, d\tau = x(t_0) + \int_{t_0}^{t} [x(\tau)]^2\, d\tau, \tag{2.1.3}$$

as we saw in (IE). But both (2.1.2) and (2.1.3) define x in terms of an antiderivative of x^2, and we do not know what x is in the first place! We need to do something new.

2.1.1. Fooling around.

Here is that new thing. Math is complicated, so perhaps we should look for the *simplest* possible solutions to $\dot{x} = x^2$. And perhaps the simplest kind of function is the *constant* function. What kind(s) of constant functions $x(t) = c$, for a fixed real number c, could solve $\dot{x} = x^2$? We compute

$$\dot{x}(t) = \frac{d}{dt}[c] = 0 \quad \text{and} \quad [x(t)]^2 = c^2,$$

and so we want

$$0 = c^2,$$

and therefore

$$c = 0.$$

This argument shows that *if* x is a constant solution to $\dot{x} = x^2$, then $x(t) = 0$ for all t. Conversely, if we put $x(t) = 0$ for all t, then

$$\dot{x}(t) = \frac{d}{dt}[0] = 0 = 0^2 = [x(t)]^2,$$

so $x = 0$ is indeed a solution. As before, we call such a constant solution an **EQUILIBRIUM** solution, as it always maintains the same value and therefore stays "at equilibrium."

Are there other, nonconstant, nonequilibrium solutions? We follow the tried-and-true method of working backward. Assume that there is a solution x to $\dot{x} = x^2$ that is not constant; since the only constant solution is 0, this means that $x(t_0) \neq 0$ for some time $t = t_0$. Then continuity implies that $x(t) \neq 0$ for $t \approx t_0$.

And now we do something new and, perhaps, unexpected: assuming $x(t) \neq 0$, we may divide both sides of $\dot{x}(t) = [x(t)]^2$ by $[x(t)]^2$ to find

$$\frac{\dot{x}(t)}{[x(t)]^2} = 1. \tag{2.1.4}$$

We have "separated" the problem by collecting all of the x-dependent factors on the left side.

Now we rewrite (2.1.4) as

$$[x(t)]^{-2}\dot{x}(t) = 1. \tag{2.1.5}$$

The left side is really a piece of calculus action that we have seen before. Perhaps it will help to define a new function ℓ by $\ell(x) := x^{-2}$, with $x \neq 0$. We write ℓ since this governs what happens on the ℓeft side of (2.1.5); also, we are adopting the (unfortunately) dual mindset that the letter x denotes both our unknown function (a dependent variable of the independent variable t) and a single number that is the independent variable of ℓ. Then (2.1.5) is

$$\ell(x(t))\dot{x}(t) = 1. \tag{2.1.6}$$

In words, (2.1.6) is the product of the composition of ℓ with x and the derivative $\dot{x}$ of x. This should sound a lot like the chain rule. All we are missing is some function L such that $L'(x) = \ell(x)$, and then we will have

$$\ell(x(t))\dot{x}(t) = L'(x(t))\dot{x}(t) = \frac{d}{dt}[L(x(t))]. \tag{2.1.7}$$

Here we are writing L', not $\dot{L}$, to emphasize that L depends on x, not t.

So, we want an antiderivative L of the function $\ell(x) = x^{-2}$. This is not too hard, since we know the power rule. Just take

$$L(x) = \int x^{-2}\, dx = \frac{x^{-2+1}}{-2+1} = -x^{-1}. \tag{2.1.8}$$

We only want *one* antiderivative of ℓ, so we are scandalously omitting the constant of integration. Then (2.1.7) is true for $L(x) = -x^{-1}$, and so (2.1.6) becomes the direct integration problem

$$\dot{y}(t) = f(t), \quad \text{where} \quad y(t) := L(x(t)) \quad \text{and} \quad f(t) := 1.$$

We certainly know how to solve this, thanks to our work in Section 1.5. There must be a constant C such that

$$L(x(t)) = t + C \tag{2.1.9}$$

for all t. Since $L(x) = x^{-1}$, (2.1.9) is really the *algebraic* problem

$$-[x(t)]^{-1} = t + C. \tag{2.1.10}$$

And now we solve for x, using the algebra that we have known for ages:

$$x(t) = -(t + C)^{-1}.$$

It looks like we have found a whole family of solutions to our ODE $\dot{x} = x^2$; each constant C gives us a different function.

Have we? We have been working backward and assuming that there existed a solution x in the first place, and that this x was not always 0.

2.1.1 Problem (!). Let C be any real number and define $x(t) := -(t + C)^{-1}$.

(i) What are the largest intervals on which x is defined? [Hint: *there are two possibilities.*] Of these intervals, which contain the interval (a, ∞) for some real number a?

(ii) Check that x solves $\dot{x} = x^2$ at all times at which x is defined.

And so we have two kinds of solutions to $\dot{x} = x^2$: the equilibrium solution $x(t) = 0$ and the *family* of nonequilibrium solutions $x(t) = 1/(C - t)$, where each choice of C generates a different member of the family.

2.1.2. Getting serious.

We can cut down on a lot of the chatter above by picking out the essential steps. To solve $\dot{x} = x^2$, first we look for constant (equilibrium) solutions. Constant solutions satisfy $\dot{x} = 0$, and so we need $x^2 = 0$, thus $x = 0$ is the only constant solution. For nonequilibrium, and thus nonzero, solutions, we divided to find

$$\frac{\dot{x}}{x^2} = 1. \tag{2.1.11}$$

We then viewed (2.1.11) as an antidifferentiation problem. Since $\dot{x}/x^2$ and 1 are the same function, any antiderivative of $\dot{x}/x^2$ should equal any antiderivative of 1, up to an additive constant. We can visualize this symbolically via the cartoon

$$\boxed{\frac{\dot{x}}{x^2} = 1 \Longrightarrow \int \frac{\dot{x}(t)}{[x(t)]^2}\, dt = \int 1\, dt + C.} \tag{2.1.12}$$

Here we are adopting the useful computational perspective that each indefinite integral represents one particular antiderivative of the integrand (and so there are not two constants of integration annoyingly floating around). And (2.1.12) is a cartoon because the previous sentence has no rigorous meaning at all.

The indefinite integral on the left of the cartoon (2.1.12) should be easy:

$$\int 1 \ dt = t.$$

For the integral on the right, we substitute $u = x(t)$ and $du = \dot{x}(t) \ dt$ to find

$$\int \frac{\dot{x}(t)}{[x(t)]^2} \ dt = \int \frac{du}{u^2} = \int u^{-2} \ du = \frac{u^{-2+1}}{-2+1} = -u^{-1} = -[x(t)]^{-1}.$$

And so we expect that for some constant C and all t in the domain of x, we should have

$$-[x(t)]^{-1} = t + C.$$

This is exactly the implicit equation that we found for x before in (2.1.10), and, as before, we solve it to get

$$x(t) = -(t + C)^{-1}. \tag{2.1.13}$$

The upshot of the cartoon (2.1.12) is that it works *really well* if we neither overthink it nor underthink it, and that if we get down to a *candidate* for a solution to an ODE, we can always check if that candidate really is a solution by plugging and chugging. Shortly we will see how to do all this with definite integrals, and that will make some things less sloppy, and other things more annoying. Regardless, you should view this process as a *formal* procedure.

2.1.2 Undefinition. *In mathematical language, the "term* FORMAL *describes any plausible result or procedure which may be unjustified or unjustifiable." (Basic Partial Differential Equations by David Bleecker & George Csordas, p. 249.)*

Before proceeding with this toy problem, we make one change to the nonequilibrium solution (2.1.13). Rewritten, this solution is

$$x(t) = -\frac{1}{t + C} = \frac{1}{-t - C}.$$

Here C can be any real number, so $-C$ can be any real number, too. (Proof: let K be any real number. Take $C = -K$ to see that $-C = -(-K) = K$.)

We will therefore write the family of nonequilibrium solutions (2.1.13) as

$$x(t) = \frac{1}{C - t},$$

where C can be an arbitrary real number. This is a bit nicer notationally and algebraically.

2.1.3. Incorporating initial conditions.

Suppose that we want to solve not just the toy problem but we also want to include an initial condition, say,

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1. \end{cases} \tag{2.1.14}$$

We have two kinds of solutions to the ODE, the equilibrium solution $x(t) = 0$ and the family of nonequilibrium solutions $x(t) = 1/(C-t)$ for constants C. The equilibrium solution is not going to be helpful here, because if $x(t) = 0$ for all t, then $x(0) \neq 1$. Can we choose C correctly in a nonequilibrium solution to meet the initial condition?

We would want

$$1 = x(0) = \frac{1}{C-0} = \frac{1}{C},$$

and so $C = 1$. That is,

$$x(t) = \frac{1}{1-t}$$

solves the IVP (2.1.14).

Now we think about the domain of our solution. Certainly the formula $1/(1-t)$ is defined for $t \neq 1$, but we are working with a differential equation here. We know that the domain has to be an interval I containing $t = 0$, and $x(t) = 1/(1-t)$ has to be defined and differentiable on I with $\dot{x}$ continuous on I. Certainly x is not defined at $t = 1$ due to division by 0. However, x is defined at all other t. So, candidates for the domain of x are the intervals $(-\infty, 1)$ and $(1, \infty)$, as well as all other (smaller) subintervals of these two. But 0 is in $(-\infty, 1)$ and not in $(1, \infty)$, so the domain of x should be $(-\infty, 1)$.

This is where we finished on Friday, August 25, 2023.

The goal of this course is to predict the future, and formulas are one approach to doing that. So what happens? One of our guiding questions tasks us to study how x behaves as time approaches the boundary of the domain of x. We have

$$\lim_{t \to -\infty} \frac{1}{1-t} = 0 \quad \text{and} \quad \lim_{t \to 1^-} \frac{1}{1-t} = \infty.$$

That is, there is a "blow-up" as $t \to 1^-$.

2.1.3 Problem (!). Can a nonequilibrium solution to $\dot{x} = x^2$ of the form $x(t) = 1/(C-t)$ solve the IVP

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 0? \end{cases}$$

2.1.4. Using definite integrals.

Here is another way to solve the IVP

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$$

from (2.1.14). Imagine that we had not done the separation of variables analysis before to get the family of nonequilbrium solutions $x(t) = 1/(C - t)$. We could follow the spirit of that analysis but use *definite* integrals instead.

First, since the initial condition is $x(0) = 1 \neq 0$, the equilibrium solution $x = 0$ is not a solution to this IVP. Instead, we work backwards: if the IVP (2.1.14) has a solution, then since $x(0) = 1 \neq 0$, for times $t \approx 0$, we have $x(t) \neq 0$ by continuity, and so, once more,

$$\frac{\dot{x}(t)}{[x(t)]^2} = 1.$$

Now we integrate both sides of this equation from 0 to some time t:

$$\int_0^t \frac{\dot{x}(\tau)}{[x(\tau)]^2}\, d\tau = \int_0^t 1\, d\tau. \tag{2.1.15}$$

We are using τ as the variable of integration since t is now the upper limit of integration. *Do not overwork the variables.*

The integral on the right of (2.1.15) is

$$\int_0^t 1\, d\tau = t.$$

The left of (2.1.15) Substitute $u = x(\tau)$ in the integral on the left of (2.1.15) and change variables, using the initial condition $x(0) = 1$:

$$\int_0^t \frac{\dot{x}(\tau)}{[x(\tau)]^2}\, d\tau = \int_{x(0)}^{x(t)} \frac{du}{u^2} = \int_1^{x(t)} \frac{du}{u^2} = -u^{-1}\Big|_{u=1}^{u=x(t)} = -\big([x(t)]^{-1} - 1^{-1}\big] = 1 - [x(t)]^{-1}.$$

Then

$$1 - [x(t)]^{-1} = t,$$

and so we solve for x as, once again, $x(t) = 1/(1-t)$.

The advantage of the definite integral is that it produces at once the solution to the IVP; there is no intermediate step of finding a whole family of solutions.

2.1.4 Problem (!). Solve the IVP

$$\begin{cases} \dot{x} = x^2 \\ x(1) = 2 \end{cases}$$

using definite integrals. Determine the domain of your solution and what happens as time approaches the boundary of this domain.

2.1.5 Problem ($\star$). Let t_0 be an arbitrary real number and let $x_0 \neq 0$. Complete the following steps using definite integrals to solve

$$\begin{cases} \dot{x} = x^2 \\ x(t_0) = x_0. \end{cases}$$

This is an abstraction of all of our work on the toy problem for an arbitrary nonzero initial condition.

(i) Suppose that x solves this IVP. Since $x(t_0) \neq 0$, for all t close to t_0, by continuity we have $x(t) \neq 0$. Use this to obtain the generalization

$$\int_{t_0}^{t} [x(\tau)]^{-2}\dot{x}(\tau)\ d\tau = \int_{t_0}^{t} 1\ d\tau$$

of (2.1.15).

(ii) Put $u = x(\tau)$ and remember $x(t_0) = x_0$ to obtain

$$\int_{x_0}^{x(t)} u^{-2}\ du = \int_{t_0}^{t} 1\ d\tau.$$

(iii) Use the power rule for antiderivatives and the fundamental theorem of calculus to obtain

$$x_0^{-1} - [x(t)]^{-1} = t - t_0.$$

(iv) Solve for $x(t)$ as

$$x(t) = \frac{1}{x_0^{-1} + t_0 - t}.$$

(v) What is the largest interval of the form (a, ∞) on which x is defined and which contains the point t_0?

2.2. Separation of variables: the full story.

Let us now put away our toys and generalize our work.

2.2.1. Separation of variables for autonomous ODE.

The exponential growth model $\dot{x} = rx$, the logistic growth model $\dot{x} = rx(1 - x/N)$, and the toy problem $\dot{x} = x^2$ all have an important feature in common: their right sides depend only on the state variable x and not time t. Such ODE have a special name.

2.2.1 Definition. *An ODE of the form*

$$\dot{x} = f(x),$$

where f is a function of the single real variable x, is AUTONOMOUS.

To solve autonomous ODE via separation of variables, we should follow the procedure that worked on our toy problem (2.1.1). First, find the constant solutions by solving $f(x) = 0$. This is ultimately a root-finding problem.

2.2.2 Definition. *An* **EQUILIBRIUM SOLUTION** *to the autonomous ODE $\dot{x} = f(x)$ is any constant solution to this ODE.*

2.2.3 Example. We find all equilibrium solutions to the logistic equation $\dot{x} = x(1-x)$ by solving $x(1-x) = 0$ for constants x. This product is 0 if and only if at least one of its factors is 0, so $x(1-x) = 0$ happens precisely when $x = 0$ or $1 - x = 0$. Thus $x = 0$ and $x = 1$ are the only equilibrium solutions.

2.2.4 Problem (!). Find all equilibrium solutions to the logistic equation $\dot{x} = rx(1-x/N)$, where r, $N > 0$ are fixed parameters.

The next theorem assures us that, as we hopefully expect, the equilibrium solutions to $\dot{x} = f(x)$ are precisely the roots of f.

2.2.5 Theorem. *Suppose that f is a function of the single real variable x.*

(i) *Let x_∞ be a root of f, i.e., $f(x_\infty) = 0$. Then the constant function $x(t) := x_\infty$ for all t solves the ODE $\dot{x} = f(x)$.*

(ii) *Conversely, suppose that x solves $\dot{x} = f(x)$ and that x is a constant function. Then $f(x(t)) = 0$ for all t.*

Proof. We prove the first part and leave the second for practice. We need to show that the function $x(t) = x_\infty$ satisfies Definition 1.4.1.

First, we can take the domain of x to be the interval $(-\infty, \infty)$, since constant functions are defined at all real numbers. Second, since x is constant, x is differentiable, and $\dot{x}(t) = 0$ for all t. In particular, $\dot{x}$ is a constant function and therefore continuous.

Third, $f(x(t)) = f(x_\infty) = 0$ for all t. Thus, fourth and finally,

$$\dot{x}(t) = 0 = f(x(t)).$$

for all t. We have therefore checked all the conditions of Definition 1.4.1 for $x(t) = x_\infty$ to be a solution to $\dot{x} = f(x)$. ■

2.2.6 Problem ($\star$). Prove part (ii) of Theorem 2.2.5.

We return to solving analytically the ODE $\dot{x} = f(x)$. After finding the constant, equilibrium solutions, the toy problem teaches us how to find nonconstant, nonequilibrium solutions: work backwards. Assume that we have a nonequilibrium solution x, so $f(x(t)) \neq 0$ for all t

in the domain of x. Then divide to get

$$\frac{\dot{x}(t)}{f(x(t))} = 1. \tag{2.2.1}$$

The functions $\dot{x}/f(x)$ and 1 are therefore the same, and so any antiderivative for $\dot{x}/f(x)$ should equal any antiderivative for 1, up to an additive constant. Thus we generalize the cartoon from (2.1.12) into the following scheme:

$$\boxed{\dot{x} = f(x) \Longrightarrow \frac{\dot{x}}{f(x)} = 1 \Longrightarrow \int \frac{\dot{x}(t)}{f(x(t))}\ dt = \int 1\ dt \Longrightarrow \int \frac{dx}{f(x)} = t + C.} \tag{2.2.2}$$

Here the cartoon is even more cartoonish because we have "substituted" $u = x(t)$ after the second $\Longrightarrow$ but we kept the variable of integration on the left as x. We work through the mechanics in two examples and then outline an alternate treatment using definite integrals that avoids all the ickiness of indefinite integrals at the cost of more symbols in play.

2.2.7 Example. We use separation of variables to study

$$\dot{x} = e^{-x}.$$

(i) First, we check for equilibrium solutions by trying to solve $e^{-x} = 0$ But this is impossible, as $e^{-x} > 0$ for all x. So, there are no equilibrium solutions.

(ii) Next, we separate variables to find

$$e^{x}\dot{x} = 1.$$

We integrate both sides with respect to t:

$$\int e^{x(t)}\dot{x}(t)\ dt = \int 1\ dt.$$

On the right, we just have

$$\int 1\ dt = t + C.$$

On the left, we have

$$\int e^{x(t)}\dot{x}(t)\ dt = \int e^{x}\ dx = e^{x} + C = e^{x} + C.$$

We therefore have the implicit equation

$$e^{x} = t + C,$$

where, as always, we have combined the constants of integration. Now we take the natural logarithm of both sides:

$$\ln(e^{x}) = \ln(t + C).$$

Then

$$x = \ln(t + C).$$

(iii) What is the domain of x? Recall that $\ln(\tau)$ is only defined for $0 < \tau$, and so we want $0 < t + C$, and thus

$$-C < t.$$

That is, the domain of our solution is $(-C, \infty)$.

(iv) What happens in the future? We have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \ln(t + C) = \infty.$$

All solutions that we found, then, blow up to ∞ over very long times. The choice of C has no effect on the end behavior of x.

(v) Finally, we solve the IVP

$$\begin{cases} \dot{x} = e^x \\ x(0) = 0. \end{cases}$$

Separation of variables gives us the family of solutions $x(t) = \ln(t+C)$, defined on $(-C, \infty)$, where C is an arbitrary real number. We try to choose C to meet the initial condition, and so we want

$$0 = x(0) = \ln(0 + C) = \ln(C).$$

Either recalling properties of the natural logarithm or exponentiating, we find $C = 1$, and so a solution to the IVP is

$$x(t) = \ln(t + 1).$$

The domain of this solution is $(-1, \infty)$.

2.2.8 Problem ($\star$). We saw in Section 2.1.4 and in particular in Problem 2.1.5 how definite integrals could avoid some of the ambiguities inherent to employing indefinite integrals. Here is an outline of how to approach Example 2.2.7 with definite integrals.

(i) We want to solve the IVP

$$\begin{cases} \dot{x} = e^{-x} \\ x(t_0) = x_0. \end{cases}$$

Separate variables and obtain

$$\int_{t_0}^{t} e^{x(\tau)}\dot{x}(\tau)\, d\tau = \int_{t_0}^{t} 1\, d\tau.$$

(ii) Change variables with $u = x(t)$ on the left to obtain

$$\int_{x_0}^{x(t)} e^u\, du = t - t_0.$$

(iii) Evaluate the integral on the left and obtain

$$e^{x(t)} = e^{x_0} - t_0 + t.$$

(iv) Solve for $x(t)$ as

$$x(t) = \ln(e^{x_0} - t_0 + t).$$

Compare this to the solution in Example 2.2.7. Where in this new solution do you see the constant C from that example?

This is where we finished on Monday, August 28, 2023.

2.2.9 Example. The problem

$$\dot{x} = x^3$$

has several differences from both the toy problem and Example 2.2.7.

(i) First, unlike Example 2.2.7 (but like the toy problem), it has equilibrium solutions: we solve $x^3 = 0$ to find that $x = 0$ is an equilibrium solution (and the only one).

(ii) Second, unlike both the toy problem and Example 2.2.7, a new algebraic complication arises after separating variables. Proceeding as usual, we assume that x is a nonequilibrium solution to $\dot{x} = x^3$, divide to find

$$\frac{\dot{x}}{x^3} = 1,$$

and integrate to find

$$\int x^{-3}\, dx = t + C.$$

Here we are looking for only one antiderivative on the left, and so

$$\int x^{-3}\, dx = -\frac{x^{-2}}{2}$$

works. Then x satisfies the implicit equation

$$-\frac{x^{-2}}{2} = t + C,$$

and therefore

$$x^2 = (-2t - 2C)^{-1}$$

Our instinct should be to solve for x by taking square roots, but remember that if A and B are real numbers with $A^2 = B$ and $B > 0$, then all we know is either $A = \sqrt{B}$ or $A = -\sqrt{B}$. So, we have two solution candidates

$$x(t) = \sqrt{(-2t - 2C)^{-1}} \quad \text{and} \quad x(t) = -\sqrt{(-2t - 2C)^{-1}}$$

and we can easily check that these are actually solutions by plugging them into the ODE $\dot{x} = x^3$.

We simplify the notation slightly by observing that since C is an arbitrary constant, so is $2C$ (proof: let K be an arbitrary real number and take $C = K/2$ to get $2C = K$). We therefore replace $2C$ with just C, with the understanding that C is still arbitrary. Thus there are really *three* kinds of solutions to this problem: the one equilibrium solution $x = 0$ and the two "branches" of nonequilibrium solutions above.

$$x(t) = \sqrt{(-2t + C)^{-1}} \quad \text{and} \quad x(t) = -\sqrt{(-2t + C)^{-1}} \tag{2.2.3}$$

(iii) Since the nonequilibrium solutions involve a square root, we should check their domains carefully. The number $\sqrt{(-2t + C)^{-1}}$ is only defined when $0 \le (-2t + C)^{-1}$. Since a number and its reciprocal have the same sign (i.e., $A \le 0$ if and only if $A^{-1} \le 0$), this means we need $0 \le -2t + C$, and so $t \le C/2$. But note also that $(-2t + C)^{-1}$ is defined only when $-2t + C \neq 0$, i.e., when $t \neq C/2$. So, for both the reciprocal and the square root in the expression $\sqrt{(-2t + C)^{-1}}$ to be defined, we need $t < C/2$. Thus the domain of either function in (2.2.3) must be $(-\infty, C/2)$.

(iv) What solution should we use? It depends, chiefly on what (if any) IVP we encounter. To solve

$$\begin{cases} \dot{x} = x^3 \\ x(0) = 0, \end{cases}$$

we should use the equilibrium solution $x(t) = 0$. (See Problem 2.2.10 for how trying one of the square root branches fails.) This certainly solves the ODE and meets the initial condition $x(0) = 0$.

To solve something like

$$\begin{cases} \dot{x} = x^3 \\ x(0) = 1, \end{cases}$$

we cannot use the equilibrium solution, and we should not use the negative branch of the square root in (2.2.3), as that branch cannot return a positive initial condition. (See Problem 2.2.10 to work this out.) So, we try to find C so that $x(t) = \sqrt{(-2t + C)^{-1}}$ solves this IVP. We want

$$1 = x(0) = \sqrt{(0 + C)^{-1}} = \sqrt{C^{-1}}.$$

Squaring both sides, we find $1 = C^{-1}$ and so $C = 1$. Then the solution is

$$x(t) = \sqrt{(-2t + 1)^{-1}}.$$

2.2.10 Problem (!). **(i)** Explain why trying to use a function of the form $x(t) = \sqrt{(-2t + C)^{-1}}$ will fail to solve the IVP

$$\begin{cases} \dot{x} = x^3 \\ x(0) = 0. \end{cases}$$

(ii) Explain why trying to use a function of the form $x(t) = -\sqrt{(-2t + C)^{-1}}$ will fail to solve the IVP

$$\begin{cases} \dot{x} = x^3 \\ x(0) = 1. \end{cases}$$

2.2.11 Problem (!). Solve the IVP

$$\begin{cases} \dot{x} = x^3 \\ x(0) = -1. \end{cases}$$

What has to change from the work above on the initial condition $x(0) = 1$?

2.2.12 Problem (+). In Example 2.2.9 we studied $\dot{x} = x^3$, separated variables, and got down to

$$[x(t)]^2 = (C - 2t)^{-1}. \tag{2.2.4}$$

(Here we are writing C, not $-2C$.) We found two "branches" of solutions, $x(t) = \sqrt{(C - 2t)^{-1}}$ and $x(t) = -\sqrt{(C - 2t)^{-1}}$. Is it possible to have yet another kind of solution that somehow "unites" both branches? Could we have a solution x such that for different times t_1 and t_2 (i.e., $t_1 \neq t_2$), we have

$$x(t_1) = \sqrt{(C - 2t_1)^{-1}} \quad \text{and} \quad x(t_2) = -\sqrt{(C - 2t_2)^{-1}} \tag{2.2.5}$$

Use (2.2.5), the continuity of x, and the intermediate value theorem to conclude that $x(t_3) = 0$ for some t_3 between t_1 and t_2. How does this contradict (2.2.4)?

2.2.2. *The analytic solution for exponential growth.*

We use separation of variables to solve the exponential growth IVP

$$\begin{cases} \dot{x} = rx \\ x(0) = x_0. \end{cases} \tag{2.2.6}$$

Of course, we should expect that the solution is $x(t) = x_0 e^{rt}$, but this is a good opportunity to see more separation of variables, think carefully about the role of parameters and initial conditions, and use definite integrals.

Case 1. $r = 0$. Then the problem really is

$$\begin{cases} \dot{x} = 0 \\ x(0) = x_0. \end{cases}$$

We can solve this by direct integration:

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\, d\tau = x_0 + \int_0^t 0\, d\tau = x_0 + 0 = x_0.$$

Case 2. $r \neq 0$. Now we consider cases on x_0.

Subcase (i) $x_0 = 0$. The only equilibrium solution to $\dot{x} = rx$ is $x = 0$, as if $rx = 0$ with $r \neq 0$, then $x = 0$. So, we can take $x(t) = 0$ for all t to meet this initial condition and solve the ODE.

Subcase (ii) $x_0 > 0$. Then if x solves (2.2.6), the continuity of x at 0 implies that

$$0 < x(t) \quad \text{for} \quad t \approx 0.$$

We therefore divide to find

$$\frac{\dot{x}(t)}{rx(t)} = 1 \quad \text{for} \quad t \approx 0.$$

Then we integrate from t_0 to t, still assuming that t is sufficiently close to t_0, and find

$$\frac{1}{r}\int_0^t \frac{\dot{x}(\tau)}{rx(\tau)}\, d\tau = \int_0^t 1\, d\tau. \tag{2.2.7}$$

The integral on the right is just

$$\int_0^t 1\, d\tau = t. \tag{2.2.8}$$

On the left, we substitute $u = x(\tau)$ with $du = \dot{x}(\tau)\, d\tau$ and find

$$\int_0^t \frac{\dot{x}(\tau)}{rx(\tau)}\, d\tau = \frac{1}{r}\int_{x(0)}^{x(t)} \frac{du}{u} = \int_{x_0}^{x(t)} \frac{du}{u} = \ln(|u|)\big|_{u=x_0}^{u=x(t)} = \ln(|x(t)|) - \ln(|x_0|).$$

Here it is important to remember that $x_0 > 0$ and we are assuming $x(t) > 0$ for all t under consideration. Thus

$$\int_{t_0}^t \frac{\dot{x}(\tau)}{x(\tau)}\, d\tau = \ln(|x(t)|) - \ln(|x_0|) = \ln(x(t)) - \ln(x_0). \tag{2.2.9}$$

We then rewrite (2.2.7) using (2.2.8) and (2.2.9) to find

$$\ln(x(t)) - \ln(x_0) = rt.$$

Now we are in the happy "algebraic" situation of solving for $x(t)$. We do this in two steps:

$$\ln(x(t)) = \ln(x_0) + rt$$

and thus

$$x(t) = e^{\ln(x_0)+rt} = e^{\ln(x_0)}e^{rt} = x_0e^{rt},$$

as we expected and desired. Note that, remarkably, we can "fold in" the equilibrium solution $x(t) = 0$ with the solution $x(t) = x_0e^{rt}$ just by taking $x_0 = 0$. In other examples, the equilibrium solutions usually "stick out" formulaically from the nonequilibrium.

Subcase (iii) $x_0 < 0$. We leave this as a problem.

2.2.13 Problem ($\star$). Repeat the work in Subcase (ii) above for the case $x_0 < 0$. How does the condition $x_0 < 0$ change a specific step in this work?

2.2.3. When things go wrong.

There are two steps at which separation of variables can break down. First, it may not be possible to solve the implicit equation for x.

2.2.14 Example. Consider the ODE

$$\dot{x} = \frac{1}{x^6+6}.$$

We check for equilibrium solutions and find none, since $1/(x^6+6) \neq 0$ for all x. Then we separate variables to find

$$(x^6+6)\dot{x} = 1$$

and integrate to find

$$\int (x^6+6)\ dx = t + C.$$

The integral on the left is pretty easy:

$$\int (x^6+6)\ dx = \frac{x^7}{7} + 6x + C,$$

thus

$$\frac{x^7}{7} + 6x = t + C.$$

Good luck solving this explicitly for x!

This is where we finished on Wednesday, August 30, 2023.

The other problem with separation of variables is that it may not be possible to evaluate an antiderivative explicitly in terms of elementary functions.

2.2.15 Example. Consider the ODE

$$\dot{x} = e^{-x^2}.$$

Once again, there are no equilibrium solutions, since $e^{-x^2} > 0$ for all x. Separating variables, we have

$$e^{x^2}\dot{x} = 1,$$

and integrating we have

$$\int e^{x^2}\ dx = t + C. \tag{2.2.10}$$

We cannot evaluate the x-integral in terms of elementary functions, and so it looks like we are stuck. That is, there is no transparent way to write the implicit equation (2.2.10) that x must satisfy—and without this equation phrased cleanly, we have no hope of trying to solve for x (which, as in the preceding example, may be impossible).

2.2.16 Remark. *Artificially introducing definite integrals could give us a slightly better implicit equation in the previous example. Suppose that we want x to satisfy $x(t_0) = x_0$, where t_0 and x_0 are given numbers. In other words, we are artificially creating an initial value problem. Then we can integrate both sides of the equality*

$$e^{[x(t)]^2}\dot{x}(t) = 1$$

from t_0 to t to get

$$t - t_0 = \int_{t_0}^{t} 1 \; d\tau = \int_{t_0}^{t} e^{[x(\tau)]^2}\dot{x}(\tau) \; d\tau = \int_{x(t_0)}^{x(t)} e^{u^2} \; du = \int_{x_0}^{x(t)} e^{u^2} \; du.$$

Thus the solution x must satisfy the implicit equation

$$\int_{x_0}^{x(t)} e^{u^2} \; du = t - t_0. \tag{2.2.11}$$

The equations (2.2.10) and (2.2.11) really say the same thing, but the latter is more explicit and specifically illustrates where x "is": in (2.2.11), the solution x "lives" in the upper limit of integration. In other words, if we define $F(x) := \int_{x_0}^{x} e^{u^2} \; du$, then the solution x to the IVP

$$\begin{cases} \dot{x} = e^{-x^2} \\ x(t_0) = x_0 \end{cases}$$

must satisfy the equation $F(x(t)) = t - t_0$. Perhaps we could develop techniques to solve an implicit equation like this!

2.2.4. Separation of variables for nonautonomous ODE.

Finally, we study the full separable problem, which is really not that much different from our prior work.

2.2.17 Definition. *An ODE $\dot{x} = f(t, x)$ is* **SEPARABLE** *if there are functions g and h such that $f(t, x) = g(t)h(x)$. That is, a separable ODE has the form $\dot{x} = g(t)h(x)$.*

2.2.18 Problem (!). Suppose that the separable ODE $\dot{x} = g(t)h(x)$ is also autonomous. What do you know about h?

Solving a separable ODE $\dot{x} = g(t)h(x)$ is essentially "more of the same." First we look for equilibrium solutions by solving $h(x) = 0$.

2.2.19 Problem (!). Suppose that g is a function defined on the interval I and h is a function defined on the interval J. Let x_∞ be a root of h in J, i.e., $h(x_\infty) = 0$. Define $x(t) := x_\infty$ for all t in I. Show that x solves $\dot{x} = g(t)h(x)$. [Hint: *repeat the proof of Theorem 2.2.5.*]

Then we look for nonequilibrium solutions by separating variables to find that a nonequilibrium solution x should[6] satisfy

$$\frac{\dot{x}(t)}{h(x(t))} = g(t)$$

for all t at which $h(x(t)) \neq 0$. Thus the functions $\dot{x}/h(x)$ and g are the same, and so any antiderivative of $\dot{x}/h(x)$ equals any antiderivative of g, up to an additive constant. We summarize our symbolic approach in the following ultimate version of the cartoons (2.1.12) and (2.2.2):

$$\boxed{\dot{x} = g(t)h(x) \Longrightarrow \frac{\dot{x}(t)}{h(x(t))} = g(t) \Longrightarrow \int \frac{\dot{x}(t)}{h(x(t))}\ dt = \int g(t)\ dt + C \Longrightarrow \int \frac{dx}{h(x)} = \int g(t)\ dt + C.}$$

The only difference from the autonomous case is that the t-integral $\int g(t)\ dt$ may be more complicated to evaluate.

2.2.20 Example. Consider the ODE

$$\dot{x} = (x-1)^4(t-1)$$

This ODE is separable with

$$g(t) = t - 1 \qquad \text{and} \qquad h(x) = (x-4)^4.$$

We solve $h(x) = 0$ to find $(x-1)^4 = 0$, so $x - 1 = 0$ and therefore $x = 1$ is the only equilibrium solution.

For nonequilibrium solutions, we separate variables to find

$$\frac{\dot{x}}{(x-1)^4} = t - 1,$$

and so we antidifferentiate to find

$$\int \frac{dx}{(x-1)^4} = \int (t-1)\ dt + C.$$

On the right, we have

$$\int (t-1)\ dt = \frac{t^2}{2} - t,$$

and on the left we substitute $u = x - 1$ and $du = dx$ to find

$$\int \frac{dx}{(x-1)^4} = \int \frac{du}{u^4} = \int u^{-3}\ du = \frac{u^{-4+1}}{-4+1} = -\frac{u^{-3}}{3} = -\frac{(x-1)^{-3}}{3}.$$

[6] Turning "should" into "must" will be a substantial result of our forthcoming existence and uniqueness theory.

Thus the nonequilibrium solutions should satisfy

$$-\frac{(x-1)^{-3}}{3} = \frac{t^2}{2} - t + C,$$

and now it is just a matter of solving for x. First, we rearrange the equation above to

$$(x-1)^{-3} = -\frac{3t^2}{2} + 3t - 3C.$$

(Of course, we could replace $-3C$ by $+C$, since C is an arbitrary constant.) Then

$$x - 1 = \left(-\frac{3t^2}{2} + 3t - 3C\right)^{-1/3},$$

and so

$$x(t) = 1 + \left(-\frac{3t^2}{2} + 3t - 3C\right)^{-1/3}.$$

We conclude that the functions

$$x(t) = 0 \quad \text{and} \quad x(t) = 1 + \left(-\frac{3t^2}{2} + 3t - 3C\right)^{-1/3}.$$

solve this ODE. Note that finding the domain of the latter solutions might be quite complicated, since we would need $-(3t^2/2) + 3t - 3C \neq 0$, and this would involve the roots of a quadratic equation that depend on C.

Hopefully this example illustrates that solving the separable nonautonomous problem $\dot{x} = g(t)h(x)$ is not much more than a glorified version of solving the separable autonomous problem; the symbolic computations just get a bit more involved.

2.2.21 Problem ($\star$). Explain why the ODE $\dot{x} = f(t)$ is separable and then solve it using separation of variables. At what point did you realize that you were really just using direct integration?

2.2.22 Problem (+). This problem studies the IVP

$$\begin{cases} \dot{x} = a(t)x \\ x(t_0) = x_0, \end{cases} \tag{2.2.12}$$

where a is continuous on the interval I and t_0 is a point in I. This IVP will reappear in several key places later in the course.

(i) In Section 2.2.2, we solved the problem in the constant case $a(t) = r$ and found that the solution was $x(t) = x_0 e^{\alpha(t)}$, where $\alpha(t) = r(t - t_0)$. Note that α solves the IVP

$$\begin{cases} \dot{\alpha} = r \\ \alpha(t_0) = 0. \end{cases}$$

Motivated by this, we might guess that the solution to (2.2.12) is

$$x(t) = x_0 e^{A(t)}, \qquad \text{where} \qquad \begin{cases} \dot{A} = a(t) \\ A(t_0) = 0. \end{cases} \tag{2.2.13}$$

Check that this is indeed the case; then solve the IVP above for A.

(ii) We can also obtain the solution (2.2.13) by separating variables. If $x_0 = 0$, then the solution should be $x(t) = 0$; assume, as before, that $x_0 > 0$ and separate variables. Your work should proceed as in the case $a(t) = r$ in Section 2.2.2, except that the integral on the right will be more complicated than (2.2.8). Specifically, after integrating, what is A?

2.2.23 Problem (+). This is the ultimate perspective on separation of variables. It shows that all the nonequilibrium solutions to a separable ODE solve a related implicit equation (involving, for better or for worse, definite integrals), and, conversely, every solution to this implicit equation actually solves the separable ODE. We could possibly solve this implicit equation with symbolic, numerical, or theoretical techniques beyond the scope of our introductory differential equations course.

Let g be a continuous function on the interval I and h be a continuous function on the interval J. Let t_0 be a point in I and x_0 be a point in J. Assume that $h(x) \neq 0$ for all x in J. Let x be a differentiable function defined on I such that $x(t)$ is in J for all t in I; in particular, $h(x(t)) \neq 0$ for all t, and so x is not an equilibrium solution.

(i) Suppose that x solves the IVP

$$\begin{cases} \dot{x} = g(t)h(x) \\ x(t_0) = x_0. \end{cases} \tag{2.2.14}$$

Show that x also solves the IMPLICIT INTEGRAL EQUATION

$$\int_{x_0}^{x(t)} \frac{du}{h(u)} = \int_{t_0}^{t} g(\tau)\, d\tau. \tag{2.2.15}$$

This is the "algebraic" equation for x that we solve after integrating in separation of variables. [Hint: *separate variables and integrate both sides from $\tau = t_0$ to $\tau = t$. Then change variables on the left with $u = x(\tau)$. This is the sort of manipulation that we did in Section 2.1.4 for the toy problem and Section 2.2.2 for exponential growth.*]

(ii) Conversely, show that if x solves the integral equation (2.2.15), then x also solves the IVP (2.2.14). [Hint: *differentiate both integrals with respect to t; you will need to* *use the fundamental theorem of calculus. Use the chain rule to differentiate the integral on the left and make $\dot{x}$ appear. For the initial condition, we want $x(t_0) = x_0$; suppose instead $x(t_0) \neq x_0$. Since $h(x) \neq 0$ for all x in J and h is continuous, the function h is either strictly positive or strictly negative, and so likewise $1/h$ is either strictly positive or strictly negative. Then $\int_{x_0}^{x(t_0)} du/h(u)$ is nonzero. But this integral also equals $\int_{t_0}^{t_0} g(\tau)\, d\tau$. A contradiction must ensue, and so we were wrong to assume $x(t_0) \neq x_0$).]*

(iii) This step shows how we can solve the implicit equation (2.2.15). For x in J, define

$$F(x) := \int_{x_0}^{x} \frac{du}{h(u)}.$$

As in the hint to part (ii) above, $1/h$ is either strictly positive or strictly negative on J. Assume that $1/h$ is strictly positive; the argument when $1/h$ is strictly negative is the same. Use property ($\int 4$) of Theorem 1.5.4 to argue that if $x_0 < x_1 < x_2$, then $F(x_1) < F(x_2)$; use that property and part (ii) of Problem 1.5.6 to argue that if $x_1 < x_2 < x_0$, then $F(x_1) < F(x_2)$. Conclude that F is strictly increasing on J, and therefore F is invertible on J, i.e., there exists a function F^{-1} such that $F^{-1}(F(x)) = x$ for all x in J. Moreover, use FTC1 to check that $F'(x) \neq 0$ for all x in J. The celebrated inverse function theorem then implies that F has a differentiable inverse F^{-1}, and $(F^{-1})'(x) = 1/F'(F^{-1}(x))$. In particular, since $F(x_0) = 0$, we have $F^{-1}(0) = x_0$.

Now put

$$x(t) := F^{-1}\left(\int_{t_0}^{t} g(\tau)\, d\tau\right).$$

Use the results above to show that x solves the IVP (2.2.14).

2.2.5. Going forward.

Separation of variables allows us to treat analytically the broad class of separable ODE $\dot{x} = g(t)h(x)$, and in particular the rich class of autonomous ODE $\dot{x} = f(x)$. However, there are worthwhile ODE to study that are not separable, and we will meet some later and develop new techniques for them.

Moreover, the method of separation of variables is far from perfect, as we may not be able to evaluate one or more antiderivatives in terms of elementary functions, and/or we may fail to solve explicitly for x. Even when it is possible to evaluate the antiderivatives and solve for x, the calculus and algebra may be burdensome; for example, treating even the rescaled logistic problem $\dot{x} = x(1-x)$ requires antidifferentiating

$$\int \frac{dx}{x(1-x)},$$

which calls for the dreaded method of partial fractions, and then a ton of algebra.

Here is where the analyst's creed (AC) should console us. It is possible to understand quite well the solutions to an ODE without having formulas for them. Subsequent qualitative and numerical approaches will give us palatable, useful alternatives to separation of variables and, indeed, any analytic method.

Finally, while separation of variables can address the question of existence of solutions to ODE (in that it gives us a symbolic algorithm for generating some solutions), the technique does not really address uniqueness rigorously. The technique unfolds under assumptions like *if* a solution x to $\dot{x} = g(t)h(x)$ exists, and *if* $h(x(t)) \neq 0$ for all t in some interval, *then* x satisfies an implicit relationship obtained via antidifferentiation. But why should there be only one solution to that relationship? We will need to build some more theory to assure ourselves that the solutions that we have found to separable IVP are indeed the only ones.

2.3. Slope fields.

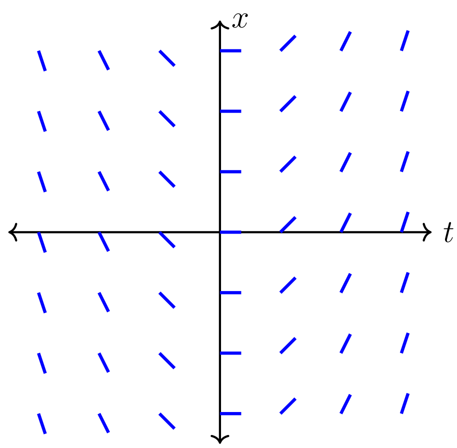
What is the derivative? Analytically, it is the limit of a difference quotient; geometrically, it is the slope of a curve. Specifically, let x be a function. Then the slope of x at the point $(t, x(t))$ in the tx-plane is $\dot{x}(t)$. So if x solves an ODE $\dot{x} = f(t, x)$, then the slope of x at $(t, x(t))$ is just $f(t, x(t))$.

Here, then, is the key geometric insight: if we know that x solves the ODE $\dot{x} = f(t, x)$, and if we know that x passes through the point $(t_\star, x_\star)$ in the tx-plane, then the slope of x at that point is $f(t_\star, x_\star)$. In other words, *we can calculate the slopes of a solution to $\dot{x} = f(t, x)$ without having a formula for x!* Next, remember the phenomenon of **LOCAL LINEARITY** from calculus: if x is differentiable at t, then "close to" t, the graph of x resembles the graph of the tangent line to x at t. In other words, *if we zoom in close enough, everything looks like a line.*

We will make these observations systematic by constructing **SLOPE FIELDS**. Given an ODE $\dot{x} = f(t, x)$, at the point (t, x) draw a small line segment with slope $f(t, x)$. If we draw enough of these segments and fill the tx-plane with a "field" of them, we will start to see a "flow" of curves in the plane. Those curves are potential solutions to $\dot{x} = f(t, x)$.

We begin with an example that does not really call for a slope field.

2.3.1 Example. All solutions to the ODE $\dot{x} = t$ are the parabolas $x(t) = t^2/2 + C$. With this in mind, we draw the slope field for this ODE "by hand" to "get a feel" for the procedure. Starting from each point (t, x) with integer coordinates for $-3 \leq t \leq 3$ and $-3 \leq x \leq 3$, we will draw a short line segment with slope t. To keep the tx-plane relatively unclutteblue, we will not label points on the axes.



If we stare at these pictures for a little while, hopefully we start to see the parabolas $x(t) = t^2/2 + C$ emerging, however crudely and inchoately.

This is where we finished on Friday, September 1, 2023.

The process above both taught us how to draw slope fields, and it reminded us of some things that we should know about the "direct integration" problem $\dot{x} = f(t)$. Namely, the following statements all mean the same thing.

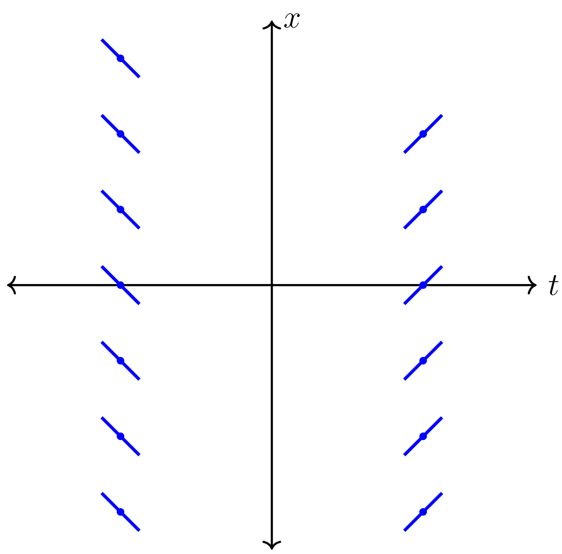
1. The function x solves the ODE $\dot{x} = f(t)$.

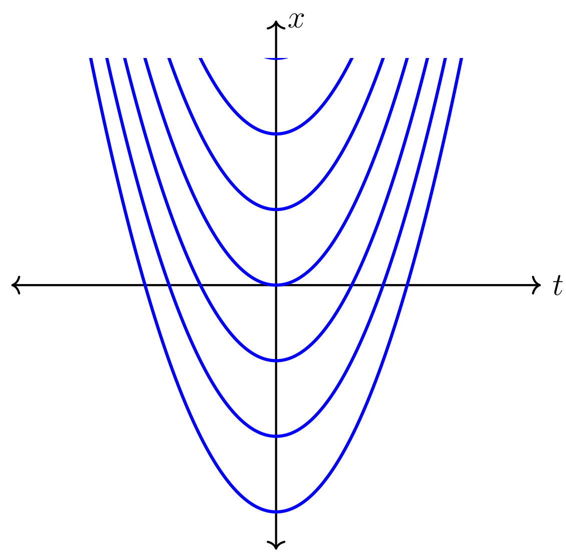
2. The slope of the curve at a point depends on the t-coordinate of that point but not the x-coordinate.

3. At two given points, the slopes are the same if the t-coordinates of both points are the same.

4. The slopes are the same along any vertical line.

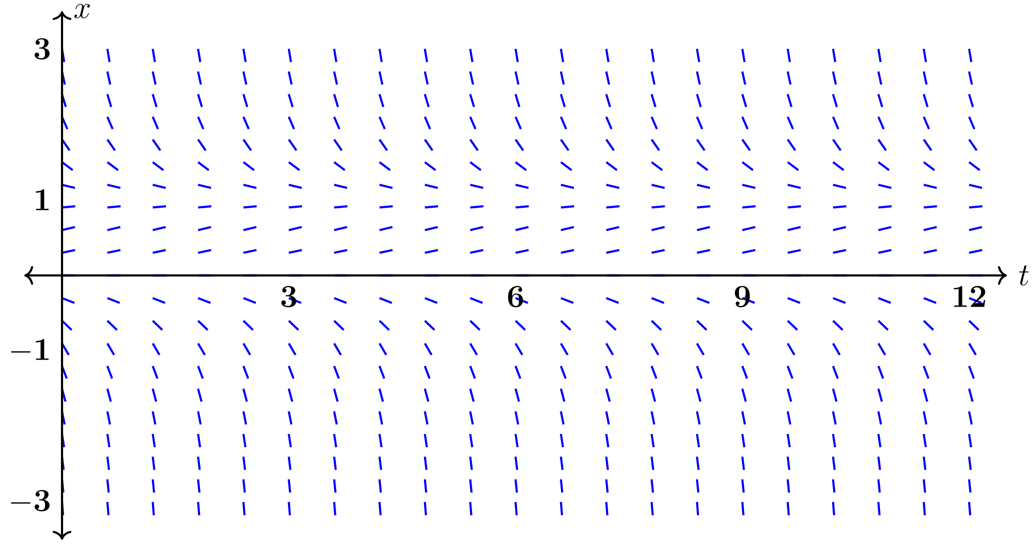
5. All solutions are just vertical translates of one fixed solution. (If F and G are antiderivatives of f, then there is a constant C such that $G(t) = F(t) + C$ for all t.)



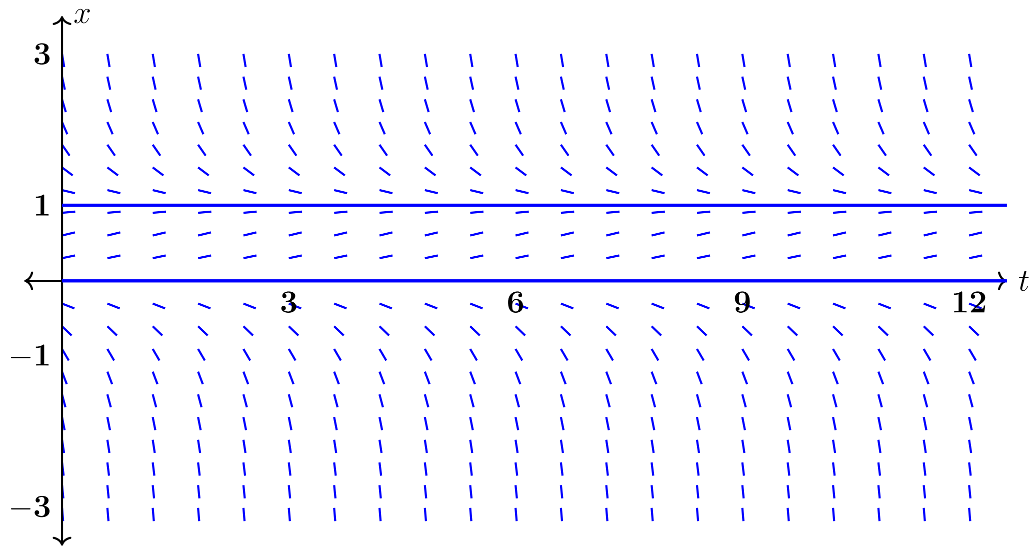


Next we study the logistic equation $\dot{x} = x(1-x)$ via slope fields. We know the two constant equilibrium solutions for this problem, but using separation of variables to find nonconstant solutions was onerous. Nonetheless, calculus gave us an idea of how, qualitatively, solutions corresponding to different initial conditions should behave, and so we will look for that qualitative behavior in the slope field.

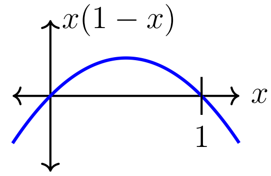
2.3.2 Example. Here is a slope field for the logistic equation $\dot{x} = x(1-x)$.



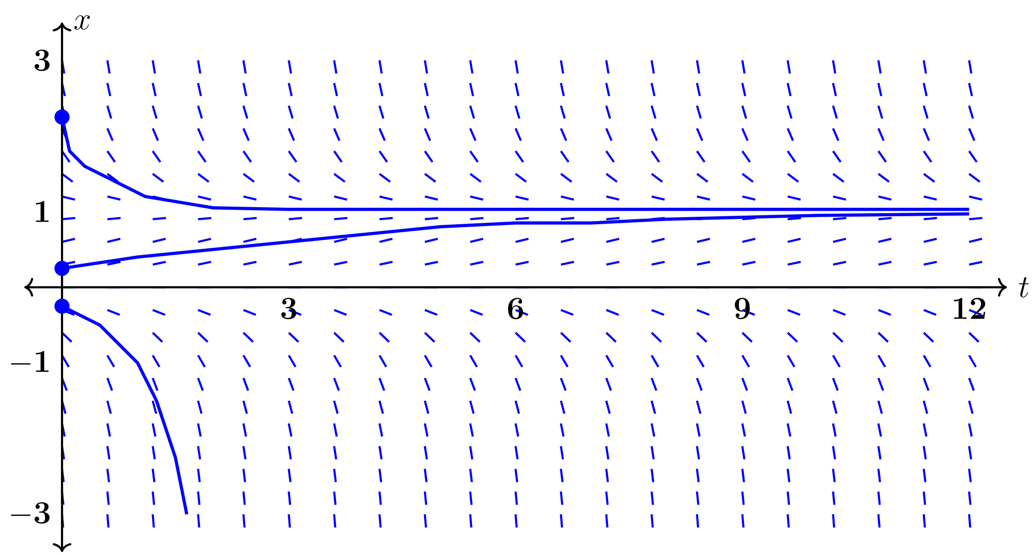
Recall that the equilibrium solutions are $x = 0$ and $x = 1$. We draw them as solid blue lines on the iteration of the slope field below.



The equilibrium solutions break up the tx-plane into three regions: below $x = 0$, between $x = 0$ and $x = 1$, and above $x = 1$. The slopes in each region are different: they are negative below $x = 0$, positive between $x = 0$ and $x = 1$, and negative above $x = 1$. This corresponds, of course, to the sign of $x(1-x)$ for these different values of x, as the following graph indicates.



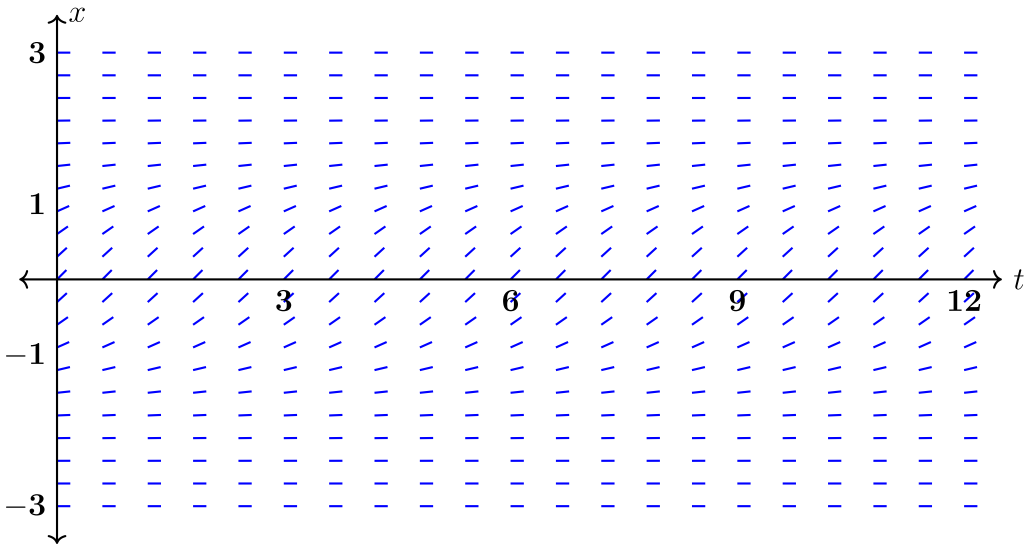
Now we try to "stitch" some of these slopes together into a continuous curve. We start drawing at $t = 0$ and do so for different values of my initial condition $x(0)$. We follow the slope segment nearest to $(0, x(0))$ and try to "jump" to the next slope, and go on like that. This is an art, not a science.



It should appear that the solution x that starts with $x(0) < 0$ ends up decreasing very rapidly towards $-\infty$. The solution x that starts with $0 < x(0) < 1$ increases up to 1. And the solution x that starts with $x(0) > 1$ decreases toward 1. In other words, where a solution starts relative to the equilibrium solutions has a profound effect on the solution's end behavior—exactly as we predicted with calculus.

2.3.3 Example. We failed to solve $\dot{x} = e^{-x^2}$ using separation of variables in Example

2.2.15. Here is a slope field for this ODE.



It looks like solutions that start between -1 and 1 increase, possibly up to a horizontal asymptote, while solutions that start below -1 or above 1 are constant. This is certainly more information than we saw in our previous encounter with this ODE (in which we learned precisely nothing), but is it correct?

First, the slope field only treats time in the interval $0 \le t \le 12$. Perhaps solutions that start between -1 and 1 really do reach a horizontal asymptote as $t \to \infty$ (if they are even defined for that long), but maybe they just grow really, really slowly, like the natural log. We need a bigger picture, and probably more work, to be sure about the asymptotics.

Second, constant solutions are equilibrium solutions: if x is a constant solution of $\dot{x} = e^{-x^2}$, then $\dot{x}(t) = 0$ for all t, and thus $e^{-[x(t)]^2} = 0$ for all t. But $e^{-[x(t)]^2} > 0$ for all t. So, the "constant" solutions that the slope field might predict are actually not there. Instead, we might note that when x is "large," e^{-x^2} is "very small," and so a more precise graph might reveal that the "horizontal" slopes at $x = \pm 3$ and thereabouts really are not horizontal.

