Day 1: Monday, August 14. Page vii and the first two paragraphs of page viii in the preface discuss the philosophy of the textbook. Pages xv–xvi continue this discussion. Since the philosophy of the textbook (and of our course) is probably different from your calculus experience, and certainly different from the approach of many ODE courses, these pages will provide some additional “orientation” for you.

Day 2: Wednesday, August 16. We derived and analyzed some population models. Pages 2–4 give a broad overview of how mathematical models are constructed. Pages 4–9 discuss exponential growth. These pages treat our analytic and qualitative approaches in somewhat different language.

Day 3: Friday, August 18. We primarily discussed the logistic equation. The book constructs this model on pp. 9–10 and then performs a detailed qualitative analysis of the model (similar to Wednesday’s work with the exponential) on pp. 11–12. We will come back to this qualitative analysis rather later when we have some deeper theoretical tools to make it more rigorous.

Day 4: Monday, August 21. We talked about fundamental definitions and direct integration problems. The book gives definitions and examples of checking solutions (even if you don’t know how to find them from scratch) on pp. 21–24. Pages 23–24 present a direct integration problem. We are working through direct integration problems in somewhat more precise and technical detail with definite integrals in Section 1.5 of the lecture notes. It is essential that you read those notes carefully, including the properties of the definite integral in Theorem 1.5.3; we didn’t talk about all of those properties in class. The definite integral will be **the** key tool for solving almost all of our differential equations!

Day 5: Wednesday, August 23. We continued talking about direct integration. There isn’t anything else in the textbook on this very important topic, so you should study the lecture notes extra carefully. Remark 1.5.19 expands on my many complaints about the ambiguities of indefinite integrals and why we should only use $\int$ to denote the symbolic procedure of antidifferentiation. (Down with the indefinite integral; long live the definite integral!) We did not work through Example 1.5.20 in class; this reviews the important technique of u-substitution with indefinite integral notation.

Day 6: Friday, August 25. We defined separable ODE as on pp. 24–25. The book starts solving separable ODE with a very different example from ours. You may want to read pp. 25–27 now, as it will give you the big picture of how separation of variables works for the general problem $\dot{x} = g(t)h