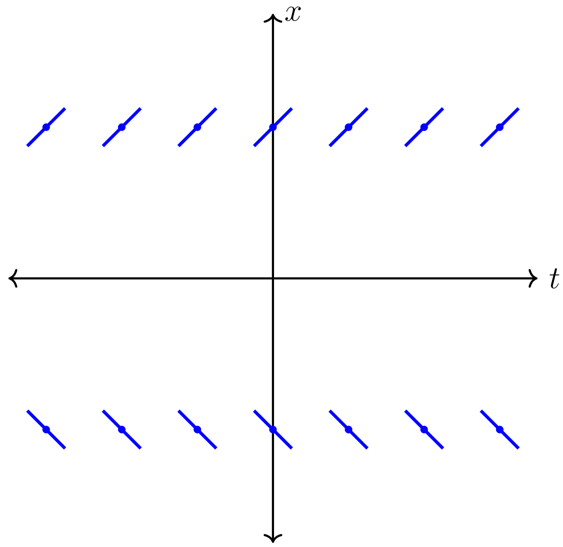
Here is another observation from the preceding examples. There is a lot of repetition in each slope field: many of those slope marks are the same. Specifically, they are repeated horizontally (unlike vertically, as we saw with the direct integration problem). This suggests that the following different statements all mean the same.

1. The function x solves the ODE $\dot{x} = f(x)$.

2. The slope of the curve at a point depends on the x-coordinate of that point but not the t-coordinate.

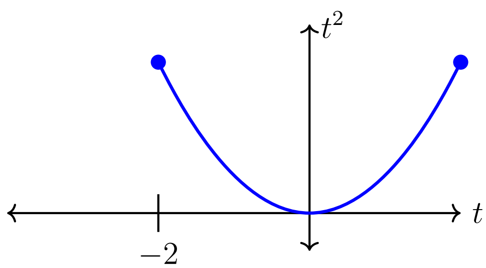
3. At two given points, the slopes are the same if the x-coordinates of both points are the same.

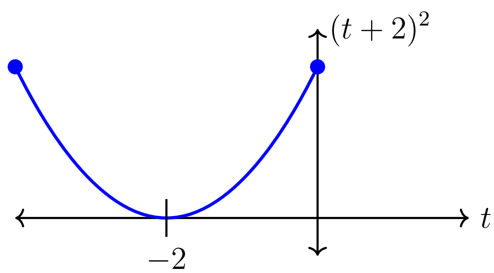
4. The slopes are the same along any horizontal line.



5. If we translate the graph of one solution horizontally, then we get another solution.

The last insight here is probably something new. Recall that if x is a function, then the graph of $y(t) := x(t + t_\star)$ is just the graph of x shifted to the left (if $t_\star > 0$) or to the right (if $t_\star < 0$).





We formalize this insight in a theorem.

2.3.4 Theorem. *Suppose that x solves $\dot{x} = f(x)$. Fix a number $t_\star$ and define $y(t) := x(t + t_\star)$. Then y also solves $\dot{y} = f(y)$. In particular, if the domain of x is the interval (a, b), then the domain of y is the "shifted" interval $(a - t_\star, b - t_\star)$.*

Proof. We need to show that $\dot{y}(t) = f(y(t))$ for all t. First,

$$\dot{y}(t) = \frac{d}{dt}[x(t + t_\star)] = \dot{x}(t + t_\star)\frac{d}{dt}[t + t_\star] = \dot{x}(t + t_\star)$$

by the chain rule. Since $\dot{x}(\tau) = f(x(\tau))$ for all τ, we can take $\tau = t + t_\star$ to find

$$\dot{y}(t) = \dot{x}(t + t_\star) = f(x(t + t_\star)) = f(y(t)). \qquad \blacksquare$$

2.3.5 Problem (!). Suppose that x solves the IVP

$$\begin{cases} \dot{x} = f(x) \\ f(0) = 1, \end{cases}$$

and that the domain of x is $(-1, 4)$. Use Theorem 2.3.4 to construct a solution y to the IVP

$$\begin{cases} \dot{y} = f(y) \\ y(10) = 1 \end{cases}$$

and state the domain of y. [Hint: *your function y will, of course, involve x.*]

2.3.6 Problem ($\star$). Separation of variables suggested that all solutions to $\dot{x} = x$ have the form $x(t) = Ce^{rt}$ for some constant C. Show that if $x_1(t) = C_1 e^t$ for some constant C_1 and if we put $x_2(t) := x_1(t + t_\star)$ for some fixed time $t_\star$, then $x_2(t) = C_2 e^{rt}$ for some constant C_2. What is C_2? Conclude that horizontally translating solutions to $\dot{x} = rx$ preserves the analytic solution structure that we expect.

2.3.7 Problem ($\star$). Horizontal translates of solutions to autonomous problems remain solutions, but other geometric operations on solutions do not necessarily yield solutions.

(i) If x solves $\dot{x} = x$, does $y(t) := x(t) + 1$ solve $\dot{y} = y$?

(ii) If x solves $\dot{x} = x^2$, does $y(t) := 2x(t)$ solve $\dot{y} = y^2$?

Slope fields are chiefly valuable for their versatility and universality. They do not require any calculus to implement (unlike separation of variables), nor do they demand any special structure of the underlying ODE (also unlike separation of variables). However, slope fields are nightmarish to draw by hand, and interpreting slope fields can depend greatly on one's own point of view. Often to see the right pattern in a slope field, we need to know what we are looking for before we see it. One rarely uses a slope field alone to study an ODE; in particular, since we are likely using a computer to generate the slope field, we may as well go further and program a numerical solver, so that we can get actual approximate graphs and values for our solutions.

This is where we finished on Wednesday, September 6, 2023.

2.4. Euler's method.

We now have two techniques for studying ODE. Separation of variables can give us a formula from which we could compute exact values, but sometimes the integration trips us up—and not all problems may have the special separable form, anyway. Slope fields help us make qualitative predictions about the behavior of solutions to ODE, but these predictions are often crude—and slope fields do not tell us the exact values of solutions at particular moments in time.

We now need a third tool beyond the analytic and qualitative methods: the numeric. A broad array of numerical methods can approximate solutions to ODE; we will study just one, called **Euler's method**.

2.4.1. Derivation of Euler's method.

As usual, we start by working backward. Suppose that we have a solution x to the IVP

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0. \end{cases} \tag{2.4.1}$$

If we want to learn about x, and we know absolutely nothing specific about f, one good way to make x appear is to integrate. This was the wrong idea at the start of the course in Section 1.1, but now it is the right idea. The fundamental theorem of calculus gives, as always,

$$x(t) - x(t_0) = \int_{t_0}^t \dot{x}(\tau)\ d\tau = \int_{t_0}^t f(\tau, x(\tau))\ d\tau,$$

and so

$$x(t) = x(t_0) + \int_{t_0}^t f(\tau, x(\tau))\ d\tau = x_0 + \int_{t_0}^t f(\tau, x(\tau))\ d\tau. \tag{2.4.2}$$

This is exactly what we obtained in (FTC) at the start of the course, and we complained then, because this equation defines x in terms of x, which is not very helpful.

However, we can turn this into a good *approximation* for the value $x(t)$ by recalling the "left endpoint rule" for approximating integrals:

$$\int_a^b g(\tau)\ d\tau \approx (b-a)g(a), \tag{LHR}$$

at least if a and b are "close" (whatever that means). Thus

$$\int_{t_0}^t f(\tau, x(\tau))\ d\tau \approx (t-t_0)f(t_0, x(t_0)) = (t-t_0)f(t_0, x_0) \tag{2.4.3}$$

when t is "close" to t_0. The "=" on the right really is genuine; it is the initial condition $x(t_0) = x_0$. And so if we combine (2.4.2) and (2.4.3), we get

$$x(t) \approx x_0 + (t-t_0)f(t_0, x_0) \quad \text{for} \quad t \approx t_0. \tag{2.4.4}$$

Now we make "close" a little more precise (but not a lot). Fix a small positive number h, maybe with $0 < h < 1$. Define $t_1 := t_0 + h$. Then (2.4.4) just says

$$x(t_1) \approx x_0 + hf(t_0, x_0).$$

We abbreviate

$$x_1 := x_0 + hf(t_0, x_0).$$

Then $x(t_1) \approx x_1$. Note that we calculated x_1 just using the given information of f and the initial data t_0 and x_0. We did not do any calculus.

Now we jump a bit forward into the future. Put $t_2 := t_1 + h = t_0 + 2h$, so t_2 is not too far away from t_1. Then

$$\begin{aligned}
x(t_2) &= x(t_1) + \int_{t_1}^{t_2} \dot{x}(\tau)\, d\tau \text{ by the fundamental theorem of calculus} \\
&= x(t_1) + \int_{t_1}^{t_2} f(\tau, x(\tau))\, d\tau \text{ since } x \text{ solves } \dot{x} = f(t,x) \\
&\approx x_1 + \int_{t_1}^{t_2} f(\tau, x(\tau))\, d\tau \text{ since } x(t_1) \approx x_1 \\
&\approx x_1 + (t_2 - t_1) f(t_1, x(t_1)) \text{ by the left-hand approximation for integrals} \\
&= x_1 + hf(t_1, x_1) \text{ since } t_2 = t_1 + h.
\end{aligned}$$

We abbreviate

$$x_2 := x_1 + hf(t_1, x_1).$$

Then $x(t_2) \approx x_2$, and we calculated x_2 just by using the given information of f and the previously calculated data t_1 and x_1. Again, we did not do any calculus. Note that there were two uses of $\approx$ above: when we replaced $x(t_1)$ with x_1 and when we approximated the integral.

These two steps suggest a scheme for numerically approximating the solution to the IVP (2.4.1). First, fix a small **TIME STEP** $h > 0$. For integers $k \geq 0$, define

$$t_k := \begin{cases} t_0, & k = 0 \\ t_{k-1} + h, & k \geq 1, \end{cases} \qquad \text{equivalently} \qquad t_k := t_0 + kh, \ k \geq 0$$

and

$$x_k := \begin{cases} x_0, & k = 0 \\ x_{k-1} + hf(t_{k-1}, x_{k-1}), & k \geq 1. \end{cases}$$

Then we expect that the true solution x to the IVP (2.4.1) enjoys the approximation

$$x(t_k) \approx x_k.$$

If we run this iteration some $n \geq 1$ times, then we generate $n + 1$ approximations to the value of x on the interval $[t_0, t_0 + nh]$. These are

$$x(t_0) = x_0, \qquad x(t_1) \approx x_1, \qquad \ldots, \qquad x(t_n) \approx x_n.$$

Conversely, to approximate a solution on the interval $[t_0, T]$ for some $T > t_0$, assuming that the solution is defined on an interval of that length, choose the number of iterations n and put $h := (T - t_0)/n$.

2.4.1 Problem. This problem is an "iterative justification" that the kth step in Euler's method is a good approximation; it is essentially an abstraction of the discussion above. Suppose that we have run Euler's method with time step $h > 0$ up to the $(k-1)$st step. Then we define

$$x_k := x_{k-1} + hf(t_{k-1}, x_{k-1}).$$

Justify each equality or approximation in the chain below:

$$\begin{aligned}
x(t_k) &\overset{(1)}{=} x(t_{k-1}) + \int_{t_{k-1}}^{t_k} \dot{x}(\tau)\, d\tau \\
&\overset{(2)}{=} x(t_{k-1}) + \int_{t_{k-1}}^{t_k} f(\tau, x(\tau))\, d\tau \\
&\overset{(3)}{\approx} x_{k-1} + \int_{t_{k-1}}^{t_k} f(\tau, x(\tau))\, d\tau \\
&\overset{(4)}{\approx} x_{k-1} + (t_k - t_{k-1})f(t_{k-1}, x(t_{k-1})) \\
&\overset{(5)}{=} x_{k-1} + hf(t_{k-1}, x(t_{k-1})) \\
&\overset{(6)}{\approx} x_{k-1} + hf(t_{k-1}, x_{k-1}).
\end{aligned}$$

Another derivation of Euler's method, which we do not discuss here, relies on tangent lines as local linear approximations to functions. This is a perfectly valid way of proceeding, but the approach above offers extra practice with the all-important FTC and later allows for straightforward improvements by getting a better approximation to the definite integral.

2.4.2. Pseudocode and sample implementations.

Here is a summary of Euler's method.

Define the function f.
Define the starting time t_0.
Define the initial value x_0.

Choose a time step $h > 0$.
Choose a number of iterations $n \geq 1$.

For $k = 1, \ldots, n$, iterate

$$\begin{cases} t_k := t_0 + kh \\ x_k := x_{k-1} + hf(t_{k-1}, x_{k-1}). \end{cases}$$

2.4.2 Example. We know that the solution to the IVP

$$\begin{cases} \dot{x} = t \\ x(0) = 0 \end{cases}$$

is $x(t) = t^2/2$, and all we needed to get it was direct integration. Nonetheless, we implement Euler's method for this problem with five iterations ($n = 5$) and time step $h = 0.2$ to see the arithmetic in an "easy" context and to compare the numerical result to the exact analytic solution. with five iterations ($n = 5$) and the time step $h = 0.2$. In the notation of our pseudocode above, we are taking $f(t, x) = t$, $t_0 = 0$, and $x_0 = 1$.

We fill in the following table.

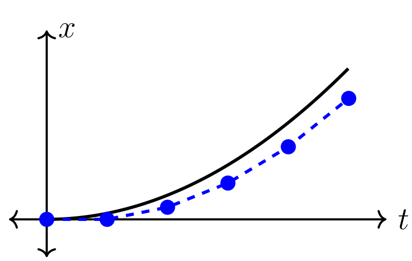
k	t_k	x_k	$f(t_k, x_k)$	$x_{k+1} = x_k + hf(t_k, x_k) = x_k + ht_k$
0	0	0	0	$0 + (0.2 \cdot 0) = 0$
1	0.2	0	0.2	$0 + (0.2 \cdot 0.2) = 0.04$
2	0.4	0.04	0.4	$0.04 + (0.2 \cdot 0.4) = 0.12$
3	0.6	0.12	0.6	$0.12 + (0.2 \cdot 0.6) = 0.24$
4	0.8	0.24	0.8	$0.24 + (0.2 \cdot 0.8) = 0.4$
5	1	0.4	1	$0.4 + (0.2 \cdot 1) = 0.6$

Now we compare the approximations to the exact value of the known solution $x(t) = t^2/2$ at the values t_k.

k	t_k	x_k	$x(t_k) = t_k^2/2$
0	0	0	0
1	0.2	0	0.02
2	0.4	0.04	0.08
3	0.6	0.12	0.18
4	0.8	0.24	0.32
5	1	0.4	0.5

It looks like our Euler's method results consistently under-approximate the true solution, but the values are definitely strictly increasing.

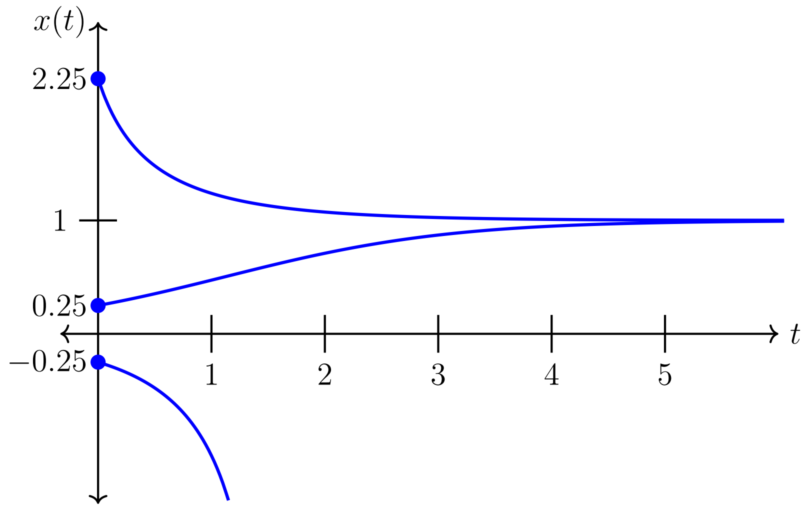
We can see this with plots. We graph the true solution $x(t) = t^2/2$ in solid black and plot the points (t_k, x_k) and connect them by dotted lines, both in blue.



2.4.3 Example. We use Euler's method to approximate solutions to the logistic IVP

$$\begin{cases} \dot{x} = x(1-x) \\ x(0) = x_0 \end{cases}$$

with initial values $x_0 = -.25$, $x_0 = .25$, and $x_0 = 2.25$. Here are the results.

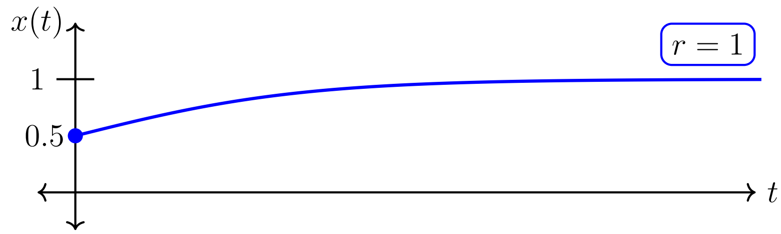


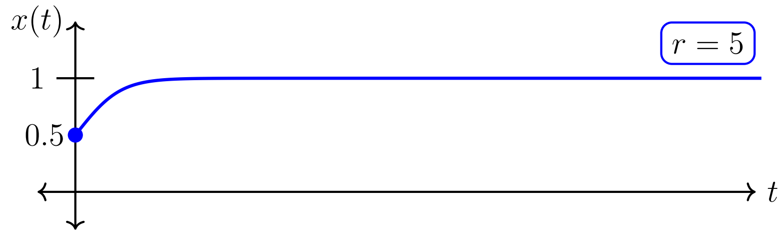
These results look quite similar to our sketches in Example 2.3.2, but much smoother and more confident. We see the three kinds of end behavior predicted by that example: a rapid decrease to $-\infty$ for the initial condition $x(0) < 0$, an increase to 1 for the solution with initial condition $x(0)$ satisfying $0 < x(0) < 1$, and a decrease to 1 for the solution with initial condition $x(0) > 1$. The numerical results, therefore, bolster our qualitative intuition from slope fields and, perhaps, are less prone to our subjective human error in interpreting the slope fields. But the numerical results do require a nontrivial amount of computing power, more so than, in principle, slope fields. Additionally, slope fields indicate the behavior of *multiple* solutions simultaneously, whereas Euler's method just approximates the solution to *one* IVP.

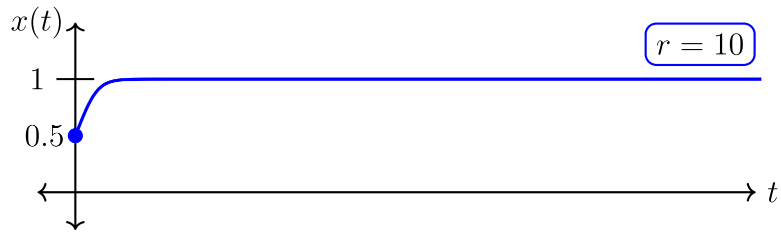
2.4.4 Example. We use Euler's method to approximate the solution to the logistic IVP

$$\begin{cases} \dot{x} = rx(1-x) \\ x(0) = 0.5 \end{cases}$$

for the different growth rates $r = 1$, 5, 10. Here are the graphs.







Over long times, all three graphs look the same: the solutions all increase to 1. However, in the "short run," things are different. The solution for $r = 1$ takes comparatively longer to get "close" to 1 than the solutions for $r = 5$ and $r = 10$. The difference between the $r = 5$ and the $r = 10$ solution is less dramatic but still present.

We might conjecture, then, that the parameter r controls the "rate" at which solutions to $\dot{x} = rx(1-x)$ converge to their end behavior. This gives us deeper insight into the role of the parameters r and N in the general logistic problem $\dot{x} = rx(1-x/N)$. It is easy to see that $x = N$ is an equilibrium solution, but it is not as easy to see, at first glance, what role r plays. Nonetheless, if we think that r is a positive number, and if $0 < x < N$, then $rx(1-x/N)$ will also be positive. Taking r to be an ever-larger positive number will just make the quantity $rx(1-x/N)$ larger. Remember that $rx(1-x/N)$, in this context, really is the value of a derivative. This suggests that taking r to be larger increases the slopes of the solution to $\dot{x} = rx(1-x/N)$. That is exactly what we are seeing in the passage from $r = 1$ to $r = 10$ in these solutions as the convergence to 1 gets "faster."

We could confirm this with analytic techniques by finding a formula for the solution to the logistic IVP.

This is where we finished on Friday, September 8, 2023.

2.4.3. Outlook on numerics.

Euler's method is the only numerical method that we will study in this course, but we will use it regularly (almost every graph of a solution to an ODE has been drawn with it), and it even generalizes rather easily to systems. Nonetheless, it is far from the only numerical method available for differential equations. One natural way to "improve" Euler's method is to consider how we derived it: by approximating a definite integral with the left-hand rule. This is a fairly crude integral approximation; in calculus we learn plenty of others. We could therefore return to the derivation of Euler's method and use a different, better integral approximation; this would no doubt change the pseudocode and implementation, but it might improve the results—ideally, we would get *better* approximations *faster*. Indeed, there are a number of ways in which our numerical results may not be optimal, and we should watch out for them. Here are some problems that lead to suboptimal numerical results.

1. *Problems with the computer.* Arithmetic and other calculations on a computer inherently involve roundoff errors.

2. *Problems with the method.* Numerical methods inherently involve *choices* of how discretizations and approximations are made. Some of those choices (say, the left-hand rule for approximating an integral) may be less ideal than others. Furthermore, numerical methods may be *unstable* in the sense that small errors early in the method propagate and expand to large errors later on.

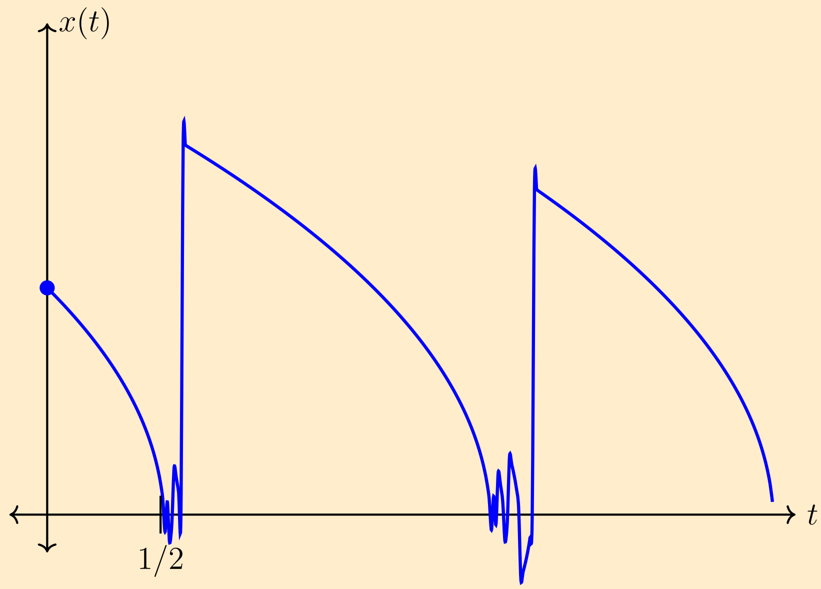
3. *Problems with the problem.* Some ODE are simply badly behaved! Perhaps in the ODE $\dot{x} = f(t, x)$, the function f very rapidly increases or decreases in either t or x (or both), and so any numerical method must contend with very large (or very small) values of f.

Remember, no one method in differential equations works all the time or gives all the desired information. We would never use only a simple slope field to study an ODE, and we will never be content with just what Euler's method, or any numerical method, says about solutions.

2.4.5 Problem ($\star$). The following is the output produced when Euler's method is run for the IVP

$$\begin{cases} \dot{x} = -1/x \\ x(0) = 1 \end{cases}$$

with $h = 0.01$ and $n = 300$.



Around $t = 1/2$, the output should look very strange.

(i) Note that when $t \approx 1/2$, Euler's method suggests that $x(t) \approx 0$. Given that the ODE reads $\dot{x} = 1/x$, why could this be a cause for concern?

(ii) Use separation of variables to solve this IVP. What insight does the resulting formula give you into the behavior of Euler's method around $t = 1/2$?

(iii) Do you want to trust the results of Euler's method for $t > 1/2$?

2.5. Existence and uniqueness theory.

Our analytic, qualitative, and numerical work so far has not guaranteed that solutions always exist to particular ODE (from Example 2.2.15, does $\dot{x} = e^{-x^2}$ *really* have solutions?), nor that they are unique. Indeed, we probably expect by now that when solutions to ODE do exist, they are not unique—all of the solutions that we found from direct integration and separation of variables (which really subsumes direct integration—recall Problem 2.2.21) came with an arbitrary constant (usually C) somewhere in their formulas. For example, all solutions to $\dot{x} = 2t$ are $x(t) = t^2 + C$.

Moreover, while we have a nice formulaic procedure for solving separable ODE, it easily breaks (again, what is happening with $\dot{x} = e^{-x^2}$?) and leaves us clueless about the existence of solutions. Slope fields and Euler's method give us insight into the long(ish)-time behavior of solutions to effectively arbitrary ODE (not just separable ones), but they do not assure us that solutions exist in the first place. If we are going to spend time talking about the properties of (if not formulas for) solutions to a problem, we should be sure that solutions actually exist. Here we discuss the general existence and uniqueness of solutions to separable problems.

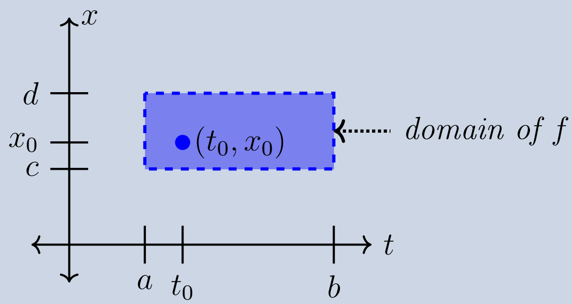
We need a piece of technical terminology first.

2.5.1 Definition. *A function h on an interval J is* **CONTINUOUSLY DIFFERENTIABLE** *if h is differentiable on J and if h' is continuous on J.*

Most functions that we meet in practice in this course or in calculus are continuously differentiable, and it takes some work to find a differentiable function that is not continuously differentiable.

2.5.2 Theorem (Existence and uniqueness for separable ODE). *Suppose that g is a continuous function on the interval (a, b) and h is a continuous, differentiable function on the interval (c, d) with h' continuous on (c, d). The numbers a and c may be $-\infty$ and the numbers b and d may be ∞. Let*

$$f(t, x) := g(t)h(x). \tag{2.5.1}$$



(i) [Existence] *Let t_0 be a point in (a, b) and x_0 be a point in (c, d). There exist numbers $\alpha > 0$ and $\omega > 0$ and a function x defined on $(t_0 - \alpha, t_0 + \omega)$ such that x solves the IVP*

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0. \end{cases}$$

In particular, the values of x satisfy $c < x(t) < d$, while α and ω satisfy $a < t_0 - \alpha$ and $t_0 + \omega < b$. (recall Definition 1.4.1).

(ii) [Uniqueness] *Suppose that y is another function on $(t_0 - \alpha, t_0 + \omega)$ that solves the IVP*

$$\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = x_0. \end{cases}$$

Then $x(t) = y(t)$ for all t.

The existence result is just that: existence. It does not give us a procedure for finding the solution x, nor does it tell us anything about ϵ. It does not tell us anything about the behavior of x. So, we do not know for how long we can predict the future, nor do we really know what happens in the future. All that existence tells us is that we *can* predict the future. If we want to understand a specific problem better, we have lots more work to do.

However, the uniqueness result should be comforting. Not only can we predict the future, if we impose initial data, then we can only predict one future. All hail the Sacred Timeline!

2.5.3 Remark. *Do the results of Theorem 2.5.2 still feel underwhelming? First, we have only stated the theorem for separable equations, and there are many equations that are not separable. We will meet one such class, the linear ODE, in the near future, and for linear ODE, remarkably, we can prove from scratch an existence and uniqueness theorem. There are far more general statements of Theorem 2.5.2 for far more general classes of ODE, but they demand, naturally, some more technical hypotheses.*

One such hypothesis involves the partial derivative of f with respect to x, but that requires knowledge of multivariable calculus. Perhaps a more accessible generalization demands that f satisfies a **Lipschitz estimate** *of the following form: there is a constant $C > 0$ such* *that for all t_1, t_2, x_1, and x_2 in the domain of f,*

$$|f(t_1, x_1) - f(t_2, x_2)| < C|t_1 - t_2| + C|x_1 - x_2|.$$

If f satisfies such an estimate, then the conclusions of Theorem 2.5.2 remain true.

Since we will not really encounter first-order ODE in this course that are not separable or linear, and since we are not presuming a multivariable calculus background, we will not go on to state further existence and uniqueness hypotheses for more complicated problems.

2.5.4 Example. **(i)** The existence and uniqueness theorem implies that the IVP

$$\begin{cases} \dot{x} = e^{-x^2} \\ x(t_0) = x_0, \end{cases}$$

which we failed to solve in Example 2.2.15, has a unique solution for any choice of t_0 and x_0. Here is why. Put $g(t) = 1$ and $h(x) = e^{-x^2}$. Then this ODE has the form $\dot{x} = g(t)h(x)$, and g is continuous on $(-\infty, \infty)$, while h is continuously differentiable on $(-\infty, \infty)$. Consequently, we can select any starting time t_0 and any initial condition x_0 that we like, and the IVP will have a unique solution.

(ii) The same is true for the logistic IVP

$$\begin{cases} \dot{x} = x(1-x) \\ x(t_0) = x_0. \end{cases} \tag{2.5.2}$$

Here $g(t) = 1$ again but now $h(x) = x(1-x)$, and so g is continuous on $(-\infty, \infty)$, while h is continuously differentiable on $(-\infty, \infty)$.

(iii) The autonomous problem

$$\begin{cases} \dot{x} = \sqrt{x} \\ x(0) = 0 \end{cases} \tag{2.5.3}$$

is not so nicely behaved. Solving $\sqrt{x} = 0$ gives the only equilibrium solution $x(t) = 0$, and this certainly solves the IVP. But separating variables leads to (see Problem 2.5.5) another solution of the IVP: $x(t) = t^2/4$. The issue here is that $h(x) = \sqrt{x}$ is not differentiable at $x = 0$, and we have set the initial state to be $x_0 = 0$. Thus the existence and uniqueness theorem simply does not apply!

2.5.5 Problem ($\star$). **(i)** Carry out the separation of variables to which part (iii) of Example 2.5.4 refers to obtain the solution $x(t) = t^2/4$ of the IVP (2.5.3).

(ii) Use the fact that $\sqrt{A^2} = |A|$ for any real number A to show that if $x(t) = t^2/4$, then $\dot{x}(t) \neq \sqrt{x(t)}$ for $t < 0$. Conclude that the domain of $x(t) = t^2/4$ when considered *as a solution to the IVP* (2.5.3) is only $[0, \infty)$, even though *as a function* this x is defined on

$(-\infty, \infty)$. Contrast this domain with what the existence and uniqueness theorem promises under better circumstances.

(iii) Use the definition of the derivative to remind yourself that $h(x) := \sqrt{x}$ is not differentiable at $x = 0$.

2.5.6 Problem ($\star$). Let

$$g(t) := \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

We will show that there cannot exist a solution to the IVP

$$\begin{cases} \dot{x} = g(t) \\ x(0) = 0 \end{cases}$$

in the sense of Definitions 1.4.1 and 1.4.4.

(i) Show that if x solves the ODE

$$\dot{x} = g(t),$$

then x must have the form

$$x(t) = \begin{cases} C_1, & t < 0 \\ t + C_2, & t \geq 0 \end{cases}$$

for some constants C_1 and C_2.

(ii) Since the solution to an ODE should be continuous, we want

$$\lim_{t \to 0^-} x(t) = \lim_{t \to 0^+} x(t).$$

Use this to show that $C_1 = C_2$, and so x really has the form

$$x(t) = \begin{cases} C, & t < 0 \\ t + C, & t \geq 0 \end{cases}$$

for some constant C.

(iii) Choose C to meet the initial condition $x(0) = 0$.

(iv) What part of Definitions 1.4.1 or 1.4.4 is violated by this form of x?

(v) Why does this not violate the existence and uniqueness theorem?

2.5.7 Problem ($\star$). This problem offers a different perspective on the exponential. Usually one starts by assuming that $e^{t+\tau} = e^t e^\tau$ for all t and τ and from there it is possible to derive familiar results like "the exponential is its own derivative." Suppose instead that we know only that—assume that there is a differentiable function E defined on $(-\infty, \infty)$ that solves the IVP

$$\begin{cases} \dot{E} = E \\ E(0) = 1. \end{cases}$$

We can show that $E(t + \tau) = E(t)E(\tau)$ for all t and τ as follows. Fix τ and put $x(t) := E(t + \tau) - E(t)E(\tau)$. Check that x solves the IVP

$$\begin{cases} \dot{x} = x \\ x(0) = 0, \end{cases}$$

and use the existence and uniqueness theorem to conclude that $x(t) = 0$ for all t.

Here is a less technical but (hopefully) no less helpful way of thinking about the existence and uniqueness theorem.

2.5.8 Example. We can paraphrase the uniqueness part of the existence and uniqueness theorem as

"Two distinct solutions of an ODE can't be in the same place at the same time."

More positively, this could read

"If two solutions are in the same place at the same time, then they are the same."

Here is why. Suppose that f satisfies the existence and uniqueness hypotheses and that x and y both solve $\dot{x} = f(t, x)$ and $\dot{y} = f(t, y)$. Say that $x(t_*) = y(t_*)$ for some t_*. Write $x(t_*) = x_0$. Then x and y solve the same IVP:

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_*) = x_0 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{y} = f(t, y) \\ y(t_*) = x_0. \end{cases}$$

The existence and uniqueness theorem then says that $x(t) = y(t)$ for all t that belong to the domain of both x and y.

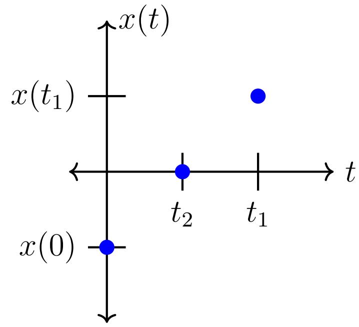
This is where we finished on Monday, September 11, 2023.

Some of the most transparent and helpful solutions to differential equations are equilibrium solutions. The preceding paraphrase reminds us that if a solution intersects an equilibrium solution only once, then that solution *is* an equilibrium solution.

2.5.9 Example. Suppose that x solves the logistic equation $\dot{x} = x(1 - x)$ and $x(0) < 0$. Qualitative (Example 2.3.2) and numerical (Example 2.4.3) evidence suggests that $x(t) < 0$ for all t. We can now prove this using the (paraphrase of) the existence and uniqueness theorem.

Suppose instead that $x(t_1) \geq 0$ for some $t_1 > 0$. Let $y(t) = 0$ for all t; then y is an equilibrium solution of the logistic equation, and certainly $y(t_1) = 0$. If $x(t_1) = 0$, then x and y are two solutions of the same ODE that are in the same place at the same time, and so $x(t) = y(t)$ for all t. In particular, $x(0) = y(0) = 0$. But $x(0) < 0$, so we cannot have $x(0) = 0$. Thus we were wrong to assume that $x(t_1) = 0$.

How about $x(t_1) > 0$? Then $x(0) < 0 < x(t_1)$, and x is continuous (being the solution of an ODE), so by the intermediate value theorem there is t_2 such that $0 < t_2 < t_1$ and $x(t_2) = 0$. (See the picture below.) The same contradiction as in the case $x(t_1) = 0$ then results.



This example hints at a deeper value of equilibrium solutions than just being more solutions to an ODE or helping to solve an IVP that the solutions from separation of variables do not (recall Problem 2.2.10). The equilibrium solution $y(t) = 0$ "fences in" solutions x with $x(0) < 0$. If a solution starts below 0, then it remains below 0 on all of its domain.

2.5.10 Problem ($\star$). We know that two different solutions to the same ODE cannot be in the same place at the same time. But can two solutions be in the same place at *different* times? The answer, as is often the case in math, is "sort of."

(i) Find two different solutions x_1 and x_2 to $\dot{x} = x$ such that $x_1(1) = 1$ and $x_2(2) = 1$. Explain why x_1 and x_2 are not the same function. Draw pictures.

(ii) Let f be continuously differentiable on $(-\infty, \infty)$ and suppose that x_1 and x_2 are solutions to $\dot{x} = f(x)$ defined on $(-\infty, \infty)$ with $x_1(t_1) = x_2(t_2)$ for some times t_1 and t_2. (Taking the domains of f, x_1, and x_2 all to be $(-\infty, \infty)$ is not strictly necessary, but it eliminates some notational complications that could otherwise obscure the point.) Define

$$y(t) := x_1(t + t_1 - t_2)$$

and check that y solves

$$\begin{cases} \dot{y} = f(y) \\ y(t_2) = x_2(t_2). \end{cases}$$

Conclude that $y = x_2$ and therefore $x_2(t) = x_1(t + t_1 - t_2)$ for all t. That is, x_2 is just a horizontal translate of x_1.

(iii) How is this result both similar to and different from Theorem 2.3.4?

(iv) Any two solutions to the exponential growth problem $\dot{x} = rx$ have the form $x_1(t) = c_1e^{rt}$ and $x_2(t) = c_2e^{rt}$ for some constants c_1 and c_2. Suppose that $x_1(t_1) = x_2(t_2)$ for some t_1 and t_2. Obtain $c_2 = e^{r(t_1-t_2)}$ and use properties of exponentials to check explicitly that $x_2(t) = x_1(t + t_1 - t_2)$ for all t.

One very useful consequence of the existence and uniqueness theorem, and a substantial generalization of the situation in Example 2.5.9, is the following "comparison test." Informally, it states that if two functions solve the same ODE, and if one function "starts below" the other, then that function "stays below" the other forever.

2.5.11 Theorem (Comparison theorem). *Suppose that f satisfies the hypotheses of the existence and uniqueness theorem. Let x and y solve the IVP*

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{y} = f(t,y) \\ y(t_0) = y_0. \end{cases}$$

If $x_0 < y_0$, then $x(t) < y(t)$ for all t in the domain of both x and y.

Proof. What goes wrong if the inequality $x(t) < y(t)$ does not hold for all t common to the domain of x and y? Say that $y(t_*) \le x(t_*)$ for some t_*. Then either $y(t_*) = x(t_*)$ or $y(t_*) < x(t_*)$.

In the first case of $y(t_*) = x(t_*)$, we see that two solutions to the same ODE are in the same place at the same time. Then $x(t) = y(t)$ for all t in their common domain. But then $x(t_0) = y(t_0)$, which cannot be true if $x(t_0) = x_0 < y_0 = y(t_0)$.

In the second case, $y(t_*) < x(t_*)$, we define an auxiliary function $z(t) = x(t) - y(t)$. Then $z(t_0) < 0$ and $z(t_*) > 0$, so the intermediate value theorem gives a time t_1 such that $z(t_1) = 0$. But then $x(t_1) = y(t_1)$, and we are back in the case of two solutions being in the same place at the same time. ■

2.5.12 Problem (!). Draw a picture illustrating the statement of the comparison theorem and another theorem illustrating its proof. Label everything clearly.

2.5.13 Problem (!). Reinterpret the results of Example 2.5.9 in light of the comparison theorem. What does the comparison theorem say about a function x such that $\dot{x} = x(1-x)$ with $1 < x(0)$?

2.5.14 Problem (!). How should Theorem 2.5.11 be adjusted in the case that $y_0 < x_0$? Make the adjustment and then prove your new theorem.

2.6. Phase line analysis for autonomous ODE.

Recall that, as we have said multiple times, an **AUTONOMOUS ODE** has the form

$$\dot{x} = f(x),$$

where f is a function of a single real variable, and its **EQUILIBRIUM SOLUTIONS** are the constant functions defined as the roots (zeros) if f (if any exist). It turns out that the equilibrium solutions to an autonomous ODE control in a special way the long-time behavior of its nonequilibrium solutions, and so we should always try to find the equilibrium solutions first by solving $f(x) = 0$. After that, we can also try separating variables to find nonequilibrium solutions, and we obtain

$$\frac{\dot{x}}{f(x)} = 1.$$

However, there is no guarantee that we can evaluate the auxiliary antiderivative

$$\int \frac{dx}{f(x)},$$

or that we could, in the end, solve for x algebraically. We discussed separation of variables for autonomous problems thoroughly in Section 2.2.1 and lamented some frustrations in Section 2.2.3.

However, we can say a lot about autonomous ODE *qualitatively*. By combining the existence and uniqueness theorem and doing a bit of calculus, we will see how the choice of initial condition alone affects what a solution to an autonomous ODE does at the extreme boundaries of its domain.

2.6.1. *Maximal existence for autonomous ODE.*

We can first assure ourselves that solutions to autonomous ODE exist. The following is a direct consequence of Theorem 2.5.2.

2.6.1 Theorem (Existence and uniqueness for autonomous ODE). *Let f be continuously differentiable on the interval (a, b). (The values $a = -\infty$ and/or $b = \infty$ are allowed.) Let x_0 be a point in (a, b). Then there are numbers $\alpha > 0$ and $\omega > 0$ such that the IVP*

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

has a unique solution x defined on $(-\alpha, \omega)$.

2.6.2 Problem (!). Show that this really is a direct consequence of Theorem 2.5.2. [Hint: *take $t_0 = 0$ and remember that in an autonomous ODE, $g(t) = 1$.*]

The only difference with the IVP here compared to those in Theorem 2.5.2 is that we have taken $t_0 = 0$, purely for convenience. That is, when studying autonomous ODE, we will always place our initial time at $t_0 = 0$. This really makes no difference other than simplifying notation (and according with our gut instinct that time begins at 0). After all, the ODE $\dot{x} = f(x)$ is independent of time!

While this result is comforting, its use by itself is limited. The theorem tells us nothing about the long-time properties of x, and nothing, in principle, about ϵ. For just how long in time does x exist?

2.6.3 Example. Here are five different autonomous (and hence separable) IVP. All have the form

$$\begin{cases} \dot{x} = f(x) \\ x(0) = 1, \end{cases}$$

and finding the formulas below is a good exercise. In each case, a domain ("Dom.") for the solution ("Soln.") is given. This is not necessarily the largest interval on which the solution is defined *as a function* but rather the largest open interval on which the solution is defined *and on which it solves the IVP*; in particular, this interval must contain the point $t = 0$. The limit ("Lim.") as t approaches the right endpoint of the domain from the left is given.

	(1)	**(2)**	**(3)**	**(4)**	**(5)**
ODE	$\dot{x} = x$	$\dot{x} = x(1-x)$	$\dot{x} = x^2$	$\dot{x} = -\dfrac{1}{x}$	$\dot{x} = \sqrt{5-x}$
IC	$x(0) = 1$	$x(0) = 1$	$x(0) = 1$	$x(0) = 1$	$x(0) = 1$
Soln.	$x(t) = e^t$	$x(t) = 1$	$x(t) = \dfrac{1}{1-t}$	$x(t) = \sqrt{1-2t}$	$x(t) = 5 - \dfrac{(t-4)^2}{4}$
Dom.	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, 1)$	$\left(-\infty, \dfrac{1}{2}\right)$	$(-\infty, 4]$
Lim.	∞	1	∞	0	5

The first two IVP are not very exciting anymore; they are exponential and logistic growth, and so we know their solutions are defined for all time. In particular, the solution for logistic growth here is the equilibrium solution $x(t) = 1$.

The other three IVP are more interesting. Each has the form $\dot{x} = f(x)$ for a relatively "tame" function f, and each can be solved with separation of variables. But the given domains all have a finite right endpoint.

(3) We studied this IVP in Sections 2.1.3 and 2.1.4 and observed the "catastrophic" situation

$$\lim_{t \to 1^-} x(t) = \lim_{t \to 1^-} \frac{1}{1-t} = \infty.$$

That is, x "blows up in finite time." There is no way for us to extend x beyond the time $t = 1$ as a continuous function; x has a vertical asymptote at $t = 1$.

(4) Now the solution $x(t) = \sqrt{1-2t}$ is defined and continuous at time $t = 1/2$, and yet we exclude $t = 1/2$ from the domain in (4). Why? Here we need to think carefully about the square root: *the square root is defined and continuous at 0 but not differentiable at 0.* (See part (iii) of Problem 2.5.5.) Recall from Definitions 1.4.1 and 1.4.4 that a solution to an ODE/IVP must not only be defined and continuous but also differentiable at all points of its domain. Thus $t = 1/2$ cannot belong to the domain of $x(t) = \sqrt{1-2t}$ if we are considering x not merely as a function but as a function that solves $\dot{x} = -1/x$.

It is also interesting to note in (4) that if we put $f(x) = -1/x$, then f is not defined at $x = 0$. But the solution $x(t) = \sqrt{1-2t}$ tends to 0 as t approaches $1/2$ from the left. Not only is the solution not differentiable at $t = 1/2$, it approaches a value outside the domain of f. How can we possibly plug this solution into $\dot{x} = -1/x$ at time $t = 1/2$ and get a numerical result that makes sense?

(5) As a function of t, ignoring the ODE/IVP context, the function $x(t) = 5-(t-4)^2/4$ is defined for all t. After all, it is just a quadratic polynomial. However, defining x in this way gives

$$\dot{x} = 2 - \frac{t}{2} \qquad \text{and} \qquad \sqrt{5-x} = \left|2 - \frac{t}{2}\right|.$$

(Here we need the rule $\sqrt{A^2} = |A|$ for any real number A.) And so to have $\dot{x(t)} = \sqrt{5-x(t)}$, we need

$$2 - \frac{t}{2} = \left|2 - \frac{t}{2}\right|,$$

and thus $2 - t/2 \geq 0$, hence $t \leq 4$. Last, taking $f(x) = \sqrt{5-x}$ in (5), we note that f is defined but not differentiable at $x = 5$, which is the limit of our solution as t approaches 4 from the left (and, for that matter, from the right).

2.6.4 Problem (!). Use separation of variables to solve the illustrative IVP

$$\begin{cases} \dot{x} = -1/x \\ x(0) = 1 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{x} = \sqrt{5-x} \\ x(0) = 1 \end{cases}$$

that appeared in Example 2.6.3.

This is where we finished on Wednesday, September 13, 2023.

Here are the common patterns in the previous example. All of the solutions to the different ODE $\dot{x} = f(x)$ with were defined on (at least) an open interval of the form (T_1, T_{end}) for some numbers $T_1 < 0 < T_{\text{end}}$. (Why 0? Because the initial time was $t_0 = 0$.) If $T_{\text{end}} = \infty$, then we could predict the future forever, and we saw that some solutions had infinite limits as $t \to \infty$ and others finite limits.

Much more interesting were the cases when $T_{\text{end}} < \infty$. In some of those cases, we had $\lim_{t \to T_{\text{end}}^-} x(t) = \infty$, which we called a “blow-up in finite time.” From the modeling point of view, in which the ODE $\dot{x} = f(x)$ represents some physical process, this is a catastrophic failure; in only a finite amount of time, the values of the solution became arbitrarily large. This either suggests that the model is wrong or that the underlying physical process is extremely delicate and subtle.

In other cases of $T_{\text{end}} < \infty$, the limit $L := \lim_{t \to T_{\text{end}}^-} x(t)$ existed as a finite real number. But this L was no arbitrary real number: we saw that either the function f that "governed' the ODE $\dot{x} = f(x)$ was not defined at L, or that f was not differentiable at L. Since the mode $\dot{x} = f(x)$ only makes sense for those x at which f is defined, and since a solution to the model is only guaranteed when the function f is continuously differentiable, the approach of x to L as $t \to T_{\text{end}}^-$ suggests that the solution is leaving the "domain of validity" of the model. In other words, the solution is starting to exhibit behavior that the model is not designed to predict, and so, once again, we should look more carefully at how good our model is.

Hopefully the following is becoming apparent: when a solution to $\dot{x} = f(x)$ fails to be defined for the entire interval $[0, \infty)$, something "interesting" happens at the finite time beyond which the solution cannot be continued. Either the solution explodes to $\pm\infty$ at the endpoint of its domain and has a vertical asymptote, or its values leave the domain on which f is continuously differentiable. The moral is that solutions to ODE do not simply stop after a finite time or "vanish" into thin air at a particular moment—*something has to happen.*

The following theorem is a precise statement of that moral. This statement is technical and worth parsing slowly and carefully.

2.6.5 Theorem (Maximal existence). *Let f be continuously differentiable on the interval (a, b), and let x_0 be a point in (a, b). There exist numbers T_1 and T_{end}, with $T_1 < 0 < T_{\text{end}}$, and a unique solution x to the IVP*

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

such that x is defined on the interval (T_1, T_{end}), and this interval (T_1, T_{end}) is "maximal" in the sense x cannot be defined outside this interval and remain a solution to the IVP. More precisely, one, and only one, of the following three alternatives holds for T_{end}.

(Max1) $T_{\text{end}} = \infty$.

(Max2) $T_{\text{end}} < \infty$ *and either* $\lim_{t \to T_{\text{end}}^-} x(t) = \infty$ *or* $\lim_{t \to T_{\text{end}}^-} x(t) = -\infty$.

(Max3) $T_{\text{end}} < \infty$ *and either* $\lim_{t \to T_{\text{end}}^-} x(t) = a$ *or* $\lim_{t \to T_{\text{end}}^-} x(t) = b$.

Identical statements hold for T_1 if we replace $T_{\text{end}} = \infty$ with $T_1 = -\infty$ in part (Max1) and the left limit $\lim_{t \to T_{\text{end}}^-}$ with the right limit $\lim_{t \to T_1^+}$ in parts (Max2) and (Max3).

This is a demanding theorem, so we paraphrase its conclusions more informally.

2.6.6 Remark. **(i)** *The possibility $T_{\text{end}} = \infty$ in part (Max1) of Theorem 2.6.5 is a sort of "ideal" result. It says that we can predict the future forever in our given model. However, it does not help us predict the behavior of our solution at ∞; we have no statement about $\lim_{t\to\infty} x(t)$ in part (Max1).*

(ii) *The possibilities $\lim_{t\to T_{\text{end}}^-} x(t) = \pm\infty$ in part (Max2) of Theorem 2.6.5 are a sort of "catastrophic" result. Our solution simply explodes! This phenomenon is often called a* BLOW-UP IN FINITE TIME. *This might represent a natural and expected result—say, the unbounded growth of a species given certain ideal environmental conditions—or maybe a flaw in our model.*

(iii) *The possibilities $\lim_{t\to T_{\text{end}}^-} x(t) = a$ or $\lim_{t\to T_{\text{end}}^-} x(t) = b$ in part (Max3) of Theorem 2.6.5 are, perhaps, the most subtle. Recall that the ODE under consideration is $\dot{x} = f(x)$, and f is guaranteed to be continuously differentiable only on the interval (a, b). Recall also that such nice behavior of f is a hypothesis of the existence and uniqueness theorem (Theorem 2.6.1). Saying that $x(t)$ tends to a or b as t approaches T_{end} from the left means that $x(t)$ is leaving the domain of f. The domain of f is the value of "states" for which the model is valid. Once the solution leaves this realm of validity, we can no longer make predictions about its behavior from our original model. (By the way, if $a = -\infty$ or $b = \infty$, then some of the limits in parts (Max2) and (Max3) are the same, and so there is some intentional redundancy in the statement of the theorem.)*

2.6.7 Problem (!). For each of the five solutions in Example 2.6.3, determine T_{end} and the limit as $t \to T_{\text{end}}^-$. Which solutions experience a blow-up in finite time?

2.6.8 Problem (!). Check that the function $x(t) := e^{-t}$ solves the IVP

$$\begin{cases} \dot{x} = -|x| \\ x(0) = 1. \end{cases}$$

How would you interpret the fact that $\lim_{t\to\infty} x(t) = 0$ in light of Theorem 2.6.5? [Hint: *where does $f(x) = |x|$ fail to be differentiable?*]

Theorem 2.6.5 is powerful, because it finally answers our question of "What happens in the future?" However, it does not give a definite answer: there are three possibilities, and there is no "test" presented to determine which one happens for a given IVP. It is possible to present such tests in fairly refined and excruciating detail via some demanding mathematical rigor (including an analysis of certain delightful improper integrals). Instead, we will develop a somewhat less excruciating tool to determine maximal domains and end behavior, with somewhat less rigor.

2.6.9 Problem ($\star$). The following results will be critical to our subsequent analysis. Let f be continuously differentiable on the interval (a, b). Suppose that $f(x_\infty) = 0$ for some point x_∞ in (a, b). Let x solve $\dot{x} = f(x)$. Use Theorems 2.5.11 and 2.6.1 to establish the following.

(i) If $x(0) < x_\infty$, then $x(t) < x_\infty$ for all t.

(ii) If $x(0) = x_\infty$, then $x(t) = x_\infty$ for all t.

(iii) If $x_\infty < x(0)$, then $x_\infty < x(t)$ for all t.

2.6.2. The phase line for the logistic equation.

We introduce the phase line by studying that most versatile model, the logistic equation. This analysis will confirm the behavior that we first saw in the logistic equation's slope field in Example 2.3.2 and in our implementation of Euler's method in Example 2.4.3. This analysis also parallels our qualitative treatment of exponential growth in Section 1.2.4. What is different is that we now have rigorous existence and uniqueness theory to back up our intuitive claims and assure us that solutions really are there and do not cross. Throughout, we consider the IVP

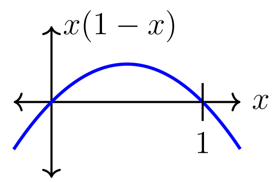
$$\begin{cases} \dot{x} = x(1-x) \\ x(0) = x_0. \end{cases} \tag{2.6.1}$$

As in Examples 2.3.2 and 2.4.3, our choices of the initial condition x_0 will make all the difference. Put

$$f(x) := x(1-x)$$

and recall, as the following graph indicates, that

$$\begin{cases} f(x) < 0, \ x < 0 \\ f(x) > 0, \ 0 < x < 1 \\ f(x) < 0, \ 1 < x. \end{cases}$$



1. Suppose that x solves (2.6.1) with $0 < x_0 < 1$. The domain of x is some interval (T_1, T_{end}), with $-\infty \le T_1 < 0 < T_{\text{end}} \le \infty$. We will chiefly be concerned with T_{end}. Since 0 and 1 are equilibrium solutions of the logistic equation, and since $0 < x(0) < 1$, the comparison theorem tells us $0 < x(t) < 1$ for all t.

2. First we study the behavior of x at time $t = 0$:

$$\dot{x}(0) = f(x(0)) = f(x_0) > 0,$$

so x is increasing at $t = 0$. We claim that x is increasing on all of its domain. Suppose not, so instead $\dot{x}(t) \le 0$ for some $t > 0$. If $\dot{x}(t) = 0$, then $f(x(t)) = \dot{x}(t) = 0$, and so either $x(t) = 0$ or $x(t) = 1$. But we saw in Step 1 that $0 < x(t) < 1$ for all t.

If $\dot{x}(t) < 0$, then since $\dot{x}(0) > 0$, there is a time s in $(0, t)$ such that $\dot{x}(s) = 0$. This s exists by the intermediate value theorem applied to the continuous function $\dot{x}$ (recall Definition 1.4.1!). Then we have the same contradiction as above from $\dot{x}(s) = 0$. And so $\dot{x}(t) > 0$ for all t in the domain of x, and therefore x is strictly increasing on its domain.

This is where we finished on Friday, September 15, 2023.

3. We have shown that x is increasing on (T_1, T_{end}). Moreover, we know that x is bounded above on this interval in the sense that $x(t) < 1$ for all t. A deep (but hopefully intuitive) theorem from calculus then tells us that the limit

$$L := \lim_{t \to T_{\text{end}}^-} x(t)$$

exists. Moreover, since $0 < x(t) < 1$, we have $0 \le L \le 1$. (Limits do not necessarily preserve *strict* inequalities.)

Specifically, here is that theorem.

2.6.10 Theorem. *Suppose that x is strictly increasing on the interval (T_1, T_{end}) in the sense that $x(t) < x(\tau)$ if $T_1 < t < \tau < T_{\text{end}}$. Suppose also that x is bounded above on (T_1, T_{end}) in the sense that there is $M > 0$ such that $x(t) \le M$ for all t in (T_1, T_{end}). Then the limit*

$$L := \lim_{t \to T_{\text{end}}^-} x(t)$$

exists and, moreover, $L \le M$.

4. The existence of L as a finite real number rules out alternatives (Max2) from Theorem 2.6.5. It also rules out (Max3) from that theorem, as here (in the notation of that theorem) we are working with $f(x) = x(1-x)$, $a = -\infty$, and $b = \infty$. So, T_{end} cannot be finite, and so $T_{\text{end}} = \infty$. We have thus ensured that the solution continues for all time, and so we can predict the future forever!

5. We can say more about L. Since $0 < x(0)$, and since x is increasing, we really have $0 < L \le 1$. Moreover, since $T_{\text{end}} = \infty$, we also have

$$\lim_{t \to \infty} x(t) = L.$$

That is, x has the horizontal asymptote L as $t \to \infty$. Next, since x solves $\dot{x} = f(x)$, we calculate

$$\lim_{t \to \infty} \dot{x}(t) = \lim_{t \to \infty} f(x(t)) = f\left(\lim_{t \to \infty} x(t)\right) = f(L). \tag{2.6.2}$$

The third equality follows from the continuity of f.

Now, horizontal asymptotes should call to mind “flat” graphs, and we should expect that the slope of x gets close to 0 as $t \to \infty$. In other words, we expect

$$\lim_{t \to \infty} \dot{x}(t) = 0. \tag{2.6.3}$$

Assuming this to be true[7], we combine (2.6.2) and (2.6.3) to obtain $f(L) = 0$, which means that either $L = 0$ or $L = 1$. But we also know $0 < L \le 1$. The only possibility left is $L = 1$.

[7] This is surprisingly tricky. It turns out that a differentiable function may have a horizontal asymptote, but its derivative may not limit to 0 at ∞. Put $x(t) := \cos(e^t)/t$. Use the squeeze theorem to show

6. We conclude that if x solves the logistic IVP (2.6.1) with $0 < x(0) < 1$, then x is defined for all time in $[0, \infty)$, x is strictly increasing, $0 < x(t) < 1$ for all t, and $\lim_{t\to\infty} x(t) = 1$. We can even figure out the concavity of x.

2.6.11 Problem (+). Suppose that x solves the logistic equation $\dot{x} = x(1-x)$. First show that

$$\ddot{x} = x(1-x)(1-2x).$$

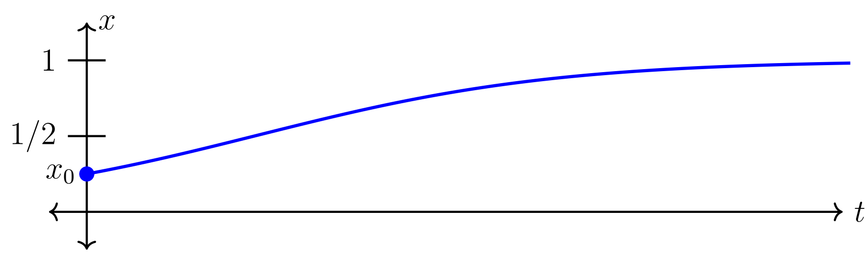
This tells us the value of the second derivative $\ddot{x}(t)$ in terms of the value $x(t)$; note that we are *not* saying $\ddot{x}(t) = t(1-t)(1-2t)$. Nonetheless, we can eke out the concavity of x from this formula.

(i) Show that if $x(t) < 1/2$, then x is concave up at time t.

(ii) Show that if $1/2 < x(t) < 1$, then x is concave down at time t.

(iii) Show that if $1 < x(t)$, then x is concave up at time t.

This is a staggering amount of information about x, and we figured it all out without having an explicit formula for x. Let us celebrate with a graph of x.



7. Now suppose that $x_0 > 1$ and let x solve the logistic IVP (2.6.1) with maximal domain (T_1, T_{end}). The comparison theorem then implies that $x(t) > 1$ for all t. Also, $\dot{x}(0) = f(x(0)) = f(x_0) < 0$, and so x is decreasing at time $t = 0$. Using exactly the same reasoning as in Step 2, we can show that x is decreasing at all t.

2.6.12 Problem (⋆). Use exactly the same reasoning as in Step 2 to show this.

Moreover, since $x(t) > 1$ for all t, x is decreasing and bounded below. Another great theorem of calculus implies that $L := \lim_{t\to T_{\text{end}}^-} x(t)$ exists with $1 \le L$. Specifically, this theorem.

$\lim_{t\to\infty} x(t) = 0$. Then compute $\dot{x}$ and show that if $t_k := \ln((\pi + 4\pi k)/2)$, then $\lim_{k\to\infty} \dot{x}(t_k) = -\infty$. The problem here is that x is oscillating too rapidly for its derivative to vanish at ∞.

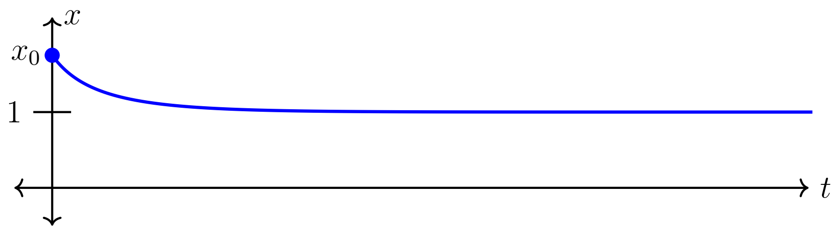
Instead, it turns out that if $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} \dot{x}(t)$ both exist as finite real numbers, then $\lim_{t\to\infty} \dot{x}(t) = 0$. That is the situation here with the logistic equation. First, we have argued that $\lim_{t\to\infty} x(t)$ exists because x is increasing and bounded above. Next, the calculations in (2.6.2) show that $\lim_{t\to\infty} \dot{x}(t)$ exist.

2.6.13 Theorem. *Suppose that x is strictly decreasing on the interval (T_1, T_{end}) in the sense that $x(\tau) < x(t)$ if $T_1 < t < \tau < T_{\text{end}}$. Suppose also that x is bounded below on (T_1, T_{end}) in the sense that there is $m > 0$ such that $m \le x(t)$ for all t in (T_1, T_{end}). Then the limit*

$$L := \lim_{t \to T_{\text{end}}^-} x(t)$$

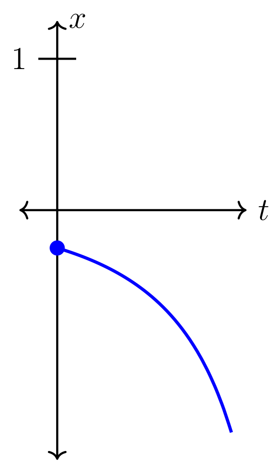
exists and, moreover, $m \le L$.

As before, the only possibility from the maximal existence theorem is that $T_{\text{end}} = \infty$. We then have the limits (2.6.2) and (2.6.3) exactly as before, and so $L = 0$ or $L = 1$; the inequality $1 \le L$ forces $L = 1$.

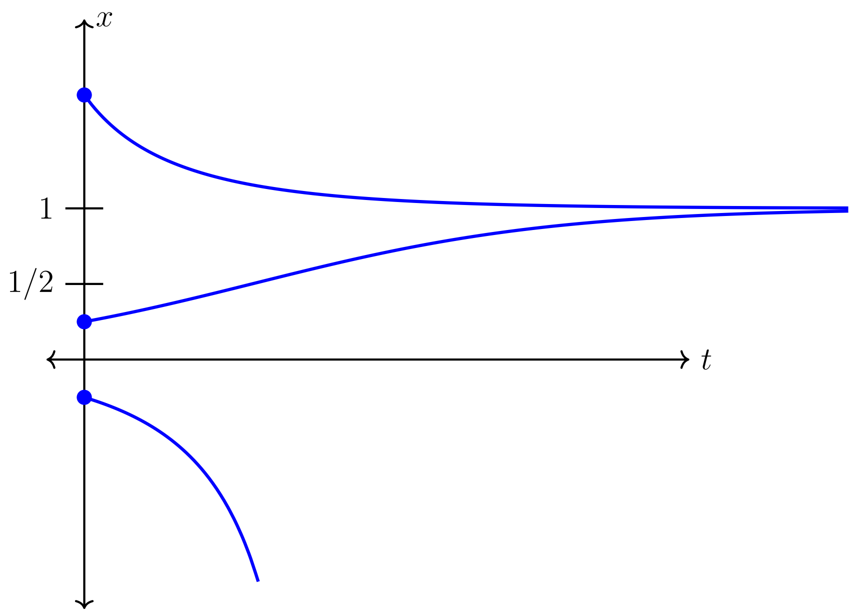


8. Finally, consider the (physically unrealistic but mathematically interesting) case $x_0 < 0$. Adapting the arguments above, we can show that x is decreasing on its domain. Consequently, we either have $\lim_{t \to T_{\text{end}}^-} x(t) = -\infty$ or, if the limit exists as a finite real number, then the limit is negative.

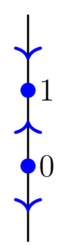
However, we no longer have an equilibrium solution below 0, and so we cannot try to argue as in the previous work that x is bounded below; in particular, the tools above do not apply to tell us if $T_{\text{end}} = \infty$ or $T_{\text{end}} < \infty$. (It is possible to develop more robust tools to determine the value of T_{end}; we will not do so here.) However, in either case we can say that $\lim_{t \to T_{\text{end}}^-} x(t) = -\infty$. First, if $T_{\text{end}} < \infty$, this follows from parts (Max2) or (Max3) of the maximal existence theorem, using $f(x) = x(1-x)$, $a = -\infty$, and $b = \infty$. Next, if $T_{\text{end}} = \infty$ and $\lim_{t \to T_{\text{end}}^-} x(t)$ is finite, then we can use the limits (2.6.2) and (2.6.3) to conclude that f has a root below 0, which is false. So, whether T_{end} is finite or infinite, we must have $\lim_{t \to T_{\text{end}}^-} x(t) = -\infty$.



9. We summarize our work by putting plots of solutions for all three cases of nonequilibrium initial conditions on the same graph. The results are exactly what Example 2.3.2 predicted for the logistic equation via slope fields and what Example 2.4.3 predicted via Euler's method. This is a nice harmony among numerical, qualitative, and theoretical methods!



There is an efficient, compact way of summarizing the behavior of solutions to the logistic equation based on their initial conditions. Solutions that start at 0 or 1 stay there forever; solutions that start below 0 decrease to $-\infty$; solutions that start between 0 and 1 increase to 1; and solutions that start above 1 decrease to 1. We represent these behaviors on a vertical line, called the **PHASE LINE** for the logistic equation, by marking the equilibrium solutions with dots and placing arrows in the segments marked by those dots to indicate increasing (a right-pointing arrow) or decreasing (a left-pointing arrow) behavior of solutions starting in those segments.



Going forward, we will work in reverse order to our treatment of the logistic equation here. First we will find the equilibrium solutions (this is how we start every autonomous problem), then we will draw the phase line, and finally we will draw solutions and predict the future based on the phase line. *The moral is that we can predict the end behavior of solutions to $\dot{x} = f(x)$ based on where they start relative to the equilibrium solutions. The equilibrium solutions control everything.*

2.6.14 Problem (+). Let f be continuously differentiable on (a, b). Prove that any nonequilibrium solution to $\dot{x} = f(x)$ is either strictly increasing or strictly decreasing on all of its domain. We used this fact several times in developing the phase line for the logistic equation. [Hint: *suppose instead that x is increasing at time t_1 and decreasing at time t_2. Adapt the argument of Step 2 above to conclude that $\dot{x}(t_3) = 0$ for some time t_0 between t_1 and t_2; conclude that $x(t_3)$ is an equilibrium solution. Why is this wrong?*]

2.6.3. Constructing and interpreting phase lines.

We can distill the work of the previous section (in particular, the rather demanding calculus ideas behind the analysis that boiled down to the phase line) into some fairly simple "tests" for the behavior of solutions to autonomous ODE depending on initial conditions. Essentially, the proof of this theorem involves taking our arguments for the specific function $f(x) = x(1-x)$ in the previous section and adapting them to the more general hypotheses below.

2.6.15 Theorem. *Suppose that f is continuously differentiable on an interval I. Let x_0 be a point in I and suppose that x solves the IVP*

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Then the relation of x_0 to the other roots of f determines the long-time behavior of x in the following cases.

(i) $f(x_0) = 0$. *Then $x(t) = x_0$ is an equilibrium solution and is defined for all t.*

(ii) $f(x_0) > 0$. *Then x strictly increases up to the next equilibrium solution above x_0 and is defined for all $t \geq 0$; if there is no equilibrium solution above x_0, then x increases to ∞ but may not be defined for all $t \geq 0$.*

(iii) $f(x_0) < 0$. *Then x strictly decreases down to the next equilibrium solution below x_0 and is defined for all $t \geq 0$; if there is no equilibrium solution below x_0, then x decreases to $-\infty$ but may not be defined for all $t \geq 0$.*

2.6.16 Problem (!). **(i)** Paraphrase case (ii) of Theorem 2.6.15 more technically as follows. If $f(x_0) > 0$ and there is $x_\infty > x_0$ such that $f(x_\infty) = 0$ but $f(x) \neq 0$ for $x_0 < x < x_\infty$, then x is defined on an interval of the form (T_1, ∞) with $T_1 < 0$ and $\lim_{t \to \infty} x(t) = x_\infty$.

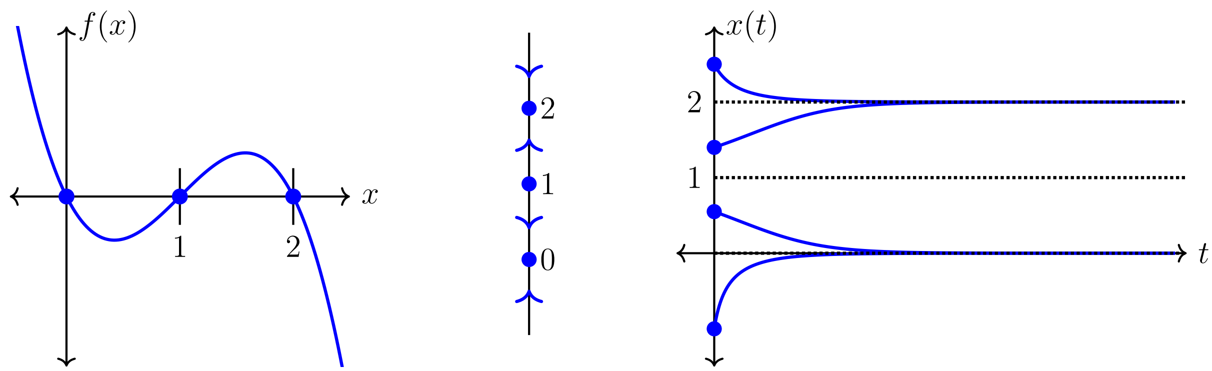
(ii) Paraphrase case (iii) of Theorem 2.6.15 more technically as follows. If $f(x_0) < 0$ and there is $x_\infty < x_0$ such that $f(x_\infty) = 0$ but $f(x) \neq 0$ for $x_\infty < x < x_0$, then x is defined on an interval of the form (T_1, ∞) with $T_1 < 0$ and $\lim_{t \to \infty} x(t) = x_\infty$.

This is where we finished on Monday, September 18, 2023.

2.6.17 Example. We study the ODE

$$\dot{x} = x(x-1)(2-x).$$

First, the equilibrium solutions are $x = 0$, $x = 1$, and $x = 2$. To draw the phase line, we need to understand the behavior of $f(x) := x(x-1)(2-x)$ around these three roots. We could figure out that behavior by evaluating f at values of x in the intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$, as f is either strictly positive or strictly negative on those intervals, or we could look at a graph of f (as sketched below). Either way, we determine the following behavior of f and, consequently, the corresponding behavior of solutions.



Hopefully the consonance among the graph of f and its roots and positive/negative behavior, the dots and arrows on the phase line, and the sketches of sample solutions (equilibrium solutions indicated by dotted lines) is clear. Here is a description of how solutions are behaving in words.

1. $x < 0$. Here $f(x) > 0$, so solutions that start below 0 increase to 0.

2. $0 < x < 1$. Here $f(x) < 0$, so solutions that start between 0 and 1 decrease to 0.

3. $1 < x < 2$. Here $f(x) > 0$, so solutions that start between 1 and 2 increase to 2.

4. $2 < x$. Here $f(x) < 0$, so solutions that start above 2 decrease to 2.

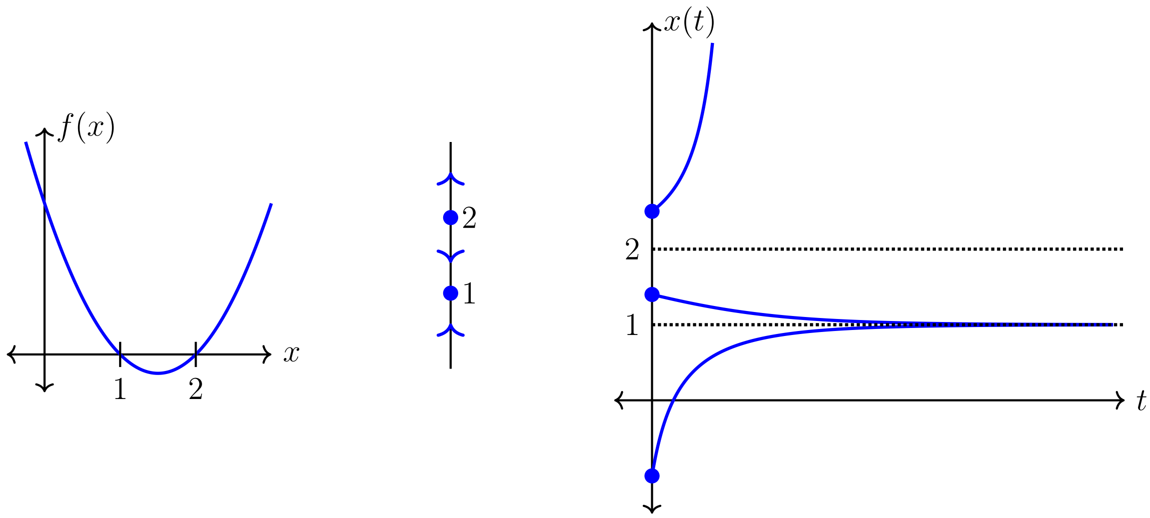
All solutions are defined on (at least) the interval $[0, \infty)$ by Theorem 2.6.15.

2.6.18 Problem (!). Draw the phase line and sketch solutions for

$$\dot{x} = x(x-1)(x-2).$$

How does everything compare to the previous example? Use the fact that $x(x-1)(x-2) = -x(x-1)(2-x)$ in your discussion.

2.6.19 Example. We study
$$\dot{x} = (x-1)(x-2).$$
The equilibrium solutions are $x = 1$ and $x = 2$, and analysis of the sign of $f(x) = (x-1)(x-2)$ reveals the following phase line and sketches of solutions.



So, solutions that start below 1 are increasing and tend to 1 over long times; solutions that start between 1 and 2 are decreasing and (also) tend to 1 over long times; and solutions that start above 2 are increasing and tend to ∞ over long times. Solutions starting at or below 2 are defined for all $t \geq 0$, but maybe a solution starting above 2 is defined only up to some time $T_{\text{end}} < \infty$, and there could be a blow-up in finite time.

2.6.20 Problem ($\star$). In the previous example, define a new function u by $u(t) := x(t) - 1$. Show that if x solves $\dot{x} = (x-1)(x-2)$, then u solves $\dot{u} = u(u-1)$. Recognize this as the "negative" of the logistic equation. How do the graphs of the solutions in Example 2.6.19 compare to those for the logistic equation in Section 2.6.2?

2.6.21 Problem (!). A phase line may have one or zero equilibrium points on it.

(i) Draw the phase line for $\dot{x} = x$. Compare your results to Step 6 from Section 1.2.4 long ago.

(ii) Draw the phase line for $\dot{x} = e^{-x^2}$. How does the new information from phase lines address the issues left unresolved by Example 2.3.3?

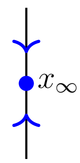
This is where we finished on Wednesday, September 20.

2.6.4. Stability.

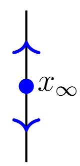
The moral of the phase line is that the equilibrium solutions to an autonomous ODE, far from being "one more thing" that we have to find when solving separable ODE, actually control the destiny of *all* solutions. In particular, we are (hopefully) seeing that equilibrium solutions either "attract" or "repel" all other solutions that start "nearby."

2.6.22 Definition. *Let f be continuously differentiable on the interval (a, b) and let x_∞ be an equilibrium solution of $\dot{x} = f(x)$, i.e., x_∞ is a point in (a, b) such that $f(x_\infty) = 0$.*

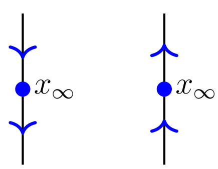
(i) *The equilibrium solution x_∞ for the ODE $\dot{x} = f(x)$ is* **STABLE** *if solutions that start near x_∞ tend to x_∞ over very long times. A stable equilibrium is also called a* **SINK**.



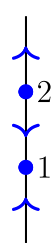
(ii) *The equilibrium solution x_∞ for the ODE $\dot{x} = f(x)$ is* **UNSTABLE** *if solutions that start near x_∞ tend away from x_∞ over very long times. An unstable equilibrium is also called a* **SOURCE**.



(iii) *The equilibrium solution x_∞ is* **SEMISTABLE** *for the ODE $\dot{x} = f(x)$ if it is neither a source nor a sink. A semistable equilibrium is also called a* **NODE**.



2.6.23 Example. We drew the phase line for $\dot{x} = (x - 1)(x - 2)$ in Example 2.6.19 and found that it had the following form.



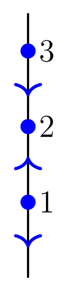
We see that solutions that start near 1 tend to 1 over long times, while solutions that start near 2 tend away from 2 over long times. Specifically, solutions that start with $1 < x(0) < 2$ tend to 1 over long times, while solutions that start with $2 < x(0)$ tend to ∞ over long times. Thus 1 is a stable equilibrium or sink, while 2 is an unstable equilibrium or source.

2.6.24 Problem (!). Classify the equilibria of Example 2.6.17.

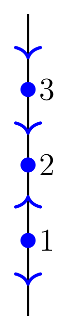
2.6.25 Problem (!). Explain why we could paraphrase the phenomenon of "stable equilibrium" as "a small change in the initial conditions does not change the long-time behavior of the solutions."

2.6.26 Example. While the ODE in our course so far have arisen from concrete population models (however tenuously), it is also a worthwhile skill to start with a mathematical, not physical, phenomenon and distill it into a model. Since we have not yet seen an ODE with a semistable equilibrium, we build one. Specifically, we look for an ODE $\dot{x} = f(x)$ for which 1 is an unstable equilibrium, 2 is stable, and 3 is semistable.

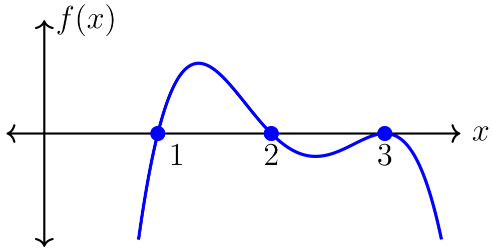
We therefore want solutions that start near 2 to tend to 2, while solutions that start near 1 tend away from 1. The situation with 3 is less clear, so we leave it out for a first pass at the phase line and obtain the following.



We see that solutions that start between 2 and 3 must tend toward 2, and therefore away from 3. Since we want 3 to be semistable, solutions that start above 3 cannot also tend away from 3; otherwise, 3 would be unstable. So, we want solutions that start above 3 to tend to 3, and therefore the complete phase line is the following.



We can use the phase line to find a function f that governs this ODE by recalling that increasing solutions start where f is positive, and decreasing solutions start where f is negative. So, f should be negative on $(-\infty, 1)$, positive on $(1, 2)$, negative on $(2, 3)$, and also negative on $(3, \infty)$. A graph of f, then, might be the following.



There are many possible formulas for such an f; one option (after some fooling around with a graphing program) is

$$f(x) = -(x-1)(x-2)(x-3)^2.$$

2.6.27 Problem (!). On the same set of axes, draw the graphs of solutions to $\dot{x} = -(x-1)(x-2)(x-3)^2$ that satisfy $x(0) = 0.5$, $x(0) = 1.5$, $x(0) = 2.5$, and $x(0) = 3.5$.

2.6.28 Remark. *We can make the terminology of Definition 2.6.22 more precise as follows.*

(i) *The equilibrium solution x_∞ for the ODE $\dot{x} = f(x)$ is stable if there is $\epsilon > 0$ such that if $x_\infty - \epsilon < x_0 < x_\infty + \epsilon$, then the solution x to*

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

is defined on $[0, \infty)$ and $\lim_{t \to \infty} x(t) = x_\infty$.

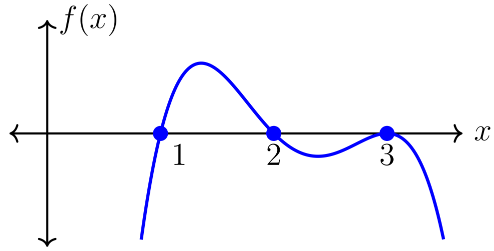
(ii) *The equilibrium solution x_∞ for the ODE $\dot{x} = f(x)$ is unstable if there is $\epsilon > 0$ such that if $x_\infty - \epsilon < x_0 < x_\infty + \epsilon$, then for the solution x to*

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0, \end{cases}$$

there are t_+, $\delta > 0$ such that if $t > t_+$, then either $x(t) < x_\infty - \delta$ or $x_\infty + \delta < x(t)$.

2.6.5. Linearization.

So far, we have constructed phase lines and classified equilibria for $\dot{x} = f(x)$ based on the *sign* of f in the intervals between equilibrium solutions. It would be nice if we could classify equilibria (and therefore draw the phase line) just from knowledge of how f is behaving *at* the equilibrium solution. The results of Example 2.6.26 suggest one way to do this. Below is, again, the graph of f.



We built f so that 1 would be an unstable equilibrium and 2 would be stable, and we note from the graph that $f'(1) > 0$ and $f'(2) < 0$. This turns out to be enough to determine whether an equilibrium is stable or unstable.

2.6.29 Theorem. *Let f be continuously differentiable on the interval (a, b) and let x_∞ be a point in (a, b) with $f(x_\infty) = 0$.*

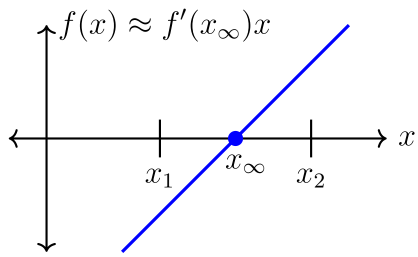
(i) *If $f'(x_\infty) < 0$, then x_∞ is stable.*

(ii) *If $f'(x_\infty) > 0$, then x_∞ is unstable.*

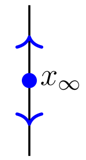
Proof. We prove part (ii) and leave part (i) as an exercise. Since $f'(x_\infty) < 0$, the function f is decreasing near x_∞. So, for $x_1 < x_\infty < x_2$, with x_1 and x_2 sufficiently close to x_∞, we have

$$f(x_1) < f(x_\infty) < f(x_2).$$

And $f(x_\infty) = 0$, so we really have $f(x_1) < 0 < f(x_2)$. Here is a visualization of this, in which we use the fact that $f(x_\infty) = 0$ and $f'(x_\infty) > 0$ to draw the local linear (i.e., the tangent line) approximation $f(x) \approx f'(x_\infty)x$ to the graph of f for x near x_∞.



Now we can apply Theorem 2.6.15. Solutions that start below but close to x_∞, say with initial value x_1 decrease away from x_∞, while solutions that start above but close to x_∞, say with initial value x_2, must increase away from x_∞. That is, a snippet of the phase line for $\dot{x} = f(x)$ looks like the following.



And so all solutions that start close to x_∞ must tend away from x_∞. ■

2.6.30 Problem ($\star$). Adapt the proof of part (ii) of Theorem 2.6.29 above to explain why part (i) is true.

2.6.31 Problem (!). Use Theorem 2.6.29 to check the stability of the equilibria in Examples 2.6.17 and 2.6.19 and compare your results to Example 2.6.23 and Problem 2.6.24.

2.6.32 Problem ($\star$). Theorem 2.6.29 omits the case $f'(x_\infty) = 0$, as here x_∞ could be stable, unstable, or semistable. For the following three ODE, classify the equilibrium at 0:

$$\dot{x} = x^3, \qquad \dot{x} = -x^3, \qquad \text{and} \qquad \dot{x} = x^2.$$

2.6.33 Problem (+). Let f be continuously differentiable on the interval (a, b) and suppose that $f(x_\infty) = 0$ for some x_∞ in (a, b).

(i) Suppose that x_∞ is a stable equilibrium for $\dot{x} = f(x)$. Is x_∞ always an unstable equilibrium for $\dot{x} = -f(x)$?

(ii) Suppose that x_∞ is an unstable equilibrium for $\dot{x} = f(x)$. Is x_∞ always a stable equilibrium for $\dot{x} = -f(x)$?

2.6.6. Complaints.

Phase lines convey useful and concise information about predicting the future, chiefly the range of solutions (are they bounded between equilibrium points or unbounded?) and their long-time limits. However, phase lines by themselves also lack lots of information. Here are some complaints.

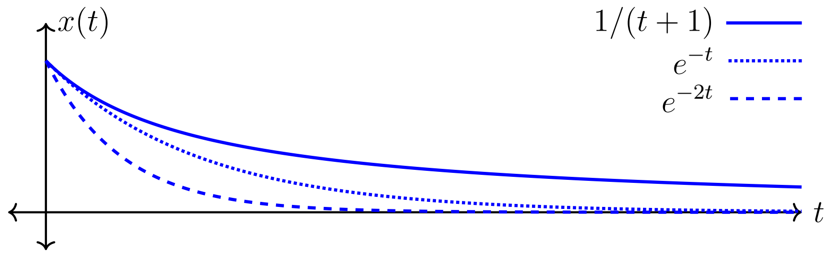
1. The domain of a solution, in particular if a solution is defined on all times into the future with domain containing $[0, \infty)$ or just up to a finite time with the domain not extending beyond some $T_{\text{end}} < \infty$, may not be apparent from a phase line. See Theorem 2.6.15 for situations in which we can guarantee that the domain is infinite.

2. The concavity of a solution is definitely not apparent from the phase line.

3. If a solution converges to a finite limit over long time, the *rate* of convergence may not be apparent from a phase line. For example,

$$\lim_{t\to\infty} \frac{1}{t+1} = 0, \qquad \lim_{t\to\infty} e^{-t} = 0, \qquad \text{and} \qquad \lim_{t\to\infty} e^{-2t} = 0, \tag{2.6.4}$$

but the functions converge to 0 at different "rates," which is obvious from the graphs below.



4. The exact value of a solution at a particular moment in time is definitely not something that a phase line can provide unless that solution is an equilibrium solution.

2.6.34 Problem ($\star$). For each of the three functions in (2.6.4), find an autonomous ODE that it solves. That is, if x is one of those three functions, come up with another function f such that $\dot{x} = f(x)$.

As we have often remarked, no one tool will tell us everything that we could possibly want to know about an ODE and its solutions.

This is where we finished on Monday, September 25, 2023.

2.7. Linear ODE and variation of parameters.

All of our tools for studying differential equations have limitations. Slope fields (a qualitative tool) and Euler's method (a numerical tool) apply to first-order ODE in their most general form $\dot{x} = f(t, x)$, but both are subject to misinterpretation. Separation of variables (an analytic tool) applies to ODE in the fairly specific form $\dot{x} = g(t)h(x)$, but even the most innocent-looking functions h can lead to nasty complications. Phase lines (another qualitative tool) provide detailed information about the asymptotics of solutions to autonomous problems $\dot{x} = h(x)$ but little to no quantitative data, and phase lines do not work for nonautonomous problems. As we have lamented, there is no one method that works for all problems, and each method will, at times, have severe disadvantages.

Practically speaking, there is only one other kind of first-order ODE in addition to separable for which analytic techniques routinely work. In fact, for this class of problems, analytic methods have, on average, better success than separation of variables does for separable problems—the technique that we are going to develop *always* gives a formula for x. This new class of problems also arises naturally in many modeling scenarios and is far more than a mathematical curiosity.

2.7.1. Harvesting.

Life was good for our population models in prior examples. With the exception of a certain bad regime in the modified logistic equation, either our populations always exploded to ∞, or they happily leveled out around a carrying capacity. Either way, they survived, and probably prospered.

The good times are over! Famine, pestilence, and peril are on the horizon! Suppose that we have a population that, in the absence of external malice, grows exponentially. (Why exponential? First, it makes the math and the model "easy." Second, it gives the right kind of ODE for us to study—namely, the linear ODE. We could definitely study the harvesting of a logistically growing population, but then the math would be harder, and the model would not be linear.) With $x(t)$ as the population at time t, we expect $\dot{x} = rx$. Now, however, because this population is useful, annoying, and/or delicious, we decide to harvest (or hunt, or, more antiseptically, remove) some members of the population. Specifically, suppose that we harvest $h(t)$ members of the population per unit time t.

We will turn these assumptions into an ODE using the following general principal. Suppose that a certain quantity changes via both an "input" source and an "output" source. In terms of population, "input" could be births and "output" could be deaths (due here to harvesting). Then the rate of change of that quantity satisfies

$$\textit{Rate of change} = \textit{Rate in} - \textit{Rate out}. \tag{RI–RO}$$

Thus the population satisfies

$$\dot{x} = rx - h(t). \tag{2.7.1}$$

If h is not constant, then this problem is not separable; in particular, it is not autonomous. Of course, we could use slope fields and Euler's method to analyze it, given a specific formula for h, but eventually we will need some new tools to study (2.7.1) if we want to say anything profound. Developing those tools will now be our top priority.

First, though, how might we choose h? There are lots of valid harvesting schemes, but we will work with *periodic* harvesting—the harvesting rate can vary in a periodic fashion. After all, this is probably how we harvest crops and hunt game on a seasonal or annual basis. So, we want h to be a nonnegative periodic[8] function.

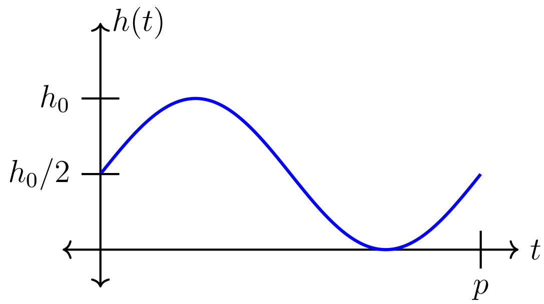
There are many such functions, but maybe the most familiar nonconstant periodic function is the sine. However, the sine can be negative, and in (2.7.1) we are subtracting h; if we subtract $h(t)$ when $h(t) < 0$, then we are really *adding* members back into the population instead of removing them. So, we want to modify the original sine to make our harvesting term nonnegative. Since $-1 \le \sin(t) \le 1$, we add 1 to get $0 \le 1 + \sin(t) \le 2$. If we take $h(t) = 1 + \sin(t)$, then, yes, we get a nonconstant, nonnegative periodic function. But this would only allow us to harvest with a rate ranging between 0 and 2.

If, as well, we multiply by a number $h_0 > 0$, then taking $h(t) = h_0(1 + \sin(t))$ allows us to remove anywhere between 0 and $2h_0$ members of the population per unit time. We refine this further to $h(t) = h_0(1 + \sin(t))/2$ so that we are harvesting between 0 and h_0 members of the population per unit time. Finally, the sine is 2π-periodic, and we may not want to harvest only with 2π-periodicity. To give us control over the frequency of harvesting, we incorporate a "frequency" parameter p and define

$$h(t) = \frac{h_0}{2}\left[1 + \sin\left(\frac{2\pi t}{p}\right)\right]. \tag{2.7.2}$$

Here is a graph of h.

[8] A function h is **PERIODIC** if there is a number $p \neq 0$ such that $h(t + p) = h(t)$ for all t.



2.7.1 Problem ($\star$). **(i)** Chase through the algebra and check that the function h defined in (2.7.2) satisfies $0 \le h(t) \le h_0$ and $h(t+p) = h(t)$ for all t.

(ii) Recall that if $a < b$ and h is continuous on $[a, b]$, then the integral

$$\frac{1}{b-a}\int_a^b h(t)\ dt$$

gives a good measure of the "average value" of h on $[a, b]$. What is the average value of h in (2.7.2) over $[0, p]$? Is this what you expected?

2.7.2 Problem (+). Graph the function h defined in (2.7.2) for several different values of h_0 and p. (Use a computer.) What happens to the graph as you change h_0 and/or p? How would you interpret those changes physically in the context of harvesting?

Assuming that at time $t = 0$ the initial population is x_0, the IVP that governs our exponentially growing population subject to harvesting is now

$$\begin{cases} \dot{x} = rx - \dfrac{h_0}{2}\left[1 + \sin\left(\dfrac{2\pi t}{p}\right)\right] \\ x(0) = x_0. \end{cases} \tag{2.7.3}$$

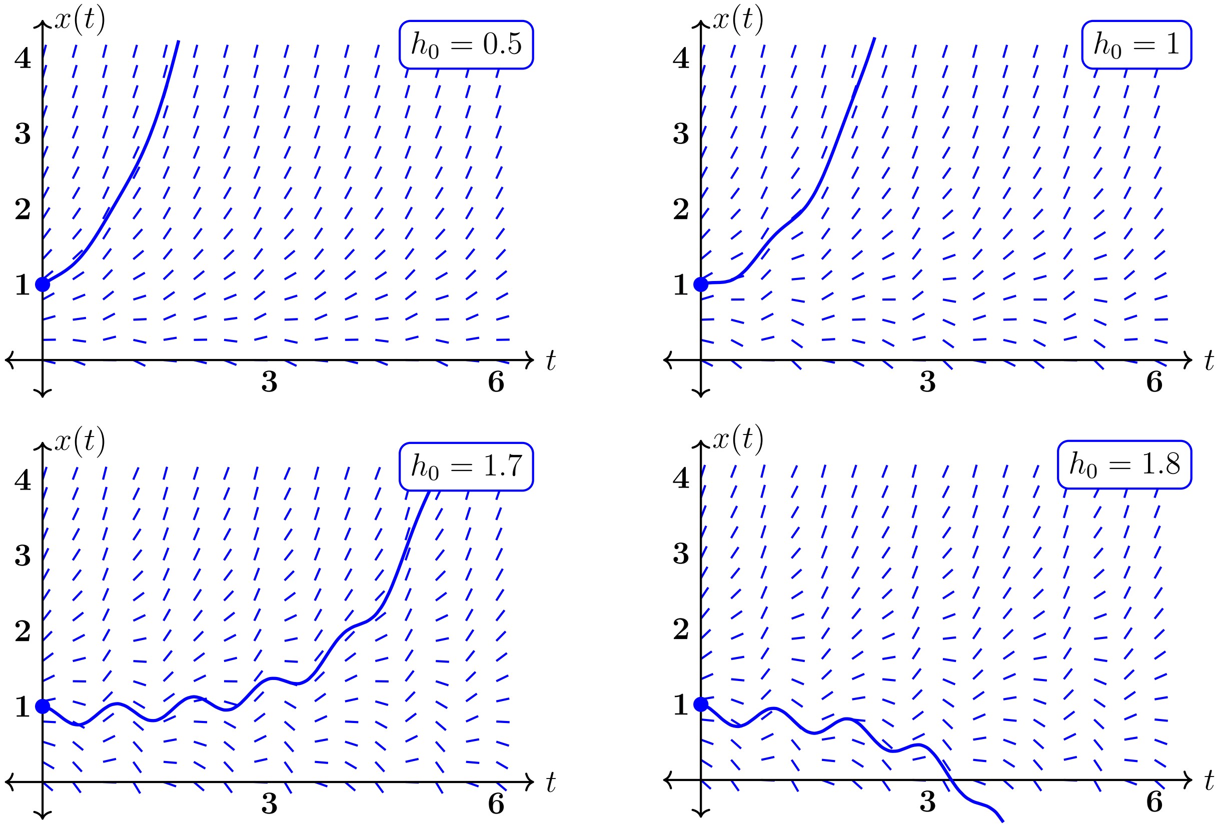
There are three parameters in this ODE—the positive numbers r, h_0, and p—and we probably also want to consider the effect of the initial condition x_0.

We might wonder what effect tweaking the values of these three parameters has on the solutions. Of course, we could go to slope fields and/or numerics and make observations. But suppose we wanted to answer definitively the following question: *are there values of the parameters p, h_0, and r that cause the population to go extinct?* That is, can we harvest in such a way that we kill off the population, and, if so, how "sensitive" is the population's behavior to the values of p, h_0, and r? We might think that because the population is growing exponentially in the absence of harvesting, and because exponentially growing populations explode to ∞ over long times, we can harvest it however we want. Perhaps we are wrong.

When faced with an unfamiliar differential equation, a good strategy is always to turn to numerics to gain some intuition. Here are the results of Euler's method for the IVP (2.7.3) with $x_0 = r = p = 1$ and several values of h. We take these values to be 1 just for simplicity, which is wholly unrealistic, but it does make the computing time quick. (Picture one really scared rabbit that can also duplicate itself.) That is, we are simulating the IVP

$$\begin{cases} \dot{x} = x - \dfrac{h_0}{2}\big(1 + \sin(2\pi t)\big) \\ x(0) = 1. \end{cases}$$

We also provide the slope fields to get a more general sense of how other solutions (i.e., other scenarios with different initial populations) are behaving. By the way, note that these slope fields are not identical as we proceed across any one of them horizontally; these are definitely not slope fields for autonomous ODE.



As we increase h_0, it appears that the population growth slows (compare $h_0 = 1$ and $h_0 = 1.7$), and eventually the population dies off when h_0 is large enough ($h_0 = 1.8$). The slope field at points (t, x) with $x < 2$ becomes substantially more "wiggly" as h_0 increases, which suggests that populations with a variety of initial conditions (not just $x_0 = 1$) will, at best, fluctuate noticeably as the harvesting rate increases. It appears that even exponential growth cannot survive under sufficiently greedy harvesting. These numerics, however, do not indicate the relationship among x_0, h_0, p, and r—in particular, is there a "threshold" for h_0 below which the population will continue to grow despite harvesting and above which the population will go extinct? This is exactly the sort of situation for which analytic techniques can give precise, rigorous answers, and to those analytic techniques we now turn.

2.7.3 Problem (!). Here is an indication that we cannot harvest an exponentially growing population however we like without threat to the population's survival. Suppose that a population grows exponentially with rate r, so that, undisturbed by our harvesting, the population satisfies $\dot{x} = rx$. Now suppose that we harvest the population at a rate of $qx(t)$ members per unit time t. Then the harvesting equation (2.6.1) is $\dot{x} = rx - qx$. Show that if $r < q$, then $\lim_{t\to\infty} x(t) = 0$, and so the population goes extinct in the long run. In words, we harvested faster than the population could grow.

2.7.4 Problem (+). Instead of our development in Section 1.3.2, we could construct the logistic equation via the paradigm of (RI–RO). Assume that a population inherently grows exponentially but that interactions within the population (violence, competition, other happy things) cause the population to decrease. If there are x members of the population, then each member can interact with $x - 1$ other members, and so there are $x(x-1)$ interactions possible. However, because the "order" of interactions is irrelevant (member A interacting with member B is the same as member B interacting with member A), there are really $x(x-1)/2$ distinct interactions possible. Thus the "rate in" should be αx, where α is a constant of proportionality, and the "rate out" is $\beta x(x-1)/2$, where β is another constant of proportionality, and so the population should satisfy

$$\dot{x} = \alpha x - \frac{\beta x(x-1)}{2}. \tag{2.7.4}$$

Chase through the algebra to see that if

$$r := \frac{2\alpha + \beta}{2} \quad \text{and} \quad N := \frac{2\alpha + \beta}{\beta},$$

then (2.7.4) is really the familiar logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{N}\right).$$

2.7.2. The structure of linear ODE.

The harvesting ODE (2.7.3) has the following form.

2.7.5 Definition. *An ODE $\dot{x} = f(t,x)$ is* **LINEAR** *if f has the special form*

$$f(t,x) = a(t)x + b(t)$$

for functions a and b. That is, a linear ODE is an equation of the form

$$\dot{x} = a(t)x + b(t). \tag{2.7.5}$$

The function a is the **COEFFICIENT** *and the function b is the* **FORCING** *or* **DRIVING** *term.*

We call a the coefficient because a multiplies x, and coefficients are supposed to multiply things; calling b the forcing/driving term is a convention that stems from terminology for *second*-order ODE, which naturally involve things being forced or driven by external influences. Note that for fixed t, the function $f(t,x) = a(t)x + b(t)$ is a linear function of x, thus the name "*linear* differential equation."

2.7.6 Example. Our harvesting ODE (2.7.3) read

$$\dot{x} = rx - \frac{h_0\big(1 + \sin(2\pi t/p)\big)}{2}.$$

Here the coefficient function is $a(t) := r$; note that a is a constant function. We will say quite a bit about "constant-coefficient" linear ODE in the future. The driving or forcing term is $b(t) := -h_0(1 + \sin(\omega t))/2$.

2.7.7 Problem (!). Although the only first-order ODE for which we will find analytic solutions in this course are separable (Section 2.2) and linear (forthcoming!), there are plenty of first-order ODE that are neither. Give an example of a first-order ODE that is neither separable nor linear, and clearly explain why it fails to be either.

The two essential pieces of data in a linear ODE are, of course, the coefficient function and the forcing function. If one of these is identically zero, then the problem becomes (somewhat) simpler.

2.7.8 Problem (!). Let I be an interval. Suppose that $a(t) = 0$ for all t in I and that b is a continuous function on I. Explain why solving the linear ODE $\dot{x} = a(t)x + b(t)$ is just a direct integration problem.

2.7.3. Homogeneous linear ODE.

Rather more interesting (and ultimately more important) than the case of $a = 0$ in $\dot{x} = a(t)x + b(t)$ is the case $b = 0$.

2.7.9 Definition. *A linear ODE $\dot{x} = a(t)x + b(t)$ is* HOMOGENEOUS *if $b(t) = 0$ for all t. That is, a homogeneous ODE is an equation of the form*

$$\dot{x} = a(t)x. \tag{2.7.6}$$

A linear ODE $\dot{x} = a(t)x + b(t)$ is NONHOMOGENEOUS *or* INHOMOGENEOUS *if it is not homogeneous, i.e., if $b(t) \neq 0$ for at least one t.*

2.7.10 Example. The ODE $\dot{x} = \sin(t)x$ is homogeneous, but $\dot{x} = \sin(t)x + e^{-\cos(t)}$ is nonhomogeneous.

2.7.11 Problem ($\star$). It is always nice to be able to build new solutions to an ODE out of existing ones. The structure of the homogeneous linear ODE

$$\dot{x} = a(t)x \tag{2.7.7}$$

makes this very easy.

(i) Suppose that x solves (2.7.7). Show that the function $y(t) := cx(t)$ also solves (2.7.7) for any constant c.

(ii) Suppose that x_1 and x_2 both solve (2.7.7). Show that the function $y(t) := x_1(t)+x_2(t)$ also solves (2.7.7).

2.7.12 Problem (!). What more can you say about the ODE

$$\dot{x} = a(t)x + b(t)$$

if you know one of the following?

(i) This ODE is autonomous.

(ii) This ODE is separable.

(iii) This ODE is both separable and autonomous.

2.7.13 Problem ($\star$). Suppose that the linear ODE

$$\dot{x} = a(t)x + b(t)$$

has an equilibrium solution. Show that this ODE is really separable. [Hint: *if a constant function x solves this ODE, then a and b are related via the equation $a(t)x + b(t) = 0$ for all t. How does this help?*]

The good news is that we already know how to solve homogeneous linear ODE. There are several ways of proceeding. First, we guessed how to solve homogeneous linear ODE all the way back in Problem 1.3.2. Since $x(t) = e^{rt}$ solves $\dot{x} = rx$ for any constant r, and since $y(t) = e^{-\cos(t)}$ solves $\dot{y} = \sin(t)y$ (recall Example 1.3.1), we can guess that if A is an antiderivative of a (i.e., $\dot{A} = a$), then

$$x(t) = e^{A(t)}$$

will solve $\dot{x} = a(t)x$. And this is easy to check:

$$\dot{x}(t) = \frac{d}{dt}[e^{A(t)}] = e^{A(t)}\dot{A}(t) = e^{A(t)}a(t) = a(t)x(t).$$

We also worked this out using separation of variables in Problem 2.2.22; the resulting implicit equation for x involves a tricky absolute value (since we have to contend with $\int x^{-1}\,dx$), and it is perhaps easier to avoid that absolute value by working with initial value problems.

However we get there, the following is true.

2.7.14 Theorem. *Let a be continuous on the interval I. Then every solution x to the linear homogeneous ODE $\dot{x} = a(t)x$ on I has the form*

$$x(t) = Ce^{A(t)} \tag{2.7.8}$$

for some real number C and some antiderivative A of a on I. This solution x is defined on all of I.

2.7.15 Problem (+). Prove this theorem. There are two things to do.

(i) First, check that functions of the form (2.7.8) really do solve the problem. That is, if $x(t) = Ce^{A(t)}$, then $\dot{x}(t) = a(t)x(t)$ for all t. This involves the chain rule and the assumption $\dot{A} = a$.

(ii) Next, explain why all solutions x to $\dot{x} = a(t)x$ have the form (2.7.8). This requires uniqueness theory. Start with any solution to $\dot{x} = a(t)x$, pick any t_0 in I, and let $y(t) = x(t_0)e^{\int_{t_0}^{t} a(\tau)\,d\tau}$. Show that x and y solve the same IVP (what is that IVP?). How does this help?

2.7.16 Problem (!). Do the results of Theorem 2.7.14 agree with the results of Problem 2.7.11? For example, if x_1 and x_2 both solve $\dot{x} = a(t)x$, then, per that problem, so should $y := x_1 + x_2$. By Theorem 2.7.14, we can write $x_1(t) = C_1e^{A(t)}$ and $x_2(t) = C_2e^{A(t)}$ for some constants C_1 and C_2. Does $x_1 + x_2$ have the form predicted by the theorem? How about cx_1 for some constant c?

2.7.17 Example. **(i)** Since $A(t) = t^2$ is an antiderivative of $a(t) = 2t + 1$, all solutions to $\dot{x} = t^2x$ have the form

$$x(t) = Ce^{t^2}$$

for some constant C.

(ii) We can solve

$$\begin{cases} \dot{x} = t^2x \\ x(0) = 2 \end{cases}$$

by first noting that the solution has to be of the form $x(t) = Cet^2$ for some constant C, and then applying the initial condition to find

$$2 = x(0) = Ce^0 = C$$

Thus $C = 2$, and the solution is $x(t) = 2e^{t^2}$.

2.7.4. Nonhomogeneous linear ODE: variation of parameters.

Let I be an interval and let a and b be continuous functions on I. We have seen that all solutions to the homogeneous linear ODE

$$\dot{x} = a(t)x$$

have the form

$$x(t) = Ce^{A(t)} \tag{2.7.9}$$

for some constant C and some antiderivative A of a. What about the nonhomogeneous ODE $\dot{x} = a(t)x + b(t)$? There are many ways to approach this problem, and all of them rely on some kind of unexpected insight that has been passed down to us through the centuries. It is completely normal if this insight feels strange, unnatural, or unexpected—what matters is that you reread and rework the following enough *until* it feels normal, natural, and expected. (And then reread and rework it a few more times for good measure.)

The key idea is the following. Take $C = 1$ in (2.7.9) to the **ASSOCIATED HOMOGENEOUS PROBLEM** $\dot{x} = a(t)x$ are *constant* multiples of the function

$$\phi(t) := e^{A(t)},$$

where A is an antiderivative of a. That is, solutions to $\dot{x} = a(t)x$ have the form

$$x(t) = C\phi(t) = Ce^{A(t)}$$

for some number C, where $\dot{A}(t) = a(t)$. What if we look for solutions to the full nonhomogeneous problem that are *variable-coefficient* multiples of ϕ? That is, we will *guess* that solutions x have the form

$$x(t) = u(t)\phi(t) = u(t)e^{A(t)},$$

where $\dot{A}(t) = a(t)$. In other words, we have replaced the constant C with the (possibly nonconstant!) function u. What is u?

We will figure it out.

This is where we finished on Wednesday, September 27, 2023.

2.7.18 Example. We met the ODE

$$\dot{x} = \sin(t)x + 2te^{-\cos(t)}$$

back in Example 1.4.2, where all we could do was check that a given function solved it. Now we actually find all solutions.

1. The associated homogeneous equation is

$$\dot{x} = \sin(t)x,$$

and we now know that all its solutions are $x(t) = Ce^{-\cos(t)}$ for some constant C. In the notation of the paragraph preceding this example, $\phi(t) = e^{-\cos(t)}$. Now we look for solutions that are variable-coefficient multiples of ϕ. That is, we guess that the solution to $\dot{x} = \sin(t)x + 2te^{-\cos(t)}$ has the form $x(t) = u(t)e^{-\cos(t)}$ for some function u. We will figure out what u is.

2. What u does determines what u is. This is a course in differential equations, so what things do typically is solve differential equations. How do we get a differential equation involving u? As the old saying goes, "If it moves, differentiate it," and so we compute (with the product rule)

$$\dot{x}(t) = \frac{d}{dt}[u(t)e^{-\cos(t)}] = \dot{u}(t)e^{-\cos(t)} + u(t)\frac{d}{dt}[e^{-\cos(t)}] = \dot{u}(t)e^{-\cos(t)} + u(t)e^{-\cos(t)}\sin(t).$$

3. We want x to satisfy $\dot{x}(t) = \sin(t)x(t) + 2te^{-\cos(t)}$ for all t, so we replace $\dot{x}(t)$ with the expression calculated above and $x(t)$ with the assumption $x(t) = u(t)e^{-\cos(t)}$ to find

$$\dot{u}(t)e^{-\cos(t)} + u(t)e^{-\cos(t)}\sin(t) = \sin(t)u(t)e^{-\cos(t)} + 2te^{-\cos(t)}.$$

The same term $u(t)e^{-\cos(t)}\sin(t)$ appears on both sides, so we can cancel it to find

$$\dot{u}(t)e^{-\cos(t)} = 2te^{-\cos(t)}.$$

4. This is almost a differential equation for u, except for the exponential factor on the left. Since exponentials are never 0, we can divide both sides by $e^{-\cos(t)}$ to conclude

$$\dot{u}(t) = 2t.$$

This is a direct integration ODE for u, which we solve to find

$$u(t) = t^2 + C$$

for some constant C. Then we have

$$x(t) = u(t)e^{-\cos(t)} = (t^2 + C)e^{-\cos(t)} = t^2e^{-\cos(t)} + Ce^{-\cos(t)}.$$

5. Of course, we can always check our work by differentiating this formula for x and seeing if it satisfies $\dot{x}(t) = \sin(t)x(t) + e^{-\cos(t)}$ for all t. (It does!)

6. Moreover, our solution strategy really produced *all* solutions to this ODE. To recap, if x solves this ODE, then we can write

$$x(t) = x(t) \cdot 1 = x(t)e^{\cos(t)}e^{-\cos(t)} = u(t)e^{-\cos(t)}, \qquad u(t) := x(t)e^{\cos(t)}.$$

The work above showed that if $x(t) = u(t)e^{-\cos(t)}$ solved $\dot{x} = \sin(t)x + e^{-\cos(t)}$, then u had to have the form $u(t) = t^2 + C$ for some constant C. Thus x had to have the form $x(t) = (t^2 + C)e^{-\cos(t)}$.

7. If we want to, we can use this formula for x in general to solve IVP, say,

$$\begin{cases} \dot{x} = \sin(t)x + e^{-\cos(t)} \\ x(0) = 2. \end{cases}$$

We know that all solutions to this ODE have the form

$$x(t) = t^2 e^{-\cos(t)} + Ce^{-\cos(t)}$$

for some constant C. We apply the initial condition to see that C needs to satisfy

$$2 = x(0) = Ce^{-\cos(0)} + (0^2 \cdot e^{-\cos(t)}) = Ce^{-1},$$

and so $C = 2e$. That is, the solution to the IVP is

$$x(t) = 2e(e^{-\cos(t)}) + t^2 e^{-\cos(t)} = (2e + t^2)e^{-\cos(t)}.$$

8. And if we look hard, we should see something special in the structure of solutions to this ODE. Again, all solutions have the form

$$x(t) = t^2 e^{-\cos(t)} + Ce^{-\cos(t)}.$$

If we take $C = 0$, we get a (comparatively) simple solution

$$x_\star(t) := t^2 e^{-\cos(t)}.$$

That is, $x_\star$ solves the nonhomogeneous problem $\dot{x}_\star = \sin(t)x_\star + e^{-\cos(t)}$. Remember also that $\phi(t) = e^{-\cos(t)}$ solves the homogeneous problem $\dot{\phi} = \sin(t)\phi$. Now we say the same thing in three different ways.

- Every solution x to the nonhomogeneous problem is the sum of $x_\star$ and a constant multiple of ϕ.

- Every solution x to the nonhomogeneous problem has the form $x(t) = x_\star(t) + C\phi(t)$.

- Every solution to the nonhomogeneous problem is the sum of one "particular" solution to the nonhomogeneous problem and a constant multiple of one (nonzero!) solution to the homogeneous problem.

This one example teaches us a lot, and everything above completely generalizes to the arbitrary nonhomogeneous problem. Say that x solves

$$\dot{x} = a(t)x + b(t),$$

and let ϕ solve the associated homogeneous problem $\dot{\phi} = a(t)\phi$. Specifically, we take $\phi(t) = e^{A(t)}$, where A is any antiderivative of a. We might ask—and this is the strange, unnatural, and unexpected insight—how ϕ could "show up" in the solution to the nonhomogeneous problem. Since ϕ is an exponential and therefore is never 0, we can multiply and divide by ϕ to find

$$x(t) = \phi(t)\left(\frac{x(t)}{\phi(t)}\right) = \phi(t)u(t), \qquad \text{where} \qquad u(t) := \frac{x(t)}{\phi(t)}. \tag{2.7.10}$$

And since we already know what ϕ is, if we just figure out what u is, then we will know x.

The product rule tells us

$$\dot{x} = \frac{d}{dt}[\phi u] = \dot{\phi}u + \phi\dot{u} = a(t)\phi u + \phi\dot{u}. \tag{2.7.11}$$

And the identity $\dot{x} = a(t)x + b(t)$ tells us

$$\dot{x} = a(t)\phi u + b(t). \tag{2.7.12}$$

Then (2.7.11) and (2.7.12) are the same, so we have

$$a(t)\phi u + \phi\dot{u} = a(t)\phi u + b(t).$$

Both sides have the same term $a(t)\phi u$, so we cancel that and are left with

$$\phi\dot{u} = b(t).$$

Once again, ϕ is never 0, so we may divide and solve for $\dot{u}$:

$$\dot{u} = \frac{b(t)}{\phi(t)}. \tag{2.7.13}$$

This is a direct integration problem for u.

Are we sure that we can solve it? If we assume that a and b are both continuous on some interval I, then a has a (necessarily continuous) antiderivative A on I, so ϕ is continuous on I, and then b/ϕ is continuous on I. Consequently, u has an antiderivative on I, and the whole thing works. In particular, we conclude that if a and b are continuous on I, then the ODE $\dot{x} = a(t)x + b(t)$ has solutions on all of I. (This is a much more explicit result about the domain of a solution than we ever got with separation of variables.)

This procedure is often called **VARIATION OF PARAMETERS** or **VARIATION OF CONSTANTS**, since we went from having $x(t) = C\phi(t)$ in the homogeneous solution, with C constant, to $x(t) = u(t)\phi(t)$ in the nonhomogeneous case, with u not necessarily constant.

2.7.19 Example. The ODE

$$\dot{x} = -\frac{2}{t}x + t - 1$$

is linear and nonhomogeneous, with $a(t) = -2/t$ and $b(t) = t - 1$. The role of $t = 0$ will probably be ticklish in the solution.

The associated homogeneous equation is

$$\dot{x} = -\frac{2}{t}x,$$

and all solutions to this ODE have the form

$$x(t) = Ce^{\int(-2/t)\ dt} = Ce^{-2\ln(|t|)}.$$

We can simplify this further by using properties of logarithms and absolute value to calculate

$$-2\ln(|t|) = \ln(|t|^{-2}) = \ln(t^{-2}),$$

and therefore all solutions to the homogeneous problem are

$$x(t) = Ce^{\ln(t^{-2})} = Ct^{-2}.$$

Note that the exponential has vanished from this formula, since the logarithm canceled it.

Now we look for solutions to the nonhomogeneous equation of the form

$$x(t) = u(t)t^{-2}.$$

This is where we finished on Friday, September 29, 2023.

We compute

$$\dot{x}(t) = \dot{u}(t)t^{-2} + u(t)(-2)t^{-3},$$

and so we want

$$\dot{u}(t)t^{-2} + u(t)(-2)t^{-3} = -\frac{2}{t}x + t - 1.$$

That is, u needs to satisfy

$$\dot{u}(t)t^{-2} + u(t)(-2)t^{-3} = -\frac{2}{t}u(t)t^{-2} + t - 1.$$

This simplifies to

$$\dot{u}(t)t^{-2} + u(t)(-2)t^{-3} = -2u(t)t^{-3} + t - 1,$$

and therefore

$$\dot{u}(t)t^{-2} = t - 1,$$

and thus

$$\dot{u}(t) = t^2(t-1).$$

At last we arrive at the direct integration problem

$$\dot{u}(t) = t^3 - t^2$$

for u, and we solve this to get

$$u(t) = \frac{t^4}{4} - \frac{t^3}{3} + C.$$

We conclude that the solution x to our original ODE is

$$x(t) = u(t)t^{-2} = \left(\frac{t^4}{4} - \frac{t^3}{3} + C\right)t^{-2} = \frac{t^2}{4} - \frac{t}{3} + Ct^{-2}.$$

We must exclude $t = 0$ from the domain of x, and so the largest domains possible are $(-\infty, 0)$ or $(0, \infty)$. Note that these intervals are the largest domains on which the coefficient function $a(t) = -2/t$ is defined (whereas the driving term $b(t) = t - 1$ is defined at all t).

Once again, we see that the solution has a special structure. Put

$$x_\star(t) := \frac{t^2}{4} - \frac{t}{3} \quad \text{and} \quad \phi(t) := t^{-2},$$

so $x_\star$ solves the nonhomogeneous problem $\dot{x} = (-2/t)x+t+1$, and ϕ solves the homogeneous problem $\dot{x} = (-2/t)x$. Our work shows that every solution x to the nonhomogeneous problem has the form

$$x(t) = x_\star(t) + C\phi(t).$$

That is, every solution to the nonhomogeneous problem is the sum of one particular solution to the nonhomogeneous problem and a constant multiple of one solution to the homogeneous problem.

We can generalize these observations about the structure of solutions to linear ODE.

2.7.20 Theorem (Linearity principle). *Let a and b be functions and suppose that $x_\star$ solves the nonhomogeneous problem $\dot{x}_\star = a(t)x_\star + b(t)$. Then every other solution x to the nonhomogeneous problem $\dot{x} = a(t)x + b(t)$ has the form*

$$x(t) = x_\star(t) + C\phi(t),$$

where C is a constant and ϕ solves the homogeneous problem $\dot{\phi} = a(t)\phi$.

Proof. We are assuming that x and $x_\star$ are functions such that

$$\dot{x}(t) = a(t)x(t) + b(t) \quad \text{and} \quad \dot{x}_\star(t) = a(t)x_\star(t) + b(t).$$

To measure how alike x and $x_\star$ are, we subtract them: put

$$y(t) := x(t) - x_\star(t).$$

Then we study y by differentiating it:

$$\begin{aligned}
\dot{y}(t) &= \dot{x}(t) - \dot{x}_\star(t) \\
&= a(t)x(t) + b(t) - \big(a(t)x_\star(t) + b(t)\big) \\
&= a(t)x(t) + b(t) - a(t)x_\star(t) - b(t) \\
&= a(t)x(t) - a(t)x_\star(t) \\
&= a(t)\big(x(t) - x_\star(t)\big) \\
&= a(t)y(t).
\end{aligned}$$

So, y solves the homogeneous ODE $\dot{y} = a(t)y$, and therefore $y(t) = Ce^{A(t)}$ for some antiderivative A of a. Put $\phi(t) = e^{A(t)}$, so $y(t) = C\phi(t)$. Then

$$x(t) = x_\star(t) + y(t) = x_\star(t) + C\phi(t). \qquad \blacksquare$$

The point of this theorem is that if we know one particular solution to the nonhomogeneous linear ODE (which sometimes takes a bit of work to find), then all we have to do is add a constant multiple of the solution to the associated homogeneous problem (which we find by antidifferentiating and exponentiating) to get *all* solutions to the nonhomogeneous problem. This affords us a tremendous amount of control over solutions to linear ODE!

2.7.21 Remark. *Do not confuse the linearity principle with the linearization theorem for classifying equilibria (Theorem 2.6.29). The linearity of limits, and thus of derivatives, is the foundation of the linearity principle, while local linear approximations are the foundation of the linearization theorem. Linearity is everything!*

2.7.22 Problem (!). Variation of parameters hinges on evaluating two antiderivatives: the one that produces ϕ to solve the associated homogeneous problem, and the one that produces u. Either antiderivative may be difficult or impossible to evaluate in terms of elementary functions. This was a problem that we encountered in separation of variables (although there we had the added *algebraic* complication of solving for x after antidifferentiating).

(i) Consider the ODE

$$\dot{x} = -2x + t^2.$$

Variation of parameters suggests that since the homogeneous problem $\dot{x} = -2x$ has solutions $x(t) = Ce^{-2t}$, we guess that $x(t) = u(t)e^{-2t}$ for the nonhomogeneous problem. Make this guess and determine a direct integration problem for u. What annoys you about this direct integration problem?

(ii) Consider the ODE

$$\dot{x} = -2x + e^{-2t}\sin(t^2).$$

Again, guess $x(t) = u(t)e^{-2t}$. What do you find challenging (maybe impossible) about the direct integration problem for u now?

2.7.5. Existence and uniqueness theory for linear IVP.

Problem 2.7.22 suggests that variation of parameters can fail, or at least be stymied, by a challenging antiderivative in one or two places: in constructing the solution to the homogeneous problem, or in solving the direct integration problem for u. As always, one way to get around antidifferentiation difficulties is to represent things with definite integrals. The theorem below does just that for a linear IVP.

2.7.23 Theorem (Existence and uniqueness for linear IVP). *Let a be a continuous function on the interval I and let t_0 be a point in I. Let x_0 be any real number. Then the only solution to the IVP*

$$\begin{cases} \dot{x} = a(t)x + b(t) \\ x(t_0) = x_0 \end{cases} \tag{2.7.14}$$

is the function

$$x(t) = x_0\phi(t) + x_\star(t), \tag{2.7.15}$$

where ϕ and $x_\star$ solve the two IVP

$$\begin{cases} \dot{\phi} = a(t)\phi \\ \phi(t_0) = 1 \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_\star = a(t)x_\star + b(t) \\ x_\star(t_0) = 0. \end{cases} \tag{2.7.16}$$

Specifically,

$$\phi(t) = e^{\int_{t_0}^t a(\tau)\ d\tau} \qquad \text{and} \qquad x_\star(t) = \phi(t)\int_{t_0}^t \frac{b(\tau)}{\phi(\tau)}\ d\tau. \tag{2.7.17}$$

More informally, we might say that the linear IVP (2.7.14) "splits" into a "linear combination" of the two "simpler" IVP in (2.7.16). Here is a cartoon of that "splitting."

$$\boxed{\begin{cases} \dot{x} = a(t)x + b(t) \\ x(t_0) = x_0 \end{cases} = \left(x_0 \cdot \begin{cases} \dot{x} = a(t)x \\ x(t_0) = 1 \end{cases}\right) + \begin{cases} \dot{x} = a(t)x + b(t) \\ x(t_0) = 0. \end{cases}}$$

2.7.24 Problem (!). Does the result of Theorem 2.7.23 agree with the result of Theorem 2.7.20? (It better!)

2.7.25 Problem (+). This problem outlines the proof of Theorem 2.7.23, which is really just variation of parameters with definite integrals in two places.

(i) It is possible to check that x as defined in (2.7.15) solves the IVP (2.7.14); this is just a matter of the definitions of ϕ and $x_\star$ in (2.7.17) and the fundamental theorem of calculus. Go ahead and check that x does solve the IVP.

(ii) It is more enlightening to derive the formula (2.7.15) by assuming that x solves (2.7.14) and, after some work, concluding that this is the right form for x. Along the way, we will learn that *only* x as defined in (2.7.15) can solve the IVP. This proves a uniqueness result for linear IVP that our version of the existence and uniqueness theorem in Theorem 2.5.2 cannot deliver.

To begin, suppose that x solves (2.7.14). Explain why we can write x in the form

$$x(t) = u(t)\phi(t)$$

for some function u. [Hint: $u = x/\phi$.]

(iii) Show that if $x(t) = u(t)\phi(t)$ solves (2.7.14), then u solves the IVP

$$\begin{cases} \dot{u} = b(t)/\phi(t) \\ u(t_0) = x_0. \end{cases}$$

(iv) Solve this IVP for u by direct integration.

(v) Conclude that x has the form given by (2.7.15).

2.7.26 Remark. *If we take $b(t) = 0$ for all t, then Theorem 2.7.23 proves that the homogeneous linear IVP has a unique solution. This gives another proof of Theorem 2.7.14 without invoking the general existence and uniqueness theorem. To avoid a circular argument, note* *that we did not use the general existence and uniqueness theory at all in this section; in particular, we used the fact that ϕ solves the linear homogeneous IVP in* (2.7.16) *but did not make any claims about the uniqueness of solutions to this IVP.*

2.7.27 Problem (!). Part (ii) of Problem 2.7.22 studied the ODE $\dot{x} = -2x + e^{-2t}\sin(t^2)$ and should have lead to frustration with an intractable indefinite integral. Now, however, we can just get from Theorem 2.7.23. Since the forcing term is defined on all of $(-\infty, \infty)$, for simplicity we take $t_0 = 0$. Suppose that the initial condition is $x(0) = x_0$. Show that

$$x(t) = x_0 e^{-2t} + e^{-2t}\int_0^t \sin(\tau^2)\ d\tau.$$

2.7.28 Problem (+). In the preceding problem, use the triangle inequality $\left|\int_a^b f(t)\ dt\right| \le \int_a^b |f(t)|\ dt$ for integrals and a familiar bound on the sine to justify the estimate

$$\left|\int_0^t \sin(\tau^2)\ d\tau\right| \le |t|.$$

Then use L'Hospital's rule and the squeeze theorem to explain why all solutions to $\dot{x} = -2x + e^{-2t}\sin(t^2)$ satisfy $\lim_{t\to\infty} x(t) = 0$.

2.7.29 Problem. The definite integral is the ideal tool for solving the harvesting ODE (2.7.3), which motivated our study of linear ODE. Recall that this ODE was

$$\begin{cases} \dot{x} = rx - \dfrac{h_0}{2}\left[1 + \sin\left(\dfrac{2\pi t}{p}\right)\right] \\ x(0) = x_0. \end{cases}$$

The parameter $r > 0$ controls the growth rate of the population in the absence of harvesting; the parameter $h_0 > 0$ controls the maximum rate of harvesting; the parameter p controls the frequency of harvesting; and $x_0 > 0$ is the initial population.

(i) Use Theorem 2.7.23 to show that the solution to this IVP is

$$x(t) = x_0 e^{rt} - \frac{h_0 e^{rt}}{2}\int_0^t e^{-r\tau}\left[1 + \sin\left(\frac{2\pi\tau}{p}\right)\right]\ d\tau. \tag{2.7.18}$$

(ii) What part of evaluating the definite integral here do you expect to be challenging, but not impossible?

2.7.30 Example. We continue the study of the harvesting problem from Section 2.7.1 using the result of Problem 2.7.29. We will proceed imagining that r and x_0 are fixed and given to us (since they are data about the original population), but we get to choose h

and p (since we are the ones doing the harvesting). Instead fighting through it by hand, a good idea is to go to a computer algebra system (which is particularly helpful since we are keeping all of the parameters abstract right now).

After *a lot* of work on the computer, it turns out that

$$x(t) = \left(x_0 - h_0\left[\frac{1}{2r} + \frac{\pi p}{(pr)^2 + (2\pi)^2}\right]\right)e^{rt} + \psi(t), \tag{2.7.19}$$

where

$$\psi(t) := \frac{ph_0}{2\sqrt{(pr)^2 + (2\pi)^2}}\left(\sin\left(\frac{2\pi t}{p} + \arctan\left(\frac{2\pi}{pr}\right)\right) + \sqrt{1 + \left(\frac{2\pi}{pr}\right)^2}\right).$$

What does all of this mean, and what impact do these analytic results have on our original question? Is it possible to choose the harvesting rate h_0 and the harvesting frequency p so that we overharvest the population into extinction?

We analyze our results and answer these questions in the following steps (and two subsequent problems).

1. First, the function ψ is p-periodic and bounded. This is because $-1 \le \sin(\tau) \le 1$ for all τ, but $\sqrt{1 + (2\pi/pr)^2} > 1$. Thus there are positive numbers m and M (which, by the way, depend on p and r) such that

$$0 < m < \psi(t) < M$$

for all t. The long-time dynamics of our population, therefore, will not be affected by ψ, which does not blow up to $+\infty$ nor down to $-\infty$ as $t \to \infty$. (In fact, ψ has no limit as $t \to \infty$.)

2. Instead, and unsurprisingly, the exponential term in x dominates the long-time behavior. Since $r > 0$, if

$$0 < x_0 - h_0\left[\frac{1}{2r} + \frac{\pi p}{(pr)^2 + (2\pi)^2}\right], \tag{2.7.20}$$

then

$$\lim_{t\to\infty}\left(x_0 - h_0\left[\frac{1}{2r} + \frac{\pi p}{(pr)^2 + (2\pi)^2}\right]\right)e^{rt} = \infty,$$

and so $\lim_{t\to\infty} x(t) = \infty$, too. We rearrange (2.7.20) to see that if

$$h_0 < \frac{x_0}{\dfrac{1}{2r} + \dfrac{\pi p}{(pr)^2 + (2\pi)^2}} = \frac{2rx_0\left[(pr)^2 + (2\pi)^2\right]}{(pr)^2 + (2\pi)^2 + 2\pi pr} =: \Omega(p, r, x_0). \tag{2.7.21}$$

then the population survives and continues to grow even with harvesting. This is because our harvesting rate is suitably "small" in terms of the initial population and its growth rate, and also in terms of the frequency with which we are harvesting.

3. However, if we take $\Omega(p, r, x_0) < h_0$, then (2.7.20) implies that we will have

$$\lim_{t\to\infty}\left(x_0 - h_0\left[\frac{1}{2r} + \frac{\pi p}{(pr)^2 + (2\pi)^2}\right]\right)e^{rt} = -\infty,$$

and so $\lim_{t\to\infty} x(t) = -\infty$, too. Realistically, this means that at some time $t > 0$ we have $x(t) = 0$, and so the population will go extinct from overharvesting.

The analytic solution to this model has given us precise quantitative criteria under which overharvesting can occur. While the numerics earlier predicted that extinction would result from overharvesting at a certain extreme, those graphs did not give us these precise quantitative criteria. We are now able to "tune" our harvesting parameters h_0 and p so that we can take as much as possible without risking extinction.

2.7.31 Problem (!). Pick your favorite (positive) values of x_0, r, and p and use a computer to graph the harvesting solution x defined in (2.7.19) for different values of h_0 greater than and less than $\Omega(p, r, x_0)$, as defined in (2.7.21). Describe in words how the graph changes for these different values of h_0. What does this mean for the population being harvested?

2.7.32 Problem ($\star$). In the harvesting model above, what happens over long times if we take $h_0 = \Omega(p, r, x_0)$ as defined in (2.7.21)? With the concern of overharvesting in mind, is this a good idea?

2.7.33 Problem (+). We might ask what changing the harvesting period p does to our conclusions in the harvesting model above.

(i) Check that $\Omega(p, r, x_0) < 2rx_0$ for all $p > 0$.

(ii) Check that $\lim_{p\to\infty} \Omega(p, r, x_0) = 2rx_0$; in taking the limit, r and x_0 are fixed positive numbers (as they always are in the harvesting problem).

(iii) Conclude that as $p \to \infty$, the interval of permissible h_0 gets larger. Explain why the following is true: we can *increase* the maximum amount that we are harvesting per unit time if we *decrease* the frequency with which those harvesting extremes occur.

2.7.34 Problem (+). In general, it is difficult to predict the asymptotic behavior as $t \to \infty$ of solutions to $\dot{x} = a(t)x + b(t)$, assuming that a and b are defined on $[0, \infty)$ or some other infinite interval; after all, the formula (2.7.15) is rather complicated. However, there are some situations in which if we assume some more information about a and b, then the asymptotics become clearer. Here is one of those situations; see Problem 2.8.5 for another.

Let a and b be continuous functions on $[0, \infty)$. For given real numbers x_0 and y_0, let x

and y solve the IVP

$$\begin{cases} \dot{x} = a(t)x + b(t) \\ x(0) = x_0 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{y} = a(t)y + b(t) \\ y(0) = y_0. \end{cases}$$

Show that if

$$\int_0^\infty a(\tau) \, d\tau := \lim_{t \to \infty} \int_0^t a(\tau) \, d\tau = -\infty, \tag{2.7.22}$$

then

$$\lim_{t \to \infty} \big(x(t) - y(t)\big) = 0.$$

Conclude that as $t \to \infty$, any two solutions have exactly the same behavior. [Hint: *Theorem 2.7.23 tells us how to solve each of these IVP. With* ϕ *as defined in that theorem, what does the improper integral condition* (2.7.22) *say about* $\lim_{t \to \infty} \phi(t)$*?*]

This is where we finished on Monday, October 2, 2023.

2.8. Constant-coefficient linear ODE.

Variation of parameters is a complete success—possibly the only complete success in this course. We have established the existence and uniqueness of solutions to linear IVP, and we even have an explicit formula for those solutions. There are, however, at least two downsides to variation of parameters. First, it requires integration, and the underlying antidifferentiation can range from hard to impossible (recall Problem 2.7.22). Second, the formula produced by variation of parameters does not always yield *insight*. For example, consider the ODE

$$\dot{x} = 2x + 3\cos(4t). \tag{2.8.1}$$

This could, with some imagination and elbow grease, model a population that is subject to periodic influx and removal (not quite harvesting, since the forcing term is not strictly negative); the numbers 2, 3, and 4 here are just chosen because they are simple. We can solve this ODE with variation of parameters, but along the way we would need to evaluate the indefinite integral

$$\int e^{-2t} \cos(4t) \, dt, \tag{2.8.2}$$

which requires integration by parts and an algebraic trick.

2.8.1 Problem (!). Work through enough of variation of parameters for the ODE (2.8.1) to convince yourself that the integral (2.8.2) eventually shows up.

By itself, the indefinite integral (2.8.2) is just an annoying integral, which was the first downside. But here is how the second downside also appears in (2.8.1). The population in question is subject to *periodic* influx and removal via the forcing term $3\cos(4t)$. *How* *does this forcing term manifest itself in the solution?* Should a periodic forcing term induce periodicity in the solution? One manifestation of the periodic forcing term is the indefinite integral (2.8.2), but how much insight does that integral alone give?

It turns out that for certain classes of linear ODE we can avoid variation of parameters entirely and effectively reduce the calculations from *antidifferentiation* to *algebra* and gain insight into how the forcing term manifests itself in the solution—at least if the forcing term has the right form. Such ODE look like the following.

2.8.2 Definition. *A* **CONSTANT-COEFFICIENT FIRST-ORDER LINEAR ODE** *is an ODE of the form*

$$\dot{x} = ax + b(t),$$

where a is a real number and b is a function.

2.8.3 Example. The ODE $\dot{x} = 2x + e^t$ and $\dot{x} = e^t$ are constant-coefficient, but $\dot{x} = \sin(t)x + 2$ is not constant-coefficient. As the name implies, only the *coefficient* has to be constant; the forcing term can certainly be nonconstant.

We can certainly use variation of parameters to solve any constant-coefficient linear ODE, but the integration involved will require antidifferentiating the product of an exponential and some other function (specifically, the forcing term). Frequently, when that function is polynomial or trigonometric, it is necessary to integrate by parts. This can be wearisome, and distracting.

2.8.4 Problem ($\star$). Here is that solution via variation of parameters. Let a be a real number and let b be continuous on $(-\infty, \infty)$. Use Theorem 2.7.23 to show that the only solution to the IVP

$$\begin{cases} \dot{x} = ax + b(t) \\ x(0) = x_0 \end{cases}$$

is the function

$$x(t) = x_0\phi(t) + \int_0^t \phi(t-\tau)b(\tau)\,d\tau, \qquad \text{where} \qquad \begin{cases} \dot{\phi} = a\phi \\ \phi(0) = 1. \end{cases}$$

2.8.5 Problem (+). In Problem 2.7.34, we discussed some hypotheses on a linear ODE that allow us to control the long-time behavior of its solutions. Here is another situation that gives us such control. Suppose that $a > 0$ and b is a continuous function on $[0, \infty)$. Let $x_\star$ solve $\dot{x}_\star = -ax_\star + b(t)$, and let x solve

$$\begin{cases} \dot{x} = -ax + b(t) \\ x(0) = x_0. \end{cases} \tag{2.8.3}$$

Use variation of parameters to compute

$$|x(t) - x_\star(t)| = |x_0|e^{-at}$$

and conclude from the squeeze theorem that

$$\lim_{t\to\infty} |x(t) - x_\star(t)| = 0.$$

Conclude that *all* solutions to (2.8.3) have the same long-time behavior.

Instead of relying solely on variation of parameters, we will develop a technique for certain special classes of forcing functions which will replace antiderivatives (which can be annoying) with algebra (which will be annoying in a different way). Along the way, we will gain insight into how specific forcing functions "manifest" themselves in natural ways in the solution. Our new method is called "undetermined coefficients," and (*caveat lector!*) it is most frequently taught in the context of *second*-order differential equations, not first-order. But it works just as well here, and much of the algebra is simpler, too.

In the following we will make more use of the linearity theorem than we did with variation of parameters. Specifically, we will find one particular solution $x_\star$ to the nonhomogeneous problem $\dot{x} = ax + b(t)$. Then we will have to remember to add a constant multiple of the solution $x(t) = e^{at}$ to the homogeneous problem $\dot{x} = ax$.

2.8.1. Exponential forcing.

Here we study constant-coefficient ODE forced by an exponential, i.e., problems of the form

$$\dot{x} = ax + be^{rt}$$

for some constants a, b, and r. What function x differentiates into a sum of a multiple of itself and an exponential? Quite possibly another exponential! Equivalently, if the ODE is forced by an exponential, a good guess is that a similar-looking exponential appears in the solution.

2.8.6 Example. Consider the ODE

$$\dot{x} = x + 2e^{3t}.$$

To what extend can the exponential e^{3t} "manifest itself" in the solution x? Perhaps the simplest such manifestation is that x has the form $x(t) = \alpha e^{3t}$ for some constant α. If so, then

$$\dot{x}(t) = 3\alpha e^{3t}.$$

We plug this and the guess $x(t) = \alpha e^{3t}$ into the ODE $\dot{x}(t) = x(t) + 2e^{3t}$ to find

$$3\alpha e^{3t} = \alpha e^{3t} + 2e^{3t}.$$

We can cancel the factor e^{3t} from each term, and so we find

$$3\alpha = \alpha + 2.$$

This is really an algebraic problem for α. It rearranges to

$$2\alpha = 2,$$

and thus

$$\alpha = 1.$$

Consequently, one solution to this ODE is

$$x_\star(t) := e^{3t}.$$

Since all solutions to the associated homogeneous problem $\dot{x} = x$ are $x(t) = Ce^t$ for some constant C, the linearity theorem implies that all solutions to the nonhomogeneous ODE are

$$x(t) = e^{3t} + Ce^t.$$

2.8.7 Problem (!). Solve $\dot{x} = x + 2e^{3t}$ using variation of parameters. Do you prefer that or the method of the previous example?

2.8.8 Remark. *Here is the moral of the previous example: to solve $\dot{x} = ax + b(t)$, a worthwhile guess is that x should be a constant multiple of b. This guess will sometimes be wrong.*

2.8.9 Example. Motivated by the previous example, we guess that a solution to

$$\dot{x} = 3x + 2e^{3t}$$

is $x(t) = \alpha e^{3t}$, which means $\dot{x}(t) = 3\alpha e^{3t}$. Then we need

$$3\alpha e^{3t} = 3\alpha e^{3t} + 2e^{3t},$$

and so

$$2e^{3t} = 0.$$

This is impossible (for many reasons), and so our guess was wrong.

Of course, we could always use variation of parameters and guess that, in general, the solution has the form $x(t) = u(t)e^{3t}$. Then we would find $u(t) = 2t + C$. (Go ahead and check it.) So, the solution is

$$x(t) = (2t + C)e^{3t} = 2te^{3t} + Ce^{3t}.$$

Taking $C = 0$, we see that one particular solution to the problem is $x(t) = 2te^{3t}$.

What if, instead of guessing $x(t) = \alpha e^{3t}$, we guessed $x(t) = \alpha te^{3t}$? Then we would have

$$\dot{x}(t) = \alpha e^{3t} + 3\alpha te^{3t},$$

and so we would find

$$\alpha e^{3t} + 3\alpha te^{3t} = 3\alpha te^{3t} + 2e^{3t}.$$

Then

$$\alpha e^{3t} = 2e^{3t},$$

and so $\alpha = 2$. That is, a particular solution is $x_\star(t) = 2te^{3t}$, and then all solutions are $x(t) = 2te^{3t} + Ce^{3t}$. This agrees with variation of parameters.

2.8.10 Remark. *Here is the moral of the previous example: if a guess for the solution of an ODE fails, see if multiplying that guess by t helps.*

2.8.11 Problem. This problem is the general approach to undetermined coefficients for linear ODE forced by an exponential. Let a, b, and r be real numbers.

(i) Suppose that $a \neq r$. Guess that $x(t) = \alpha e^{rt}$ solves $\dot{x} = ax + be^{rt}$ and determine the value of α (in terms of a, b, and/or r). Where specifically in your work did you use the assumption that $a \neq r$?

(ii) Suppose that $a = r$. Guess now that $x(t) = \alpha te^{rt}$ solves $\dot{x} = ax + be^{rt}$ and determine the value of α (in terms of a, b, and/or r).

(iii) What goes wrong if you guess that $x(t) = \alpha e^{rt}$ solves $\dot{x} = rx + be^{rt}$?

(iv) Solve $\dot{x} = ax + be^{rt}$ using variation of parameters and make sure you get the same result as in parts (i) and (ii). How does your work have to change when $a = r$ compared to when $a \neq r$?

2.8.2. Sinusoidal forcing.

Here we study constant-coefficient ODE forced by a sinusoid, i.e., problems of the form

$$\dot{x} = ax + b\sin(\omega t) \qquad \text{or} \qquad \dot{x} = ax + b\cos(\omega t)$$

for some constants a, b, and ω. What function x differentiates into a sum of a multiple of itself and a sinusoid? Quite possibly another sinusoid...or maybe a *sum* of sinusoids, as we shall see. Equivalently, if the ODE is forced by a sinusoid, a good guess is that a similar-looking (linear combination of) sinusoid(s) appears in the solution.

2.8.12 Example. We study the ODE

$$\dot{x} = -x + 2\cos(3t).$$

Based on Remark 2.8.8, we first guess

$$x(t) = \alpha\cos(3t).$$

Then

$$\dot{x} = -3\alpha\sin(3t),$$

and so we need

$$-3\alpha\sin(3t) = -\alpha\cos(3t) + 2\cos(3t).$$

We combine everything into

$$(2-\alpha)\cos(3t) - 3\alpha\sin(3t) = 0.$$

This equation has to be true for all t, and so the result of Problem 2.8.13 tells us

$$2-\alpha = 0 \quad \text{and} \quad 3\alpha = 0.$$

But then $\alpha = 2$ and $\alpha = 0$, which is impossible.

Instead, a better guess, which takes into account the presence of the sine when differentiating the cosine, is

$$x(t) = \alpha\cos(3t) + \beta\sin(3t).$$

Then

$$\dot{x}(t) = -3\alpha\sin(3t) + 3\beta\cos(3t),$$

and so we want

$$-3\alpha\sin(3t) + 3\beta\cos(3t) = -\alpha\cos(3t) - \beta\sin(3t) + 2\cos(3t).$$

This rearranges to

$$(3\beta+\alpha-2)\cos(3t) + (\beta-3\alpha)\sin(3t) = 0,$$

and Problem 2.8.13 tells us

$$\begin{cases} 3\beta+\alpha-2 = 0 \\ \beta - 3\alpha = 0. \end{cases}$$

This is a linear system of equations for α and β, and there are many ways to solve it. For example, we have $\beta = 3\alpha$, and then

$$9\alpha + \alpha - 2 = 0,$$

from which we find $\alpha = 1/5$ and so $\beta = 3/5$.

Then a particular solution to the nonhomogeneous ODE is

$$x_\star(t) = \frac{\cos(3t)}{5} + \frac{3\sin(3t)}{5},$$

and therefore all solutions are

$$x(t) = \frac{\cos(3t)}{5} + \frac{3\sin(3t)}{5} + Ce^{-t}.$$

As time goes on, regardless of the value of C, the term Ce^{-t} becomes very small and contributes very little to the solution, so the dominant terms in the solution (for large t) are the sinusoidal ones. That is, all solutions settle down into the same oscillatory state (something of the reverse of the harvesting problem—or if we ran the harvesting problem "backwards in time"). This is exactly what Problem 2.8.5 is saying.

2.8.13 Problem ($\star$). Let ω be a real number. Suppose that A and B are real numbers such that

$$A\cos(\omega t) + B\sin(\omega t) = 0$$

for all t. Show that $A = 0$ and $B = 0$. [Hint: *what happens at* $t = \pi/\omega$? *Then pick some other* t *such that* $\cos(\omega t) = 0$ *and* $\sin(\omega t) \neq 0$.]

2.8.14 Problem ($\star$). This problem is the general approach to undetermined coefficients for linear ODE forced by a sinusoid. Let a, b_1, b_2, and ω be real numbers.

(i) Find real numbers α and β such that the function

$$x(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$$

solves

$$\dot{x} = ax + b_1\cos(\omega t) + b_2\sin(\omega t).$$

(ii) What goes wrong in the work above if you guess only $x(t) = \alpha\cos(\omega t)$ or $x(t) = \beta\sin(\omega t)$?

2.8.3. Polynomial forcing.

Here we study linear constant-coefficient ODE forced by a polynomial, i.e., problems of the form

$$\dot{x} = ax + b(t), \qquad b(t) := \sum_{k=0}^{n} c_k t^k,$$

where a and c_k are real numbers. A good idea, naturally, is to guess that a is some polynomial of the same degree as b.

2.8.15 Example. We study the ODE

$$\dot{x} = -2x + t^2,$$

which appeared, to our annoyance, in part (i) of Problem 2.7.22. The forcing term $b(t) = t^2$ is a degree-2 polynomial (i.e., quadratic), and so we guess

$$x(t) = \alpha_2 t^2 + \alpha_1 t + \alpha_0$$

for some constants α_2, α_1, and α_0.

Then

$$\dot{x}(t) = 2\alpha_2 t + \alpha_1,$$

and so we need

$$2\alpha_2 t + \alpha_1 = -2\alpha_2 t^2 - 2\alpha_1 t - 2\alpha_0 + t^2.$$

This rearranges into

$$(2\alpha_2 - 1)t^2 + 2(\alpha_2 + \alpha_1) + (\alpha_1 + 2\alpha_0) = 0.$$

It is a fact (which we will not prove) that a polynomial is identically zero if and only if all of its coefficients are 0. That is,

$$\sum_{k=0}^{n} \alpha_k t^k = 0 \text{ for all } t \iff \alpha_0 = 0, \ldots, \alpha_n = 0.$$

So, we get the linear system of equations

$$\begin{cases} 2\alpha_2 - 1 = 0 \\ \alpha_2 + \alpha_1 = 0 \\ \alpha_1 + 2\alpha_0 = 0. \end{cases}$$

One way to solve this is to note that $\alpha_2 = 1/2$, so

$$\frac{1}{2} + \alpha_1 = 0,$$

and therefore $\alpha_1 = -1/2$. Then

$$-\frac{1}{2} + 2\alpha_0 = 0,$$

and so $\alpha_0 = 1/4$.

Thus a particular solution to the nonhomogeneous ODE is

$$x_\star(t) = \frac{t^2}{2} - \frac{t}{2} + \frac{1}{4},$$

and therefore all solutions to this ODE are

$$x(t) = \frac{t^2}{2} - \frac{t}{2} + \frac{1}{4} + Ce^{-t}.$$

2.8.4. *Other kinds of forcing.*

The methods above generalize vastly to more complicated forcing terms at the cost of vastly more algebra. In particular, if the forcing term is the sum of more tractable terms, we can use another consequence of linearity to handle things "term by term."

2.8.16 Problem ($\star$). **(i)** Suppose that x_1 solves $\dot{x}_1 = ax_1 + b_1(t)$ and x_2 solves $\dot{x}_2 = ax_2 + b_2(t)$. That is, the coefficient is the same in each ODE, but the forcing terms are different. Put $x(t) := x_1(t) + x_2(t)$ and show that x solves

$$\dot{x} = ax + b_1(t) + b_2(t).$$

This result is sometimes called the **SUPERPOSITION PRINCIPLE**, because "superposition" is a synonym for "adding," and "adding principle" sounds less cool.

(ii) Find all solutions to the ODE

$$\dot{x} = -2x + \cos(t) + e^t.$$

More generally, one can use the ideas of undetermined coefficients to solve

$$\dot{x} = ax + b(t),$$

where b is the product of an exponential, a sinusoid, and a polynomial by guessing that x is a product of an exponential with the same "rate," a sum of sinusoids with the same frequency, and a polynomial of the same degree with unknown coefficients. Thus if

$$b(t) = e^{rt}\big(b_1 \cos(\omega t) + b_2 \sin(\omega t)\big) \sum_{k=0}^{n} c_k t^k, \tag{2.8.4}$$

then one should guess

$$x(t) = e^{rt} \cos(\omega t) \sum_{k=0}^{n} \alpha_k t^k + e^{rt} \sin(\omega t) \sum_{k=0}^{n} \beta_k t^k$$

if $a \neq r$ and otherwise guess t times this guess if $a = r$. Then solve for the coefficients α_k and β_k, making sure to pause once in a while and rediscover the will to live.

2.8.17 Example. A good guess for one particular solution to the ODE

$$\dot{x} = 5t + e^{4t} \cos(3t) t^2$$

is

$$x(t) = e^{4t} \cos(3t)\big(\alpha_2 t^2 + \alpha_1 t + \alpha_0\big) + e^{4t} \sin(3t)\big(\beta_2 t^2 + \beta_1 t + \beta_0\big),$$

while a good guess for one particular solution to the ODE

$$\dot{x} = 4t + e^{4t} \cos(3t) t^2$$

is

$$x(t) = te^{4t} \cos(3t)\big(\alpha_2 t^2 + \alpha_1 t + \alpha_0\big) + te^{4t} \sin(3t)\big(\beta_2 t^2 + \beta_1 t + \beta_0\big).$$

Both guesses require a lot of differentiation and a lot of algebra.

Just because we *can* do something does not mean we *should*, and problems with the kind of forcing term like (2.8.4) are what give the first course in ODE the bad reputation of being a "cookbook" class for learning formulas. If the way of proceeding involves a large amount of algebra for finding coefficients, it is probably more expedient just to use a computer to do the symbolic integration for variation of parameters.

This is where we finished on Wednesday, October 4, 2023.

3. FIRST-ORDER SYSTEMS

Our models so far have studied the evolution of a single quantity (in all cases, a population) over time. While the quantity may experience an external influence (such as harvesting), the quantity has not *interacted* with any other evolving quantity. This is not wholly physically realistic: a single population does not grow in a vacuum but likely interacts with other populations that are growing alongside it. A population of rabbits, for example, may experience harvesting due to a population of foxes hunting them, but the fox population will grow or decline in part based on the availability of its rabbity food source; additionally, a given rabbit population (say, Warren #1) may compete[9] with another rabbit population (Warren #2) for food and territory, even if the rabbits are not eating each other. The joint evolution of two or more time-dependent quantities can often be modeled by a *system* of differential equations, the study of which we now take up. The adjective "joint" is key: we will be interested not merely in how the related quantities evolve separately but how they evolve *together*.

While population models will continue to propel our studies, there is another kind of model that, at the level of systems, leads to arguably more tractable and ubiquitous models. Various incarnations of Newton's second law govern the behavior of moving objects. One quantity important to the study of a moving object is its position relative to some origin or destination point, or, equivalently, its *displacement* from a fixed reference point. But that is not the only quantity important when studying motion—for example, when we are driving somewhere, we certainly care about our "position" in the sense of "distance to destination," but we also care very much about our *speed*. Naturally, displacement and velocity are related (among other things, the latter is the derivative of the former). Coupling displacement and velocity together turns out to yield very important systems of differential equations that are also enjoy very fruitful analytic solution techniques.

3.1. Predator-prey models.

As motivation for some terminology and concepts of systems, we construct a simple model of one species eating another. Let $x(t)$ be the population at time t of a species that will be our "prey"—say, the familiar rabbits. Suppose that, in the absence of predators, the prey population grows at a rate proportional to its current state; thus in a predator-free world, x satisfies

$$\dot{x} = ax$$

for some $a > 0$. Now, however, suppose that a population of "predators"—say, foxes—is eating the delicious, nutritious rabbits. At time t there are $y(t)$ predators in the environment. We may imagine that the number of prey eaten per unit time is proportional to the number of interactions between the predators and the prey. If there are $x(t)$ prey at time t and $y(t)$ predators at time t, then there are $x(t)y(t)/2$ total number of interactions possible. Following the compartmental rule (RI–RO) that "Rate of change = rate in minus rate out," the rate of change of the prey population in the presence of predators is

$$\dot{x} = ax - bxy$$

[9] This, plus the foxes, is 90% of the plot of the excellent novel *Watership Down*.

for some $b > 0$.

3.1.1 Problem (!). Does this remind you of how we (re)derived the logistic equation in Problem 2.7.4?

Suppose next that the predators are badly dependent on this single prey species (unlikely, but not impossible), and in the total absence of prey the predator population *decreases* at a rate proportional to its current state; thus in a prey-free world, y satisfies

$$\dot{y} = -cy$$

for some $c > 0$. (These are some pretty antisocial foxes.) However, interactions between the predator and prey population feed the former and cause it to grow, so with prey present we may assume

$$\dot{y} = -cy + dxy$$

for some constant $d > 0$. Thus our PREDATOR-PREY model is

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = -cy + dxy. \end{cases} \tag{3.1.1}$$

Unsurprisingly, we should also incorporate initial conditions, say,

$$x(0) = x_0 \quad \text{and} \quad y(0) = y_0 \tag{3.1.2}$$

for the initial populations. A SOLUTION to this system is a *pair* of differentiable functions x and y both defined on the same interval I such that

$$\begin{cases} \dot{x}(t) = ax(t) - bx(t)y(t) \\ \dot{y}(t) = -cy(t) + dx(t)y(t) \end{cases}$$

for all t in I and also the initial conditions (3.1.2) are met.

3.1.2 Problem (!). In the absence of predators ($y = 0$), what does the model (3.1.1) predict will happen over long times to the prey? What happens if, instead, there are no prey ($x = 0$)? Remember, $a > 0$ and $c > 0$.

Algebraically, this is a pretty simple model; surely we have seen "worse" functions appear in studying linear ODE. Analytically, this is a pretty tough model! There are no general analytic techniques that will work on a problem of the form (3.1.1). (This is a negative statement and, in principle, rather hard to prove: we would have to show conclusively that we could not come up with formulas for all solutions to (3.1.1).) Nonetheless, we can use some old ideas to get a few solutions, at least.

3.1.3 Example. For concreteness, put $a = 1$, $b = 2$, $c = 3$, and $d = 4$, and consider the system

$$\begin{cases} \dot{x} = x - 2xy \\ \dot{y} = -3y + 4xy. \end{cases}$$

We can find equilibrium solutions to this system by looking for *constant* functions x and y that solve it. (Such a procedure worked often in the past, so we may as well try to adapt it here.) Then $\dot{x} = \dot{y} = 0$, so x and y are constants satisfying the pair of nonlinear equations

$$\begin{cases} x - 2xy = 0 \\ -3y + 4xy = 0. \end{cases}$$

This is really a factoring problem, and we rewrite the system as

$$\begin{cases} x(1 - 2y) = 0 \\ y(-3 + 4x) = 0. \end{cases}$$

Taking $x = 0$ and $y = 0$ then certainly produces a solution, but also requiring

$$1 - 2y = 0 \quad \text{and} \quad -3 + 4x = 0$$

gives a solution. In this latter case, we see that another equilibrium solution is

$$x = \frac{3}{4} \quad \text{and} \quad y = \frac{1}{2}.$$

Physically, we can interpret these equilibrium solutions as situations in which the populations modeled by this predator-prey system are in perfect balance. In the case $x = y = 0$, there are no predators and no prey, and nothing interesting ever happens. In the case $x = 3/4$ and $y = 1/2$, there are exactly as many prey as needed to sustain the predator population without allowing the predator population to grow or starve, and there are exactly as many predators needed to keep the prey from ever growing or decreasing. (Here we probably should assume that there are some meaningful units underlying the problem, so we avoid having 0.75 rabbits or half a fox.)

3.1.4 Problem ($\star$). Suppose that a, b, c, and d are nonzero numbers. Find all equilibrium solutions to the system

$$\begin{cases} \dot{x} = ax + bxy \\ \dot{y} = cy + dxy. \end{cases}$$

3.1.5 Problem ($\star$). **(i)** By factoring, show that if we know the predator population y, then the prey equation in (3.1.1)

$$\dot{x} = ax - bxy$$

is really a homogeneous linear ODE in x. In principle, then, if we know everything about the predators, we know everything about the prey; the same is true if we know everything about the prey.

(ii) Can the predators ever eat the prey into extinction?

3.1.6 Problem ($\star$). **(i)** How would the predator-prey model (3.1.1) change if we assumed that the prey grew logistically, not exponentially, in the absence of predators? [Hint: *what is the new "rate in" in the "rate in minus rate out" paradigm?*]

(ii) Suppose the predators and the prey broker a truce and decide to cooperate. (The foxes become vegetarians and in particular do not eat the plants that the rabbits eat.) How would the predator-prey model change if we assumed that interaction among populations helped *each* population grow?

3.2. First-order planar systems.

We will study systems of differential equations that involve two, and only two, unknowns. It is possible to generalize this to problems involving an arbitrary, finite number of unknowns, but the notation gets messier, while the concepts stay more or less the same. Sticking with two unknowns is therefore the least challenging (relatively speaking) way to meet nontrivial systems. Motivated by the structure of the predator-prey model, we introduce some basic terminology and then quickly proceed to numerical and qualitative methods (since the analytic approach will not be very fruitful here).

3.2.1 Definition. **(i)** *A* FIRST-ORDER (PLANAR) SYSTEM *of differential equations is a pair of equations of the form*

$$\begin{cases} \dot{x} = f(t,x,y) \\ \dot{y} = g(t,x,y), \end{cases} \tag{3.2.1}$$

where f and g are functions of three variables that are defined for t in some interval (a,b), x in some interval (c_1,d_1), and y in some interval (c_2,d_2).

(ii) *A* SOLUTION *to the system* (3.2.1) *is a pair of differentiable functions x and y defined on the same interval I such that*

$$\begin{cases} \dot{x}(t) = f(t,x(t),y(t)) \\ \dot{y}(t) = g(t,x(t),y(t)) \end{cases}$$

for all t in I and that $\dot{x}$ and $\dot{y}$ are continuous on I. We may augment the system (3.2.1) *to an* INITIAL VALUE PROBLEM *by demanding that*

$$x(t_0) = x_0 \quad \text{and} \quad y(t_0) = y_0$$

for some t_0 in (a,b) and some points x_0 in (c_1,d_1) and y_0 in (c_2,d_2).

(iii) *The system* (3.2.1) *is* AUTONOMOUS *if f and g are independent of time, that is, if* *the system has the form*

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{3.2.2}$$

for functions f and g of only two variables.

For contrast, we will often call a first-order ODE $\dot{x} = f(t, x)$ a **SCALAR ODE** to emphasize that it involves a single equation and is not a system.

3.2.2 Example. The predator-prey model

$$\begin{cases} \dot{x} = x - 2xy \\ \dot{y} = -3y + 4xy \end{cases}$$

is a system with

$$f(t, x, y) = x - 2xy \quad \text{and} \quad g(t, x, y) = -3y + 4xy.$$

This is an autonomous problem.

3.2.3 Example. We can modify the predator-prey system with harvesting in one or both components For example, the rabbits (prey) could be hunted by humans for food, but the foxes (predators) could also be hunted by humans for sport (because humans are the worst). The system

$$\begin{cases} \dot{x} = x - 2xy - 2(1 + \sin(t)) \\ \dot{x} = -3y + 4xy - (1 + \sin(t)) \end{cases}$$

is a simple example of including harvesting in each component. Note that the rabbits are harvested at a greater rate than foxes because of the factor of 2 on the sinusoidal term in the first equation. This is not an autonomous system, as with

$$f(t, x, y) := x - 2xy - 2(1 + \sin(t)) \quad \text{and} \quad g(t, x, y) := -3y + 4xy - (1 + \sin(t))$$

the functions f and g depend explicitly on time.

3.2.4 Problem ($\star$). Explain why the predator-prey with harvesting model from the previous example has no equilibrium solutions. [Hint: *if there are numbers x_∞ and y_∞ such that $f(t, x_\infty, y_\infty) = 0$ for all t, then the sine is constant. Can that happen?*] Explain why physically this means that the predator and prey populations are never in perfect balance. Given the nontrivial influence of an external force (whoever is doing the harvesting), is this what you expected?

3.2.5 Problem (!). Suppose that f and g are functions of the two variables x and y and that x_∞ and y_∞ are numbers with

$$f(x_\infty, y_\infty) = g(x_\infty, y_\infty) = 0.$$

Prove that the functions $x(t) := x_\infty$ and $y(t) := y_\infty$ solve the autonomous system (3.2.2). These functions are **EQUILIBRIUM SOLUTIONS** of the system (just as in Definition 2.2.2 for scalar problems).

This is where we finished on Friday, October 6, 2023.

3.3. Phase portraits and direction fields.

To say anything concrete about the scalar ODE $\dot{x} = f(t, x)$, we had to be pretty explicit with what f was, and in the end we only worked with two kinds of f, separable ($f(t, x) = g(t)h(x)$) and linear ($f(t, x) = a(t)x + b(t)$). The same is true for systems of equations. There are no universal techniques that work for all systems. It is possible to develop a broad existence and uniqueness theory, but that requires more specialized properties of f and g that deploy the language of multivariable calculus; we will not study such theories (although we will prove an existence and uniqueness result for one very special kind of system). It is also possible to adapt Euler's method (and other numerical methods for scalar problems) to systems. Essentially, one works componentwise and treats each component as a scalar ODE to which the original Euler's method applies.

3.3.1 Problem (+). Suppose that x and y solve

$$\begin{cases} \dot{x} = f(t, x, y) \\ \dot{y} = g(t, x, y), \end{cases} \qquad \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0. \end{cases} \tag{3.3.1}$$

(i) Argue that for any t in the domain of x, we have

$$\begin{aligned} x(t) = x(t_0) + \int_{t_0}^t \dot{x}(\tau) \, d\tau = x_0 + \int_{t_0}^t f(\tau, x(\tau), y(\tau)) \, d\tau &\approx x_0 + (t - t_0) f(t_0, x(t_0), y(t_0)) \\ &= x_0 + (t - t_0) f(t_0, x_0, y_0). \end{aligned}$$

If we take $t_1 = t_0 + h$ for some small $h > 0$, we should then have $x(t_1) \approx x_1$, where

$$x_1 := x_0 + h f(t_0, x_0, y_0).$$

(ii) From this kind of integral approximation and the dim memory of the development of Euler's method in Section 2.4 (encoded somewhere in your reptilian hindbrain), argue that the following pseudocode should produce a meaningful numerical approximation to the system (3.3.1).

We will not dwell on programming numerical methods for systems; instead, we assume that we have a numerical solver for systems and proceed to study its output. We will mostly work with the one predator-prey model

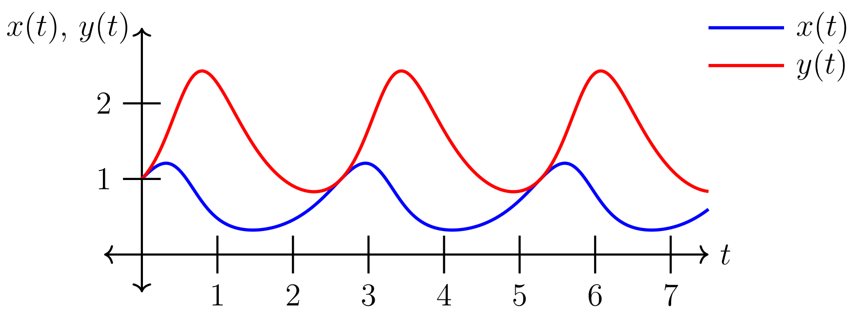
$$\begin{cases} \dot{x} = 3x - 2xy \\ \dot{y} = -2y + 3xy \end{cases}$$

with a variety of initial conditions. While we will not pursue an existence and uniqueness theory for this model, we do mention that for any choice of initial conditions on $x(0)$ and $y(0)$, the model has a unique solution.

3.3.2 Example. Consider the predator-prey system

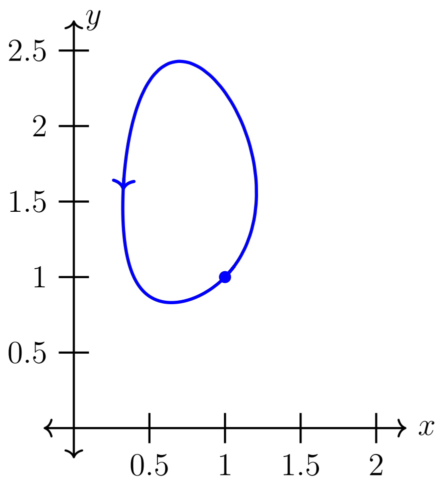
$$\begin{cases} \dot{x} = 3x - 2xy \\ \dot{y} = -2y + 3xy \end{cases} \qquad \text{with} \qquad x(0) = 1 \text{ and } y(0) = 1.$$

The coefficients and initial values here are chosen not to be physically realistic but to give easy-to-read numerical results. While we could find equilibrium solutions (as in Example 3.1.3), there are no analytic methods for producing solutions in general. Instead, we graph below the functions x (in blue) and y (in red) as approximated by Euler's method for systems.



The behaviors of both predators and prey appear to be periodic. For a very short time after time $r = 0$, both the predator and prey populations increase, but then the predator population keeps increasing, while the prey population starts to decrease. (Does this feel realistic?) The predators reach their maximum population around time $t = 1$, while the prey reach their minimum around time $t = 1.5$. That is, the minimum prey population occurs shortly after the maximum predator population. (Does this feel realistic?) Then the predator population reaches its minimum a little after time $t = 2$, and the prey population reaches its maximum around time $t = 3$, with the predator population still well below its maximum. (Does this feel realistic?) The cycle appears to continue.

In studying a system of ODE we are often interested in the *joint* evolution of its component unknowns. That is, how are x and y changing *simultaneously*? One way to see this is to plot x and y against time t as above, but this is not the only way. We could think of the ordered pairs $(x(t), y(t))$ as defining a parametric curve in the two-dimensional plane. We plot this curve below, using the data from Euler's method presented above. Note that the axes are now x and y and time t is suppressed; however, we mark the initial point $(1, 1)$ and put an arrow on the curve to indicate its "trajectory" as time increases from $t = 0$.



Starting from the initial condition $(1, 1)$, the solution curve (x, y) briefly "bulges out" to the right—this is the simultaneous increase of predators and prey—and then moves upwards and to the left. This is the increase in predators and the decrease in prey. After reaching its highest point, the solution curve drops down almost vertically—the simultaneous decrease in predators and prey—and then moves upwards and to the right—another simultaneous increase.

Whether the first graph with x and y plotted separately as functions of time or the second graph with (x, y) plotted parametrically is easier to read is largely a matter of taste. Both graphs reveal different kinds of information about the solution, which may or may not be useful.

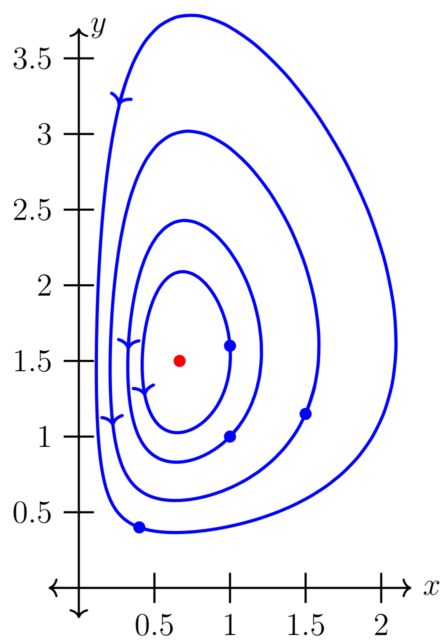
Previously we saw that the phase line was an effective way of representing qualitative information about solutions to scalar autonomous ODE $\dot{x} = f(x)$; in particular, the phase line encoded equilibrium solutions and the long-time behavior (monotonicity and asymptotes) of solutions that did not start at an equilibrium value. Moreover, we constructed phase lines using only algebraic and calculus knowledge of f—no formulas for x, no numerics. We might try to draw a "phase plane" or "phase portrait" of a homogeneous planar system in a similar way: mark the equilibrium solutions in the plane and then draw enough trajectories of solutions (x, y) that we have an idea of how solutions behave relative to those equilibrium points.

We will typically need a computer/numerical solver to do this, and there is nothing wrong with us for that—it is highly unlikely that you will ever solve a meaningful ODE without a computer, except maybe on an in-class exam, but remember that computers cannot *interpret* their output for you (not yet, anyway). In general, with systems, while finding equilibrium solutions is still a matter with algebra (solving often *nonlinear* systems of equations), predicting the long-time behavior of nonequilibrium solutions can be very difficult. We do not have facile notions of monotonicity in the joint evolution of two functions f and g of two variables! Nonetheless, the value of the phase plane is that it captures the general behavior of solutions relative to equilibrium points and allows us to see the behavior of multiple solutions simultaneously.

3.3.3 Example. Here is the phase plane (in Quadrant I) for the system

$$\begin{cases} \dot{x} = 3x - 2xy \\ \dot{y} = -2y + 3xy. \end{cases}$$

We plotted four solutions with initial conditions at $(.4, .4)$, $(1, 1)$, $(1, 1.5)$, and $(1, 1.6)$. The rationale behind these initial conditions was just to get a representative sample of solutions whose trajectories were nicely spaced out from each other for visibility purposes. We also plotted (in red) the equilibrium solution at $(2/3, 3/2)$, which can be found from the results of Problem 3.1.4. These four solutions are enough to suggest that solutions that start with $x(0) > 0$ and $y(0) > 0$ will remain positive-valued in both components (in particular, avoiding the other equilibrium point at $(0, 0)$) and spiral periodically around the equilibrium point at $(2/3, 3/2)$. The solutions can get close to the axes, but they never touch the axes; doing so implies that either x or y is 0, and then at least one of the populations has gone extinct. The models do not permit extinction (recall Problem 3.1.5).



When studying scalar ODE, we discussed slope fields before numerics, and separation of variables before slope fields. For systems, our analytic options are so limited that we went right to the numerics. It is also possible to develop an analogue of slope fields. Recall that the chief virtue of the slope field for the scalar problem $\dot{x} = f(t, x)$ is that the slope field indicates the behavior of solutions without requiring any calculus or intense numerical computation. In short, the slope field worked "on the cheap." We can exploit the structure of a system in a similar way.

To do this, we need some other ideas from calculus. It is a fact that if the functions x

and y determine a parametric curve (x, y) in the two-dimensional plane, then the slope of the tangent line to this curve at all times t such that $\dot{x}(t) \neq 0$ is

$$\frac{\dot{y}(t)}{\dot{x}(t)}.$$

And if $\dot{x}(t) = 0$, then the tangent line is vertical. So, if x and y satisfy the system

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y), \end{cases}$$

then the slope of the tangent line to the curve (x, y) at time t is, assuming $\dot{x}(t) \neq 0$,

$$\frac{\dot{y}(t)}{\dot{x}(t)} = \frac{g(x(t), y(t))}{f(x(t), y(t))}.$$

And so if the parametric solution curve (x, y) passes through the point $(x_\star, y_\star)$, and if $f(x_\star, y_\star) \neq 0$, then the slope of the tangent line to the curve through $(x_\star, y_\star)$ is

$$\frac{g(x_\star, y_\star)}{f(x_\star, y_\star)}.$$

We can therefore approximate the shape of parametric solution curves (x, y) to the system ?? by filling the two-dimensional plane with small line segments of slope $g(x_\star, y_\star)/f(x_\star, y_\star)$ through points $(x_\star, y_\star)$ at which $f(x_\star, y_\star) \neq 0$ and otherwise drawing vertical line segments through points $(x_\star, y_\star)$ at which $f(x_\star, y_\star) = 0$. As with drawing slope fields, this is a task best left to a computer; our job is to *interpret and critique* the outputs.

3.3.4 Example. Consider the predator-prey model

$$\begin{cases} \dot{x} = 3x - 2xy \\ \dot{y} = -2y + 3xy \end{cases}$$

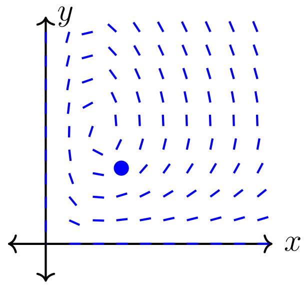
from Example 3.3.2. Through each point $(x_\star, y_\star)$ in the two-dimensional plane, we draw a line segment of slope

$$\frac{-3y_\star + 2x_\star y_\star}{3x_\star - 2x_\star y_\star}$$

if $3x_\star - 2x_\star y_\star \neq 0$, and otherwise we draw a vertical line segment. The result is a sort of slope field that approximates the trajectory in Example 3.3.2, along with plenty of others. Note that the line segment through the point $(1, 1)$ has slope

$$\frac{-2(1) + 3(1)(1)}{3(1) - 2(1)(1)} = \frac{1}{1} = 1,$$

while the line segments through points on the y-axis are vertical, since there the points have the form $(0, y)$, and $3(0) - 2(0)(0) = 0$.



We are missing, however, one piece of information that the slope field for a scalar ODE inherently provides and that the parametric curve in Example 3.3.2 demonstrates. Recall that slope fields for scalar ODE are drawn with time t as the horizontal axis; thus there is always a notion of "direction" or "trajectory" for the solution approximated by a slope field—moving to the right indicates increasing time. In the parametric setting for systems, time is suppressed, which is why we drew the arrow on the curve in Example 3.3.2.

The way to incorporate "direction" into the slope field—and thereby augment it to a "*direction* field"—is to think about what the signs of $\dot{x}$ and $\dot{y}$ tell us *jointly* about the trajectory of the parametric curve (x, y). For example, if $\dot{x}(t) > 0$ and $\dot{y}(t) > 0$, then both x and y are increasing at time t. We therefore should expect the parametric curve (x, y) to be moving "up and to the right" at this time t. Thus if $f(x_\star, y_\star) > 0$ and $g(x_\star, y_\star) > 0$, we expect that if the parametric curve (x, y) passes through the point $(x_\star, y_\star)$, then the curve is moving "up and to the right" at this point. And so we would augment the positively sloped line segment through $(x_\star, y_\star)$ with an arrow pointing up and to the right.

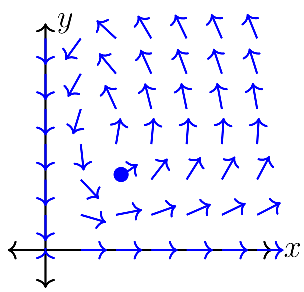
Similarly, if $\dot{x}(t) > 0$ and $\dot{y}(t) < 0$, then x is increasing but y is decreasing at time t. Then (x, y) should move "down and to the right" at time t. More generally, we can consider eight (!) possible cases on the signs of $\dot{x}$ and $\dot{y}$ that control the direction of the parametric curve (x, y). We detail these signs in the table below and upgrade the former slope field into a direction field with appropriately oriented arrows.

$\dot{x}$	$\dot{y}$	x-behavior	y-behavior	(x, y)-behavior
$+$	$+$	$\longrightarrow$	$\uparrow$	$\nearrow$
$+$	$-$	$\longrightarrow$	$\downarrow$	$\searrow$
$-$	$-$	$\longleftarrow$	$\downarrow$	$\swarrow$
$-$	$+$	$\longleftarrow$	$\uparrow$	$\nwarrow$
$+$	0	$\longrightarrow$	$\bullet$	$\longrightarrow$
$-$	0	$\longleftarrow$	$\bullet$	$\longleftarrow$

0	+	$\bullet$ $\quad\uparrow\quad\uparrow$
0	$-$	$\bullet$ $\quad\downarrow\quad\downarrow$

3.3.5 Example. Here is the *direction* field for the system

$$\begin{cases} \dot{x} = 3x - 2xy \\ \dot{y} = -2y + 3xy. \end{cases}$$

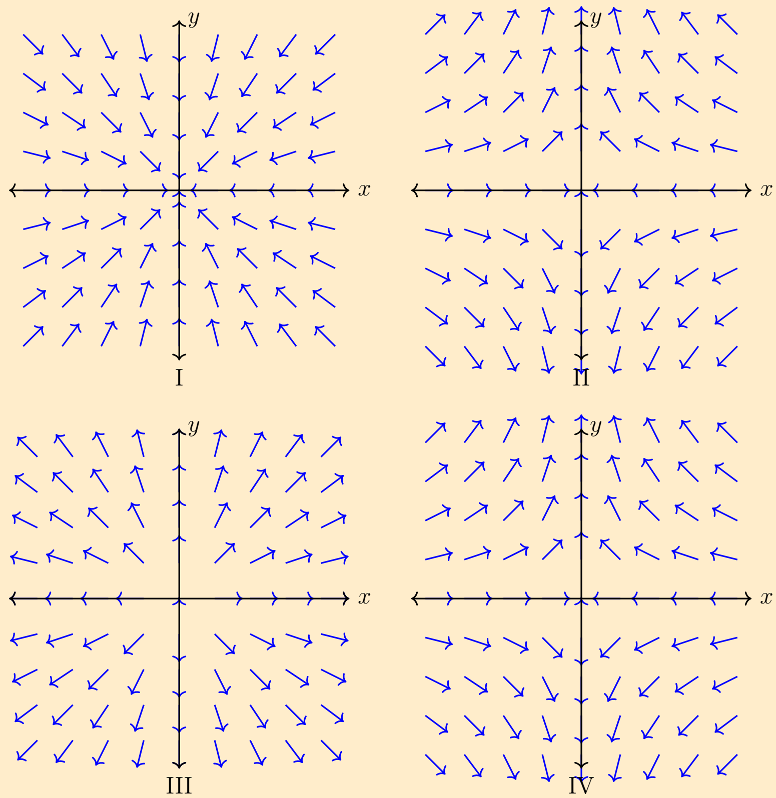


The key upgrade, compared to the output in Example 3.3.4 is the inclusion of the arrows at the end of the line segments indicating direction. Now we can see exactly the sort of "counterclockwise" orientation that we saw in Example 3.3.2 for the specific initial condition $x(0) = y(0) = 1$. The difference here is that we can see multiple trajectories (i.e., solutions with different initial conditions) at once; this was also the advantage of a slope field (breadth) over Euler's method (depth) for scalar problems.

3.3.6 Problem ($\star$). Below are four direction fields which correspond to the four systems

$$\begin{cases} \dot{x} = x \\ \dot{y} = y, \end{cases} \qquad \begin{cases} \dot{x} = -x \\ \dot{y} = -y, \end{cases} \qquad \begin{cases} \dot{x} = x \\ \dot{y} = -y, \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{x} = -x \\ \dot{y} = y. \end{cases}$$

Based on what you know about exponential growth, determine which direction field corresponds to which system. For each system, describe the long-time behavior of the x- and y-components separately, and then describe the "joint" behavior of the pair (x, y) over long times.



This is where we finished on Monday, October 9, 2023.

3.4. The SIR model.

We study a model of disease propagation within a population as an illustration of the various concepts and tools introduced for systems so far (and because it is a valuable and painfully relevant model in and of itself). There are many, many ways to model the spread of disease, and this is just one; we will discuss some of its virtues, flaws, and augmentations along the way.

We will humanize the situation and refer to members of the population as "people," rather than just "members." We divide the population into three categories.

1. People who have the disease. These people are **INFECTED**. In this model, "infected" is synonymous with "infectious" or "contagious" but maybe not "symptomatic" (as is unfortunately the case in real life).

2. People who have had the disease but no longer have it (possibly because they have died from the disease). These people are **RECOVERED**. We assume that no one has recovered at the starting time $t = 0$ of our modeling of this population, since we want to model the disease right from its very entrance into the population.

3. Everyone else. We assume that the entire population can get the disease, but that a person cannot get the disease twice. So, we call people this third category **SUSCEPTIBLE**, since they are neither currently infected nor recovered (and therefore they are "susceptible" to getting the disease). We emphasize that a person that is neither infected nor recovered must be susceptible. (There are no vaccines or natural immunities in this model.)

We also assume that the population is constant in the sense that no new people enter the population and no one leaves the population. Even if a person dies from the disease, we just consider them recovered and do not remove them from the total population count; rather, they have just changed categories within the population. (This probably requires us to consider the disease over short enough a timeframe that the population is not changing noticeably due to births, moving in/away, or deaths.) Say that the population, then, is always N. Let $x(t)$ be the number of susceptible people in the population at time t, $y(t)$ the number of infected, and $z(t)$ the number of recovered. Then

$$x(t) + y(t) + z(t) = N \tag{3.4.1}$$

for all t. Also, we assume

$$y(0) \neq 0 \quad \text{and} \quad z(0) = 0, \tag{3.4.2}$$

as if no one is sick at the start, there is no disease to spread, and also no one has recovered from the disease at the start of our modeling process.

Susceptible people can become infected, and infected people can become recovered, but recovered cannot become susceptible or infected again. We therefore expect that the number of susceptible people can only decrease, never increase, and the number of recovered people can only increase. There are many ways that a disease might spread and thereby convert susceptibles into infecteds, and the simplest is through direct interaction. If there are $x(t)$ susceptibles and $y(t)$ infecteds, then there are

$$\frac{x(t)y(t)}{2}$$

distinct ways for the two groups to interact. We assume that a certain proportion of those interactions lead to infection, and so we take

$$\dot{x} = -\frac{axy}{2}$$

for some constant $a > 0$. That is, the rate of change of susceptible *decay* is directly proportional to the number of susceptible-infected interactions. This is reminiscent of our alternate derivation of the logistic equation in Problem 2.7.4 and also of the "rate of change = rate in minus rate out" principle (RI–RO) from which we derived the harvesting model in Section 2.7.1. Here, however, there is no influx of susceptibles, so the "rate in" is 0.

Since this "rate out" of susceptibles yields infecteds, it is the "rate in" for infecteds, and so we would expect

$$\dot{y} = \frac{axy}{2}.$$

However, people do not stay infected forever but become recovered, and so the "rate out" for infecteds should be the "rate in" for recovereds. What should this rate be?

Probably the simplest growth or decay rate is exponential, so we might say that the "rate out" for infecteds is $-by$ for some $b > 0$. Then we expect

$$\dot{y} = \frac{axy}{2} - by \quad \text{and} \quad \dot{z} = by.$$

However, is it realistic to assume that the rate of recovery is directly proportional to the current number of infecteds? Recent painful experience might suggest that when more people are infected, it is *harder* for them to recover because of stress on healthcare services. It turns out that if a population decays exponentially with rate r, i.e., the population u satisfies $\dot{u} = -ru$ for some $r > 0$, then $1/r$ can be interpreted as the "average lifespan" of a member in this population; see Problem 3.4.9 (which we defer to the end of this section to avoid interrupting our flow here); conversely, if a population is decreasing, and if the average lifespan of a member in that population is $1/r$, then the model $\dot{u} = -ru$ is a decent representation of that population's decrease. With this in mind, we can say that $1/b$ is the average length of time that a person is sick; then the "rate out" of infecteds is reasonably by.

With this (quite possibly debatable) justification out of the way, we will adopt

$$\begin{cases} \dot{x} = -axy/2 \\ \dot{y} = axy/2 - by \\ \dot{z} = by \end{cases}$$

as the model for our spread of disease, but we make one change. While it is possible that the precise *number* of people in each category is important (Spring 2020 COVID cases, number of students in a class of 36 who have the flu), it may be more meaningful and manageable to consider the *fraction* of the total population in each category. With N as the total population, we put

$$S := \frac{x}{N}, \qquad I := \frac{y}{N}, \qquad \text{and} \qquad R := \frac{z}{N} \tag{3.4.3}$$

and define

$$\alpha := \frac{aN}{2} \quad \text{and} \quad \beta := b.$$

Then we obtain the **SIR MODEL**

$$\begin{cases} \dot{S} = -\alpha SI \\ \dot{I} = \alpha SI - \beta I \\ \dot{R} = \beta I. \end{cases} \tag{3.4.4}$$

3.4.1 Problem (!). Check this. Also, use (3.4.1) to show

$$S(t) + I(t) + R(t) = 1$$

for all t, and use (3.4.2) to show

$$I(0) \neq 0 \quad \text{and} \quad S(0) + I(0) = 1. \tag{3.4.5}$$

We call α the **infection coefficient** and β the **recovery coefficient**. A good exercise is to conjecture how varying α or β (or both simultaneously) affects the behavior of solutions to this model.

The symbol R_0 will later denote an important constant related to the SIR model that is *not* the initial value $R(0)$, so we will not write S_0, I_0, or R_0 for initial conditions. Instead, we note that since $R(0) = 0$ and $S(0)+I(0)+R(0) = 1$, we have $S(0) = 1-I(0)$: everyone who is not infected at the start is susceptible. As with the predator-prey model, we assume, but do not prove, that the SIR model has a unique solution for any choice of the initial condition $(S(0), I(0), R(0))$.

We want to predict epidemics with this model. What precisely counts as an "epidemic" is somewhat subjective, but for the purposes of this model we will equate "epidemic" with "the disease is spreading." That is, an epidemic happens if I is increasing.

3.4.2 Example. If I is constant, then I will not be increasing, and so we may suspect that equilibrium solutions to the SIR model will not be very interesting. Suppose that

$$\begin{cases} SI = 0 \\ (\alpha S - \beta)I = 0 \\ I = 0. \end{cases}$$

So, taking $I = 0$ and S and R to be any constants always works. However, we assume $R(0) = 0$ for the purpose of disease modeling (which is more specific than just solving equations), and since we saw $S(0) = 1 - I(0)$ above in that case, we really have $S(t) = S(0) = 1 - I(0) = 1$. Thus the only *meaningful* equilibrium solution is $S(t) = N$ and $I(t) = R(t) = 0$. In that happy case, no one is infected ever, and there is no spread of disease; everyone remains susceptible but never becomes infected. Would that we lived in such a world.

The third equation in (3.4.4) is really a direct integration problem once we know I with

$$R(t) = R(0) + \int_0^t \dot{R}(\tau)\, d\tau = \beta \int_0^t I(\tau)\, d\tau, \tag{3.4.6}$$

since $R(0) = 0$, and so we can pare down the model to just two equations:

$$\begin{cases} \dot{S} = -\alpha SI \\ \dot{I} = \alpha SI - \beta I. \end{cases} \tag{3.4.7}$$

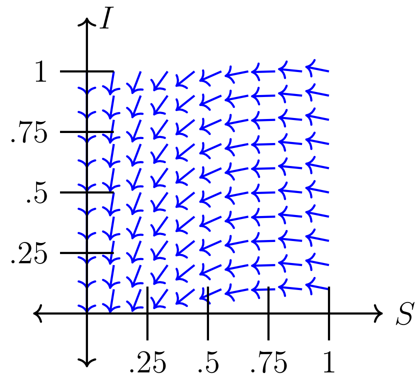
3.4.3 Problem (!). How does (3.4.7) resemble the predator-prey models in Section 3.1, and how is it different?

3.4.4 Example. To determine the behavior of nonequilibrium solutions to the SIR model, we look at some qualitative and numerical data.

(i) We first study the problem

$$\begin{cases} \dot{S} = -0.25SI \\ \dot{I} = 0.25SI - 0.2I. \end{cases}$$

We begin with the direction field.



It looks like all parametric solution plots (S, I) move "down and to the left." More precisely, if a parametric solution starts with roughly $S(0) < 0.75$, then it seems that both the S- and I-values are always decreasing. In particular, the I-values always tend to 0, although the S-values appear to tend to some positive number as time goes on. (If $S(0) = 0$, then it appears that $S(t) = 0$ for all t, and I just decreases down to 0.) If a solution starts with $S(0) > 0.75$, then it appears that the parametric solution moves upwards until the S-values hit 0.75.

These geometric observations suggest the following about the model.

• Assume that solutions are defined on $[0, \infty)$; this can be proved, but we will not do it. Then

$$\lim_{t \to \infty} I(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} S(t) =: S_\infty,$$

with $S_\infty > 0$ unless $S(0) = 0$. Thus the fraction of infecteds is eventually reduced to 0, and no one stays infected forever. The fraction of susceptibles does not tend to 0, and so some portion of the population always remains susceptible and never gets the disease.

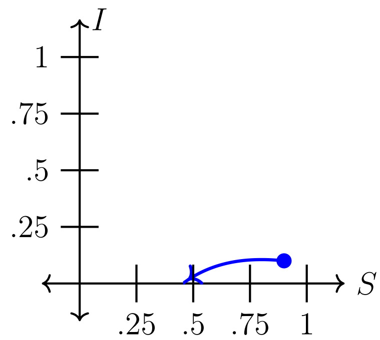
• The susceptible function S is always strictly decreasing. This is expected, since infecteds only become recovereds, and recovereds do not become susceptible again.

• Once the susceptible fraction decreases below a certain threshold (maybe 0.75?), the infected function strictly decreases. Thus if there are too few susceptibles to infect, the disease cannot spread further.

Now we look at the initial value problem

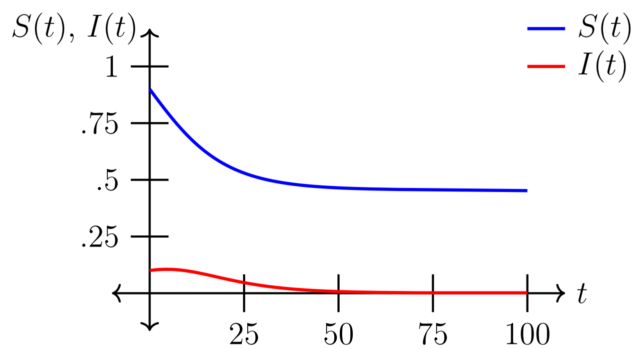
$$\begin{cases} \dot{S} = -0.25SI \\ \dot{I} = 0.25SI - 0.1I, \end{cases} \qquad \begin{cases} S(0) = 0.9 \\ I(0) = 0.1. \end{cases}$$

It is probably fair to say that "most" people are susceptible here, not infected, at the start. First we plot the parametric solution (S, I).



This seems to bear out our prediction from the direction field. There is a steady decrease in S and a minute increase in I at the start, but then a steady decrease once S dips below 0.75 (or thereabouts). The I-coordinate decreases to 0, while the S-coordinate limits to a value around 0.5.

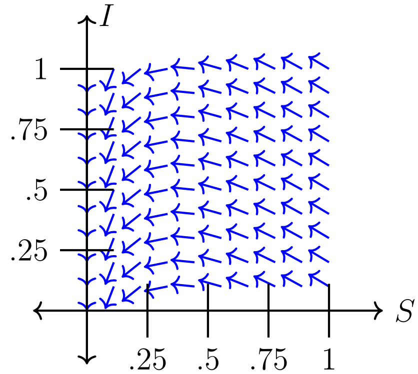
Now we look at the numerical results from the individual components. Hopefully this just further confirms our observations above with a greater emphasis on the separate behavior of S and I.



(ii) There are really four numbers that we can vary in the set-up of the SIR model: the infection coefficient α, the recovery coefficient β, the initial fraction of susceptibles $S(0)$, and the initial fraction of infecteds $I(0)$. Here we double the infection coefficient to 0.5 and leave the other three values the same as above. That is, we look at the system

$$\begin{cases} \dot{S} = -0.5SI \\ \dot{I} = 0.5SI - 0.2I. \end{cases}$$

Here is the direction field.

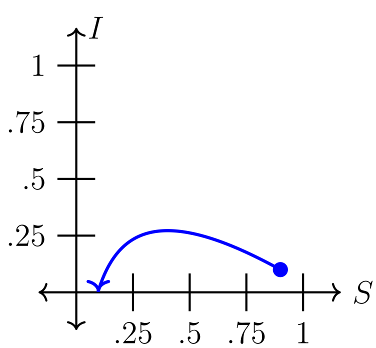


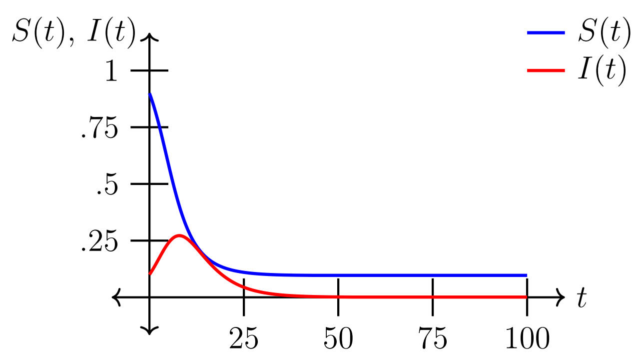
The major difference from the case $\alpha = 0.25$ before is that more solutions appear to be increasing in the I-component. Specifically, solutions that start with $S(0) > 0.5$ definitely have increasing I-components, and it is only when the S-values decrease to somewhere between 0.5 and 0.25 that the I-values start to decrease. In particular, the disease can keep spreading even when there is a smaller fraction of susceptibles than before. Moreover, the S-values seem to decrease to limiting values much closer to 0 than in the previous case. Thus many more people get infected in this model (although the I-values also decrease to 0, and so eventually all infecteds recover). This is unsurprising (though unpleasant): the infection coefficient is higher, so the same number of interactions as before have a higher chance of yielding infection than before.

Now we look at data for the IVP

$$\begin{cases} \dot{S} = -0.5SI \\ \dot{I} = 0.5SI - 0.2I, \end{cases} \qquad \begin{cases} S(0) = 0.9 \\ I(0) = 0.1. \end{cases}$$

We present the parametric solution curve and then the individual components.



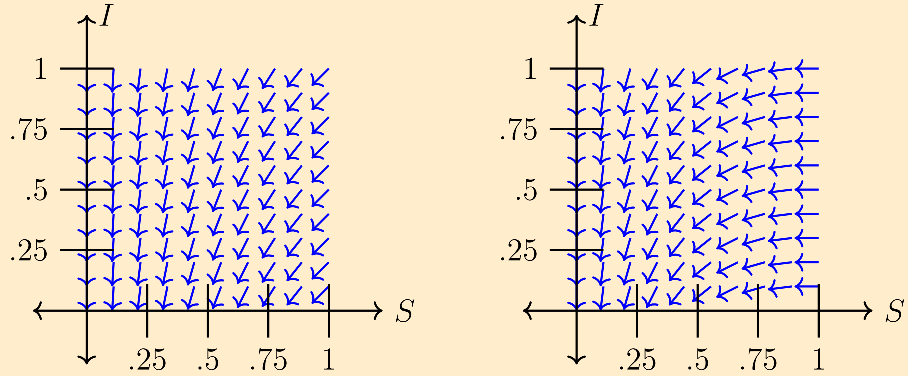


As we expected from the direction field, the infected fraction reaches a larger maximum and the susceptible fraction decreases to a smaller limit.

3.4.5 Problem (!). One direction field below corresponds to the SIR model

$$\begin{cases} \dot{S} = -0.1SI \\ \dot{I} = 0.1SI - \beta I \end{cases}$$

with $\beta = 0.1$ and the other to the model with $\beta = 0.2$. Which is which (and why)?



3.4.6 Example. We can generalize and justify many of the observations in Example 3.4.4 using the structure of the SIR system and calculus.

1. *S is always decreasing.* We have $\dot{S} = -\alpha SI$. Since $\alpha > 0$ and since populations must be nonnegative, if we assume that $S(t) \neq 0$ and $I(t) \neq 0$ for all t, then $\dot{S}(t) < 0$ for all t, and so S is always decreasing.

2. *$S(t) > 0$ for all t.* First, $\dot{S} = -\alpha SI$ is a homogeneous problem for S (assuming that we know I), and so

$$S(t) = S(0) \exp\left(-\alpha \int_0^t I(\tau)\, d\tau\right).$$

Thus $S(t) = 0$ if and only if $S(0) = 0$. In that case, since $S(0) + I(0) + R(0) = 1$ and $R(0) = 0$, we would have $I(0) = 1$, and the population would be completely infected at the start. If everyone is sick from the beginning, there is nothing really to model about the spread of the disease. So, we should assume $S(0) \neq 0$, in which case $S(t)$ is always positive.

3. *$I(t) > 0$ for all t.* Likewise, we have $\dot{I} = (\alpha S - \beta)I$. This is now a linear homogeneous problem for I, and its solution is

$$I(t) = I(0) \exp\left(\int_0^t \big(\alpha S(\tau) - \beta\big)\, d\tau\right).$$

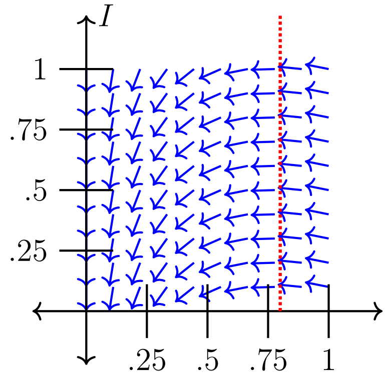
It is not immediately clear whether I is always increasing or decreasing, but, unless $I(0) = 0$, we can see that I is always positive. Indeed, we are assuming $I(0) > 0$, so I is always positive. This is boring: there was never anyone infected to spread pestilence in the first place.

4. *R is always increasing.* We could either use the equation $\dot{R} = \beta I$ to see that $\dot{R}$ is positive and thus R is increasing, or we could use the formula (3.4.6) for R, the positivity of I, and the monotonicity of integrals (since $I(\tau) > 0$ for $0 \le \tau \le t$, we also have $\int_0^t I(\tau)\, d\tau > 0$) to see that R is always increasing.

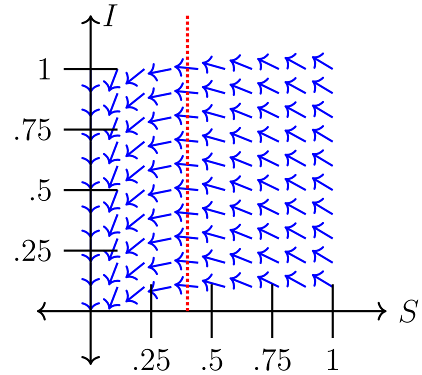
$\boxed{\text{This is where we finished on Wednesday, October 11, 2023.}}$

5. *If $S(t_\star) < \beta/\alpha$ for some $t_\star \ge 0$, then I is decreasing for $t \ge t_\star$.* Since $\dot{I} = (\alpha S - \beta)I$, we know that $\dot{I}(t) < 0$ whenever $\alpha S(t) - \beta < 0$, equivalently, whenever $S(t) < \beta/\alpha$. Since S is decreasing, if $S(t_\star) < \beta/\alpha$ for some $t_\star$, then $S(t) < S(t_\star) < \beta/\alpha$ whenever $t > t_\star$. And so $\dot{I}(t) < 0$ for $t \ge t_\star$.

With this in mind, we revisit the two SIR models from Example 3.4.4 and draw in dotted red the vertical line $S = \beta/\alpha$. We can clearly see how arrows through points (S, I) with $S < \beta/\alpha$ point down and to the left, indicating that I is decreasing at those points.



$\alpha = 0.25$, $\beta = 0.2$, $\beta/\alpha = 0.8$



$\alpha = 0.5$, $\beta = 0.2$, $\beta/\alpha = 0.4$

In particular, if $S(0) < \beta/\alpha$, then $S(t) < \beta/\alpha$ for all t, so I is always decreasing, and the disease never spreads. In other words, if the initial fraction of susceptibles is too small relative to the parameters of the disease, there is no epidemic! (We are not saying anything about the initial fraction of infecteds because of (3.4.5): if we know $S(0)$, then since $R(0) = 0$, we have $I(0) = 1 - S(0)$.) Equivalently, if $R_0 := \alpha S(0)/\beta < 1$, then the disease never spreads. This result is a precise quantitative statement relating the initial conditions and the parameters of the model to the long-term behavior of its solutions—what more could we ask from differential equations?

3.4.7 Problem (+). Suppose that S, I, and R solve the SIR model for a given initial condition $(S(0), I(0), R(0))$ with $0 < S(0) < 1$, $0 < I(0) < 1$, and $R(0) = 0$. Suppose that all three functions are defined on $[0, \infty)$. We know that S is strictly decreasing on $[0, \infty)$ and $0 \le S(t)$ for all t; Theorem 2.6.13 implies that $S_\infty := \lim_{t\to\infty} S(t)$ exists, and we have $0 \le S_\infty \le 1$. Likewise, since R is strictly increasing on $[0, \infty)$ and $R(t) \le 1$ for all t, Theorem 2.6.10 implies that $R_\infty := \lim_{t\to\infty} R(t)$ exists, and we have $0 \le R_\infty \le 1$.

(i) Use the fact that $S(t) + I(t) + R(t) = 1$ for all t to conclude that $I_\infty := \lim_{t\to\infty} I(t)$ exists.

(ii) Suppose that $I_\infty \neq 0$. Estimate

$$R(t) = R(0) + \beta \int_0^{t_0} I(\tau)\ d\tau + \beta \int_{t_0}^{t} I(\tau)\ d\tau \geq R(0) + \beta \int_0^{t_0} I(\tau)\ d\tau + \frac{\beta I_\infty (t - t_0)}{2}$$

and deduce that $\lim_{t\to\infty} R(t) = \infty$. [Hint: *remember that if* $f(\tau) \leq g(\tau)$ *on* $[t, t_0]$, *then* $\int_{t_0}^{t} f(\tau)\ d\tau \leq \int_{t_0}^{t} g(\tau)\ d\tau$.] This is impossible because $R_\infty \leq 1$, and so we must have $I_\infty = 0$.

3.4.8 Problem ($\star$). Suppose that S and I solve the system (3.4.7). Show that the function

$$J(t) := \frac{\beta}{\alpha} \ln(S(t)) - S(t) - I(t)$$

is constant. [Hint: *what is* $\dot{J}(t)$*?*] Broadly speaking, if a function constructed out of solutions to a (system of) differential equation(s) is constant, then that function is a **FIRST INTEGRAL** for the problem. First integrals encode quantities related to the problem that do not change or are "conserved."

3.4.9 Problem (+). Suppose that x models a decreasing population: at time t, there are $x(t)$ members in this population, and if $0 \leq s < t$, then $x(t) < x(s)$. Moreover, suppose that $\lim_{t\to\infty} x(t) = 0$. At some point in time, then, every member of the population will be removed from it (e.g., via death or transition to a different population). Each member of the population, therefore, has a finite "lifespan." What is the average lifespan of the population? Since there were, at the start, $x(0)$ members of the population, if we knew the lifespan of each and every member—say, ℓ_k for the kth member of the population, then this average would be

$$\frac{\text{total number of time units lived by the whole population}}{\text{the whole population}} = \frac{1}{x(0)} \sum_{k=1}^{x(0)} \ell_k.$$

It is unlikely, however, that we would know all this data.

Here is what we do instead. At time $s \geq 0$, there are $x(s)$ members of the population, and at time $t > s$, there are $x(t)$ members, so $x(s) - x(t)$ members are removed between times s and t. (Recall that x is strictly decreasing, so $x(s) - s(t) > 0$.) Observe that by the mean value theorem,

$$x(s) - x(t) = \left(\frac{x(t) - x(s)}{t - s}\right)(s - t) = \dot{x}(\tau)(s - t) = |\dot{x}(\tau)|(t - s)$$

for some τ in (s, t). We have also used the fact that since x is decreasing, $\dot{x}(\tau) < 0$. If t and s are close, then $t \approx \tau$, and so we can say

$$x(s) - x(t) \approx |\dot{x}(t)|(t - s).$$

Moreover, if t and s are close, then living s time units is roughly the same as living t time units (although we would probably prefer to live t units than $s < t$ units) and so the total number of time units lived by the members who were removed between times s and t is appproximately

$$t|\dot{x}(t)|(t-s).$$

Now let $R > 0$ be a large number. Divide the interval $[0, R]$ into n small subintervals $[t_{k-1}, t_k]$. Then the total number of time units lived by the members who were removed between times t_{k-1} and t_k is approximately

$$t_k|\dot{x}(t_k)|(t_k - t_{k-1}),$$

and so the total number of time units lived by members who were removed between times 0 and R is approximately

$$\sum_{k=1}^{n} t_k|\dot{x}(t_k)|(t_k - t_{k-1}).$$

This is a Riemann sum approximation for the integral $\int_0^R t|\dot{x}(t)|\ dt$. If we take $R \to \infty$, then the improper integral $\int_0^\infty t|\dot{x}(t)|\ dt$ should capture the total number of time units lived across the population, and so the average lifespan should be

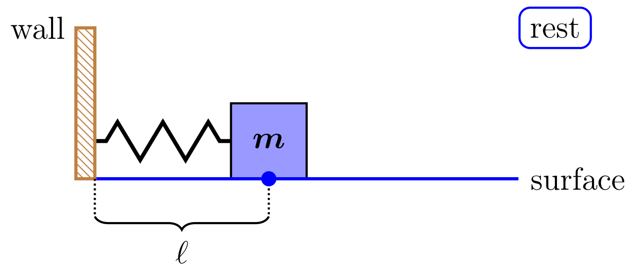
$$\frac{1}{x(0)} \int_0^\infty t|\dot{x}(t)|\ dt.$$

With this definition of average lifespan, show that if a population decays exponentially at rate $r > 0$ (i.e., $\dot{x} = -rx$) then its average lifespan is $1/r$.

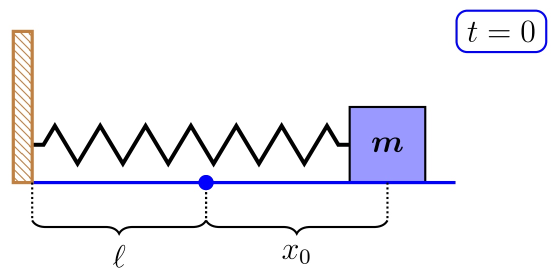
3.5. The harmonic oscillator.

We now discuss possibly the most important model in the course. It induces not a first-order system of ODE but rather a single second-order linear ODE and so serves as a bridge between systems and second-order linear equations, which appear frequently in applications. This model will be one of our few "complete success stories" in the sense that we can always solve it analytically. The actual solution formulas will be relatively easy and rely ultimately more on algebra (the quadratic formula) than calculus. However, the ideas underlying them are deep and versatile.

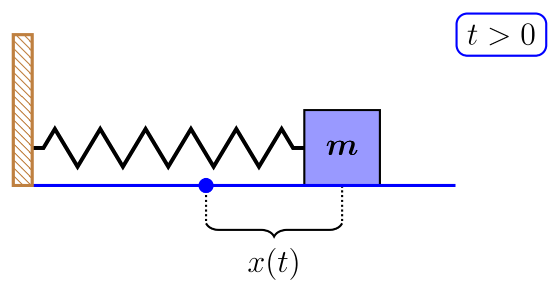
Consider the following physical situation. Place an object of uniform mass $m > 0$ along a horizontal surface. Connect the object to a wall on the left by a spring. At rest the object is ℓ units from the wall. This is a **harmonic oscillator**.



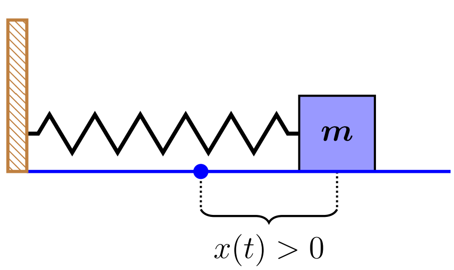
Pull the oscillator some x_0 units to the right (or push to the left; we will interpret "right" as $x_0 > 0$ and "left" as $x_0 < 0$) and let it go, maybe with a little extra oomph, maybe not. What happens? How does the oscillator move?

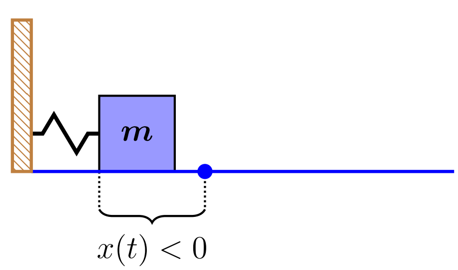


We make the fundamental assumption that the oscillator can only move to the left or the right along the surface, i.e., its motion is effectively one-dimensional. This allows us to introduce a coordinate system: denote the oscillator's displacement from its equilibrium position at time t by $x(t)$.



For example, since at the very start we pulled the oscillator a distance x_0 from equilibrium, we have $x(0) = x_0$. Assume that the displacement is positive if the oscillator is to the right of its equilibrium position, negative if the oscillator is to the left of its equilibrium position, and zero if the oscillator is exactly at its equilibrium position. (This is unlike our population models: now $x(t) < 0$ makes physical sense.)





Newton's law will give us an ODE governing the behavior of the oscillator. Here mass is m and acceleration at time t is $\ddot{x}(t)$. Suppose that we can measure all the forces acting on the oscillator at time t by $\mathsf{F}(t)$, where F is some function. Then

$$m\ddot{x}(t) = \mathsf{F}(t).$$

The precise choice of F will determine the precise ODE governing the oscillator (and also our course of study for the foreseeable future). One force that will always be present arises from the spring. Experience[10] teaches that the further we pull a spring, the more force we have to exert. If we stretch a spring a distance x, the spring pulls back with the force $\mathsf{F}_{\mathsf{spr}}(x)$. Experience probably also teaches us that we want $\mathsf{F}_{\mathsf{spr}}(x)$ to (1) be proportional to x and (2) act in the opposite direction to x. So, we define

$$\mathsf{F}_{\mathsf{spr}}(x) := -\kappa x \tag{3.5.1}$$

for some $\kappa > 0$. The definition (3.5.1) is **Hooke's law**.

3.5.1. The undamped harmonic oscillator.

Assume, for the moment, and wholly unrealistically, that the *only* force experienced by the oscillator is the spring force—no friction against the surface, no air resistance, no cats coming by to play. Such an oscillator is called **undamped**. Then $\mathsf{F} = \mathsf{F}_{\mathsf{spr}}$, and so Newton's law says

$$m\ddot{x}(t) = \mathsf{F}_{\mathsf{spr}}(x(t)) = -\kappa x(t),$$

which we more typically write[11] as

$$m\ddot{x} + \kappa x = 0. \tag{3.5.2}$$

This is, of course, a **second-order** ODE, as $\ddot{x}$ appears in the equation, but no higher derivatives of x are there.

So, what happens? Since there is no friction, we might expect the oscillator, once put in motion, to move forever. Nothing is there to slow it down, or speed it up. In particular, it definitely should not settle down to stay motionless at some fixed distance from equilibrium, and so we expect that $\lim_{t\to\infty} x(t)$ does not exist; this presumes that (3.5.2) even has a solution x (of course it does), but we will come to that presently.

We might make a more precise conjecture in one particular physical situation. Suppose that we do not move the oscillator at all from equilibrium: $x(0) = x_0 = 0$. Suppose that we do not even touch the oscillator, so that at time $t = 0$, it is motionless: $\dot{x}(0) = 0$. Then the oscillator should never move and thus stay at equilibrium for all time, i.e., we expect $x(t) = 0$ for all t. More formally, we expect that

$$\begin{cases} m\ddot{x} + \kappa x = 0 \\ x(0) = 0 \\ \dot{x}(0) = 0 \end{cases} \Longrightarrow x = 0.$$

[10] In the lab, with Slinkies...

[11] While we usually wrote first-order problems in the form $\dot{x} = f(t, x)$, we usually do *not* isolate the highest derivative in second-order problems or beyond.

Our inclusion of $\dot{x}(0)$ in the set-up here hints at what will be a significant difference in our study of the harmonic oscillator from population models. Knowledge of position (x) alone will ultimately be insufficient. We will also need to consider velocity ($\dot{x}$) to have a sufficient amount of data to predict the oscillator's motion. This will ultimately lead to a system in which the two unknowns are x and $\dot{x}$. But first we need to augment the oscillator model further.

This is where we finished on Friday, October 13, 2023.

3.5.2. The damped harmonic oscillator.

Suppose now that the oscillator experiences friction or air resistance in addition to the spring force—but otherwise there are no other forces (no cats coming by to play, not yet...). We now say that the oscillator is **DAMPED**.

Experience suggests that friction is proportional to velocity; say that if the oscillator is moving with velocity $\dot{x}$, then the frictional force that it experiences at time t is $\mathsf{F}_{\mathsf{fr}}(t) = -b\dot{x}(t)$ for some $b > 0$. Then the total force that the oscillator experiences is the sum of the spring force and the friction force: $\mathsf{F} = \mathsf{F}_{\mathsf{spr}} + \mathsf{F}_{\mathsf{fr}}$, and so Newton's law now reads

$$m\ddot{x} = -\kappa x - b\dot{x},$$

or, as we will more often write it,

$$m\ddot{x} + b\dot{x} + \kappa x = 0. \tag{3.5.3}$$

In the absence of other forces, then, we expect that friction will slow down the oscillator over long times and cause it to return to its rest position. Thus we expect the long time behavior

$$m\ddot{x} + b\dot{x} + \kappa x \text{ with } b,\ \kappa > 0 \Longrightarrow \lim_{t\to\infty} x(t) = 0.$$

3.5.3. The driven harmonic oscillator.

The oscillators considered so far, whether undamped or damped, have experience no "external" forces. That is, to set up the oscillator, there is always a spring connecting the oscillator to a wall and a surface over which the oscillator moves. The spring always contributes a spring force (what else would we call it?), and the surface sometimes contributes a damping force (and sometimes the surface is magical and does not). These two kinds of forces are "internal" or "inherent" to the oscillator. But maybe a force "external" to the oscillator influences its motion—an earthquake, shaking the wall to which the oscillator is attached; a microlocalized black hole pulling the oscillator in one direction; a cat walking by and whacking the oscillator with her beefy paw.

If there is an external force, then the total force experienced by the oscillator at time t has the form $\mathsf{F}(t) = \mathsf{F}_{\mathsf{spr}}(t) + \mathsf{F}_{\mathsf{fr}}(t) + f(t)$, where $\mathsf{F}_{\mathsf{spr}}$ is the spring force, F_{fr} is the friction force (we now allow $\mathsf{F}_{\mathsf{fr}} = 0$, so $b = 0$, to incorporate the undamped oscillator), and f is a catch-all term for "all the other forces." The displacement of the oscillator then is

$$m\ddot{x} + b\dot{x} + \kappa x = f(t).$$

An oscillator experiencing an external force is called **DRIVEN**[12] or (unsurprisingly) **FORCED**; an oscillator without an external force is **FREE**.

3.5.4. Guiding questions.

Here is a summary of our work on the ODE governing the displacement of a harmonic oscillator and its attendant terminology.

$m\ddot{x} + b\dot{x} + \kappa x = f(t)$	
Damped: $b > 0$	Undamped: $b = 0$
Free: $f(t) = 0$	Driven: $f(t) \neq 0$

We will refer to the ODE

$$m\ddot{x} + b\dot{x} + \kappa x = f(t) \tag{3.5.4}$$

as the **EQUATION OF MOTION** for the harmonic oscillator with mass $m > 0$, damping coefficient $b \geq 0$, spring constant $\kappa > 0$, and driving term f (which may be 0).

Here are some questions that will guide our work. The first three are reiterations of ones that we previously asked.

1. If friction is present and $b > 0$, do we have $\lim_{t\to\infty} x(t) = 0$?

2. If friction is not present and $b = 0$, how can we quantify the idea that "the oscillator keeps moving forever and does not slow down"?

3. What are the roles of initial displacement $x(0)$ and velocity $\dot{x}(0)$ in the long-term behavior of the oscillator? In particular, if $x(0) = \dot{x}(0) = 0$, do we have $x(t) = 0$ for all t?

4. How might changing the parameters m, b, and κ affect the solution? We might expect that a "heavier" mass moves "more slowly" than a "lighter" one. We might expect that if we "turn up" the friction, the oscillator returns to equilibrium "more quickly." We might expect that a "stiffer" spring pulls back "more quickly" than a "looser" spring. Overall, *how can we quantify these questions in terms of the parameters m, b, and κ, and how can we see their effects in the solution?*

5. How might a particular driving term f (say, a regular, periodic whacking of the oscillator by our jerk of a cat) manifest itself in the solution x? How can we see explicitly the dependence of a solution on f?

Finally, here are some harder questions, which we will not pursue in this course, but which can nonetheless be addressed with enough work.

6. What if the oscillator and its environment "change" over time? Maybe the mass leaks, the surface (over which the oscillator moves) gets rougher, or the spring stiffens. In these cases, we would want the "material" data m, b, and κ to depend on time, and so we would need to solve problems of the form

$$m(t)\ddot{x} + b(t)\dot{x} + \kappa(t)x = f(t).$$

[12] Look back at Definition 2.7.5 right now. This is why we used the terms "forcing" and "driving" in that definition.

This turns out to be quite hard!

7. What if the spring is stretched over "long" distances? Hooke's law (3.5.1) is really only valid when the spring is stretched a "short" length from equilibrium. Otherwise, we might need to incorporate a nonlinear term

More precisely, if $\mathsf{F}_{\mathsf{spr}}$ is the spring force, we might expand $\mathsf{F}_{\mathsf{spr}}$ in a Taylor series; we expect $\mathsf{F}_{\mathsf{spr}}(0) = 0$, since stretching the spring 0 units from its equilibrium length requires no work, and no force. Taking $\mathsf{F}_{\mathsf{spr}}(x) \approx -\kappa x$ is just using the first term of that Taylor series, and that is only a decent approximation to $\mathsf{F}_{\mathsf{spr}}$ when $x \approx 0$. Perhaps we might now include another term in the Taylor approximation and use a *nonlinear* force $\mathsf{F}_{\mathsf{spr}}(x) = -\kappa x - \beta x^2$. Here the spring force has a quadratic term; it could be even more complicated. Then the equation of motion for the oscillator is

$$m\ddot{x} + b\dot{x} + \kappa x + \beta x^2 = f(t).$$

This too is quite hard!

While we know by now that there is more to life than formulas—pause and recite the Analyst's Creed (AC)—a good initial attempt at answering these questions might be finding some formulas and playing with them. Specifically, we want to solve the IVP

$$\begin{cases} m\ddot{x} + b\dot{x} + \kappa x = f(t) \\ x(0) = x_0 \\ \dot{x}(0) = y_0 \end{cases} \tag{3.5.5}$$

for given numbers m, b, κ, x_0, and y_0 and a given function f. We now proceed to do just that.

3.5.5. The harmonic oscillator as a system.

Except we do not solve exactly (3.5.5) right away. While starting with this IVP is a valid life decision, we can make a clever change of variables that converts (3.5.5) into a special kind of system.

As with so many substitutions, the following is a trick, which you need not have anticipated. First, since we assume $m > 0$, we may divide to find

$$m\ddot{x} + b\dot{x} + \kappa x = f(t) \iff \ddot{x} + \frac{b\dot{x}}{m} + \frac{\kappa x}{m} = \frac{f(t)}{m} \iff \ddot{x} = -\frac{b\dot{x}}{m} - \frac{\kappa x}{m} + \frac{f(t)}{m}.$$

Now we define

$$y := \dot{x}$$

and compute

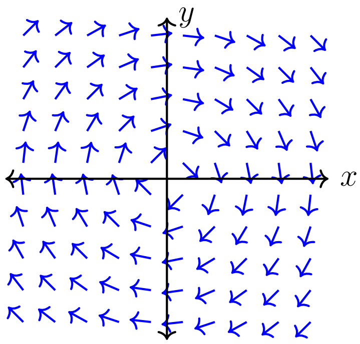
$$\dot{y} = \ddot{x} = -\frac{b\dot{x}}{m} - \frac{\kappa x}{m} + \frac{f(t)}{m} = -\frac{by}{m} - \frac{\kappa x}{m} + \frac{f(t)}{m}.$$

And so the second-order IVP (3.5.5) is equivalent to the first-order system

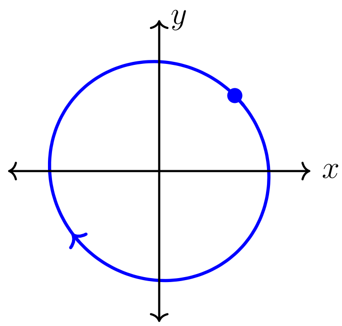
$$\begin{cases} \dot{x} = y \\ \dot{y} = -by/m - \kappa x/m + f(t)/m, \end{cases} \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0. \end{cases} \tag{3.5.6}$$

3.5.1 Example. Here are qualitative and numerical results for three undriven harmonic oscillators. The mass in each oscillator is $m = 1$, and the spring constant is $\kappa = 1$. We vary the damping coefficient from $b = 0$ (undamped) to $b = 1$ and $b = 2$ to see the effects of increased damping.

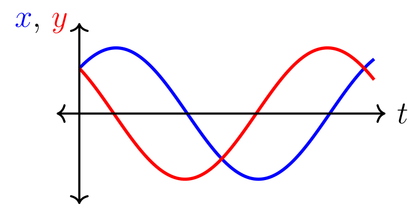
(i) First we take $b = 0$. We present the direction field and then plot parametric and componentwise solutions for the IVP with $x(0) = 1$ and $\dot{x}(0) = 1$ (i.e., $y(0) = 1$).



$m = 1,\ b = 0,\ \kappa = 1$



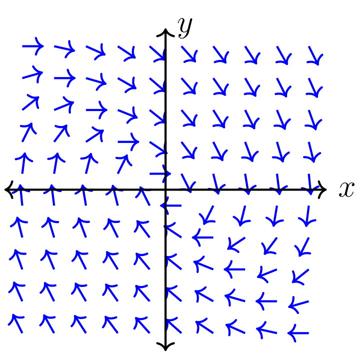
$m = 1,\ b = 0,\ \kappa = 1$



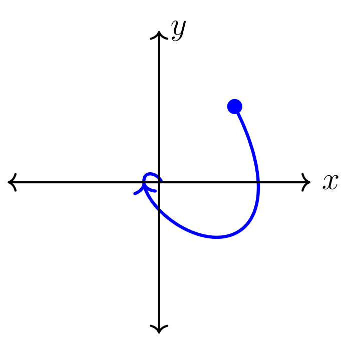
$m = 1,\ b = 0,\ \kappa = 1$

The vectors in the direction field seem to spiral around the origin. This is reminiscent of the predator-prey direction field in Example 3.3.5, and that should call to mind *periodic* (or *oscillatory*?) behavior. It looks like the x and y values just repeat in some periodic fashion. Indeed, this is what the parametric and componentwise plots for the specific IVP shown. Note that whenever $x(t) = 0$, the oscillator is at its equilibrium position. So, the undamped oscillator would move back and forth around its equilibrium position eternally without coming arbitrarily closer to it forever (displacement x is periodic), and its velocity would never This is probably what we expect to happen in the (fictional!) case that there is no damping force to slow the oscillator down.

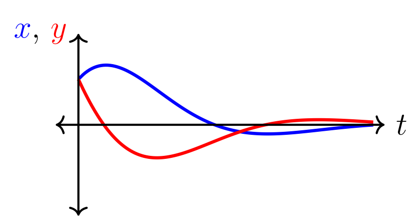
(ii) Here are results for the damped case with $b = 1$.



$m = 1,\ b = 1,\ \kappa = 1$



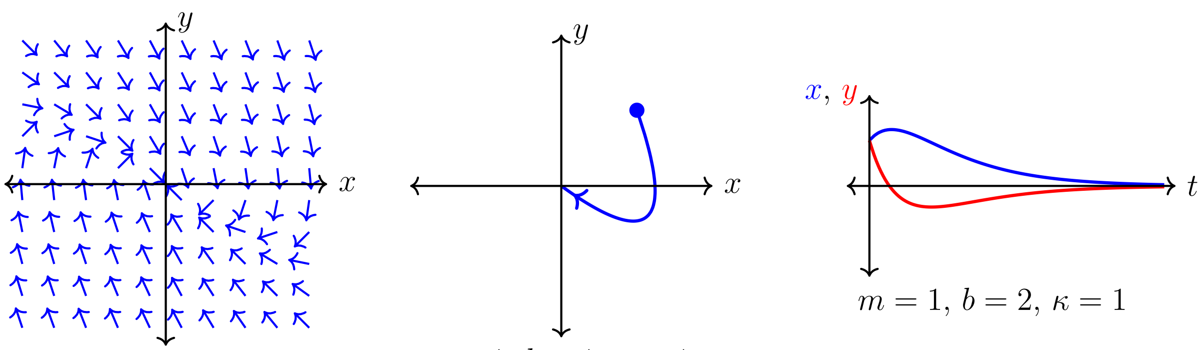
$m = 1,\ b = 1,\ \kappa = 1$



$m = 1,\ b = 1,\ \kappa = 1$

The vectors in the direction field seem instead to spiral into the origin. (We might want to zoom in a bit closer to the origin to be sure that the vectors are not just spiraling tightly around the origin in a variation on the first direction field.) This suggests that x and y both go to 0 over long times, and so the oscillator both slows down (velocity $\dot{x} = y$ goes to 0) and approaches its equilibrium position closely (x goes to 0). This is what we expect damping to do: over long times motion slows down and effectively stops. More precisely, from the parametric and componentwise plots for the solution to the IVP, both displacement and velocity approach 0, but displacement x is 0 at least once, possibly more than once (if we continued the plots for more time). This means that as the oscillator slows down, it passes through its equilibrium position at least once. In particular, since $x(0) = 1$, the oscillator starts out to the right of equilibrium and then (since the graph of x becomes negative) passes to the left of equilibrium.

(iii) Finally, we double the damping to $b = 2$.



$m = 1,\ b = 2,\ \kappa = 1$ $m = 1,\ b = 1,\ \kappa = 1$ $m = 1,\ b = 2,\ \kappa = 1$

The situation in this direction field is quite similar to the one for $b = 1$ in that the vectors seem to spiral or twist into the origin. However, the vectors are now slightly steeper than in the second field, which suggests that x and y are approaching 0 more quickly. That is, the oscillator slows down more quickly and approaches its equilibrium position more quickly; after all, the damping is more powerful. Moreover, the x-coordinate in the parametric plot and in the componentwise plot is never 0; indeed, x appears to stay strictly positive ($y = \dot{x}$ definitely does not), and so the oscillator does not appear to ever *reach* its equilibrium position. Instead, it stays to the right forever.

This direction field has one other curious feature: it looks as though some vectors fall precisely on the line $y = -x$. This suggests that a solution pair (x, y) to this system satisfies $y(x) = -x(t)$ for all t. Such a solution is really a "one-dimensional object" since it only involves the one function x. We will exploit such "straight-line solutions" tremendously in the near future.

As with the predator-prey and SIR models, numerical and qualitative results for the harmonic oscillator help us make conjectures about this model's behavior. However, we ultimately need calculus to verify those conjectures, and maybe a few other tools.

It turns out that we can find exact analytic formulas for *all* solutions to the harmonic oscillator model, to a degree impossible for the predator-prey and SIR models. The most fruitful way to think about this system for the harmonic oscillator involves the language of linear algebra: matrices and vectors. To that we now turn.

4. LINEAR SYSTEMS

4.1. Vectors, matrices, and linear systems.

Our goal is to develop an efficient notation for encoding problems like the system (3.5.6) for the harmonic oscillator.

4.1.1. Vectors.

4.1.1 Definition. *A* **VECTOR** *is an ordered pair of two real numbers. If x and y are real numbers, then we may write either $\mathbf{x} := (x, y)$ or*

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Both ways of writing $\mathbf{x}$ mean the same, and at times one may be easier to read than the other.

The word "vector," of course, generalizes vastly beyond ordered pairs—any n-tuple of numbers is a vector and, through the right lens, functions themselves are vectors. In this course vectors will be strictly ordered pairs (since we will only discuss systems with two components).

We perform arithmetic on vectors componentwise.

4.1.2 Definition. **(i)** *Let $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$. Then*

$$\mathbf{x}_1 + \mathbf{x}_2 = (x_1, y_1) + (x_2, y_2) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = (x_1 + x_2, y_1 + y_2).$$

(ii) *If $\mathbf{x} = (x, y)$ and if c is any real number, then*

$$c\mathbf{x} = c(x, y) = c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}.$$

4.1.3 Example. We have

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad \text{and} \quad 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 \\ 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

4.1.4 Problem (!). The **ZERO VECTOR** is the vector

$$\mathbf{0} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Check that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for any vector $\mathbf{x}$.

4.1.5 Remark. *We do not multiply two vectors (at least, not in this course). When multiplying a vector by a real number, we always write the real number first: $c\mathbf{x}$, never $\mathbf{x}c$.*

We do perform calculus on vectors componentwise.

4.1.6 Definition. **(i)** *Let x and y be differentiable functions on the interval $[a,b]$ and let $\mathbf{x}(t) := (x(t), y(t))$. We define*

$$\dot{\mathbf{x}}(t) := \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

(ii) *Let x and y be continuous functions on the interval I and let $\mathbf{x}(t) := (x(t), y(t))$. We define*

$$\int_a^b \mathbf{x}(t)\ dt := \begin{pmatrix} \int_a^b x(t)\ dt \\ \int_a^b y(t)\ dt \end{pmatrix}.$$

4.1.7 Example. If

$$\mathbf{x}(t) = \begin{pmatrix} t^2 \\ \cos(\pi t) \end{pmatrix},$$

then

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 2t \\ -\pi \sin(\pi t) \end{pmatrix}$$

and

$$\int_0^1 \mathbf{x}(t)\ dt = \begin{pmatrix} \int_0^1 t^2\ dt \\ \int_0^1 \cos(\pi t)\ dt \end{pmatrix} = \begin{pmatrix} (t^3/3)\big|_{t=0}^{t=1} \\ (\sin(\pi t)/\pi)\big|_{t=0}^{t=1} \end{pmatrix} = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}.$$

4.1.2. Matrices.

A matrix (at least in this course—like vectors, there are matrices of many sizes beyond the following) is a square array of four numbers; this is a terrible definition, so terrible that we do not even package it as an "undefinition." Here is an example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We do not attempt to define "square array," but we do note that if we put

$$\mathbf{a}_1 := \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 := \begin{pmatrix} 2 \\ 4 \end{pmatrix},$$

then we should think of $\mathbf{a}_1$ and $\mathbf{a}_2$ as the "columns" of A.

Perhaps, then, we could consider A as an ordered pair of vectors. Like ordered pairs, what matters is that everything in a matrix is determined entrywise or componentwise; whatever the symbol

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.1.1}$$

means, we should have

$$A = B, \qquad \text{where} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

if and only if the corresponding entries or components are equal:

$$A = B \iff a_{11} = b_{11},\ a_{12} = b_{12},\ a_{21} = b_{21},\ \text{and } a_{22} = b_{22}.$$

Also, with A from (4.1.1), we will sometimes put

$$\mathbf{a}_1 := \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \qquad \text{and} \qquad \mathbf{a}_2 := \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

and abbreviate

$$A := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}.$$

This is where we finished on Monday, October 16, 2023.

We are so very close to encoding the system version (3.5.6) of the harmonic oscillator in matrix-vector notation. We just need one more bit of arithmetic—possibly the strangest, at first glance, in all of mathematics.

4.1.8 Definition. *Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be vectors,*

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}, \qquad \textit{and} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the MATRIX-VECTOR PRODUCT *$A\mathbf{x}$ is the vector*

$$A\mathbf{x} := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{a}_1 + y\mathbf{a}_2.$$

4.1.9 Example. We compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 6\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} + \begin{pmatrix} 12 \\ 24 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}.$$

4.1.10 Remark. *When multiplying a matrix and a vector, we always write the matrix first: $A\mathbf{x}$, never $\mathbf{x}A$.*

4.1.11 Example. Let

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \qquad c = 6, \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}.$$

Then

$$A\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 4\begin{pmatrix} 0 \\ -2 \end{pmatrix} + 5\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \end{pmatrix} + \begin{pmatrix} 5 \\ -15 \end{pmatrix} = \begin{pmatrix} 5 \\ -23 \end{pmatrix}$$

and

$$c\mathbf{y} = 6\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 42 \\ 48 \end{pmatrix},$$

so

$$A\mathbf{x} + c\mathbf{y} = \begin{pmatrix} 5 \\ -23 \end{pmatrix} + \begin{pmatrix} 42 \\ 48 \end{pmatrix} = \begin{pmatrix} 47 \\ 23 \end{pmatrix}.$$

4.1.12 Problem (!). Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Check that

$$A\mathbf{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

and conclude that if y_1 and y_2 are numbers, then the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

means the same as the **MATRIX-VECTOR EQUATION**

$$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{b} := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

4.1.13 Problem ($\star$). Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be vectors, and define $\mathbf{e}_1 := (1, 0)$ and $\mathbf{e}_2 := (0, 1)$. Show that if $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$, then

$$A\mathbf{e}_1 = \mathbf{a}_1 \quad \text{and} \quad A\mathbf{e}_2 = \mathbf{a}_2.$$

In other words, multiplying A against $\mathbf{e}_1$ or $\mathbf{e}_2$ "selects" the columns of A.

Matrix-vector multiplication interacts with vector arithmetic in a crucially important manner.

4.1.14 Theorem (Linearity of matrix-vector multiplication). *Let A be a matrix.*

(i) $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$ *for any vectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$.

(ii) $A(c\mathbf{x}) = cA\mathbf{x}$ *for any real number* c *and vector* $\mathbf{x}$.

4.1.15 Problem (+). Prove this theorem. [Hint: *give names to the components of A and the vectors and grind it all out from the definitions.*]

Here is a particularly useful matrix.

4.1.16 Definition. *The* **IDENTITY MATRIX** *is*

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Context will always make clear whether we are using I for the identity matrix or for an interval.

4.1.17 Problem (!). Here is why the identity matrix is the identity matrix: $I\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$. Check this.

4.1.18 Problem ($\star$). We can do calculus on matrices as we do calculus on vectors: componentwise (or entrywise). Or *columnwise*—say that $\mathbf{a}_1$ and $\mathbf{a}_2$ are functions and

$$A(t) := \begin{bmatrix} \mathbf{a}_1(t) & \mathbf{a}_2(t) \end{bmatrix}.$$

Then we define

$$\dot{A}(t) := \begin{bmatrix} \dot{\mathbf{a}}_1(t) & \dot{\mathbf{a}}_2(t) \end{bmatrix},$$

with $\dot{\mathbf{a}}_1$ and $\dot{\mathbf{a}}_2$ defined as in Definition 4.1.6.

(i) If $\mathbf{a}_1(t) = (t, \cos(t))$ and $\mathbf{a}_2(t) = (e^t, t^2)$, and if $A(t) = \begin{bmatrix} \mathbf{a}_1(t) & \mathbf{a}_2(t) \end{bmatrix}$, what is $\dot{A}$?

(ii) Prove the following **PRODUCT RULE FOR MATRIX-VECTOR MULTIPLICATION**: if A is a matrix with differentiable entries and if $\mathbf{x}$ is a vector with differentiable entries, then the function $\mathbf{f}(t) := A(t)\mathbf{x}(t)$ satisfies

$$\dot{\mathbf{f}}(t) = \dot{A}(t)\mathbf{x}(t) + A(t)\dot{\mathbf{x}}(t).$$

[Hint: *This is a thankless, componentwise, brute-force calculation. Brute force is the best force!*]

4.1.3. Linear systems.

Here is the virtue of this (possibly bizarre) definition of matrix-vector multiplication. Recall that the harmonic oscillator IVP

$$\begin{cases} m\ddot{x} + b\dot{x} + \kappa x = f(t) \\ x(0) = x_0 \\ \dot{x}(0) = y_0 \end{cases}$$

is, on putting $y := \dot{x}$, equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -by/m - \kappa x/m + f(t)/m, \end{cases} \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0. \end{cases} \tag{4.1.2}$$

Now we vectorize. Put

$$\mathbf{x} := \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 := \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Note that $\mathbf{x}$ is a function, but $\mathbf{x}_0$ is a constant vector. Then we compute

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \\
&= \begin{pmatrix} y \\ -by/m - \kappa x/m + f(t)/m \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -\kappa x/m \end{pmatrix} + \begin{pmatrix} y \\ -by/m \end{pmatrix} + \begin{pmatrix} 0 \\ f(t)/m \end{pmatrix} \\
&= x\begin{pmatrix} 0 \\ -\kappa/m \end{pmatrix} + y\begin{pmatrix} 1 \\ -b/m \end{pmatrix} + \begin{pmatrix} 0 \\ f(t)/m \end{pmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -\kappa/m & -b/m \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t)/m \end{pmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -\kappa/m & -b/m \end{bmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ f(t)/m \end{pmatrix}.
\end{aligned}$$

Thus with

$$A := \begin{bmatrix} 0 & 1 \\ -\kappa/m & -b/m \end{bmatrix} \quad \text{and} \quad \mathbf{b}(t) := \begin{pmatrix} 0 \\ f(t)/m \end{pmatrix},$$

the system (4.1.2) is equivalent to the problem

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases}$$

This problem will be our central object of study for some time to come, and it should optimistically remind us of constant-coefficient linear ODE (Section 2.8).

4.1.19 Definition. *Suppose that A is a matrix and b_1 and b_2 are functions. The* LINEAR SYSTEM *with* COEFFICIENT MATRIX *A and* FORCING *or* DRIVING *term $\mathbf{b} = (b_1, b_2)$ is* *the problem*

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t). \tag{4.1.3}$$

Its **SOLUTION** *is a vector* $\mathbf{x} = (x, y)$ *of differentiable functions defined on some interval* I *such that*

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}(t)$$

for all t *in* I*. The system* (4.1.3) *is* **HOMOGENEOUS** *if* $\mathbf{b}(t) = \mathbf{0}$ *for all* t *and otherwise* **NONHOMOGENEOUS***.*

Our optimism is well-founded: like the scalar linear ODE (recall Theorem 2.7.23), solutions to linear systems exist and are unique.

4.1.20 Theorem. *Let* b_1 *and* b_2 *be continuous on an interval* I *containing the point* t_0*, let* A *be a matrix, and let* $\mathbf{x}_0$ *be a vector. Then the IVP*

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

has a unique solution defined on all of I*.*

We will eventually prove this theorem, but that will take some time, and quite some machinery. Developing that machinery, and understanding that eventual proof, will be among our most important accomplishments in this course.

4.1.21 Problem (!). Find functions f and g of the variables t, x, and y such that the linear system (4.1.3) has the form

$$\begin{cases} \dot{x} = f(t, x, y) \\ \dot{y} = g(t, x, y). \end{cases}$$

4.1.22 Problem (!). Let A be a matrix. Use the linearity of matrix-vector multiplication to prove that the **LINEARITY PRINCIPLE** that if $\mathbf{x}_1$ and $\mathbf{x}_2$ solve $\dot{\mathbf{x}} = A\mathbf{x}$ and if c_1 and c_2 are constants, then $\mathbf{x}(t) := c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ also solves $\dot{\mathbf{x}} = A\mathbf{x}$.

4.1.23 Example. It is often convenient to assume that a harmonic oscillator has mass 1, damping coefficient $q \geq 0$, and spring coefficient $p > 0$. Suppose that the oscillator is free. Then the equation of motion (3.5.4) for this oscillator then reads

$$\ddot{x} + p\dot{x} + qx = 0,$$

and consequently the linear system for this oscillator is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \mathbf{x}.$$

We will spend a great deal of time understanding linear systems with this especially nice matrix.

4.1.24 Problem (!). Suppose that every entry of A is 0. Find all solutions to the linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$.

4.2. Equilibrium solutions to autonomous linear systems.

The linear system $\dot{\mathbf{x}} = A\mathbf{x}+\mathbf{b}(t)$ is particularly simple when the function $\mathbf{b}$ is constant (and it is especially simple when $\mathbf{b} = \mathbf{0}$). Here we can find equilibrium (constant) solutions to $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ using only some simple ideas from linear algebra and a tool that will reappear in many places to come.

4.2.1. Equilibrium solutions for homogeneous systems.

First we work with the homogeneous problem $\dot{\mathbf{x}} = A\mathbf{x}$. An equilibrium solution is a constant solution, so if $\mathbf{x}$ is an equilibrium solution, then $\dot{\mathbf{x}} = \mathbf{0}$, and so $A\mathbf{x} = \mathbf{0}$. This is just a compressed system of two linear equations. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then (by Problem 4.1.12), with $\mathbf{x} = (x, y)$, $A\mathbf{x} = \mathbf{0}$ if and only if

$$\begin{cases} ax + by = 0 \\ cx + dy = 0. \end{cases}$$

There are many, many ways to solve a problem like this—which is why linear algebra exists. However, we do not need fancy tools to find at least one solution.

4.2.1 Problem (!). Check that the TRIVIAL SOLUTION $\mathbf{x}(t) := \mathbf{0}$ always solves the linear homogeneous problem $\dot{\mathbf{x}} = A\mathbf{x}$.

4.2.2 Example. **(i)** Put

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

We find equilibrium solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ by solving

$$\begin{cases} x + 2y = 0 \\ x + y = 0. \end{cases}$$

There are many ways to do this; one is to use the second equation to rewrite $x = -y$ and plug that into the first to get $-y + 2y = 0$, thus $y = 0$. And then $x = 0$. So, the only equilibrium solution here is the trivial solution $\mathbf{x}(t) = \mathbf{0}$.

(ii) Put

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

We find equilibrium solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ by solving

$$\begin{cases} x + 2y = 0 \\ 2x + 4y = 0. \end{cases}$$

If we factor 2 out of the second equation, it becomes $x + 2y = 0$, which is the same as the first. Thus this problem is really redundant: there is one equation present, not two. And if $x + 2y = 0$, then $x = -2y$.

This says that if we select y, then we know what x has to be to get a solution. In particular, all equilibrium solutions $\mathbf{x} = (x, y)$ must have form

$$\mathbf{x} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

So, each choice of y gives a new equilibrium solution, and there are infinitely many of them.

This is where we finished on Wednesday, October 18, 2023.

There is a particularly easy way to determine if the trivial solution $\mathbf{x}(t) = \mathbf{0}$ is the only equilibrium solution to a homogeneous linear system of ODE, or if the system has infinitely many nontrivial equilibrium solutions.

4.2.3 Definition. *The* DETERMINANT *of the matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the number

$$\det(A) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := ad - bc.$$

4.2.4 Example. The determinants of the matrices in Example 4.2.2 are

$$\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right) = (1 \cdot 1) - (2 \cdot 1) = 1 - 2 = -1$$

and

$$\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = (1 \cdot 4) - (2 \cdot 2) = 4 - 4 = 0.$$

Here is the utility of the determinant. First we state a result about linear systems, and then we paraphrase it for differential equations.

4.2.5 Theorem. *Let A be a matrix.*

(i) *If $\det(A) \neq 0$, then the only solution to $A\mathbf{x} = \mathbf{0}$ is the "trivial" solution $\mathbf{x} = \mathbf{0}$.*

(ii) *If the only solution to $A\mathbf{x} = \mathbf{0}$ is the "trivial" solution $\mathbf{x} = \mathbf{0}$, then $\det(A) \neq 0$.*

(iii) *If $\det(A) = 0$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many "nontrivial" solutions. In particular, if A has at least one nonzero entry, there is a vector $\mathbf{x}_\infty$ such that every solution to $\mathcal{A}\mathbf{x} = \mathbf{0}$ has the form $\mathbf{x} = c\mathbf{x}_\infty$ for some constant c. Otherwise, if all entries of A are 0, then any vector $\mathbf{x}$ solves $A\mathbf{x} = \mathbf{0}$.*

(iv) *If $A\mathbf{x} = \mathbf{0}$ has a "nontrivial" solution $\mathbf{x} \neq \mathbf{0}$, then $\det(A) = 0$.*

The proof of this theorem would not teach us anything new about differential equations, so we do not give it here. Instead, here is the useful paraphrase.

4.2.6 Corollary. *Let A be a matrix.*

(i) *If $\det(A) \neq 0$, then the only equilibrium solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is the "trivial" solution $\mathbf{x}(t) = \mathbf{0}$.*

(ii) *If $\det(A) = 0$, then $\dot{\mathbf{x}} = A\mathbf{x}$ has infinitely many "nontrivial" equilibrium solutions. In particular, if A has at least one nonzero entry, there is a vector $\mathbf{x}_\infty$ such that every equilibrium solution to $\dot{\mathbf{x}} = A\mathbf{x}$ has the form $\mathbf{x}(t) = c\mathbf{x}_\infty$ for some constant c. Otherwise, if all entries of A are 0, then all vectors are equilibrium solutions.*

4.2.7 Problem (!). Use Theorem 4.2.5 to prove Corollary 4.2.6.

The two possibilities $\det(A) = 0$ and $\det(A) \neq 0$, and the two separate conclusions, are exactly the results of Example 4.2.2.

4.2.8 Example. Consider a free harmonic oscillator with mass 1, damping coefficient $p \geq 0$, and spring constant $q > 0$. By Example 4.1.23, its equation of motion is $\ddot{x} + p\dot{x} + qx = 0$, and its linear system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \mathbf{x},$$

where $\mathbf{x} = (x, \dot{x})$. We compute

$$\det\left(\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\right) = (0 \cdot (-p)) - (1 \cdot (-q)) = q > 0,$$

so this system has no nontrivial equilibrium solution.

In turn, this means that the problem $\ddot{x} + p\dot{x} + qx = 0$ has no constant solution other than $x(t) = 0$. We did not need matrix methods to see this: if x solves this ODE and is constant, then $\dot{x} = 0$ and $\ddot{x} = 0$, so we would have $qx = 0$ and therefore $x = 0$. Physically, this means that the only way for a free harmonic oscillator to have constant displacement from equilibrium is for it to have *zero* displacement from equilibrium and thus to stay at equilibrium for all time.

4.2.2. Equilibrium solutions for nonhomogeneous systems.

To find equilibrium solutions for the nonhomogeneous problem $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$, we need to solve $A\mathbf{x} + \mathbf{b} = \mathbf{0}$, equivalently,

$$A\mathbf{x} = -\mathbf{b}.$$

This is a nonhomogeneous system of linear equations. Mechanically, trying to solve this is quite similar to the previous homogeneous systems.

4.2.9 Example. **(i)** We look for equilibrium solutions to

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This demands that we solve

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{0},$$

equivalently,

$$\begin{cases} x + 2y + 1 = 0 \\ x + y = 0. \end{cases}$$

The second equation tells us $x = -y$, so the first becomes

$$-y + 2y + 1 = 0,$$

thus $y + 1 = 0$, and so $y = -1$. Then $x = 1$, and the only equilibrium solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(ii) We look for equilibrium solutions to

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

We need

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \mathbf{0},$$

and thus

$$\begin{cases} x + 2y + 3 = 0 \\ 2x + 4y + 6 = 0. \end{cases}$$

The second equation here is twice the first, so we just need to solve $x + 2y + 3 = 0$. We can do this by taking $x = -2y - 3$, and so all equilibrium solutions $\mathbf{x} = (x, y)$ are

$$\mathbf{x} = \begin{pmatrix} -2y-3 \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

There are infinitely many equilibrium solutions, which, not incidentally, was exactly the situation with the same matrix in part (ii) of Example 4.2.2.

(iii) We look for equilibrium solutions to

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

As before, this leads to a linear system:

$$\begin{cases} x + 2y + 3 = 0 \\ 2x + 4y + 5 = 0. \end{cases}$$

Now the second equation is not a multiple of the first, and the problem looks harder to solve. We could try to use one of the equations to write x in terms of y, or y in terms of x, but a faster way to fail is to subtract 2 times the first equation from the second:

$$2x + 4y + 5 - 2(x + 2y + 3) = 0,$$

and thus we need

$$5 - 6 = 0,$$

which is impossible. Thus the problem has no solutions.

The situations in the preceding example are characteristic of what happens when we look for equilibrium solutions to nonhomogeneous linear systems: either there is exactly one, or infinitely many, or none. We discuss without proof when the first situation happens in general.

4.2.10 Theorem. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose that $\det(A) \neq 0$ *and let* $\mathbf{y}$ *be a vector. Then the only solution* $\mathbf{x}$ *to* $A\mathbf{x} = \mathbf{y}$ *is* $\mathbf{x} = A^{-1}\mathbf{y}$, *where*

$$A^{-1} := \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a. \end{bmatrix}$$

The matrix A^{-1} *is the* **INVERSE** *of* A*. In particular,* A^{-1} *satisfies*

$$A(A^{-1}\mathbf{y}) = \mathbf{y} \quad \text{and} \quad A^{-1}(A\mathbf{y}) = \mathbf{y}$$

for any vector $\mathbf{y}$.

4.2.11 Remark. *A mnemonic for remembering the structure of* A^{-1} *is "interchange the diagonals, negate the off-diagonals, and divide by the determinant."*

The following is an immediate corollary of the preceding theorem.

4.2.12 Corollary. *If* $\det(A) \neq 0$*, then the only equilibrium solution to* $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ *is* $\mathbf{x}(t) = -A^{-1}\mathbf{b}$.

Proof. In Theorem 4.2.10, take $\mathbf{y} = -\mathbf{b}$. ■

4.2.13 Example. We revisit part (i) of Example 4.2.9 by first computing

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(1 \cdot 1) - (2 \cdot 1)} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

and then

$$-\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} (-1 \cdot 1) + (2 \cdot 0) \\ (1 \cdot 1) + (-1 \cdot 0) \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Corollary 4.2.12 says that this is the only equilibrium solution to

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which is exactly what we found in part (i) of Example 4.2.9.

4.2.14 Problem (+). If $\det(A) = 0$ and $\mathbf{b} \neq \mathbf{0}$, then the problem $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ may or may not have equilibrium solutions. In particular, if $\det(A) = 0$ and one equilibrium solution is known to exist, there are, as we said above, infinitely many. Here is why. Suppose that $\mathbf{x}_\star$ is a vector with $A\mathbf{x}_\star = -\mathbf{b}$ and $\mathbf{x}_\infty$ is a vector with $A\mathbf{x}_\infty = \mathbf{0}$. Show that $\mathbf{x}(t) := c\mathbf{x}_\infty + \mathbf{x}_\star$ is an equilibrium solution to $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ for any constant c.

This is where we finished on Friday, October 20, 2023.

4.2.3. Linear independence.

We have used the determinant to test if we can solve a linear system. We can also cast things in the language of vectors, not matrices and linear systems, and that will be useful in the near future.

From time to time we will be given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{y}$, and we will want to find numbers c_1 and c_2 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{y}. \tag{4.2.1}$$

It will be advantageous if we can find these numbers *uniquely*, too. That is, we want there to be only one choice of the constant c_1 and only one choice of the constant c_2 that makes the vector equation (4.2.1) true.

By definition of matrix-vector multiplication, solving (4.2.1) is equivalent to solving

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \mathbf{y}.$$

And, by Theorem 4.2.10, we can do that uniquely when the determinant of the matrix is nonzero. Such a matrix is invertible, and its columns also have a special name.

4.2.15 Definition. *The vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are* **LINEARLY INDEPENDENT** *if*

$$\det\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\right) \neq 0.$$

The following theorem is an immediate consequence of this definition, the reasoning preceding this definition, and Theorems 4.2.5 and 4.2.10.

4.2.16 Theorem. *Let* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *be vectors.*

(i) *Suppose that* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are linearly independent. For any vector* $\mathbf{y}$, *there are unique numbers* c_1 *and* c_2 *such that*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{y}. \tag{4.2.2}$$

In particular, the only c_1 *and* c_2 *that solve* $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ *are* $c_1 = 0$ *and* $c_2 = 0$.

(ii) *Suppose that for any vector* $\mathbf{y}$, *there is a unique choice of* c_1 *and* c_2 *that solves* (4.2.2). *Then* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are linearly independent.*

(iii) *Suppose that the only* c_1 *and* c_2 *that solve*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

are $c_1 = 0$ *and* $c_2 = 0$. *Then* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are linearly independent.*

4.2.17 Problem (+). Prove this theorem using Theorems 4.2.5 and 4.2.10.

4.2.18 Example. The vectors

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

are linearly independent, because

$$\det\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\right) = \det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = (1\cdot 1) - (0\cdot 0) = 1,$$

but the vectors

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

are linearly dependent, because

$$\det\left(\begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} \right) = \det\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right) = (1 \cdot 0) - (0 \cdot 2) = 0.$$

4.2.19 Problem (!). Suppose that λ_1 and λ_2 are numbers with $\lambda_1 \neq \lambda_2$. Show that the vectors

$$\mathbf{v}_1 := \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 := \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

are linearly independent.

4.3. The lessons of totally and partially decoupled systems.

We now know how to find equilibrium solutions to autonomous linear systems; it boiled down to a problem of linear algebra. What about nonautonomous systems and nonequilibrium solutions? How can we find *all* solutions, and what do they do?

We start by studying two special kinds of homogeneous linear systems $\dot{\mathbf{x}} = A\mathbf{x}$ that will teach us a great deal about all linear systems. (Unfortunately, neither of these systems can represent a harmonic oscillator, and so we still have work to do.)

4.3.1 Example. It is hardly fair to call the linear system

$$\begin{cases} \dot{x} = 2x \\ \dot{y} = -y \end{cases} \tag{4.3.1}$$

a "system," since the equation with the x derivative does not involve y, and the equation with the y derivative does not involve x. Such a system is **TOTALLY DECOUPLED**. We solve it and discuss some enlightening aspects of the solution.

1. The problem (4.3.1) consists of just two exponential growth equations placed together, and we know how to solve them:

$$x(t) = c_1 e^{2t} \quad \text{and} \quad y(t) = c_2 e^{-t},$$

where c_1 and c_2 are arbitrary constants. Note that we do not say

$$x(t) = ce^{2t} \quad \text{and} \quad y(t) = ce^{-t}$$

with the same constant c appearing in both functions; there is no reason that the constant for x should be the same as the constant for y. Indeed, if $c_1 = c_2 = c$, then $x(0) = y(0) = c$, and nothing in the original problem (4.3.1) specifies that constraint.

2. This form of the solution makes it easy to solve an IVP like

$$\begin{cases} \dot{x} = 2x \\ \dot{y} = -y, \end{cases} \qquad \begin{cases} x(0) = 1 \\ y(0) = 2. \end{cases}$$

We want to choose c_1 and c_2 so that

$$1 = x(0) = c_1 e^{2 \cdot 0} = c_1 \quad \text{and} \quad 2 = y(0) = c_2 e^{-0} = c_2.$$

The solution to the IVP is therefore

$$x(t) = e^{2t} \quad \text{and} \quad y(t) = 2e^{-t}.$$

More generally, we could use our knowledge of scalar IVP to argue that the problem

$$\begin{cases} \dot{x} = 2x \\ \dot{y} = -y, \end{cases} \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

has a *unique* solution for *any* choice of initial conditions x_0 and y_0. In particular, we do not need Theorem 4.1.20 to come to this conclusion.

3. None of the work above needed the notation of vectors and matrices, but this language does streamline some things and make some other things more obvious. Put

$$A := \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

The work above shows that every solution $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ has the form

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-t} \end{pmatrix}$$

for some constants c_1 and c_2.

There is another way to write this solution $\mathbf{x}$ that, while not necessarily obvious, exposes the arbitrary constants c_1 and c_2 in a useful way. Namely, we have

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-t} \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{-t} \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Put

$$\mathbf{x}_1(t) := e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) := e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then every solution to $\dot{\mathbf{x}} = A\mathbf{x}$ has the form

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) \tag{4.3.2}$$

for some constants c_1 and c_2. Take $c_2 = 0$ in (4.3.2) to see that $\mathbf{x}_1$ is a solution to $\dot{\mathbf{x}} = A\mathbf{x}$, and take $c_1 = 0$ in (4.3.2) to see that $\mathbf{x}_2$ is also a solution to $\dot{\mathbf{x}} = A\mathbf{x}$.

4. Now suppose that we want to solve the more general IVP

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \qquad \mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{4.3.3}$$

Since all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ have the form $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ for some c_1 and c_2, we just have to choose c_1 and c_2 so that

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0. \tag{4.3.4}$$

That is, we need to solve the matrix-vector equation

$$\begin{bmatrix} \mathbf{x}_1(0) & \mathbf{x}_2(0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{x}_0$$

for c_1 and c_2. This is always possible, since

$$\det\left(\begin{bmatrix} \mathbf{x}_1(0) & \mathbf{x}_2(0) \end{bmatrix} \right) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 1 \neq 0.$$

That is, $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent, so Theorem 4.2.16 applies, and we can always find unique c_1 and c_2 that make (4.3.4) true.

In this case, the form of $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ makes computing c_1 and c_2 in terms of x_0 and y_0 a snap, but the deeper property in play is the fact that $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.

4.3.2 Problem (!). Let a and d be real numbers. Find all solutions to the totally decoupled system

$$\begin{cases} \dot{x} = ax \\ \dot{y} = dy. \end{cases}$$

Then solve the IVP with initial conditions

$$x(0) = x_0 \quad \text{and} \quad y(0) = y_0.$$

If we write this problem in the form $\dot{\mathbf{x}} = A\mathbf{x}$, what is A?

Example 4.3.1 has two major lessons for us. Here is the first: we want to look for solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ that do something special.

4.3.3 Definition. *Let A be a matrix. A* **FUNDAMENTAL SOLUTION SET** *for the system $\dot{\mathbf{x}} = A\mathbf{x}$ is a pair of functions $\mathbf{x}_1$ and $\mathbf{x}_2$ defined on $(-\infty, \infty)$ with the following properties.*

(i) *Both $\mathbf{x}_1$ and $\mathbf{x}_2$ solve $\dot{\mathbf{x}} = A\mathbf{x}$.*

(ii) *The vectors $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.*

If we have a fundamental solution set for a homogeneous linear system (*constructing* that set is another matter, which we will take up presently), then, quite simply, we win.

4.3.4 Theorem. *Let A be a matrix and suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ solve $\dot{\mathbf{x}} = A\mathbf{x}$. Suppose also that $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent. Then for any vector $\mathbf{x}_0$, the unique solution of the IVP*

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{4.3.5}$$

has the form

$$\mathbf{x}(t) := c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \tag{4.3.6}$$

for some constants c_1 and c_2. In particular, there is only one way to choose c_1 and c_2 so that $\mathbf{x}$ as defined in (4.3.6) solves (4.3.5).

4.3.5 Problem (!). Prove this. [Hint: *Theorem 4.1.20 says that this IVP has a unique solution. Problem 4.1.22 says that $\mathbf{x}$ as defined in (4.3.6) solves (4.3.5). And linear independence says that there are unique c_1 and c_2 such that $c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0$.*]

So, how do we find fundamental solution sets? Do they always exist? And how will understanding the homogeneous problem $\dot{\mathbf{x}} = A\mathbf{x}$ help us with our goal of understanding the nonhomogeneous problem $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$? We have some work to do.

4.3.6 Problem (!). Explain why the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ can never be part of a fundamental solution set for a linear system $\dot{\mathbf{x}} = A\mathbf{x}$.

4.3.7 Problem (+). Let $\boldsymbol{\phi}_1 = (\phi_{11}, \phi_{21})$ and $\boldsymbol{\phi}_2 = (\phi_{12}, \phi_{22})$ form a fundamental solution set for the linear system $\dot{\mathbf{x}} = A\mathbf{x}$. Prove that the vectors $\boldsymbol{\phi}_1(t)$ and $\boldsymbol{\phi}_2(t)$ are linearly independent for all t as follows.

(i) Put

$$W(t) := \det\left(\begin{bmatrix}\boldsymbol{\phi}_1(t) & \boldsymbol{\phi}_2(t)\end{bmatrix}\right).$$

Explain why we want $W(t) \neq 0$ for all t.

(ii) If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

show that

$$\dot{\phi}_{11} = a\phi_{11} + b\phi_{12}.$$

Find similar expressions for the derivatives of the other three components of $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$.

(iii) Use those expressions to calculate

$$\dot{W}(t) = (a+d)W(t).$$

(iv) Conclude that
$$W(t) = W(0)e^{(a+d)t},$$
and deduce from this that $W(t) \neq 0$ for all t.

We said that Example 4.3.1 taught us two lessons about solving $\dot{\mathbf{x}} = A\mathbf{x}$. One of them was the role of fundamental solution sets in solving IVP. The other lesson involves the *form* of the fundamental solution set. It will be clearer after another, slightly more complicated example.

4.3.8 Example. The system
$$\begin{cases} \dot{x} = 2x + y \\ \dot{y} = -y \end{cases}$$
is not totally decoupled, since the equation involving the x derivative also involves y; rather, it is **PARTIALLY DECOUPLED**. Nonetheless, it is still fairly easy to solve. First, the second equation just yields
$$y(t) = c_2e^{-t},$$
where c_2 is an arbitrary constant. Then the first equation becomes the first-order linear ODE
$$\dot{x} = 2x + c_2e^{-t}. \tag{4.3.7}$$
We can solve this with variation of parameters or, better, undetermined coefficients (see Problem 4.3.9) to find
$$x(t) = c_1e^{2t} - \frac{c_2e^{-t}}{3}, \tag{4.3.8}$$
where c_1 is another arbitrary constant.

So, if we put
$$A := \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix},$$
then all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are
$$\mathbf{x}(t) = \begin{pmatrix} c_1e^{2t} - c_2e^{-t}/3 \\ c_2e^{-t} \end{pmatrix} = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} -1/3 \\ 1 \end{pmatrix}.$$

Taking, successively, $c_1 = 0$ and $c_2 = 0$ shows that the functions
$$\mathbf{x}_1(t) := e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) := e^{-t}\begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$$
both solve $\dot{\mathbf{x}} = A\mathbf{x}$. We compute
$$\det\left(\begin{bmatrix} \mathbf{x}_1(0) & \mathbf{x}_2(0) \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & -1/3 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0$$
to conclude that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental solution set for $\dot{\mathbf{x}} = A\mathbf{x}$.

4.3.9 Problem (!). Let c_2 be an arbitrary real number. Use undetermined coefficients to obtain the solution (4.3.8) for the ODE (4.3.7).

4.3.10 Problem ($\star$). Let a, b, and d be real numbers with $a \neq d$.

(i) Find a fundamental solution set for the general partially decoupled problem

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = dy. \end{cases}$$

Where explicitly in your solution are you using the assumption $a \neq d$? (We will think about the case $a = d$ later.)

(ii) Explain why you basically have to do no new work to solve the partially decoupled problem

$$\begin{cases} \dot{x} = ax \\ \dot{y} = cx + dy, \end{cases}$$

where now c is any real number, and still $a \neq d$. [Hint: *does the order in which you write a pair of equations really matter?*]

This is where we finished on Monday, October 23, 2023.

Here now is the second lesson of Example 4.3.1, bolstered by the results of Example 4.3.8: functions in the fundamental solution sets so far have had the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ for a real number λ and a vector $\mathbf{v}$. The first lesson teaches us what solutions should do: be linearly independent at time $t = 0$. The second lesson teaches us what solutions should look like: exponentials. Hopefully the second lesson is unsurprising because of the role of exponentials in solving scalar linear ODE.

Here endeth the lessons. How do we exploit them to solve systems of the form $\dot{\mathbf{x}} = A\mathbf{x}$ that are not totally or partially decoupled? This is essential for understanding the harmonic oscillator as a system.

4.3.11 Problem (!). Explain why the system for the harmonic oscillator cannot be written as a totally or partially decoupled system.

4.4. Eigenvalues and eigenvectors.

The functions in the fundamental solution sets that we have so far seen have had the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}, \tag{4.4.1}$$

where λ is constant and $\mathbf{v}$ is a vector. Here we develop a procedure for constructing such solutions for general systems $\dot{\mathbf{x}} = A\mathbf{x}$ that are not necessarily totally or partially decoupled.

The right idea to find such solutions is, like so many things, not obvious at first glance: we *guess* that the system $\dot{\mathbf{x}} = A\mathbf{x}$ has a solution essentially of the form (4.4.1). Specifically, we guess that

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v} \tag{4.4.2}$$

solves $\dot{\mathbf{x}} = A\mathbf{x}$. Here λ is a real number and $\mathbf{v}$ is a vector, and figure out what λ and $\mathbf{v}$ have to be (or do). One thing that we can say from the start is that $\mathbf{v} \neq \mathbf{0}$. Otherwise, the result is all too easy: $\mathbf{x}$ then is just

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{0} = \mathbf{0}.$$

And the zero vector cannot be part of a fundamental solution set (go back and do Problem 4.3.6).

So, we assume that $\mathbf{x}$ is defined by (4.4.2) with $\mathbf{v} \neq \mathbf{0}$, and we compute

$$\dot{\mathbf{x}}(t) = \lambda e^{\lambda t}\mathbf{v}, \tag{4.4.3}$$

and so to have $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, we want

$$\lambda e^{\lambda t}\mathbf{v} = A(e^{\lambda t}\mathbf{v}). \tag{4.4.4}$$

We use the linearity of matrix-vector multiplication to rewrite $A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}\mathbf{v}$, so that (4.4.4) becomes

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}. \tag{4.4.5}$$

Since $e^{\lambda t} \neq 0$ regardless of λ or t, we divide to find that λ and $\mathbf{v}$ must satisfy

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{4.4.6}$$

Conversely, if λ and $\mathbf{v}$ satisfy (4.4.6), then we can just multiply both sides by $e^{\lambda t}$ for any number t to recover (4.4.5), turn that into (4.4.4), and recognize from that (4.4.3) with $\mathbf{x}$ defined by (4.4.2).

In short, if we want solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, then λ and $\mathbf{v}$ together just need to satisfy $A\mathbf{v} = \lambda\mathbf{v}$. This relationship among A, λ, and $\mathbf{v}$ is quite special.

4.4.1 Definition. *Let A be a matrix, λ be a real number, and $\mathbf{v}$ be a vector with $\mathbf{v} \neq \mathbf{0}$. Then λ is an* EIGENVALUE OF A, *and* $\mathbf{v}$ *is an* EIGENVECTOR OF A CORRESPONDING TO λ, *if*

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{4.4.7}$$

4.4.2 Problem (!). **(i)** Check that $\lambda_1 = 2$ and $\lambda_2 = -1$ are the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then look back at the results of Example 4.3.1.

(ii) Check that $\lambda_1 = 2$ and $\lambda_2 = -1$ are (also) the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}.$$

Then reread Example 4.3.8.

4.4.3 Problem ($\star$). Let A be a matrix and let λ be an eigenvalue of A. Explain why an eigenvector $\mathbf{v}$ corresponding to λ is an equilibrium solution for the system $\dot{\mathbf{x}} = (A - \lambda I)\mathbf{x}$.

Much of mathematics hinges on the fact that every matrix has eigenvalues and eigenvectors. The condition (4.4.7) is equivalent to

$$A\mathbf{v} - \lambda\mathbf{v} = 0,$$

and if I is the identity matrix (Definition 4.1.16), this reduces further to

$$(A - \lambda I)\mathbf{v} = 0.$$

Since we require $\mathbf{v} \neq \mathbf{0}$, Theorem 4.2.5 implies that this precisely when

$$\det(A - \lambda I) = 0.$$

Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

so

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix},$$

and therefore

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc).$$

Thus $\det(A - \lambda I) = 0$ if and only if

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0. \tag{4.4.8}$$

This is a quadratic equation in λ, and it always has solutions—in particular, every matrix has eigenvalues! Specifically, it may have two distinct real solutions, one "repeated" real solution, or one "complex conjugate pair" of solutions. We will examine each of these cases, and their ramifications for the dynamics of the system $\dot{\mathbf{x}} = A\mathbf{x}$, in detail.

The CHARACTERISTIC EQUATION (4.4.8) tells us how to find eigenvalues, but it does not tell us how to find eigenvectors. However, this is not too difficult. Once we know an eigenvalue λ, we just have to solve the matrix-vector equation $(A - \lambda I)\mathbf{v} = 0$ for $\mathbf{v}$.

4.4.4 Example. Let
$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}.$$
To find the eigenvalues of A, we need to find the numbers λ such that
$$\det(A - \lambda I) = 0.$$
We first compute
$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{bmatrix}.$$
Now we compute the determinant
$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{bmatrix}\right) = -\lambda(-4-\lambda) - (-3) = 4\lambda + \lambda^2 + 3.$$
So, we want to solve the quadratic equation
$$\lambda^2 + 4\lambda + 3 = 0,$$
which we can do either with the quadratic formula or by factoring
$$\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3).$$
Either way, the solutions are $\lambda = -1$ and $\lambda = -3$, and so these are the eigenvalues of A.

Now we need eigenvectors. To find an eigenvector corresponding to -1, we need to solve
$$A\mathbf{v} = -\mathbf{v},$$
equivalently, with $\mathbf{v} = (v_1, v_2)$,
$$\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$
This turns into the system of equations
$$\begin{cases} v_2 = -v_1 \\ -3v_1 - 4v_2 = -v_2. \end{cases}$$
The second equation is redundant, as it is equivalent to $-3v_1 = 3v_2$ and thus $v_2 = -v_1$, which is the first equation. So, all eigenvectors $\mathbf{v} = (v_1, v_2)$ corresponding to -1 are
$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
provided that $v_1 \neq 0$ (since $\mathbf{0}$ cannot be an eigenvector). In particular, for simplicity, we may want to take
$$\mathbf{v}_1 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as an eigenvector.

Exactly the same kind of work shows that all eigenvectors corresponding to -3 have the form

$$\mathbf{v} = v_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

for some nonzero constant v_1. For simplicity, we might use

$$\mathbf{v}_2 := \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

as the eigenvector corresponding to -3.

More precisely, we want to solve $A\mathbf{v} = -3\mathbf{v}$, which is

$$\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

and thus

$$\begin{cases} v_2 = -3v_1 \\ -3v_1 - 4v_2 = -3v_2. \end{cases}$$

Again, the second equation is redundant, as it is equivalent to $-3v_1 = v_2$, which is the first equation. So, we want $\mathbf{v} = (v_1, v_2) = (v_1, -3v_1)$, as claimed above.

4.4.5 Problem (!). The structure of the eigenvectors from the previous example was no accident. Suppose that λ is an eigenvalue of the matrix

$$\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix},$$

where c and d are any numbers. Show that the vector

$$\mathbf{v} := \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

is a corresponding eigenvector. [Hint: *you do not need to calculate the eigenvector from scratch; it is given here, and you just need to check that it satisfies the definition.*]

4.4.6 Problem (!). Consider a free harmonic oscillator with mass 1, damping coefficient $p \geq 0$, and spring coefficient $q > 0$.

(i) Referring to Example 4.1.23 as needed, show that the characteristic equation for this oscillator's linear system is $\lambda^2 + p\lambda + q = 0$. Conclude that we can just read off the characteristic equation from the oscillator's equation of motion without computing any determinants!

(ii) Show that 0 is never an eigenvalue for this linear system. [Hint: $q > 0$.]

4.4.7 Problem ($\star$). Here is an easy way to calculate the characteristic equation of a matrix A without computing $\det(A - \lambda I)$. Let

$$A =$$

The TRACE of A is

$$\operatorname{tr}(A) := a + d,$$

i.e., the trace is the sum of the diagonal entries of A. Show that any eigenvalue of A satisfies

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0.$$

Conclude from the quadratic formula that eigenvalues are functions of trace and determinant:

$$\lambda = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2}.$$

4.4.8 Problem ($\star$). Let a, b, c, and d be real numbers. Show that the eigenvalues of the three matrices below are always a and d:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

[Hint: *the formula in Problem 4.4.7 might be faster than the definition.*]

This is where we finished on Wednesday, October 25, 2023.

Remember that our underlying goal has been to produce fundamental solution sets (Definition 4.3.3) for the system $\dot{\mathbf{x}} = A\mathbf{x}$. If A has distinct eigenvalues, then we are done.

4.4.9 Theorem. *Let A be a matrix and suppose that λ_1 and λ_2 are eigenvalues of A with $\lambda_1 \neq \lambda_2$. If $\mathbf{v}_1$ is an eigenvector of A corresponding to λ_1 and $\mathbf{v}_2$ is an eigenvector of A corresponding to λ_2, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.*

We will not prove this theorem here. Instead, we will use its corollary often.

4.4.10 Corollary. *Suppose that the matrix A has distinct eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then the functions*

$$\mathbf{x}_1(t) := e^{\lambda_1 t}\mathbf{v}_1 \qquad \textit{and} \qquad \mathbf{x}_2(t) := e^{\lambda_2 t}\mathbf{v}_2$$

form a fundamental solution set for the linear system $\dot{\mathbf{x}} = A\mathbf{x}$.

Proof. The discussion preceding Definition 4.4.1 (or the definition of eigenvalue and eigenvector and brute-force calculus) shows that $\mathbf{x}_1$ and $\mathbf{x}_2$ solve $\dot{\mathbf{x}} = A\mathbf{x}$. We calculate $\mathbf{x}_1(0) = \mathbf{v}_1$

and $\mathbf{x}_2(0) = \mathbf{v}_2$ and use Theorem 4.4.9 to see that $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent. Consequently, $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the definition of fundamental solution set. ■

4.4.11 Example. **(i)** In Example 4.4.4, we computed that the eigenvalues and (one choice of) eigenvectors for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

were

$$\lambda_1 = -1 \quad \text{and} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$\lambda_2 = -3 \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Consequently, a fundamental solution set for the system $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}_1(t) := e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) := e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

All solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, then, have the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

for some constants c_1 and c_2.

(ii) Consider the free harmonic oscillator with mass $m = 1$, damping coefficient $b = 4$, and spring constant $\kappa = 3$. Then its displacement satisfies

$$\ddot{x} + 4\dot{x} + 3x = 0,$$

and so its linear system is $\dot{\mathbf{x}} = A\mathbf{x}$. In particular, all solutions to the original equation of motion for the harmonic oscillator under consideration are the first component of the general solution to this system That is, all solutions to $\ddot{x} + 4\dot{x} + 3x = 0$ are

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$

for some constants c_1 and c_2.

Observe that

$$\lim_{t\to\infty} x(t) = 0$$

for any choice of c_1 and c_2 and, likewise,

$$\lim_{t\to\infty} \dot{x}(t) = \lim_{t\to\infty} \big(- c_1 e^{-t} - 3c_2 e^{-3t} \big) = 0.$$

This is in line with our earlier conjectures that the displacement and velocity of damped harmonic oscillators should go to 0 over very long times.

Finally, recall from Example 4.4.4 that the characteristic equation of A is

$$\lambda^2 + 4\lambda + 3 = 0.$$

The coefficients on the left are exactly those in the ODE $\ddot{x} + 4\dot{x} + 3x = 0$. We will see that we can read off the characteristic equation for the system version of a second-order linear ODE just from that ODE. If all we want is formulas, we can bypass systems entirely!

But we want more than formulas.

The problem with Corollary 4.4.10 is fourfold. First, not every matrix has distinct eigenvalues. The characteristic equation is a quadratic equation, which can have repeated real roots. Then there is only one eigenvalue, and Corollary 4.4.10 simply does not apply. Second, if the characteristic equation has the complex, nonreal root λ, what does $e^{\lambda t}$ mean? (For that matter, what does $e^{\lambda t}$ really mean when λ is real?) Third, we have not professed the Analyst's Creed (AC) in a while: having a formula for something is not the same as understanding that thing. Corollary 4.4.10 gives us formulas for solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, but it does not exactly tell us what $\mathbf{x}$ and its components are doing, together or separately. In particular, fourth (and finally), what is the harmonic oscillator doing?

4.5. Fundamental solution sets and phase portraits.

4.5.1. Straight-line solutions.

We know that if the matrix A has distinct eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the functions $\mathbf{x}_1(t) := e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) := e^{\lambda_2 t}\mathbf{v}_2$ form a fundamental solution set for the system $\dot{\mathbf{x}} = A\mathbf{x}$. Such solutions might be called "straight-line solutions," because if we were to plot them parametrically, they would lie on a straight line through the origin.

Remember that a line through the origin is either a set of points of the form $(0, y)$ or (x, mx) for some number $m \neq 0$. The set with points of the form $(0, y)$ is just the y-axis, while the set with points of the form (x, mx) is the line through the origin with slope m (i.e., the line $y = mx$).

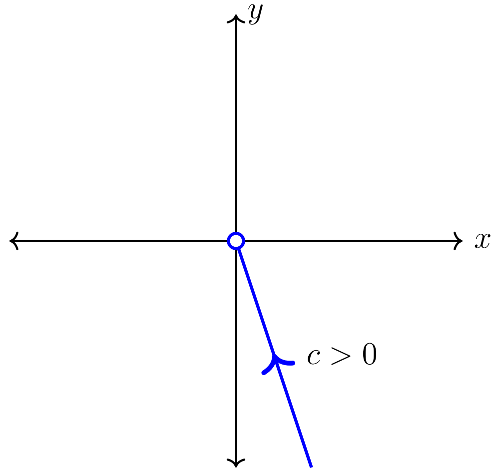
4.5.1 Example. **(i)** Let

$$\mathbf{x}(t) = e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

This was one of the functions in the fundamental solution set in Example 4.4.11. If we write $\mathbf{x}(t) = (x(t), y(t))$, then

$$x(t) = e^{-3t} \quad \text{and} \quad y(t) = -3e^{-3t}.$$

We see that $x(t) > 0$ and $y(t) < 0$, so all points $\mathbf{x}(t)$ lie in Quadrant IV. Moreover, we have $y(t) = -3x(t)$, so all points lie on the line $y = -3x$ in Quadrant IV. Finally, we have $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$. We can therefore describe the behavior of this function $\mathbf{x}$ as $t \to \infty$ as "all points $\mathbf{x}(t)$ lie on the line $y = -3x$ in Quadrant IV and tend to $\mathbf{0}$ as $t \to \infty$.

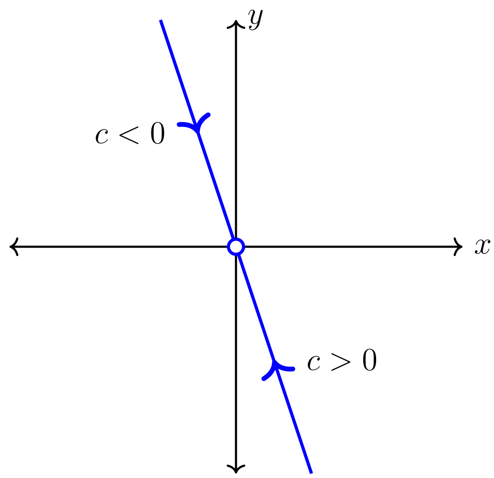


(ii) Let c be any nonzero number and let

$$\mathbf{x}(t) = ce^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

If $c > 0$, then we are in the same situation as above: with $\mathbf{x} = (x, y)$, the component x is always positive and the component y is always negative. Again, "all points $\mathbf{x}(t)$ lie on the line $y = -3x$ in Quadrant IV and tend to $\mathbf{0}$ as $t \to \infty$. If $c < 0$, however, then $x < 0$ and $y > 0$, so "all points $\mathbf{x}(t)$ lie on the line $y = -3x$ in Quadrant II and tend to $\mathbf{0}$ as $t \to \infty$. And if $c = 0$, then $\mathbf{x}(t) = \mathbf{0}$ for all t, and nothing interesting happens.

Then every point $\mathbf{x}(t)$ lies on the line $y = x$ with $x > 0$. Below we plot $\mathbf{x}(t)$ in the xy-plane; we plot $\mathbf{x}(t)$ for $t > 0$ in blue and for $t < 0$ in red. Note that $\mathbf{x}(t) \neq \mathbf{0}$ for all t. Moreover, as $t \to 0$, both components of $\mathbf{x}(t)$ go to 0, but as $t \to \infty$, both components of $\mathbf{x}(t)$ go to ∞. In the latter case of $t \to \infty$, we can be more descriptive and say that $\mathbf{x}(t)$ goes to ∞ "along the line $y = x$."



More generally, we will want to plot functions $\mathbf{x}$ of the form

$$\mathbf{x}(t) = f(t)\mathbf{v},$$

where f is a real-valued function and $\mathbf{v} = (v_1, v_2)$ is a vector. Put $x(t) = f(t)v_1$ and $y(t) = f(t)v_2$, so $\mathbf{x}(t) = (x(t), y(t))$. If $v_1 \neq 0$, then

$$y(t) = f(t)v_2 = \left(\frac{v_2}{v_1}\right) f(t)v_1 = \left(\frac{v_2}{v_1}\right) x(t),$$

and so each point $\mathbf{x}(t)$ lies on the line $y = (v_2/v_1)x$. If $v_1 = 0$, then $\mathbf{x}(t) = (0, f(t)v_2)$, and all points of this form lie on the y-axis.

4.5.2 Problem (!). Suppose $\mathbf{x}(t) = (x(t), y(t))$ is a STRAIGHT-LINE solution to $\dot{\mathbf{x}} = A\mathbf{x}$ in the sense that there is a constant m with $y(t) = mx(t)$ for all t. Equivalently, $\mathbf{x}(t) = x(t)\mathbf{v}$, where $\mathbf{v} = (1, m)$. Show that $x(t) = Ce^{\lambda t}$ for some constants C and λ. [Hint: *plug* $\mathbf{x}(t) = x(t)\mathbf{v}$ *into the system* $\dot{\mathbf{x}} = A\mathbf{x}$. *What do you learn about* x*?*]

This is where we finished on Monday, October 30, 2023.

4.5.2. The eigenvalues are real, negative, and distinct.

If the matrix A has the distinct real eigenvalues λ_1 and λ_2 (so $\lambda_1 \neq \lambda_2$), and if $\mathbf{v}_1$ is an eigenvector for λ_1 and $\mathbf{v}_2$ is an eigenvector for λ_2, then all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Analytically, this is easy ("easy"). Qualitatively, what is happening? How do the components x and y of this solution $\mathbf{x} = (x, y)$ behave, separately and together? The answer depends on the signs of the eigenvalues, and also the structure of the eigenvectors, and we encapsulate it in the phase plane. Previously, we had to rely on numerical results to draw phase planes and gain intuition on the long-time behavior of solutions to systems; now, however, we can do almost everything by hand using eigenvalues, eigenvectors, and some common sense. (Going forward, we will not present eigenvalue/eigenvector calculations in any detail but merely state the data.) The results need not be as precise as numerics—indeed, they will be essentially cartoons, but hopefully evocative ones.

First we consider the case in which both eigenvalues are negative.

4.5.3 Example. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}.$$

Its eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -3$, and corresponding eigenvectors are $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$.

1. All solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-3t}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1 e^{-t}\\c_2 e^{-3t}\end{pmatrix}.$$

2. If $\mathbf{x} = (x, y)$ is any solution, then

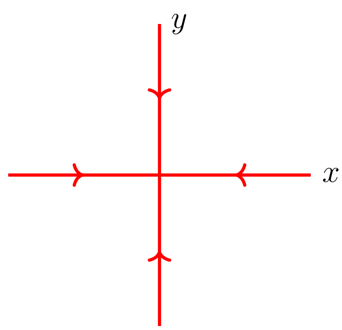
$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0.$$

But how are x and y behaving *jointly* as $t \to \infty$? Certainly

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0},$$

but *how* do the points $\mathbf{x}(t)$ approach $\mathbf{0}$? The two-dimensional plane is very large, and there are many paths of approach.

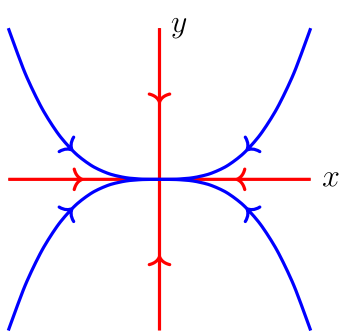
3. To keep matters simple, first suppose $c_2 = 0$. Then $\mathbf{x}(t) = (c_1 e^{-t}, 0)$ lies entirely on the x-axis. If $c_1 = 0$, then $\mathbf{x}(t) = (0, c_2 e^{-3t})$ lies entirely on the y-axis. Then the "straight-line solutions" to $\dot{\mathbf{x}} = A\mathbf{x}$ have this parametric plot.



4. What if neither c_1 nor c_2 is 0? Then $\mathbf{x}(t) = (c_1 e^{-t}, c_2 e^{-3t})$. With $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^{-3t}$, some algebraic sleight-of-hand shows

$$y(t) = c_2 e^{-3t} = \frac{c_2 c_1^3 e^{-3t}}{c_1^3} = \frac{c_2 (c_1 e^{-t})^3}{c_1^3} = \frac{c_2}{c_1^3}[x(t)]^3.$$

So, all solutions lie on cubics of the form $y = cx^3$ for some constant c. The signs of c_1 and c_2 determine the quadrant in which the cubic lies, e.g., if $c_1 > 0$ and $c_2 > 0$, then the cubic is in Quadrant I (because both coordinates of $(c_1 e^{-t}, c_2 e^{-3t})$ are positive), but if $c_1 > 0$ and $c_2 < 0$, then the cubic is in Quadrant II (because the first coordinate of $(c_1 e^{-t}, c_2 e^{-3t})$ is positive but the second is negative). Thus the full behavior of the phase plane for $\dot{\mathbf{x}} = A\mathbf{x}$ is the following picture.



5. Here is another way to see how solutions that do not lie on one of the axes behave as $t \to \infty$. Say that $\mathbf{x} = (x, y)$ is one such solution. Then $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^{-3t}$ for some nonzero c_1 and c_2. Consider the ratio

$$\frac{y(t)}{x(t)} = \frac{c_2 e^{-3t}}{c_1 e^{-t}} = \frac{c_2 e^{-2t}}{c_1}.$$

Then

$$\lim_{t\to\infty} \frac{y(t)}{x(t)} = \lim_{t\to\infty} \frac{c_2 e^{-2t}}{c_1} = 0,$$

and so over long times the ratio y/x approaches 0. That is, $y \approx 0 \cdot x$, and so not only do the points $\mathbf{x}(t)$ approach $\mathbf{0}$ as $t \to \infty$, they do so along the line $y = 0$, i.e., along the x-axis. This is the behavior that we are seeing in the phase plane cubics as $t \to \infty$.

6. Here is a third perspective on how solutions approach $\mathbf{0}$. For t large, the number e^{-3t} is much smaller than the number e^{-t} (which is still small!). So, we should expect that the term $c_2 e^{-3t}\mathbf{v}_2$ is much smaller than the term $c_1 e^{-t}\mathbf{v}_1$ in the function $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-3t}\mathbf{v}_2$, at least if $c_1 \neq 0$. (If $c_1 = 0$, then only the term with e^{-3t} is present, so comparing two terms makes no sense.) Thus for large t, if $c_1 \neq 0$, then we expect $\mathbf{x}(t) \approx c_1 e^{-t}\mathbf{v}_1$, and so as $t \to \infty$, the points $\mathbf{x}(t)$ approach $\mathbf{0}$ in the direction of $\mathbf{v}_1$.

4.5.4 Problem (!). Repeat all the work in the previous example for the matrix

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}.$$

What changes, and how does the phase portrait compare to the one that we drew above? [Hint: *interchange the role of the x- and y-axes in the previous example.*]

In general, if the matrix A has two negative distinct eigenvalues $\lambda_2 < \lambda_1 < 0$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ have the form

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2, \tag{4.5.1}$$

and so $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. The nuances are in *how* a solution $\mathbf{x}$ may approach $\mathbf{0}$ over long times. If one of the coefficients c_1 or c_2 is 0, then the solution is a "straight-line solution" that stays on a line through the origin with slope given by the vector $\mathbf{v}_1$ or $\mathbf{v}_2$ as in Section 4.5.1.

Otherwise, suppose c_1 and c_2 are nonzero. Since $\lambda_2 < \lambda_1$, then the components of $c_2 e^{\lambda_2 t}\mathbf{v}_2$ will be smaller (for large t) than the components of $c_1 e^{\lambda_1 t}\mathbf{v}_1$, and so $c_1 e^{\lambda_1 t}\mathbf{v}_1$ "dominates." That is, when $c_1 \neq 0$, we expect

$$\mathbf{x}(t) \approx c_1 e^{\lambda_1 t}\mathbf{v}_1 \tag{4.5.2}$$

for t large, and so $\mathbf{x}$ approaches $\mathbf{0}$ in the direction of the eigenvector $\mathbf{v}_1$.

More precisely, if $\mathbf{v}_1 = (v_{11}, v_{12})$ and $\mathbf{v}_2 = (v_{21}, v_{22})$, and if $\mathbf{x} = (x, y)$, then from (4.5.1) we have

$$\frac{y(t)}{x(t)} = \frac{c_1 v_{12} + c_2 e^{(\lambda_2 - \lambda_1)t} v_{22}}{c_1 v_{11} + c_2 e^{(\lambda_2 - \lambda_1)t} v_{21}}. \tag{4.5.3}$$

If $v_{11} \neq 0$, this shows that

$$\lim_{t \to \infty} \frac{y(t)}{x(t)} = \frac{v_{12}}{v_{11}}, \tag{4.5.4}$$

and so over long times we have $y(t) \approx (v_{12}/v_{11})x(t)$. That is, over long times solutions with $c_1 \neq 0$ lie roughly on the line $y = (v_{12}/v_{11})x$.

4.5.5 Problem ($\star$). **(i)** Use (4.5.1) to compute the quotient (4.5.3). [Hint: *factor* $e^{\lambda_1 t}$ *from the numerator and denominator.*]

(ii) Use (4.5.3) to prove the limit (4.5.4), assuming $v_{11} \neq 0$. [Hint: $\lambda_2 - \lambda_1 < 0$.]

(iii) With $y(t)/x(t)$ defined by (4.5.3), what is $\lim_{t \to \infty} y(t)/x(t)$ when $v_{11} = 0$? Is this what you expect from the approximation (4.5.2)?

If the matrix A has two distinct negative eigenvalues, then all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ tend to the origin over long times, and they do so tangent to the eigenvector corresponding to the larger eigenvalue. In this case, we give the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ a name reminiscent of the "attractive" behavior of certain solutions to autonomous scalar ODE.

4.5.6 Definition. *Suppose that the matrix A has two distinct negative eigenvalues. Then the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ for $\dot{\mathbf{x}} = A\mathbf{x}$ is a* SINK *or* STABLE.

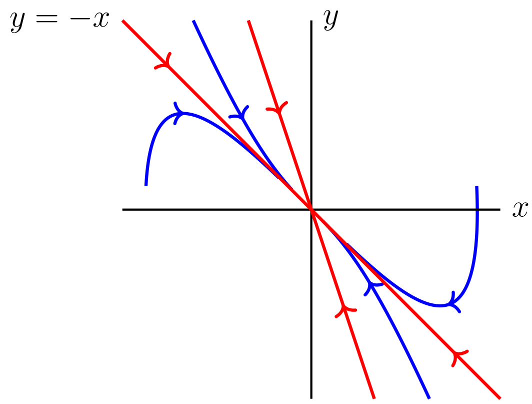
The eigenvectors in Example 4.5.3 were special because multiples of them just lie on the x- and y-axes. In general, the situation for a matrix with two negative distinct eigenvalues is just a "rotation" of the situations in Example 4.5.3 and Problem 4.5.4.

4.5.7 Example. Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \mathbf{x},$$

which we studied in Examples 4.4.4 and 4.4.11. Its eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -3$ with corresponding eigenvectors $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, -3)$. All solutions are $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-3t}\mathbf{v}_2$. If $c_1 = 0$, then the points $\mathbf{x}(t) = c_2 e^{-3t}\mathbf{v}_2$ approach the origin along the line $y = -3x$; if $c_2 = 0$, then the points $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}_1$ approach the origin along the line $y = -x$. And if $c_1 \neq 0$, then the term $c_1 e^{-t}\mathbf{v}_1$ dominates, and so the points $\mathbf{x}(t)$ for $c_1 \neq 0$

approach the origin tangent to the line $y = -x$.



4.5.3. The eigenvalues are real, positive, and distinct.

If A has two distinct positive eigenvalues, then the phase portrait for $\dot{\mathbf{x}} = A\mathbf{x}$ is completely the reverse of the above. Say that the eigenvalues are λ_1 and λ_2 with $0 < \lambda_1 < \lambda_2$ and that eigenvectors are $\mathbf{v}_1$ and $\mathbf{v}_2$. Then all solutions are $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. We expect that when t is large, the components of $\mathbf{x}(t)$ move away from 0 and are either large positive or large negative numbers—but how exactly do they move away? Between the vectors $c_1 e^{\lambda_1 t}\mathbf{v}_1$ and $c_2 e^{\lambda_2 t}\mathbf{v}_2$, there are four components, each of which could go to $+\infty$, $-\infty$, or maybe stay at 0 for large t.

To get a more meaningful sense of the phase portrait, we can run time *in reverse* and take $t \to -\infty$ (something that we usually do not do in this course!). Then $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$, and when t is a negative number with $|t|$ large, the term $c_1 e^{\lambda_1 t}\mathbf{v}_1$ is larger than the term $c_2 e^{\lambda_2 t}\mathbf{v}_2$. So, over long *negative* times, we have $\mathbf{x}(t) \approx c_1 e^{\lambda_1 t}\mathbf{v}_1$. (Equivalently, draw the phase portrait assuming the eigenvectors are the same but flip λ_1 to $-\lambda_1$ and λ_2 to $-\lambda_2$.) Then over long positive times, we just run the picture in reverse.

A few examples should make this clearer.

4.5.8 Example. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix},$$

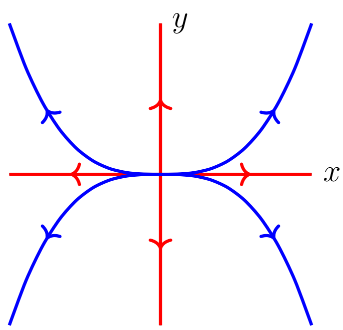
so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$ with eigenvectors $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$. This is the negative of the matrix, and also the negative of the eigenvalues, from Example 4.5.3.

All solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are

$$\mathbf{x}(t) = c_1 e^t \mathbf{v}_1 + c_2 e^{3t}\mathbf{v}_2 = \begin{pmatrix} c_1 e^t \\ c_2 e^{3t} \end{pmatrix}.$$

When $c_1 = 0$, solutions $\mathbf{x}(t) = (0, c_2 e^{3t})$ lie on the y-axis; if $c_2 > 0$, then such solutions lie on the positive y-axis and tend "up" the y-axis, away from the origin, as $t \to \infty$, while if $c_2 < 0$, such solutions lie on the negative y-axis and tend "down" the y-axis, away from the origin, as $t \to \infty$. Similarly, if $c_2 = 0$, solutions $\mathbf{x}(t) = (c_1e^t, 0)$ lie on the x-axis; if $c_1 > 0$, then such solutions lie on the positive x-axis and tend "right" along the x-axis, away from the origin, as $t \to \infty$, while if $c_1 < 0$, such solutions lie on the negative x-axis and tend "left" along the x-axis, away from the origin, as $t \to \infty$. Finally, if $c_1 \neq 0$, then, exactly as in Example 4.5.3, solutions lie on cubics $y = Cx^3$, but they tend away from the origin along those cubics. The signs of c_1 and c_2 again determine the quadrant in which that cubic lies.

Here, then, is the phase portrait, and we note that it is essentially the phase portrait from Example 4.5.3 run in reverse.



4.5.9 Problem (!). Repeat all the work in the previous example for the matrix

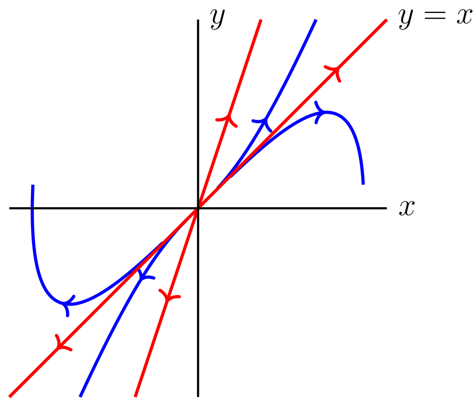
$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

What changes, and how does the phase portrait compare to the one that we drew above? Compare the results to Problem 4.5.4. [Hint: *interchange the role of the x- and y-axes in the previous example.*]

4.5.10 Example. The eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$$

are $\lambda_1 = 1$ and $\lambda_2 = 3$, and corresponding eigenvectors are $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, 3)$. All solutions are $\mathbf{x}(t) = c_1e^t\mathbf{v}_1 + c_2e^{3t}\mathbf{v}_2$. If $c_1 = 0$, then the points $\mathbf{x}(t) = c_2e^{3t}\mathbf{v}_2$ lie on the line $y = 3x$ and move away from the origin. If $c_2 = 0$, then the points $\mathbf{x}(t) = c_1e^t\mathbf{v}_1$ lie on the line $y = x$ and move away from the origin. Otherwise, if $c_1 \neq 0$, we run time in reverse and see that if $t < 0$ with $|t|$ large, then $\mathbf{x}(t) \approx c_1e^t\mathbf{v}_1$, and so in reverse time the points $\mathbf{x}(t)$ with $c_1 \neq 0$ approach the origin tangent to the line $y = x$. This gives the phase portrait below.



If the matrix A has two distinct positive eigenvalues, then all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ tend away from the origin over long times, and so we name the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ as we did solutions to scalar autonomous problems that "repelled" other nearby solutions.

4.5.11 Definition. *Suppose that the matrix A has two distinct positive eigenvalues. Then the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ for $\dot{\mathbf{x}} = A\mathbf{x}$ is a* **SOURCE** *or* **UNSTABLE**.

This is where we finished on Wednesday, November 1, 2023.

4.5.4. One eigenvalue is positive and one is negative.

As usual, we begin with an example that superficially looks much like our previous starting ones.

4.5.12 Example. The eigenvalues of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

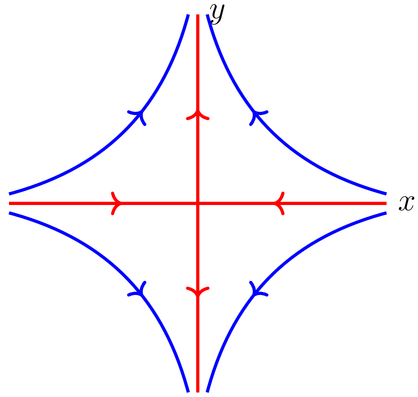
are $\lambda_1 = -1$ and $\lambda_2 = 3$, and corresponding eigenvectors are $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$. All solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are therefore $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{3t}\mathbf{v}_2$.

If $c_1 = 0$, then $\mathbf{x}(t) = c_2 e^{3t}\mathbf{v}_2$, and these solutions lie on the y-axis. They move away from the origin as $t \to \infty$ depending on the sign of c_2: if $c_2 > 0$, then these solutions move to $+\infty$ up the y-axis, and if $c_2 < 0$, then they move to $-\infty$ down the y-axis.

However, if $c_2 = 0$, then $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}_1$, and these solutions lie on the x-axis. All such solutions approach the origin as $t \to \infty$. Thus we have two different behaviors of solutions! Some are attracted to the origin, and some are repelled from it. We have never seen anything like this with systems or with scalar phase lines before.

In general, for large positive times, the term $c_1 e^{-t}\mathbf{v}_1$ is very small, so the term $c_2 e^{3t}\mathbf{v}_2$ dominates. That is, for $t > 0$ large, $\mathbf{x}(t) \approx c_2 e^{3t}\mathbf{v}_2$, and so over long times solutions *that are not on the x-axis* move away from the origin and approach the y-axis.

We put all of these observations together into the following phase portrait.



Due to the simple structure of the eigenvectors, we can say a little more about the trajectories in the phase portrait here. Put $\mathbf{x}(t) = (x(t), y(t))$, so $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^{3t}$. If $c_1 \neq 0$, then

$$y(t) = c_2 e^{3t} = \frac{c_2}{e^{-3t}} = \frac{c_2 c_1^3}{c_1^3 e^{-3t}} = \frac{c_2 c_1^3}{(c_1 e^{-t})^3} = \frac{c_2 c_1^3}{[x(t)]^3}.$$

Thus all trajectories not on the y-axis lie on the curves $y = Cx^{-3}$, which have the form sketched in our cartoon above.

4.5.13 Problem (!). Repeat all the work in the previous example for the matrix

$$\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

What changes, and how does the phase portrait compare to the one that we drew above? [Hint: *interchange the role of the x- and y-axes in the previous example.*]

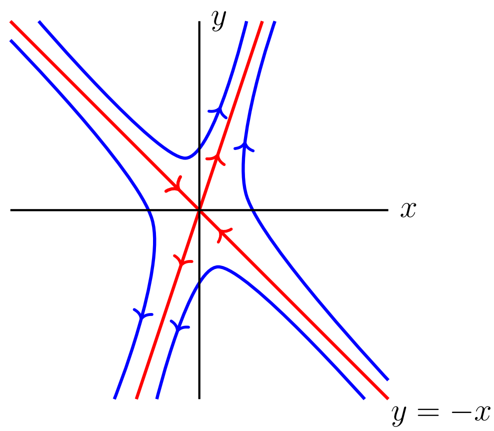
In general, if a matrix A has a positive eigenvalue $\lambda_2 > 0$ and a negative eigenvalue $\lambda_1 < 0$ (with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$) then solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ exhibit something of a dichotomy. Straight-line solutions $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1$ approach the origin as $t \to \infty$, but as $t \to \infty$, all other solutions $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$ with $c_2 \neq 0$ are dominated by the second term (since $\lambda_1 < 0$ but $\lambda_2 > 0$) so $\mathbf{x}(t) \approx c_2 e^{\lambda_2 t}\mathbf{v}_2$. Such solutions with $c_2 \neq 0$ therefore move away from the origin tangent to $\mathbf{v}_2$ as $t \to \infty$.

To get a fuller picture, it is helpful to run time in reverse. Since $\lambda_1 < 0$, the first term $c_1 e^{\lambda_1 t}\mathbf{v}_1$ dominates as $t \to -\infty$, and so for $t < 0$ with $|t|$ large, if $c_1 \neq 0$, we have $\mathbf{x}(t) \approx c_1 e^{\lambda_1 t}\mathbf{v}_1$. Thus solutions with $c_1 \neq 0$ move away from the origin tangent to $\mathbf{v}_1$ as $t \to -\infty$.

4.5.14 Example. The matrix
$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$
has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ and corresponding eigenvectors $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, 3)$. All solutions are $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{3t}\mathbf{v}_2$. If $c_1 = 0$, then solutions $\mathbf{x}(t) = c_2 e^{3t}\mathbf{v}_2$ lie on the line $y = 3x$ and move away from the origin as $t \to \infty$; if $c_2 = 0$, then solutions $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}_1$ lie on the line $y = -x$ and move toward the origin as $t \to \infty$. All other solutions with $c_2 \neq 0$ (i.e., all solutions not on the line $y = -x$) move away from the origin tangent to $\mathbf{v}_2$ as $t \to \infty$, i.e., they move away from the origin approaching the line $y = 3x$.

In reverse time, the term $c_1 e^{-t}\mathbf{v}_1$ dominates if $c_1 \neq 0$, and so for $t < 0$ with $|t|$ large, we have $\mathbf{x}(t) \approx c_1 e^{-t}\mathbf{v}_1$. Thus as $t \to -\infty$, all solutions with $c_1 \neq 0$ (i.e., all solutions not on the line $y = 3x$) move away from the origin tangent to $\mathbf{v}_1$, i.e., they move away from the origin approaching the line $y = -x$.

Here, then, is the phase portrait.



4.5.15 Problem (+). Make the remarks preceding this example precise as follows. Suppose that the matrix A has the eigenvalues $\lambda_1 < 0$ and $\lambda_2 > 0$ with corresponding eigenvectors $\mathbf{v}_1 = (v_{11}, v_{12})$ and $\mathbf{v}_2 = (v_{21}, v_{22})$. Any solution $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ has the form $\mathbf{x} = (x, y)$ with $x(t) = c_1 e^{\lambda_1 t} v_{11} + c_2 e^{\lambda_2 t} v_{21}$ and $y(t) = c_1 e^{\lambda_1 t} v_{12} + c_2 e^{\lambda_2 t} v_{22}$. Similar to the limit (4.5.4) and, more generally, the strategy of Problem 4.5.5, determine $\lim_{t\to\infty} y(t)/x(t)$ and $\lim_{t\to-\infty} y(t)/x(t)$.

When the matrix A has one positive and one negative eigenvalue, the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ for $\dot{\mathbf{x}} = A\mathbf{x}$ has a peculiar name.

4.5.16 Definition. *Suppose that the matrix A has one positive and one negative eigenvalue, Then the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ for $\dot{\mathbf{x}} = A\mathbf{x}$ is a* SADDLE*, and we also label it as* UNSTABLE.

The unique feature of a saddle is that one straight-line solution approaches the origin as $t \to \infty$ but all other solutions move away in the direction of the eigenvector corresponding to the positive eigenvalue.

4.5.5. One eigenvalue is zero and the other is nonzero.

If the matrix A has one eigenvalue equal to zero and the other is $\lambda_2 \neq 0$, then if $\mathbf{v}_1$ and $\mathbf{v}_2$ are corresponding eigenvectors, all solutions $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ still have the form

$$\mathbf{x}(t) = c_1 e^{0 \cdot t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

The difference is that now one term above is *constant*.

4.5.17 Example. The matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

has eigenvalues 0 and -1 and eigenvectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (1,-1)$. Then all solutions $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ are $\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2$. If $c_1 = 0$, then $\mathbf{x}(t) = c_2 e^{-t}\mathbf{v}_2$ lies on the line $y = -x$ and approaches the origin as $t \to \infty$. If $c_2 = 0$, then $\mathbf{x}(t) = c_1\mathbf{v}_1$ for all t, and so $\mathbf{x}$ is constant—that is, $\mathbf{x}$ is an equilibrium solution. But since c_1 can be arbitrary, there are infinitely many equilibrium solutions! In particular, since $\mathbf{v}_1 = (1,0)$, each equilibrium solution lies on the x-axis, and every point on the x-axis is an equilibrium solution.

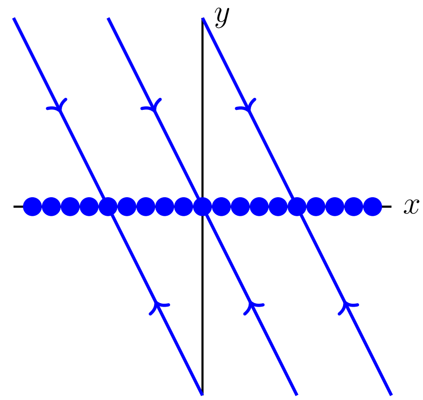
We have never seen anything like this. First, in all of our examples before with nonzero eigenvalues, the origin was the only equilibrium solution to $\dot{\mathbf{x}} = A\mathbf{x}$. Now, with a zero eigenvalue, we have a *line* of equilibrium solutions. Second, the origin is not an "isolated" equilibrium solution; by choosing c_1 appropriately, we can construct a nonzero equilibrium solution $\mathbf{x}(t) = c_1\mathbf{v}_1$ that is as close as we like to the origin. In all of our examples with phase lines (for scalar ODE $\dot{x} = f(x)$), the equilibrium solutions were always "isolated" in the sense that each was a certain minimum distance from the others. (Go back and look.) These phenomena are surprising consequences of working in two dimensions and having a zero eigenvalue!

Now we draw the phase portrait. There are straight-line solutions on $y = -x$ that approach the origin, which we draw as usual, and we mark the equilibrium solutions on the x-axis with dots. For the other solutions, note that

$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2 = \begin{pmatrix} c_1 + c_2 e^{-t} \\ -c_2 e^{-t} \end{pmatrix}.$$

If we take $\mathbf{x} = (x,y)$, then $x(t) = c_1 + c_2 e^{-t}$ and $y(t) = -c_2 e^{-t}$, thus $y(t) = -x(t) + c_1$.

That is, all other solutions lie on lines of the form $y = -x + c$ for some constant c.



4.5.18 Problem (!). Repeat the work of the previous example for the matrices

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

What is different? [Hint: *the eigenvalues will change, and also the line of equilibrium solutions—where is it in each case?*]

4.5.19 Problem ($\star$). **(i)** Suppose that 0 is an eigenvalue of the matrix A. Show that $\dot{\mathbf{x}} = A\mathbf{x}$ has infinitely many equilibrium solutions.

(ii) Suppose that $\dot{\mathbf{x}} = A\mathbf{x}$ has an equilibrium solution $\mathbf{x}(t) = \mathbf{x}_0$ with $\mathbf{x}_0 \neq \mathbf{0}$. Show that 0 is an eigenvalue of A.

(iii) Reread Corollary 4.2.6. How did that corollary predict these results?

This is where we finished on Friday, November 3, 2023.

4.5.6. The eigenvalues are the same.

Our success in constructing fundamental solution sets so far has hinged on the fact that the matrix has had two distinct eigenvalues; the precise relations of their signs determined the trajectories on the phase portrait, but, formulaically, the only thing that mattered was that the eigenvalues were real and distinct. It is possible that the matrix A has only one (necessarily real) eigenvalue. Recall that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the eigenvalues solve the characteristic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

The solutions to this quadratic equation are

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

If $(a+d) - 4(ad-bc) = 0$, then the only solution is

$$\lambda = \frac{a+d}{2}.$$

In this case, we know at least one solution to $\dot{\mathbf{x}} = A\mathbf{x}$. Let $\mathbf{v}_1$ be an eigenvector for A corresponding to λ and put $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$. Then $\mathbf{x}_1$ solves $\dot{\mathbf{x}} = A\mathbf{x}$. The problem is that $\mathbf{x}_1$ is only one function, and we need two functions for a fundamental solution set. How can we find another?

First, it is possible that λ has another eigenvector $\mathbf{v}_2$ such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. In this case, the functions

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2 \tag{4.5.5}$$

form a fundamental solution set for $\dot{\mathbf{x}} = A\mathbf{x}$.

4.5.20 Problem (!). Why?

4.5.21 Example. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The only eigenvalue of A is $\lambda = -1$, which can be seen from Problem 4.4.8, or directly, from the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$. The latter factors into $(\lambda+1)^2 = 0$, so its only root is $\lambda = -1$.

To find eigenvectors corresponding to -1, we solve $A\mathbf{v} = -\mathbf{v}$ for $\mathbf{v} = (v_1, v_2)$, which becomes the system

$$\begin{cases} -v_1 = -v_1 \\ -v_2 = -v_2. \end{cases}$$

In the past, we would try to write v_1 in terms of v_2 or v_2 in terms of v_1. Here that is not possible, and both equations above collapse to $0 = 0$, which is a true statement. What this means is that any choice of v_1 and v_2 will yield an eigenvector! That is, every vector is an eigenvector for -1. To see this, just compute

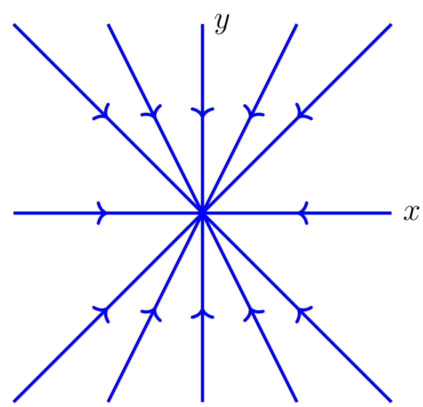
$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -\mathbf{v}.$$

Then to solve $\dot{\mathbf{x}} = A\mathbf{x}$, we could take the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to be any linearly independent vectors that we like, say, $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (0,1)$. Then all solutions $\mathbf{x}$ are

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Of course, since the problem $\dot{\mathbf{x}} = A\mathbf{x}$ is totally decoupled, we did not need any fancy linear algebra to see this.

Moreover, the form of the solution above tells us about the phase portrait. Any solution $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ has the form $\mathbf{x}(t) = e^{-t}\mathbf{c}$ for some vector $\mathbf{c}$, and every vector $\mathbf{c}$ generates a solution of this form. Thus all solutions lie on straight lines through the origin; every solution is a straight-line solution; and every straight line through the origin is the trajectory of some solution. And, of course, all solutions go to $\mathbf{0}$ over long times. Thus the phase portrait looks like the following—note that we have never seen more than two distinct straight-line trajectories in the previous arrangements of eigenvalues.



4.5.22 Problem (!). Explain how we could have predicted the structure of the phase portrait in Example 4.5.21 from the fundamental solution set (4.5.5). [Hint: *all solutions to* $\dot{\mathbf{x}} = A\mathbf{x}$ *here are* $\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$, *and this is a straight-line solution. How does the linear independence of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *allow us to capture all straight lines through the origin with this formula?*]

It turns out that a matrix with only one eigenvalue but two linearly independent eigenvectors has a very special form.

4.5.23 Problem (+). This problem shows that if the matrix A has the repeated real eigenvalue λ, and if λ has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then A is really the diagonal matrix

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

(i) Let $\mathbf{v}$ be any vector. Explain why there are constants c_1 and c_2 such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Use this to show that $A\mathbf{v} = \lambda\mathbf{v}$.

(ii) Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. Let $\mathbf{v} = (1, 0)$. Use the result above and Problem 4.1.13 to show that $\mathbf{a}_1 = \lambda\mathbf{v}$. With a difference choice of $\mathbf{v}$, obtain $\mathbf{a}_2 = (0, \lambda)$.

More interesting, ultimately, is the case in which A has one repeated real eigenvalue and only one linearly independent eigenvector. The following example will teach us a great deal about what to expect in this situation.

4.5.24 Example. The matrix

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

has one eigenvalue, $\lambda = -1$, and all eigenvectors corresponding to -1 have the form $\mathbf{v} = (v, 0)$ for some constant 0. No two of these vectors can be linearly independent, since

$$\det\left(\begin{bmatrix} v_1 & v_2 \\ 0 & 0 \end{bmatrix}\right) = (v_1 \cdot 0) - (v_2 \cdot 0) = 0,$$

no matter how v_1 and v_2 are chosen. Thus one solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where we are taking the eigenvector to be $\mathbf{v}_1 = (1, 0)$, but it is not immediately clear what the second function should be to form a complete fundamental solution set.

However, this is a partially decoupled system, and we can solve it componentwise. First, it reads

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -y, \end{cases}$$

so $y(t) = c_2 e^{-t}$, and therefore x must solve

$$\dot{x} = -x + c_2 e^{-t}.$$

We can solve this with variation of parameters or undetermined coefficients; for the latter, since the coefficient on x is -1, and since that is the same coefficient in the exponent, we should guess $x(t) = \alpha t e^{-t}$. Doing so yields $\alpha = c_2$, and so adding a constant multiple of the homogeneous solution yields

$$x(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

Then all solutions $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ are

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 t e^{-t} \\ c_2 e^{-t} \end{pmatrix}. \tag{4.5.6}$$

We might wonder how the solution $\mathbf{x}_1$ shows up in this formula. In particular, what is the role of the eigenvector $(1, 0)$? To suss this out, we expand

$$\begin{pmatrix} c_1 e^{-t} + c_2 t e^{-t} \\ c_2 e^{-t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 t e^{-t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{-t} \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 t e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now we see more clearly where $(1, 0)$ appears. The first term here is $c_1\mathbf{x}_1$ with $\mathbf{x}_1$ defined above. Define

$$\mathbf{x}_2(t) = t e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to see that all solutions have the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Taking $c_1 = 0$ and $c_2 = 1$ shows that $\mathbf{x}_2$ is a solution, and since

$$\mathbf{x}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which are linearly independent (why?), we see that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental solution set.

The deeper question is the role of this vector $(0, 1)$ in $\mathbf{x}_2$. How does it relate to A, the eigenvalue -1, and the eigenvector $(1, 0)$? How could we have expected $(0, 1)$ to show up without having calculated so explicitly the formula for all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$? Understanding $(0, 1)$ is our next task. We will return to this example later and draw the phase portrait, too.

4.5.25 Problem (!). With A as in the previous example, show that all solutions to the IVP

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

have the form

$$\mathbf{x}(t) = e^{-t}\mathbf{x}_0 + te^{-t}\mathbf{v},$$

where $\mathbf{v}$ is some eigenvector of A. [Hint: *go back to the formula* (4.5.6). *What role do the constants* c_1 *and* c_2 *play in calculating* $\mathbf{x}(0)$*?*]

This is where we finished on Monday, November 6, 2023.

Example 4.5.24 considered a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where A had one repeated eigenvalue λ with one linearly independent eigenvector $\mathbf{v}$. (That is, if $\mathbf{u}$ is another eigenvector, then $\mathbf{u} = c\mathbf{v}$ for some constant c. Since $\mathbf{u}$ is an eigenvector, $\mathbf{u} \neq \mathbf{0}$, and so $c \neq 0$. But then $(-c)\mathbf{v} + \mathbf{u} = \mathbf{0}$, so $\mathbf{v}$ and $\mathbf{u}$ are linearly dependent.) The example then showed that a fundamental solution set for $\dot{\mathbf{x}} = A\mathbf{x}$ had the form

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} \quad \text{and} \quad \mathbf{x}_2(t) = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w} \tag{4.5.7}$$

for some other vector $\mathbf{w}$. We surely expected the function $\mathbf{x}_1$ to show up in the fundamental solution set, but the function $\mathbf{x}_2$ was a surprise.

Can we generalize this work to all systems $\dot{\mathbf{x}} = A\mathbf{x}$ in which A has a repeated eigenvalue λ with only one linearly independent eigenvector $\mathbf{v}$? For this to succeed, we need the function $\mathbf{x}_2$ as defined above to solve $\dot{\mathbf{x}} = A\mathbf{x}$, and we need $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ to be linearly independent. That is, we need $\mathbf{v}$ and $\mathbf{w}$ to be linearly independent. For $\mathbf{x}_2$ to be a solution, we calculate

$$\dot{\mathbf{x}}_2(t) = e^{\lambda t}\mathbf{v} + te^{\lambda t}\lambda\mathbf{v} + e^{\lambda t}\lambda\mathbf{w}$$

and

$$A\mathbf{x}_2(t) = A\big(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}\big) = te^{\lambda t}A\mathbf{v} + e^{\lambda t}A\mathbf{w} = te^{\lambda t}\lambda\mathbf{v} + e^{\lambda t}A\mathbf{w}.$$

The first equality in the calculation of $A\mathbf{x}_2(t)$ is the linearity of matrix-vector arithmetic (Theorem 4.1.14), and the second is the identity $A\mathbf{v} = \lambda\mathbf{v}$, since λ is an eigenvalue of A with eigenvector $\mathbf{v}$.

So, to have $\dot{\mathbf{x}}_2(t) = A\mathbf{x}_2(t)$, we want

$$e^{\lambda t}\mathbf{v} + te^{\lambda t}\lambda\mathbf{v} + e^{\lambda t}\lambda\mathbf{w} = te^{\lambda t}\lambda\mathbf{v} + e^{\lambda t}A\mathbf{w}.$$

We immediately subtract the term $te^{\lambda t}\lambda\mathbf{v}$ from both sides to get

$$e^{\lambda t}\mathbf{v} + e^{\lambda t}\lambda\mathbf{w} = e^{\lambda t}A\mathbf{w},$$

and then we divide both sides by $e^{\lambda t}$ to find

$$\mathbf{v} + \lambda\mathbf{w} = A\mathbf{w},$$

or

$$A\mathbf{w} - \lambda\mathbf{w} = \mathbf{v},$$

or, most compactly,

$$(A - \lambda I)\mathbf{w} = \mathbf{v}. \tag{4.5.8}$$

So, $\mathbf{w}$ must solve the linear system (4.5.8) with data given by A, λ, and $\mathbf{v}$. (Recall that I is the identity matrix from Definition 4.1.16.) Conversely, if $\mathbf{w}$ solves (4.5.8), then reversing all of the work above leads to $\dot{\mathbf{x}}_2 = A\mathbf{x}_2$.

4.5.26 Problem (!). Go back to Example 4.5.24 and check that with

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad \lambda = -1, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

it is the case that $(A - \lambda I)\mathbf{w} = \mathbf{v}$.

There is just one catch—actually, two. First, why should it be possible to solve (4.5.8)? Second, if $\mathbf{w}$ solves (4.5.8), will $\mathbf{v}$ and $\mathbf{w}$ be linearly independent?

Here are the answers.

4.5.27 Theorem. *Suppose that the matrix A has one repeated real eigenvalue λ, and suppose that λ has only one linearly independent eigenvector $\mathbf{v}$. Then there exists a vector $\mathbf{w}$ that solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Moreover, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. The vector $\mathbf{w}$ is called a* **GENERALIZED EIGENVECTOR** *corresponding to λ.*

While not terribly difficult, the proof of this theorem relies on more linear algebra than is appropriate for this point in the course, so we will omit it. Instead, we can now conclude that if A has one repeated real eigenvalue λ, and suppose that λ has only one linearly independent eigenvector $\mathbf{v}$, then the functions $\mathbf{x}_1$ and $\mathbf{x}_2$ defined in (4.5.7) with $\mathbf{w}$ solving $(A - \lambda I)\mathbf{w} = \mathbf{v}$

do indeed form a fundamental solution set for $\dot{\mathbf{x}} = A\mathbf{x}$. Then all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ have the form

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}). \tag{4.5.9}$$

4.5.28 Problem ($\star$). Here is a different way to view the general solution (4.5.9).

(i) Suppose that A has one repeated real eigenvalue λ, and suppose that λ has only one linearly independent eigenvector. Show that we can also write any solution $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ in the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{x}(0) + te^{\lambda t}\mathbf{u}, \tag{4.5.10}$$

where $\mathbf{u} = (A - \lambda I)\mathbf{x}(0)$. Moreover, show that either $\mathbf{u} = \mathbf{0}$ or $\mathbf{u}$ is an eigenvector of A. [Hint: *with* $\mathbf{x}$ *given by* (4.5.9)*, what is* $\mathbf{x}(0)$ *in terms of* c_1, c_2, $\mathbf{v}$, *and* $\mathbf{w}$*? Then use the facts that* $(A - \lambda I)\mathbf{v} = \mathbf{0}$ *(why?) and* $(A - \lambda I)\mathbf{w} = \mathbf{v}$ *to compute* $(A - \lambda I)\mathbf{x}(0)$ *and recognize that as* $\mathbf{u}$*. Finally, note that* $c_2\mathbf{v}$ *is also an eigenvector for* $c_2 \neq 0$.]

(ii) Use (4.5.10) to solve $\dot{\mathbf{x}} = A\mathbf{x}$, where A is as in Example 4.5.24. Compare your solution here to (4.5.6).

The behavior of the solutions (4.5.9) depends on the sign of λ. First suppose $\lambda < 0$; then each term in this solution goes to $\mathbf{0}$ as $t \to \infty$. (This requires the limit $\lim_{t\to\infty} te^{\lambda t} = 0$, which is proved by L'Hospital's rule—"exponentials dominate polynomials at ∞.") And when t is large, the dominant term in (4.5.9) is $c_2 te^{\lambda t}\mathbf{v}$, if $c_2 \neq 0$. That is, if $c_2 \neq 0$, then $\mathbf{x}(t) \approx c_2 te^{\lambda t}\mathbf{v}$, and so over long times solutions that do not start on the straight-line solution $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}$ approach the origin tangent to $\mathbf{v}$.

We have seen this behavior before, and, naively, we might think that it results in a phase portrait like the ones in Examples 4.5.3 and 4.5.7. This will not quite be the case; here, the trajectories will stay only on certain "sides" of the eigenvector $\mathbf{v}$.

4.5.29 Example. The overworked Example 4.5.24 taught us that all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},$$

are

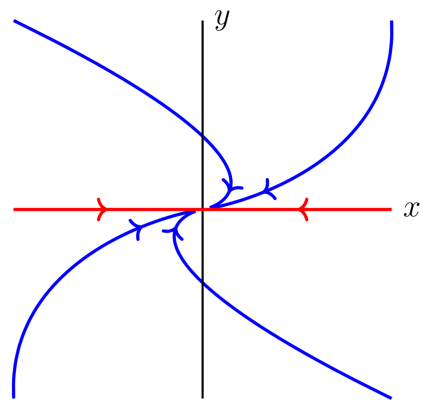
$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 te^{-t} \\ c_2 e^{-t} \end{pmatrix}.$$

For t large, then, the term $c_2 te^{-t}$ dominates the first component, and so we have

$$\mathbf{x}(t) \approx \begin{pmatrix} c_2 te^{-t} \\ c_2 e^{-t} \end{pmatrix}.$$

(This is slightly different from our saying $\mathbf{x}(t) \approx te^{\lambda t}\mathbf{v}$ above.) Since te^{-t} and e^{-t} are both positive, the components of this approximation have the same sign: they are positive for $c_2 > 0$ and negative for $c_2 < 0$. So, all solutions not on the x-axis (the case $c_2 = 0$, which we did not consider just now) eventually lie in Quadrant I (both components are positive) or Quadrant III (both components are negative). In particular, solutions that start in Quadrant II or Quadrant IV eventually move out of those quadrants; a solution starting in Quadrant II has $c_2 > 0$ and so moves into Quadrant I, while a solution starting in Quadrant IV has $c_2 < 0$ and so moves into Quadrant III. This is unlike the situation in Example 4.5.3, where the eigenvalues were negative and distinct, and trajectories could approach the origin and remain within any quadrant for all time.

Here, then, is the phase portrait; note the slight "spiral" of trajectories toward the origin.



This can be confirmed by doing numerical simulations for solutions starting in each of the four quadrants. More generally, this example illustrates how our analytic, qualitative, and numerical methods work in concert, and how each can fill in a gap left by the other: our initial qualitative guess that the phase portrait would look like a sink was wrong, and we had to go to the analytic solution (or maybe some representative numerics) to see the "twisting" behavior.

4.5.30 Problem (!). Revisit Example 4.5.24 and use the formula (4.5.6) to show that all solutions in this example not on the x-axis (i.e., with $c_2 \neq 0$) lie on curves of the form

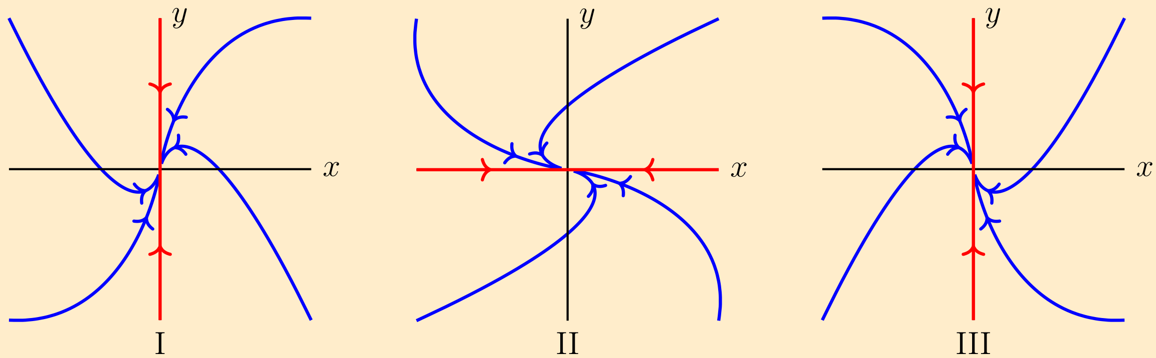
$$x = \frac{c_1 y}{c_2} - y \ln\left(\frac{y}{c_2}\right).$$

[Hint: $-t = \ln(y/c_2)$.] Plot some of these curves using graphing technology. Do you see the "twists" of the phase portrait emerging?

4.5.31 Problem (⋆). Below are phase portraits for the system $\dot{\mathbf{x}} = A\mathbf{x}$, where A is one of the following matrices, each of which has the repeated real eigenvalue -1 (which in turn has only one linearly independent eigenvector):

$$\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

Which phase portrait corresponds to which system? [Hint: *first find an eigenvector for each A; that will determine the straight-line solution. To determine the quadrants in which* *solutions eventually end up ("twist up"), you could find all solutions analytically and then repeat the analysis of Example 4.5.24. Here is a faster way: pick a point* $\mathbf{y} = (y_1, y_2)$ *not on the straight line solution (maybe on an axis?) and compute* $\mathbf{u} = A\mathbf{y}$*. Say that* $\mathbf{u}$ *has the components* $\mathbf{u} = (u_1, u_2)$*. Based on these components, use the ideas discussed after Example 3.3.4 to determine what the direction field for* $\dot{\mathbf{x}} = A\mathbf{x}$ *looks like at this point* (y_1, y_2)*. Which phase portrait points in that direction?*]



4.5.32 Problem ($\star$). How should the systems in Problem 4.5.31 be modified so that all trajectories move away from the origin over long times (i.e., so that the arrows on the phase portraits are all reversed)? [Hint: *the repeated eigenvalue should be* $\lambda = 1$.]

4.5.33 Problem (+). Suppose that 0 is a repeated eigenvalue of the matrix A.

(i) Explain why the characteristic equation of A must be $\lambda^2 = 0$ and then use Problem 4.4.7 to conclude that A has the very special form

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \quad \text{where} \quad a^2 = -bc.$$

(ii) If 0 has two linearly independent eigenvectors, use Problem 4.5.23 to show that A is the ZERO MATRIX whose entries are all 0; conclude that every vector is an equilibrium solution for $\dot{\mathbf{x}} = A\mathbf{x}$ here.

(iii) If 0 has only one linearly independent eigenvector, use Problem 4.5.28 to show that every solution to $\dot{\mathbf{x}} = A\mathbf{x}$ lies on a line (not necessarily through the origin) with the same slope. What is that slope?

4.5.7. The eigenvalues are complex, nonreal numbers.

The final case on the eigenvalues for us to consider is when they are complex and nonreal.

4.5.34 Undefinition. *Let a and b be real numbers. An expression of the form $a+bi$, where $i^2 = -1$, is a* COMPLEX NUMBER*. The* REAL PART *of $a + bi$ is $\mathrm{Re}(a + bi) := a$ and the*

IMAGINARY PART *of* $a + bi$ *is* $\text{Im}(a + bi) := b$.

We perform arithmetic with complex numbers exactly as we would with real numbers, except we allow $i^2 = -1$*. Two complex numbers are* **EQUAL** *precisely when if their corresponding real and imaginary parts are equal:* $z = w$ *is true when both* $\text{Re}(z) = \text{Re}(w)$ *and* $\text{Im}(z) = \text{Im}(w)$*.*

Every real number is a complex number (let a be real and write $a = a + 0 \cdot i$), so the interesting case here is when the eigenvalues are complex with nonzero imaginary part. To see how to find a fundamental solution set in this case, we work through one long example, with some lucky guesses and observations along the way, and then we make things systematic.

4.5.35 Example. Consider the undriven, undamped harmonic oscillator with mass and spring constant both 1. Since there are no friction and no external forces, we expect that this oscillator will oscillate forever and never permanently approach its equilibrium position. The equation of motion is then $\ddot{x} + x = 0$. Hopefully solutions will be periodic.

1. The equation of motion $\ddot{x} + x = 0$ is, with $y = \dot{x}$ and $\mathbf{x} = (x, y)$, equivalent to the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

The characteristic equation here is

$$\lambda^2 - (0 + 0)\lambda + (0 \cdot 0 - 1 \cdot (-1)) = 0,$$

which simplifies to

$$\lambda^2 + 1 = 0.$$

The roots here are $\lambda = \pm i$, and so the eigenvalues are the distinct numbers $\lambda_1 = i$ and $\lambda_2 = -i$. The eigenvalues are then (recall Problem 4.4.5, which did not require that λ be real) $\mathbf{v}_1 = (1, i)$ and $\mathbf{v}_2 = (1, -i)$. Consequently, we expect (from Corollary 4.4.10) that a fundamental solution set is

$$\mathbf{x}_1(t) = e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

2. Huh? We should quibble with this expectation for a number of reasons. First, what does e^{it} mean? The exponent it is complex and nonreal. For that matter, what does e^{τ} mean for a real number τ? We discussed this all the way back in Theorem 1.2.2: the exponential is a power series. If we use that power series definition from (1.2.5), and if we recall the power series definitions of the sine and cosine, and if we do some algebra, eventually we come to the slightly more transparent expectation

$$e^{it} = \cos(t) + i \sin(t).$$

This is **EULER'S FORMULA FOR THE COMPLEX EXPONENTIAL**, and we will henceforth adopt it as the *definition* of the symbol e^{it}.

So, we are now expecting that

$$\mathbf{x}_1(t) = \big(\cos(t) + i\sin(t)\big)\begin{pmatrix}1\\ i\end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \big(\cos(t) - i\sin(t)\big)\begin{pmatrix}1\\ -i\end{pmatrix}$$

are a fundamental solution set for our system above. (Here we used $e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$.)

3. This is still not ideal. First, what does it mean to differentiate something that contains i? For example, do we have

$$\dot{\mathbf{x}}_1(t) = \big(-\sin(t) + i\cos(t)\big)\begin{pmatrix}1\\ i\end{pmatrix}?$$

Second, our problem was posed entirely using real data. The harmonic oscillator's mass, spring constant, and friction constant were real numbers, and the matrix governing the system above contains only real entries. What *physically* does a solution formula with complex, nonreal entries mean in the context of a problem posed only using real data?

4. Answer: not much. It would be preferable if we could extract real-valued solutions from the complex-valued symbol-pushing above. Here is a clever trick: rewrite $\mathbf{x}_1$ as

$$\mathbf{x}_1(t) = \big(\cos(t) + i\sin(t)\big)\left[\begin{pmatrix}1\\ 0\end{pmatrix} + i\begin{pmatrix}0\\ 1\end{pmatrix}\right]$$

and then foil it out to find

$$\mathbf{x}_1(t) = \cos(t)\begin{pmatrix}1\\ 0\end{pmatrix} + i\cos(t)\begin{pmatrix}0\\ 1\end{pmatrix} + i\sin(t)\begin{pmatrix}1\\ 0\end{pmatrix} + i^2\sin(t)\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Now use the identity $i^2 = -1$ and factor out the remaining two appearances of i to find

$$\mathbf{x}_1(t) = \left[\cos(t)\begin{pmatrix}1\\ 0\end{pmatrix} - \sin(t)\begin{pmatrix}0\\ 1\end{pmatrix}\right] + i\left[\cos(t)\begin{pmatrix}0\\ 1\end{pmatrix} + \sin(t)\begin{pmatrix}1\\ 0\end{pmatrix}\right].$$

Put

$$\mathbf{x}_{\mathbf{r}}(t) := \cos(t)\begin{pmatrix}1\\ 0\end{pmatrix} - \sin(t)\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}\cos(t)\\ -\sin(t)\end{pmatrix}$$

and

$$\mathbf{x}_{\mathbf{i}}(t) := \cos(t)\begin{pmatrix}0\\ 1\end{pmatrix} + \sin(t)\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\sin(t)\\ \cos(t)\end{pmatrix},$$

so all of the calculations above just boil down to saying

$$\mathbf{x}_1(t) = \mathbf{x}_{\mathbf{r}}(t) + i\mathbf{x}_{\mathbf{i}}(t).$$

We can think of $\mathbf{x}_{\mathbf{r}}$ and $\mathbf{x}_{\mathbf{i}}$ as the real and imaginary parts of the vector-valued function $\mathbf{x}$.

5. We are expecting that $\dot{\mathbf{x}}_1 = A\mathbf{x}_1$, and we know the derivative distributes over sums, so we should also expect

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_{\mathbf{r}} + i\dot{\mathbf{x}}_{\mathbf{i}}.$$

And we know that matrix-vector multiplication distributes over sums, so we should further expect

$$A\mathbf{x}_1 = A\mathbf{x}_{\mathbf{r}} + iA\mathbf{x}_{\mathbf{i}}.$$

So, if all goes well, we should have

$$\dot{\mathbf{x}}_{\mathbf{r}} + i\dot{\mathbf{x}}_{\mathbf{i}} = A\mathbf{x}_{\mathbf{r}} + iA\mathbf{x}_{\mathbf{i}}.$$

Since two complex numbers are equal precisely when their real and imaginary parts are equal, we might expect that this identity implies

$$\dot{\mathbf{x}}_{\mathbf{r}} = A\mathbf{x}_{\mathbf{r}} \quad \text{and} \quad \dot{\mathbf{x}}_{\mathbf{i}} = A\mathbf{x}_{\mathbf{i}}. \tag{4.5.11}$$

6. Remember the great thing about differential equations: we can always check our work. We have concrete formulas for $\mathbf{x}_{\mathbf{r}}$, $\mathbf{x}_{\mathbf{i}}$, and A. So, we can calculate directly that (4.5.11) is true. (Do it.) And so we have two solutions to the problem $\dot{\mathbf{x}} = A\mathbf{x}$. Even better, we can compute (do this, too)

$$\mathbf{x}_{\mathbf{r}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathbf{i}}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which are linearly independent. And so $\mathbf{x}_{\mathbf{r}}$ and $\mathbf{x}_{\mathbf{i}}$ form a fundamental solution set for $\dot{\mathbf{x}} = A\mathbf{x}$.

7. Thus all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ have the form

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

We will figure out the phase portrait later. (We never saw sines and cosines for distinct or repeated real eigenvalues!) In particular, for the harmonic oscillator governed by $\ddot{x} + x = 0$, all solutions can be extracted from the first component above: they are

$$x(t) = c_1 \cos(t) + c_2 \sin(t).$$

This is exactly the oscillatory behavior of the undriven, undamped harmonic oscillator that we anticipated above.

This example had a lot of useful (if unanticipated) ideas. Our next task is to generalize those ideas to systems with complex eigenvalues of more or less arbitrary form and then to distill the phase portraits of such systems out of the eigenvalues.

This is where we finished on Wednesday, November 8, 2023.

Here is that generalization. Suppose that A has the complex eigenvalues $\alpha \pm i\beta$ with $\beta \neq 0$. Let $\mathbf{v}$ be an eigenvector for the eigenvalue $\alpha + i\beta$ and write $\mathbf{v} = \mathbf{v_r} + i\mathbf{v_i}$, where $\mathbf{v_r}$ and $\mathbf{v_i}$ are vectors with real components. In Example 4.5.35, we had

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \alpha = 0, \qquad \beta = 1, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{v_r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{v_i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we expect, formally, that a solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}(t) = e^{(\alpha+i\beta)t}(\mathbf{v_r} + i\mathbf{v_i}).$$

The first thing to manipulate is the exponential. Since addition in the exponent should be multiplication "outside," we expect

$$e^{(\alpha+i\beta)t} = e^{\alpha t + i\beta t} = e^{\alpha t}e^{i\beta t},$$

and then Euler's formula $e^{iy} = \cos(y) + i\sin(y)$ gives

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big).$$

Then we expect

$$\mathbf{x}(t) = e^{(\alpha+i\beta)t}(\mathbf{v_r} + i\mathbf{v_i}) = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)(\mathbf{v_r} + i\mathbf{v_i}).$$

When we foil out the product on the right, we obtain

$$\mathbf{x}(t) = e^{\alpha t}\big(\cos(\beta t)\mathbf{v_r} - \sin(\beta t)\mathbf{v_i}\big) + ie^{\alpha t}\big(\sin(\beta t)\mathbf{v_r} + \cos(\beta t)\mathbf{v_i}\big).$$

4.5.36 Problem (!). Foil that out.

In Example 4.5.35, defining

$$\mathbf{x_r}(t) = e^{\alpha t}\big(\cos(\beta t)\mathbf{v_r} - \sin(\beta t)\mathbf{v_i}\big) \quad \text{and} \quad \mathbf{x_i}(t) = e^{\alpha t}\big(\sin(\beta t)\mathbf{v_r} + \cos(\beta t)\mathbf{v_i}\big) \tag{4.5.12}$$

gave a fundamental solution set for $\dot{\mathbf{x}} = A\mathbf{x}$. We can check that here *directly from the definitions* of $\mathbf{x_r}$ and $\mathbf{x_i}$; we do not need any calculus involving i. Problem 4.5.37 shows how to check that $\dot{\mathbf{x}}_\mathbf{r} = A\mathbf{x_r}$ and $\dot{\mathbf{x}}_\mathbf{i} = A\mathbf{x_i}$. Next,

$$\mathbf{x_r}(0) = \mathbf{v_r} \quad \text{and} \quad \mathbf{x_i}(0) = \mathbf{v_i},$$

and Problem 4.5.37 shows how to check that $\mathbf{v_r}$ and $\mathbf{v_i}$ are linearly independent.

4.5.37 Problem (+). Suppose that the matrix A has the eigenvalue $\alpha + i\beta$ with $\beta \neq 0$.

(i) Show that $\alpha - i\beta$ is also an eigenvalue for A, and so the eigenvalues, when complex and nonreal, must come in "complex conjugate pairs." [Hint: *suppose that if* $\lambda = \alpha + i\beta$ *solves the quadratic equation* $\lambda^2 + p\lambda + q = 0$*, where* p *and* q *are real numbers. Calculate* $(\alpha - i\beta)^2 + p(\alpha - i\beta) + q$ *and show that this number is* 0.]

(ii) With $\mathbf{v} = \mathbf{v_r} + i\mathbf{v_i}$ as an eigenvector for A corresponding to $\alpha + i\beta$, and with the components of $\mathbf{v_r}$ and $\mathbf{v_i}$ as real numbers, show that $\mathbf{v_i} \neq \mathbf{0}$. [Hint: *suppose instead that* $\mathbf{v_i} = \mathbf{0}$. *Then* $\mathbf{v} = \mathbf{v_r}$ *is an eigenvector for* A *corresponding to* $\alpha + i\beta$, *so* $A\mathbf{v_r} = (\alpha + i\beta)\mathbf{v_r}$. *Since two complex numbers are equal precisely when their real and imaginary parts are equal, argue that* $\beta\mathbf{v_r} = \mathbf{0}$. *Since* $\mathbf{v} = \mathbf{v_r}$ *is an eigenvector, argue that* $\mathbf{v_r} \neq \mathbf{0}$*; conclude that* $\beta = 0$*, a contradiction.*]

(iii) Show that $\mathbf{v_r}$ and $\mathbf{v_i}$ are linearly independent. [Hint: *assume that* $c_1\mathbf{v_r} + c_2\mathbf{v_i} = \mathbf{0}$ *for some real numbers* c_1 *and* c_2*; by Theorem 4.2.16, it suffices to show* $c_1 = 0$ *and* $c_2 = 0$. *If* $c_1 = 0$*, then* $c_2\mathbf{v_i} = 0$*, and since* $\mathbf{v_i} = 0$*, this means* $c_2 = 0$. *So, assume* $c_1 \neq 0$ *and rewrite* $\mathbf{v_r} = c\mathbf{v_i}$*, where* $c = -c_2/c_1$. *Conclude that* $\mathbf{v} = (c + i)\mathbf{v_i}$ *and so, since* $\mathbf{v}$ *is an eigenvector for* A *with eigenvalue* $\alpha + i\beta$*, this means* $A\mathbf{v_i} = (\alpha + i\beta)\mathbf{v_i}$. *How does this contradict the preceding part?*]

(iv) Since $A(\mathbf{v_i} + i\mathbf{v_i}) = (\alpha + i\beta)(\mathbf{v_r} + i\mathbf{v_i})$, show that

$$A\mathbf{v_r} + iA\mathbf{v_i} = (\alpha\mathbf{v_r} - \beta\mathbf{v_i}) + i(\beta\mathbf{v_r} + \alpha\mathbf{v_i}).$$

Since two complex numbers are equal precisely when their real and imaginary parts are equal, argue that

$$A\mathbf{v_r} = \alpha\mathbf{v_r} - \beta\mathbf{v_i} \quad \text{and} \quad A\mathbf{v_i} = \beta\mathbf{v_r} + \alpha\mathbf{v_i}. \tag{4.5.13}$$

Use the definition of $\mathbf{x_r}$ in (4.5.12) to show

$$\dot{\mathbf{x}}_\mathbf{r}(t) = e^{\alpha t}\cos(\beta t)(\alpha\mathbf{v_r} - \beta\mathbf{v_i}) - e^{\alpha t}(\beta\mathbf{v_r} + \alpha\mathbf{v_i}).$$

Finally, use this calculation along with the identities in (4.5.13) and the definition of $\mathbf{x_r}$ to show that $\dot{\mathbf{x}}_\mathbf{r}(t) = A\mathbf{x_r}(t)$. Do the same for $\mathbf{x_i}$.

4.5.38 Example. Consider a free, damped harmonic oscillator with mass 1, friction constant 4, and spring constant 13. Its displacement x satisfies

$$\ddot{x} + 4\dot{x} + 13x = 0,$$

and since this oscillator is damped, we expect $\lim_{t\to\infty} x(t) = 0$.

The corresponding linear system is

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -13 & -4 \end{pmatrix}\mathbf{x},$$

and its eigenvalues are $-2 \pm 3i$. An eigenvector for $-2 + 3i$ is

$$\mathbf{v} = \begin{pmatrix} 1 \\ -2 + 3i \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

So, here we have

$$\alpha = -2, \qquad \beta = 3, \qquad \mathbf{v_r} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{v_i} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

A fundamental solution set for this system is therefore

$$\mathbf{x_r}(t) = e^{-2t}\left[\cos(3t)\begin{pmatrix}1\\-2\end{pmatrix} - \sin(3t)\begin{pmatrix}0\\3\end{pmatrix}\right]$$

and

$$\mathbf{x_i}(t) = e^{-2t}\left[\sin(3t)\begin{pmatrix}1\\-2\end{pmatrix} + \cos(3t)\begin{pmatrix}0\\3\end{pmatrix}\right].$$

All solutions $\mathbf{x}$ then have the form

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x_r}(t) + c_2\mathbf{x_i}(t) \\ &= \begin{pmatrix} c_1e^{-2t}\cos(3t) + c_2e^{-2t}\sin(3t) \\ -2c_1e^{-2t}\cos(3t) - 3c_1e^{-2t}\sin(3t) - 2c_2e^{-2t}\sin(3t) + 3c_2e^{-2t}\cos(3t)\end{pmatrix}.\end{aligned} \tag{4.5.14}$$

In particular, all solutions to the original ODE $\ddot{x} + 4\dot{x} + 13x = 0$ are

$$x(t) = c_1e^{-2t}\cos(3t) + c_2e^{-2t}\sin(3t).$$

Since

$$\lim_{t\to\infty} e^{-2t}\cos(3t) = \lim_{t\to\infty} e^{-2t}\sin(3t)$$

by the squeeze theorem, we see that $\lim_{t\to\infty} x(t) = 0$ for any solution to this ODE. This is exactly what we expect in a damped harmonic oscillator.

Our next task is to draw phase portraits for systems with complex eigenvalues. In particular, if solutions go to $\mathbf{0}$ over long times, exactly *how* do they approach the origin?

This is where we finished on Friday, November 10, 2023.

4.5.39 Example. The work in Example 4.5.35 tells us that all solutions to

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x}$$

have the form

$$\mathbf{x}(t) = \begin{pmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t)\end{pmatrix}$$

for some constants c_1 and c_2. If, for simplicity, we take $c_1 = 1$ and $c_2 = 0$, then we get the solution $\mathbf{x}(t) = (\cos(t), -\sin(t))$. The form of this solution should remind us of the unit circle but parametrized in the "clockwise direction," not the usual "counterclockwise" direction.

Indeed, if $x(t) = \cos(t)$ and $y(t) = \sin(t)$, then $(x(t))^2 + (y(t))^2 = 1$, and so this solution does lie on the unit circle. More generally, for arbitrary c_1 and c_2, we have (after some algebra)

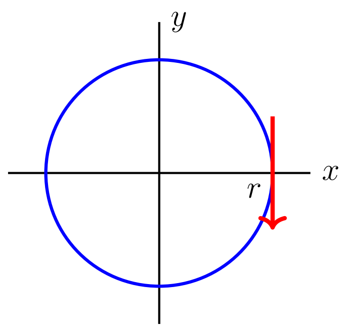
$$\big(c_1\cos(t) + c_2\sin(t)\big)^2 + \big(-c_1\sin(t) + c_2\cos(t)\big)^2 = (c_1^2 + c_2^2)(\cos^2(t) + \sin^2(t)) = c_1^2 + c_2^2.$$

So, all solutions lie on circles centered at the origin.

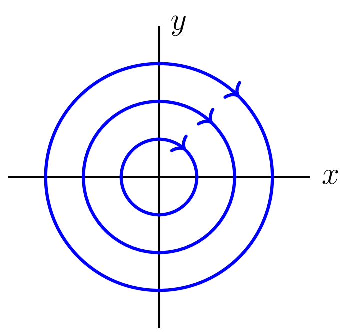
Furthermore, we can determine their orientation by thinking about the underlying direction field. Any circle centered at the origin must pass through a point $(r, 0)$ where $r > 0$. (Here r, of course, is the radius of that circle.) We can determine the structure of the direction field at that point by computing

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -r \end{pmatrix}.$$

Since $x > 0$, the ideas in the table after Example 3.3.4 tell us that the arrow in the direction field passing through $(r, 0)$ points straight down.



This indicates that all circular trajectories evolve clockwise, as we observed above with $c_1 = 1$ and $c_2 = 0$. So, the following cartoon describes the phase portrait.



More generally, if the eigenvalues of A have zero real part, i.e., if they are $\lambda = \pm\beta i$ for some $\beta \neq 0$, then all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ lie on *ellipses* centered at the origin. Recall that an ellipse centered at the origin is the set of all points (x, y) such that

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$$

for some r_1, $r_2 > 0$. To see this, observe from (4.5.12) that when $\alpha = 0$, each component of a solution $c_1\mathbf{x_r} + c_2\mathbf{x_i}$ can be written in the form

$$A\cos(\beta t) + B\sin(\beta t)$$

for some constants A and B. Now write A and B in polar coordinates:

$$A = r\cos(\theta) \quad \text{and} \quad B = r\sin(\theta), \quad \text{where} \quad r = \sqrt{A^2 + B^2} \quad \text{and} \quad \tan(\theta) = \frac{A}{B}.$$

Then

$$A\cos(\beta t) + B\sin(\beta t) = r\big[\cos(\theta)\cos(\beta t) + \sin(\theta)\sin(\beta t)\big] = r\cos(\beta t - \theta),$$

where the last equality is a trig addition formula. The first component of $c_1\mathbf{x_r} + c_2\mathbf{x_i}$ can then be written as

$$x(t) = r_1\cos(\beta t + \theta_1),$$

where θ_1 is just the negative of the θ from above; the second component can be written as

$$y(t) = r_2\sin(\beta t + \theta_2),$$

where here we have added $\pi/2$ to the phase shift from above to convert cosine into sine (recall $\sin(\tau + \pi/2) = \cos(\tau)$).

All together, solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ have the form

$$\mathbf{x}(t) = \begin{pmatrix} r_1\cos(\beta t + \theta_1) \\ r_2\sin(\beta t + \theta_2) \end{pmatrix}$$

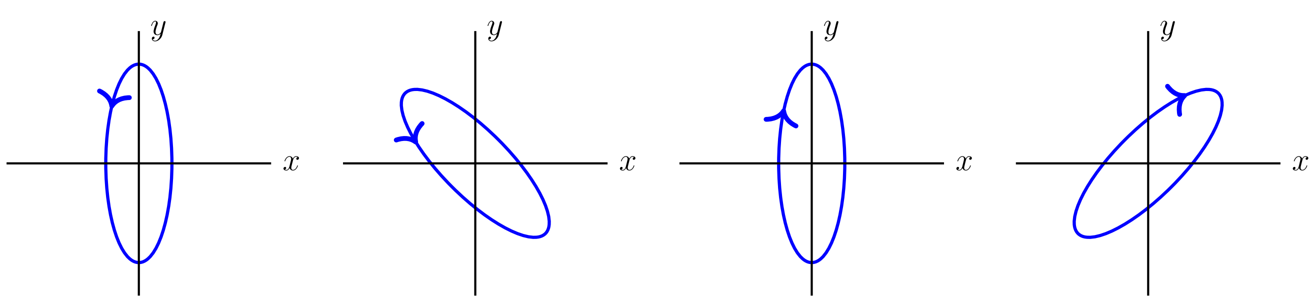
when A has the complex, purely imaginary eigenvalues $\lambda = \pm\beta i$. With $\mathbf{x} = (x, y)$, such solutions satisfy

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1 \tag{4.5.15}$$

and therefore lie on ellipses centered at the origin.

We can determine the direction of rotation by computing $A\mathbf{v}$ where $\mathbf{v} = (c, 0)$ with $c > 0$ arbitrary. If the y-component of $A\mathbf{v}$ is positive, then the trajectory passing through $(c, 0)$ is increasing in the y-direction, and therefore the rotation is counterclockwise. If the y-component is negative, then the trajectory is decreasing in the y-direction, and therefore the rotation is clockwise.

When A has purely imaginary eigenvalues, we call the origin a **CENTER** for the system $\dot{\mathbf{x}} = A\mathbf{x}$. Here are some common phase portraits of centers.



4.5.40 Problem ($\star$). Suppose that $q > 0$. Sketch the phase portrait for the linear system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -q & 0 \end{bmatrix} \mathbf{x}.$$

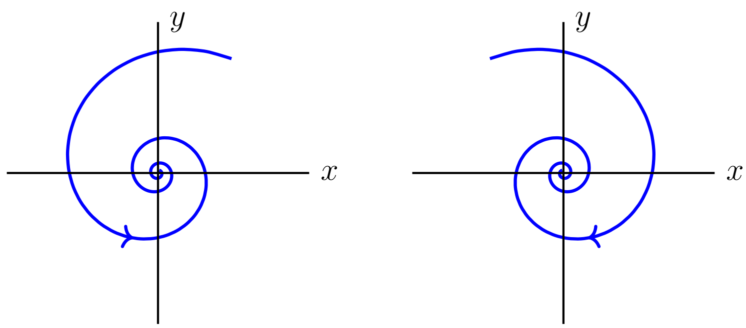
What does this say about the free undamped harmonic oscillator?

If A has complex eigenvalues with nonzero real part, i.e., the eigenvalues are $\lambda = \alpha \pm i\beta$ with both $\alpha \neq 0$ and $\beta \neq 0$, then the elliptical trajectories of the previous center case become distorted into "spirals." Specifically, using the fundamental solution set from (4.5.12), now retaining the factor of $e^{\alpha t}$, we can do the same work with polar coordinates and trig addition formulas as before to conclude that all solutions $\mathbf{x}$ to $\dot{\mathbf{x}} = A\mathbf{x}$ have the form

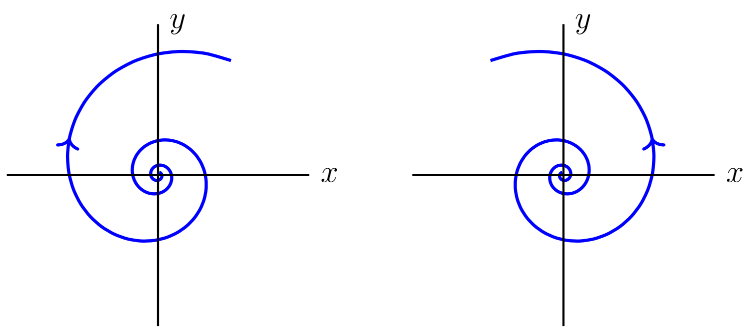
$$\mathbf{x}(t) = e^{\alpha t} \begin{pmatrix} r_1 \cos(\beta t + \theta_1) \\ r_2 \sin(\beta t + \theta_2) \end{pmatrix}.$$

The vector by itself still lies on the ellipse (4.5.15) as before, but now the factor of $e^{\alpha t}$ either expands this ellipse (in the case $\alpha > 0$) out to ∞ or shrinks it down to the origin (in the case $\alpha < 0$).

When $\alpha < 0$, we call the origin a **SPIRAL SINK** for $\dot{\mathbf{x}} = A\mathbf{x}$: all solutions tend to the origin along "spirals" like the two below.



When $\alpha > 0$, we call the origin a **SPIRAL SOURCE** for $\dot{\mathbf{x}} = A\mathbf{x}$: all solutions tend away from the origin along "spirals" like the two below.



As with centers, we can determine the orientation of the spiral (clockwise or counterclockwise) by calculating $A\mathbf{v}$ for some $\mathbf{v} = (r, 0)$, $r > 0$, and observing the sign of the y-component of $A\mathbf{v}$.

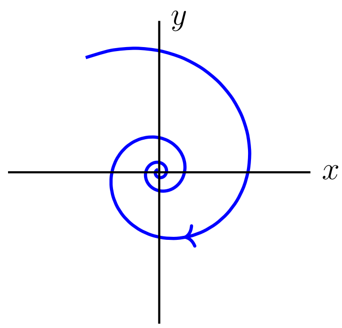
4.5.41 Example. **(i)** In Example 4.5.38, we saw that the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 \\ -13 & -4 \end{bmatrix}$$

are $\lambda = -2 \pm 3i$. Here $\alpha = -2 < 0$, so the origin is a spiral sink. We can determine the orientation of the spiral by calculating

$$\begin{bmatrix} 0 & 1 \\ -13 & -4 \end{bmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -13r \end{pmatrix}$$

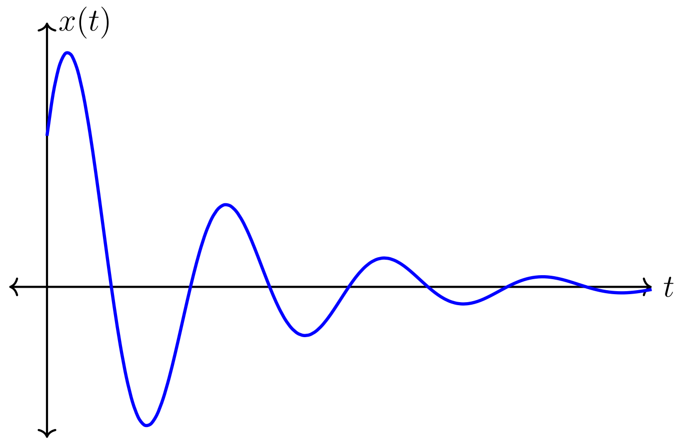
with $r > 0$. Then $-13r < 0$, so the spiral is clockwise, and the phase portrait for $\dot{\mathbf{x}} = A\mathbf{x}$ is the one below.



(ii) This linear system governs the harmonic oscillator whose equation of motion is $\ddot{x} + 4\dot{x} + 13x = 0$. This is a damped oscillator, so we expect that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} \dot{x}(t) = 0$. And this is exactly what the phase portrait is saying: both x and $y = \dot{x}$ go to 0 over long times, because the portrait spirals into the origin.

We have seen many phase portraits go to the origin, but we never saw spirals until now. Here is what the spiral means for the harmonic oscillator: each time the spiral crosses the y-axis, the x-coordinate of the solution $\mathbf{x} = (x, y)$ must be 0. So, each crossing of the y-axis corresponds to a zero of the function x, and physically that means the oscillator is passing through its equilibrium position. Since the phase portrait spirals around the origin infinitely many times (although we cannot show that in a realistic drawing), the oscillator passes through equilibrium infinitely many times. Furthermore, as the portrait spirals into the origin, the x-values become uniformly smaller (so do the y-values). This means that after each pass through equilibrium, the oscillator still moves to the left or right of equilibrium, but just to smaller and smaller displacements.

The result is that the graph of any (nonzero) solution x to $\ddot{x} + 4\dot{x} + 13x = 0$ should have the "decaying oscillatory" behavior below: the graph oscillates (like a sinusoid) but also decays (like an exponential).



Of course, we could also see this from the formula for x in (4.5.14) from Example 4.5.38, the first component of which gave us

$$x(t) = e^{-2t}\big(c_1\cos(3t) + c_2\sin(3t)\big).$$

This shows explicitly the exponential decay from e^{-2t} hitting the oscillatory behavior of the sinusoidal $c_1\cos(3t) + c_2\sin(3t)$.

4.6. Variation of parameters for linear systems.

We have developed a remarkable amount of information about the homogeneous linear system $\dot{\mathbf{x}} = A\mathbf{x}$. This information will enable us to solve the nonhomogeneous problem $\dot{\mathbf{x}} = A\mathbf{x}+\mathbf{b}(t)$ and prove the existence and uniqueness result for linear system IVP in Theorem 4.1.20.

Here is what we have explicitly developed.

4.6.1 Theorem. *Let A be a matrix. Then the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ always has a fundamental solution set: there exist differentiable functions $\mathbf{x}_1$ and $\mathbf{x}_2$ on $(-\infty,\infty)$ that solve $\dot{\mathbf{x}} = A\mathbf{x}$, and the vectors $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.*

4.6.2 Problem (!). To refresh your memory, explain how to construct $\mathbf{x}_1$ and $\mathbf{x}_2$ in all of the different cases on the eigenvalues (and eigenvectors) of A. [Hint: *there are a lot of cases.*]

We did not say in Theorem 4.6.1 that all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ have the form $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, as we did in Theorem 4.3.4. Now the point is to prove the uniqueness result.

Our first step is to get a better fundamental solution set.

4.6.3 Lemma. *There exist differentiable functions $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ on $(-\infty, \infty)$ that solve $\dot{\mathbf{x}} = A\mathbf{x}$ and that satisfy*

$$\boldsymbol{\phi}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{\phi}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

4.6.4 Problem (!). Use Theorem 4.6.1 to prove this lemma.

It will be convenient to put these functions $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ into a matrix.

4.6.5 Lemma. *With $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ from Lemma 4.6.3, define*

$$\Phi(t) := \begin{bmatrix} \boldsymbol{\phi}_1(t) & \boldsymbol{\phi}_2(t) \end{bmatrix}. \tag{4.6.1}$$

Then $\Phi(t)$ is invertible for all t. Moreover, with $\dot{\Phi}(t)$ defined componentwise (or columnwise) as in Problem 4.1.18, we have

$$\dot{\Phi}(t)\mathbf{v} = A(\Phi(t)\mathbf{v})$$

for any vector $\mathbf{v}$.

4.6.6 Problem (!). Use Problem 4.3.7, the definition of linear independence, Theorem 4.2.10, and Problem 4.1.18 to prove this lemma.

With these tools in hand, we will mimic our work on variation of parameters for scalar linear ODE in Section 2.7. There, we studied nonhomogeneous linear ODE like

$$\dot{x} = ax + b(t) \tag{4.6.2}$$

and easily solved the homogeneous problem

$$\dot{x} = ax \tag{4.6.3}$$

in the form

$$x(t) = ce^{at} \tag{4.6.4}$$

for some constant c. (In Section 2.7, we also allowed a to depend on t; here, for simplicity, and for analogy with the linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$, we keep a constant.) Then we guessed that the nonhomogeneous problem (4.6.2) had a solution of the form

$$x(t) = u(t)e^{at} \tag{4.6.5}$$

and we evaluated the nonhomogeneous ODE (4.6.2) at this guess to develop a direct integration problem for u. A key step in the calculations was that $e^{at} \neq 0$, so we could always solve for $\dot{u}$.

With systems, we are expecting that all solutions to the homogeneous system $\dot{\mathbf{x}} = A\mathbf{x}$ should have the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

for some constants c_1 and c_2. This is analogous to the solution form (4.6.4) for the scalar homogeneous problem (4.6.3). Strictly speaking, this is just an expectation, since we have not proved any uniqueness result for linear systems. (The same thing happened even for exponential growth: we never proved uniqueness of solutions for something as simple as $\dot{x} = ax$ until we did variation of parameters.)

Then, by analogy with the guess (4.6.5), we might expect that solutions to the nonhomogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$ should have the form

$$\mathbf{x}(t) = u_1(t)\boldsymbol{\phi}_1(t) + u_2(t)\boldsymbol{\phi}_2(t) \tag{4.6.6}$$

for some functions u_1 and u_2. Since we usually work with vectors, we could put

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

to rewrite the guess (4.6.6) in the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{u}(t), \tag{4.6.7}$$

where Φ was defined in (4.6.1).

4.6.7 Problem (!). Show that $\mathbf{u}(0) = \mathbf{x}(0)$. [Hint: $\Phi(0) = I$, *the identity matrix.*]

With $\mathbf{x}$ in the form (4.6.6) or (4.6.7) we can calculate that if $\mathbf{x}$ solves $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$, then

$$\Phi(t)\dot{\mathbf{u}}(t) = \mathbf{b}(t). \tag{4.6.8}$$

4.6.8 Problem (!). Calculate this. [Hint: *the product rule for matrix-vector multiplication in Problem 4.1.18 may be helpful when* $\mathbf{x}$ *is in the form* (4.6.7)*, or you could use the product rule directly on* (4.6.6) *and then look for the matrix-vector product* $\Phi(t)\dot{\mathbf{u}}(t)$ *when the dust settles.*]

Since $\Phi(t)$ is invertible for all t, we can obtain from (4.6.8) the direct integration problem

$$\dot{\mathbf{u}}(t) = \Phi(t)^{-1}\mathbf{b}(t)$$

for $\mathbf{u}$. Recall that $\Phi(t)^{-1}$ was defined in Theorem 4.2.10. We solve this direct integration problem as

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\ d\tau = \mathbf{x}(0) + \int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\ d\tau.$$

Then from the form of $\mathbf{x}$ in (4.6.7) and the linearity of matrix-vector multiplication (note that $\int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\ d\tau$ is, in the end, just a vector), we have

$$\mathbf{x}(t) = \Phi(t)\left(\mathbf{x}(0) + \int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\ d\tau\right) = \Phi(t)\mathbf{x}(0) + \Phi(t)\int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\ d\tau.$$

We have therefore shown that if $\mathbf{x}$ solves

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{4.6.9}$$

then $\mathbf{x}$ has the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\, d\tau. \tag{4.6.10}$$

(Have we really? Yes: rewrite $\mathbf{x}(t) = \Phi(t)\mathbf{u}(t)$ where $\mathbf{u}(t) = \Phi(t)^{-1}\mathbf{x}(t)$; then solve for $\mathbf{u}$ as above.) Conversely, we could differentiate (4.6.10) and use the product rule for matrix-vector multiplication (Problem 4.1.18) and Lemma 4.6.5 to conclude that defining $\mathbf{x}$ by (4.6.10) gives a solution to (4.6.9).

This result would be nicer if we knew that Φ was unique, but, unfortunately, Lemma 4.6.3 could not give that to us.

4.6.9 Problem ($\star$). Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ both solve (4.6.9). Put $\mathbf{y}(t) := \mathbf{x}_1(t) - \mathbf{x}_2(t)$ and show that $\mathbf{y}$ solves

$$\begin{cases} \dot{\mathbf{y}} = A\mathbf{y} \\ \mathbf{y}(0) = \mathbf{0}. \end{cases}$$

Conclude from the formula (4.6.10) with $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{b}(t) = \mathbf{0}$ that

$$\mathbf{y}(t) = \Phi(t)\mathbf{0} + \Phi(t)\int_0^t \Phi(\tau)^{-1}\mathbf{0}\, d\tau = \mathbf{0}$$

and therefore $\mathbf{x}_1 = \mathbf{x}_2$. This means that solutions to (4.6.9) are indeed unique.

It turns out that we can simplify (4.6.10) even further to avoid some matrix multiplication and, in particular, the matrix inverse. First, for a given t, the matrix $\Phi(t)$ is constant with respect to the variable of integration τ in the integral $\int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\, d\tau$, and so if we use the componentwise definition of the integral in Definition 4.1.6 and the definition of matrix-vector multiplication (again, for each t, the integral $\int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\, d\tau$ is a vector), we obtain

$$\Phi(t)\int_0^t \Phi(\tau)^{-1}\mathbf{b}(\tau)\, d\tau = \int_0^t \Phi(t)\big(\Phi(\tau)^{-1}\mathbf{b}(\tau)\big)\, d\tau.$$

Next, it turns out that

$$\Phi(t)\big(\Phi(\tau)^{-1}\mathbf{v}\big) = \Phi(t-\tau)\mathbf{v} \tag{4.6.11}$$

for any vector $\mathbf{v}$ and any numbers t and τ.

4.6.10 Problem (+). Prove this. Specifically, fix τ and define

$$\boldsymbol{\psi}_1(t) := \Phi(t+\tau)\big(\Phi(\tau)^{-1}\mathbf{v}\big) \quad \text{and} \quad \boldsymbol{\psi}_2(t) := \Phi(t)\mathbf{v}.$$

Show that $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ both solve

$$\begin{cases} \dot{\boldsymbol{\psi}} = A\boldsymbol{\psi} \\ \boldsymbol{\psi}(0) = \mathbf{v} \end{cases}$$

and therefore $\boldsymbol{\psi}_1(t) = \boldsymbol{\psi}_2(t)$ for all t by the uniqueness result for (4.6.9). In particular, $\boldsymbol{\psi}_1(t-\tau) = \boldsymbol{\psi}_2(t-\tau)$, which is (4.6.11). [Hint: *use Lemma 4.6.5 to differentiate* Φ.]

Here is our hard-won conclusion.

4.6.11 Theorem. *Let* $\mathbf{b}$ *be continuous on an interval* I *containing* 0. *Then the unique solution to the IVP*

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{4.6.12}$$

is

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(t-\tau)\mathbf{b}(\tau)\, d\tau,$$

where

$$\Phi(t) = \begin{bmatrix} \boldsymbol{\phi}_1(t) & \boldsymbol{\phi}_2(t) \end{bmatrix}, \qquad \begin{cases} \dot{\boldsymbol{\phi}}_1 = A\boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_1(0) = (1,0), \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{\boldsymbol{\phi}}_2 = A\boldsymbol{\phi}_2 \\ \boldsymbol{\phi}_2(0) = (0,1). \end{cases}$$

4.6.12 Problem (!). How does this result remind you of Problem 2.8.4?

Theorem 4.6.11 is a very precise result that completely answers the question of existence and uniqueness of solutions for linear system IVP and that gives an exact formula to boot. The problem is that the formula is a monstrosity: one has to construct the matrix $\Phi(t)$ by solving two linear homogeneous system IVP, then multiply the matrix and $\mathbf{b}$, and then integrate. All of the antidifferentiation problems that haunted scalar variation of parameters can reappear here, and now there are two components. In practice, one rarely cares about the precise formula for solutions to (4.6.12) in terms of elementary functions; instead, the integral representation is typically sufficient (and preferred) for deeper analysis of the system.

4.6.13 Problem (+). Theorem 4.1.20 put the initial condition at t_0, not 0. Apply Theorem 4.1.20 to the IVP

$$\begin{cases} \dot{\mathbf{y}} = A\mathbf{y} + \mathbf{b}(t+t_0) \\ \mathbf{y}(0) = \mathbf{x}_0 \end{cases}$$

and then change variables in the integral to prove Theorem 4.1.20.

This is where we finished on Monday, November 13, 2023.

5. SECOND-ORDER LINEAR EQUATIONS

We started down the long dark path of linear systems by considering first the harmonic oscillator, which is governed by a second-order linear IVP of the form

$$\begin{cases} m\ddot{x} + b\dot{x} + \kappa x = f(t) \\ x(0) = x_0 \\ \dot{x}(0) = y_0. \end{cases} \tag{5.0.1}$$

Physically, for the oscillator, we demanded $m > 0$, $b \geq 0$, and $\kappa > 0$, but as a mathematical problem (not considering any underlying model), this IVP makes sense for any values of m, b, and κ. To keep things interesting, we should suppose $m \neq 0$, as otherwise the problem reduces to a first-order linear IVP (with an extra constraint on $\dot{x}$, which possibly makes the problem unsolvable!).

We converted this single second-order problem to a system of two first-order equations by first rescaling and then introducing a new variable:

$$\begin{array}{l} m\ddot{x} + b\dot{x} + \kappa x = f(t) \\ \Big\downarrow \; p = \dfrac{b}{m},\ q = \dfrac{\kappa}{m},\ g(t) = \dfrac{f(t)}{m} \\ \ddot{x} + p\dot{x} + qx = g(t) \\ \Big\downarrow \; y = \dot{x},\ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},\ A = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix},\ \mathbf{b}(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \\ \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \end{array}$$

We have now completely solved the driven linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$, first in the homogeneous case $\mathbf{b}(t) = \mathbf{0}$ (with lots of subcases based on the eigenvalues of A and, sometimes, their eigenvectors) and then in the nonhomogeneous case via the great and terrible variation of parameters formula for systems. In theory, then, we could solve the IVP (5.0.1) by converting it to a system and applying systems methods.

In theory. In practice, we can extract some shortcuts from our diligent systems work based largely on the very special structure of A here. Note that the first row of A consists of the numbers 0 and 1. This is very special indeed. Then we will be able to analyze some aspects of the structure of solutions to (5.0.1) quite quickly.

5.0.1 Problem (+). It is also possible to do all of this in reverse. That is, suppose that we know everything there is to know about solving (**??**), and now we want to solve a linear system

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t) \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \qquad \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}. \tag{5.0.2}$$

Suppose that f_1 is differentiable. Then $\dot{x}(t) = ax(t) + by(t) + f_1(t)$, and since x and y solve the system and are therefore differentiable, $\dot{x}$ is the sum of three differentiable functions and is therefore differentiable itself. Compute

$$\ddot{x} = a\dot{x} + b\dot{y} + \dot{f}_1(t)$$

and substitute

$$\dot{y} = cx + dy \quad \text{and} \quad by = \dot{x} - ax$$

to obtain

$$\ddot{x} = (a+d)\dot{x} - (ad - bc)x + b\dot{f}_1(t) + f_2(t).$$

Conclude that the problem (5.0.2) is equivalent to

$$\begin{cases} \ddot{x} - \operatorname{tr}(A)\dot{x} + \det(A)x = b\dot{f}_1(t) + f_2(t) \\ x(0) = x_0 \\ \dot{x}(0) = ax_0 + by_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = dy + cx \\ y(0) = y_0. \end{cases}$$

Note that if $f_1 = f_2 = 0$ (i.e., if the linear system is homogeneous), then the coefficients in $\ddot{x} - \operatorname{tr}(A)\dot{x} + \det(A)x = 0$ are the same as in the characteristic equation for A.

We will learn how to solve the first IVP above from scratch shortly. Then we can view the second IVP as a first-order linear ODE in y driven by cx, and we will know x from the first IVP.

5.1. The homogeneous problem.

First we study the homogeneous problems

$$m\ddot{x} + b\dot{x} + \kappa x = 0 \quad \text{and} \quad \ddot{x} + p\dot{x} + qx = 0, \tag{5.1.1}$$

which lead to homogeneous systems of the form $\dot{\mathbf{x}} = A\mathbf{x}$; throughout this section, we will use A to mean the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}. \tag{5.1.2}$$

The characteristic equation for such A reads

$$m\lambda^2 + b\lambda + \kappa = 0 \iff \lambda^2 + p\lambda + q = 0,$$

and so its solutions (the eigenvalues of A) are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4m\kappa}}{2m} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Everything that follows depends on the sign of the discriminant $b^4 - 4m\kappa = p^2 - 4q$: is it positive, negative, or zero? Going forward, we will abuse terminology and refer to the characteristic equation for A as the **CHARACTERISTIC EQUATION FOR THE ODE** (5.1.1);

also, we will call the eigenvalues of A (i.e., the roots of the characteristic equation) the **EIGENVALUES FOR THE ODE** (5.1.1).

We will present qualitative results only for the harmonic oscillator; that is, when considering the long-time behaviors of solutions, we will assume $m > 0$, $b \geq 0$, and $\kappa > 0$, equivalently, $p \geq 0$ and $q > 0$. When we do not put such sign conditions on the ODE in (5.1.1), the phase portraits for $\dot{\mathbf{x}} = A\mathbf{x}$ can have any of the behaviors that we previously observed, except one.

5.1.1 Problem (+). Suppose that λ is a repeated eigenvalue for a matrix A in the form (5.1.2). Explain why there cannot be two linearly independent eigenvectors for λ. [Hint: *use the result of Problem 4.5.23.*]

5.1.1. There are two real, distinct eigenvalues.

Here we assume

$$b^2 - 4m\kappa > 0 \iff p^2 - 4q > 0,$$

and so the eigenvalues are

$$\lambda_1 := \frac{-b + \sqrt{b^2 - 4m\kappa}}{2m} = \frac{-p - \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \lambda_2 := \frac{-b - \sqrt{b^2 - 4m\kappa}}{2m} = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Note that $\lambda_1 > \lambda_2$ here.

Then eigenvectors for A corresponding to λ_1 and λ_2 are, respectively,

$$\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \tag{5.1.3}$$

by Problem 4.4.5, and all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.$$

Now we can extract the solution x to the ODE (5.1.1) just from the first component of $\mathbf{x}$ above:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

5.1.2 Example. The characteristic equation for the ODE

$$\ddot{x} + 4\dot{x} + 3x = 0$$

is

$$\lambda^2 + 4\lambda + 3 = 0,$$

equivalently,

$$(\lambda + 1)(\lambda + 3) = 0.$$

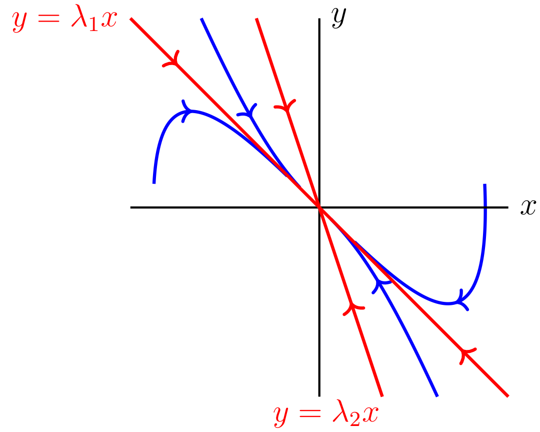
Thus its roots are $\lambda = -1$, -3, and so all solutions to this ODE are

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

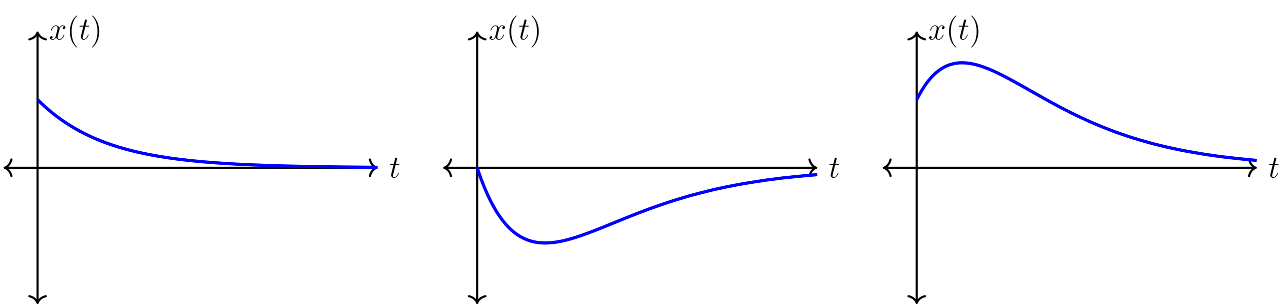
In the case of the harmonic oscillator, both eigenvalues λ_1 and λ_2 are negative.

5.1.3 Problem ($\star$). Prove this. [Hint: *for the harmonic oscillator, $p \geq 0$, so $-p \leq 0$. If $p = 0$, then we have complex, nonreal roots, so assume $p > 0$. Then it is definitely the case that $\lambda_2 < 0$. (Why?) We have $\lambda_1 < 0$ if and only if $\sqrt{p^2 - 4q} < p$. Square both sides of this inequality and use the fact that $q > 0$.*]

So, for the harmonic oscillator, we are in the case of two distinct, negative eigenvalues, and therefore the origin is a sink for $\dot{\mathbf{x}} = A\mathbf{x}$. We call this oscillator **OVERDAMPED**, chiefly in contrast to the "underdamped" case when the eigenvalues are complex and nonreal (see below). Moreover, due to the structure of the eigenvectors in (5.1.3), all straight-line solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ lie on the lines $y = \lambda_1 x$ and $y = \lambda_2 x$, which have negative slope. The phase portrait for the harmonic oscillator might then look like the following.



Here are some possible graphs for displacement x. In general, solutions when there are two distinct negative eigenvalues will either monotonically decrease (or increase) to 0 or have one local maximum (minimum) and then decrease (increase) monotonically to 0 after that extreme value.



5.1.2. There is one repeated real eigenvalue.

Here we assume

$$b^2 - 4m\kappa = 0 \iff p^2 - 4q = 0.$$

Note that since $m \neq 0$, we can solve for κ as $\kappa = b^2/4m$, and likewise for q as $q = p^2/4$. For the oscillator, this implies a precise quantitative relationship among the mass, the damping constant, and the spring constant; more generally, this case creates a specific relationship among the coefficients of the ODE (5.1.1).

The only eigenvalue is then

$$\lambda = -\frac{b}{4m} = -\frac{p}{2},$$

and an eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

All solutions to $\dot{\mathbf{x}}$ therefore have the form

$$\mathbf{x}(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} + c_2 \left[t e^{\lambda t} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} + e^{\lambda t} \mathbf{w} \right],$$

where $\mathbf{w}$ needs to satisfy

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

Remarkably, here we can take

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is due to the very special structure of A.

5.1.4 Problem ($\star$). Check this in two ways. First, with $\mathbf{w} = (0, 1)$ and $\lambda = -p/2$, directly calculate $(A - \lambda I)\mathbf{w}$ and show that this equals $\mathbf{v} = (1, \lambda)$. Second, find all solutions $\mathbf{w}$ to the system $(A - \lambda I)\mathbf{w} = \mathbf{v}$ with $\lambda = -p/2$ and $\mathbf{v} = (1, \lambda)$; show that one such solution is $\mathbf{w} = (0, 1)$.

Thus all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ here are

$$\mathbf{x}(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} + c_2 \left[t e^{\lambda t} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} + e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

Extracting the first component, all solutions to the ODE are

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

Note that the vector $\mathbf{w}$ contributes nothing to the first component!

5.1.5 Example. The characteristic equation for the ODE

$$\ddot{x} + 2\dot{x} + x = 0$$

is

$$\lambda^2 + 2\lambda + 1 = 0,$$

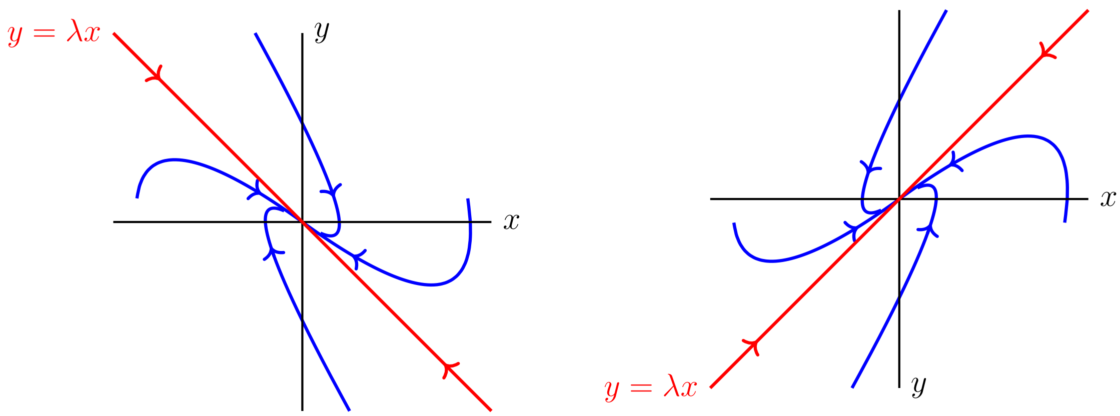
which factors into

$$(\lambda + 1)^2 = 0.$$

Thus the only root is $\lambda = -1$, so all solutions to the ODE are

$$x(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

In the case of the harmonic oscillator, we have $\lambda = -p/2 < 0$, as $p \geq 0$, and if $p = 0$, then we have complex, nonreal eigenvalues. We call this harmonic oscillator **CRITICALLY DAMPED**—the value p is "critical" in the sense that if it is made slightly larger, then we have two distinct real eigenvalues, and if it is made slightly smaller, then we have complex, nonreal eigenvalues. All straight-line solutions lie on the line $y = \lambda x$, which has negative slope. All other solutions "twist" into the origin according to one of the two phase portraits below.



One possible graph for displacement x is the following; there are other variations, depending on how the arbitrary constants c_1 and c_2 are selected, but all variations are similar to the overdamped oscillator.

5.1.3. There are two complex conjugate eigenvalues.

Here we assume

$$b^2 - 4m\kappa < 0 \iff p^2 - 4q < 0.$$

The eigenvalues are then $\lambda = \alpha \pm \beta i$, where

$$\alpha = -\frac{b}{2m} = -\frac{p}{q} \quad \text{and} \quad \beta = \frac{\sqrt{4m\kappa - b^2}}{2m} = \frac{\sqrt{4q - p^2}}{2}.$$

Note that since $b^2 - 4m\kappa < 0$, we have $4m\kappa - b^2 > 0$, and likewise $4q - p^2 > 0$, so the square root does indeed produce a positive real number. In particular, while α may be positive, negative, or 0, we always have $\beta > 0$.

An eigenvector for A corresponding to $\alpha + \beta i$ is

$$\mathbf{v} = \begin{pmatrix} 1 \\ \alpha + \beta i \end{pmatrix} = \mathbf{v_r} + i\mathbf{v_i} \quad \text{where} \quad \mathbf{v_r} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \quad \text{and} \quad \mathbf{v_i} = \begin{pmatrix} 1 \\ \beta \end{pmatrix}.$$

From the demanding structure of the fundamental solution set in (4.5.12), we conclude that all solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are

$$\mathbf{x}(t) = e^{\alpha t} \begin{pmatrix} c_1 \cos(\beta t) + c_2 \sin(\beta t) \\ -c_1 \beta \sin(\beta t) + c_2 \beta \cos(\beta t) \end{pmatrix}. \tag{5.1.4}$$

5.1.6 Problem (!). Chase through the algebra and check this.

From the first component of (5.1.4), all solutions to the ODE are then

$$x(t) = e^{\alpha t}\big(c_1 \cos(\beta t) + c_2 \sin(\beta t)\big). \tag{5.1.5}$$

This factorization emphasizes the separate roles of α and β. The parameter β, which is always positive, induces oscillations from the sine and cosine terms. The parameter α induces decay if $\alpha < 0$ or blow-up if $\alpha > 0$, whereas when $\alpha = 0$ the solutions are purely sinusoidal and thus oscillatory.

5.1.7 Example. **(i)** The characteristic equation for the ODE

$$\ddot{x} + 4x = 0$$

is

$$\lambda^2 + 4 = 0,$$

and hopefully we see immediately that its roots are

$$\lambda = \pm 2i.$$

Here $\alpha = 0$ and $\beta = 2$, so all solutions to the ODE are

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

The solutions are purely oscillatory, and there is no exponential growth or decay.

(ii) The characteristic equation for the ODE

$$\ddot{x} + 4\dot{x} + 13 = 0$$

is

$$\lambda^2 + 4\lambda + 13 = 0,$$

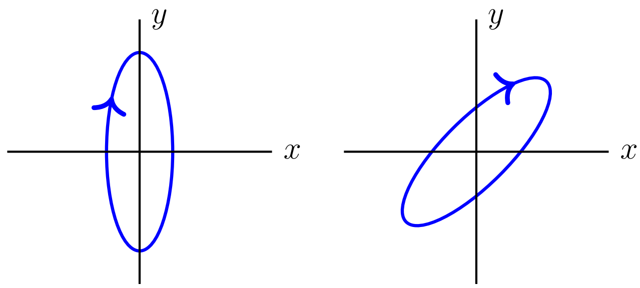
and (after going to the quadratic formula and doing some arithmetic), its roots are

$$\lambda = -2 \pm 3i.$$

Here $\alpha = -2$ and $\beta = 3$, and so all solutions to the ODE are

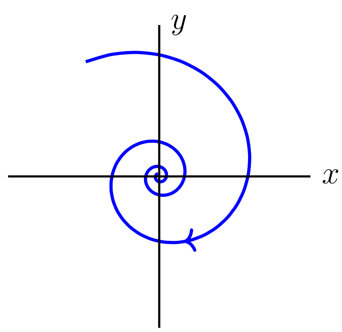
$$x(t) = e^{-2t}\big(c_1 \cos(3t) + c_2 \sin(3t)\big).$$

In the case of the harmonic oscillator, we have $\alpha = -p/2 \leq 0$. If $\alpha = p = 0$, then solutions are purely sinusoidal, and the oscillator is (as we have said many times before) UNDAMPED. In the phase portrait, solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ lie on ellipses centered at the origin, which is a center for this system. Specifically, these elliptical trajectories move clockwise.



5.1.8 Problem ($\star$). Explain why. [Hint: *in the direction field for $\dot{\mathbf{x}} = A\mathbf{x}$, in what direction does the arrow through $(r, 0)$ point when $r > 0$? See the table after Example 3.3.4 to determine the direction of the arrow from the signs of $\dot{x}$ and $\dot{y}$; remember that here $\dot{x} = y$ and $\dot{y} = -px - qy$.*]

However, if $\alpha < 0$ (i.e., $p > 0$), then solutions decay exponentially, and the oscillator is **UNDERDAMPED**. Nonetheless, even if $\alpha < 0$, we always have $\beta > 0$, and so the sinusoidal terms in (5.1.5) will contribute small (but decaying) oscillations to the graph of x. In the phase portrait, solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ lie on spirals that move into the origin, and so the origin is a spiral sink for this system. Each intersection of the spiral with the y-axis represents a root of the component x; there are infinitely many roots, but the extreme values of x between successive roots decay to 0 due to the negative exponential factor. The spirals move clockwise, like the ellipses in the center case.



5.1.9 Problem ($\star$). Use the hint for Problem 5.1.8 to explain why.

Because of the presence of these oscillations, solutions for the harmonic oscillator with $\alpha < 0$ do not monotonically decay to 0 over long times; they keep oscillating between (successively smaller!) positive and negative values. For this reason, in contrast with the case of two distinct, negative real eigenvalues, the oscillator with $\alpha < 0$ is **UNDERDAMPED**—the damping force is not strong enough to keep the oscillator strictly to the right or left of equilibrium. See the graph in Example 4.5.41.

This is where we finished on Wednesday, November 15, 2023.

5.2. The nonhomogeneous problem.

Now we study the nonhomogeneous problem

$$\ddot{x} + p\dot{x} + qx = g(t). \tag{5.2.1}$$

For convenience, we are assuming that the coefficient on $\ddot{x}$ is 1. When $p \geq 0$ and $q > 0$, we view (5.2.1) as modeling a harmonic oscillator driven by some external force (i.e., some force in addition to the ever-present spring force and the possibly present damping force), and so we will call g the **FORCING** or **DRIVING** term.

We can first say quite a bit about the structure of solutions to this ODE. Suppose that we have found one particular solution $x_\star$ to (5.2.1) and that x is any other solution. Then

$$x(t) = \phi(t) + x_\star(t),$$

where ϕ is some solution to the homogeneous problem

$$\ddot{x} + p\dot{x} + qx = 0.$$

Specifically, ϕ will have the form

$$\phi(t) = \begin{cases} c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}, & \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are the distinct real eigenvalues} \\ c_1e^{\lambda t} + c_2te^{\lambda t}, & \text{if } \lambda \text{ is the only repeated real eigenvalue} \\ e^{\alpha t}\big(c_1\cos(\beta t) + c_2\sin(\beta t)\big), & \text{if } \alpha \pm \beta i (\beta \neq 0) \text{ are complex conjugate eigenvalues.} \end{cases}$$

In other words, any solution to the nonhomogeneous problem is the sum of one particular solution to the nonhomogeneous problem and some solution to the homogeneous problem.

5.2.1 Problem (!). Prove this. Specifically, given two solutions x and $x_\star$ to (5.2.1), put $\phi(t) := x(t) - x_\star(t)$. Show that $\ddot{\phi} + p\dot{\phi} + q\phi = 0$. How does this resemble the linearity principle for first-order linear ODE (Theorem 2.7.20)?

We can always solve the nonhomogeneous problem (5.2.1), at the cost of having to work with a complicated integral.

5.2.2 Theorem. *Let g be continuous on an interval I containing 0 and let ϕ_{10} and ϕ_{01} solve the homogeneous IVP*

$$\begin{cases} \ddot{\phi}_{10} + p\dot{\phi}_{10} + q\phi_{10} = 0 \\ \phi_{10}(0) = 1 \\ \dot{\phi}_{10}(0) = 0 \end{cases} \qquad \textit{and} \qquad \begin{cases} \ddot{\phi}_{01} + p\dot{\phi}_{01} + q\phi_{01} = 0 \\ \phi_{01}(0) = 0 \\ \dot{\phi}_{01}(0) = 1. \end{cases}$$

Then the function

$$x_\star(t) := \int_0^t \phi_{01}(t - \tau)g(\tau)\, d\tau$$

satisfies

$$\begin{cases} \ddot{x}_\star + p\dot{x}_\star + qx_\star = g(t) \\ x_\star(0) = 0 \\ \dot{x}_\star(0) = 0, \end{cases}$$

and the only solution to

$$\begin{cases} \ddot{x} + p\dot{x} + qx = g(t) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

is

$$x(t) := x_0\phi_{10}(t) + y_0\phi_{01}(t) + x_\star(t).$$

5.2.3 Problem (+). **(i)** Use Theorem 4.6.11 to prove Theorem 5.2.2. [Hint: *in Theorem 4.6.11, take* $\mathbf{b}(t) = (0, g(t))$ *and* $\mathbf{x}_0 = \mathbf{0}$. *Due to the structure of the underlying A here, it will be the case that the homogeneous solutions* $\boldsymbol{\phi}_1$ *and* $\boldsymbol{\phi}_2$ *in Theorem 4.6.11 will have the form* $\boldsymbol{\phi}_1 = (\phi_{10}, \dot{\phi}_{10})$ *and* $\boldsymbol{\phi}_2 = (\phi_{01}, \dot{\phi}_{01})$. *Here the subscripts are meant to reflect the initial conditions that* $\boldsymbol{\phi}_1$ *and* $\boldsymbol{\phi}_2$ *meet.*]

(ii) How does this result resemble Theorem 2.7.23?

Unfortunately, the integral in this theorem is at best opaque, at worst impossible to evaluate in terms of elementary functions. However, we can adapt the method of undetermined coefficients from first-order problems to (5.2.1) when the forcing term g has one of several special forms. Fortunately, these forms are common in applications. Specifically, the following g are both mathematically tractable and physically relevant.

1. Constant forcing: $g(t) = A$ for some real number A. Such a g could represent a constant force applied continuously to an oscillator.

2. Exponential forcing: $g(t) = Ae^{rt}$ for some real numbers A and r. Such a g could represent a force that increases ($r > 0$) or decreases ($r < 0$) exponentially in strength over time.

3. Sinusoidal forcing: $g(t) = A\cos(\omega t)$ or $g(t) = B\sin(\omega t)$ for some real numbers A, B, and ω. Such a g could represent a force that varies periodically and continuously between two extremes; the parameters A and B are the extreme values of this force, and the parameter ω controls the period of the force, equivalently, the frequency at which the force varies between its extreme values.

We will study some of these forcing types in isolation, but combining them is easy.

5.2.4 Problem (!). Suppose that x_1 and x_2 are solutions to

$$\ddot{x}_1 + p\dot{x}_1 + qx_1 = g_1(t) \quad \text{and} \quad \ddot{x}_2 + p\dot{x}_2 + qx_2 = g_2(t).$$

Show that the function $y(t) := Ax_1(t) + Bx_2(t)$ solves

$$\ddot{y} + p\dot{y} + qy = Ag_1(t) + Bg_2(t)..$$

5.2.1. Sinusoidal forcing.

Here we study the problem

$$\ddot{x} + p\dot{x} + qx = A\cos(\omega t).$$

5.2.5 Example. Consider the ODE

$$\ddot{x} + 4x = \cos(3t).$$

This represents an undamped harmonic oscillator with a sinusoidal external force.

1. First we look at the homogeneous problem

$$\ddot{x} + 4x = 0.$$

The characteristic equation is

$$\lambda^2 + 4 = 0,$$

which, as we know well, has the complex conjugate roots $\lambda = \pm 2i$, and so all homogeneous solutions are

$$\phi(t) = c_1\cos(2t) + c_2\sin(2t).$$

Then all solutions to the original nonhomogeneous problem have the form

$$x(t) = c_1\cos(2t) + c_2\sin(2t) + x_\star(t)$$

for one particular solution $x_\star$ to the nonhomogeneous problem.

2. Based on our work with sinusoidal forcing for first-order ODE (Section 2.8.2), we should be inclined to guess that one particular solution to this ODE should have the form

$$x(t) = \alpha\cos(3t) + \beta\sin(3t)$$

for some constants α and β. After all, what sort of function x, on taking two derivatives and being added to itself, yields $\cos(3t)$? Surely only an x involving $\cos(3t)$ and maybe $\sin(3t)$; our first-order experience suggests that including the $\sin(3t)$ term at the start is necessary.

3. Under this guess, we have

$$\dot{x}(t) = -3\alpha\sin(3t) + 3\beta\cos(3t) \quad \text{and} \quad \ddot{x} = -9\alpha\cos(3t) - 9\beta\sin(3t).$$

Substituting this guess for x and this expression for $\ddot{x}$ into the ODE, we obtain

$$\big(-9\alpha\cos(3t) - 9\beta\sin(3t)\big) + 4\big(\alpha\cos(3t) + \beta\sin(3t)\big) = \cos(3t).$$

This simplifies to

$$-5\alpha\cos(3t) - 5\beta\sin(3t) = \cos(3t),$$

and then

$$(-5\alpha - 1)\cos(3t) - 5\beta\sin(3t) = 0.$$

4. Since we want this to be true for all t, Problem 2.8.13 tells us that we need

$$-5\alpha - 1 = 0 \quad \text{and} \quad -5\beta = 0.$$

The second equation gives us $\beta = 0$ and the first $\alpha = -1/5$.

5. So, our guess has yielded the particular solution

$$x_\star(t) = -\frac{\cos(3t)}{5}$$

for the ODE. Thus all solutions to the ODE are

$$x(t) = \phi(t) + x_\star(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{\cos(3t)}{5}.$$

In particular, all solutions are periodic and bounded.

5.2.6 Problem ($\star$). Adapt the work of the preceding example to find all solutions to

$$\ddot{x} + \omega_0^2 x = A\cos(\omega t)$$

when $\omega_0 > 0$ and $\omega_0 \neq \omega$ and A is any real number. Where explicitly in your work are you using the condition $\omega_0 \neq \omega$?

5.2.2. Resonance.

A remarkable result can occur if we force an undamped harmonic oscillator by a sinusoidal force with just the right frequency.

5.2.7 Example. We solved $\ddot{x} + 4x = \cos(3t)$ in Example 5.2.5 and saw that all solutions are periodic and bounded. Suppose we change the frequency of the forcing function and consider instead the ODE

$$\ddot{x} + 4x = \cos(2t).$$

1. As in Example 5.2.5, the homogeneous problem is $\ddot{x} + 4x = 0$, and so its solutions are $\phi(t) = c_1\cos(2t) + c_2\sin(2t)$. Again, and as always, we just need one particular solution to $\ddot{x} + 4x = \cos(2t)$.

2. A natural first step is to try the same guess as before: $x(t) = \alpha\cos(2t) + \beta\sin(2t)$. Then

$$\dot{x}(t) = -2\alpha\sin(2t) + 2\beta\cos(2t) \quad \text{and} \quad \ddot{x}(t) = -4\alpha\cos(2t) - 4\beta\sin(2t).$$

Substituting this into the ODE $\ddot{x} + 4x = \cos(2t)$, we obtain

$$\big(-4\alpha\cos(2t) - 4\beta\sin(2t)\big) + 4\big(\alpha\cos(2t) + \beta\sin(2t)\big) = \cos(2t),$$

and thus

$$\cos(2t) = 0,$$

which is impossible.

3. The problem with our first guess is that the functions $x_1(t) = \cos(2t)$ and $x_2(t) = \sin(2t)$ both solve the homogeneous problem $\ddot{x} + 4x = 0$. Substituting them into $\ddot{x} + 4x = \cos(2t)$ has absolutely no effect and does not afford us any control over α and β.

4. Something similar happened in Example 2.8.9, and we saw there that a good idea was to multiply the original guess by t. We try this again and guess

$$x(t) = t\big(\alpha\cos(2t) + \beta\sin(2t)\big).$$

Then

$$\dot{x}(t) = \alpha\cos(2t) + \beta\sin(2t) + t\big(-2\alpha\sin(2t) + 2\beta\cos(2t)\big)$$

and

$$\begin{aligned}\ddot{x}(t) &= -2\alpha\sin(2t) + 2\beta\cos(2t) + \big(-2\alpha\sin(2t) + 2\beta\cos(2t)\big) + t\big(-4\alpha\cos(2t) - 4\beta\sin(2t)\big)\\ &= -4\alpha\sin(2t) + 4\beta\cos(2t) + t\big(-4\alpha\cos(2t) - 4\beta\sin(2t)\big).\end{aligned}$$

5. Substituting these calculations into the ODE, we find that we need

$$-4\alpha\sin(2t) + 4\beta\cos(2t) + t\big(-4\alpha\cos(2t) - 4\beta\sin(2t)\big) + 4t\big(\alpha\cos(2t) + \beta\sin(2t)\big) = \cos(2t).$$

The terms with a factor of t on the left cancel each other out perfectly, and so this reduces to

$$-4\alpha\sin(2t) + 4\beta\cos(2t) = \cos(2t),$$

and thus

$$-4\alpha\sin(2t) + (4\beta - 1)\cos(2t) = 0.$$

Then we need

$$-4\alpha = 0 \quad \text{and} \quad 4\beta - 1 = 0,$$

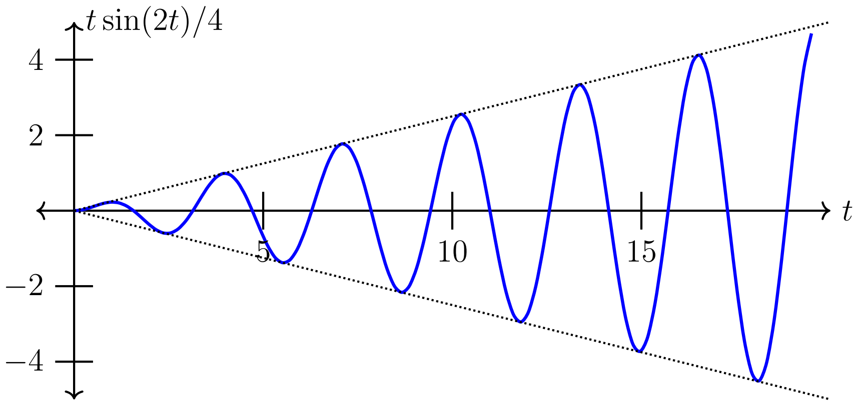
so $\alpha = 1$ and $\beta = 1/4$. Thus one particular solution to the ODE is

$$x_\star(t) = \frac{t\sin(2t)}{4},$$

and so all solutions are

$$x(t) = \phi(t) + x_\star(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{t\sin(2t)}{4}.$$

6. The key difference between solutions to this ODE $\ddot{x} + 4x = \cos(2t)$ and our previous example $\ddot{x}+4x = \cos(3t)$ is the structure of the particular solution. Here, $x_\star(t) = t\sin(2t)/4$ is neither periodic nor bounded due to the fact of t. (Consequently, no solution to $\ddot{x}+4x = \cos(2t)$ is periodic or bounded!) In fact, $x_\star$ oscillates between progressively larger values as $t \to \infty$ (which is not to say that $\lim_{t\to\infty} x_\star(t) = \pm\infty$; in fact, $x_\star$ has no limit as $t \to \infty$). Here is a graph of $x_\star$.



Note that the oscillations of $x_\star$ keep growing as t gets larger; these are the dotted lines that "envelop" the graph. We can see this analytically by taking $t = (2k+1)\pi/4$, where k is an integer, and computing

$$x_\star(t_k) = \frac{(2k+1)\pi}{16}\sin\left(\frac{(2k+1)\pi}{2}\right) = \frac{(2k+1)\pi}{16}(-1)^k + 1.$$

As $k \to \infty$, we have $x_\star(t_{2k}) \to -\infty$ and $x_\star(t_{2k+1}) \to \infty$.

7. The key difference between the structure of this ODE $\ddot{x}+4x = \cos(2t)$ and the previous example $\ddot{x}+4x = \cos(3t)$ is that the frequency of the periodic forcing term here is the same as the frequency of solutions to the homogeneous problem $\ddot{x} + 4x = 0$: both frequencies are 2. We can think of the homogeneous problem $\ddot{x}+4x = 0$ as representing material data inherent to the oscillator; here, its mass is always 1, its spring constant is always 4, and there is no friction. Given such an oscillator, we can choose to apply different forces, like $g(t) = \cos(3t)$ as before or $g(t) = \cos(2t)$ as we did now. Choosing the force to interact in a very special way with the material data of the oscillator (the number 2) led to the extreme behavior of $x_\star$.

The phenomenon of the previous example is called **RESONANCE** and broadly refers to forcing an (inherently undamped) oscillator by the same "natural" frequency at which solutions to the homogeneous problem oscillate.

5.2.8 Problem ($\star$). **(i)** Generalize the work of the preceding example to find all solutions to

$$\ddot{x} + \omega^2 x = A\cos(\omega t),$$

where A and ω are any numbers with $\omega \neq 0$. Contrast your result with that of Problem 5.2.6.

(ii) How does your work here remind you of a certain part of Problem 2.8.11?

(iii) Why did we never see resonance in the first-order problem

$$\dot{x} = ax + b\cos(\omega t)?$$

[Hint: *reread Problem 2.8.14.*]

This is where we finished on Friday, November 17, 2023.

6. The Laplace Transform

There will be no further updates to the lecture notes, as we will follow the textbook more or less verbatim. See the daily reading log.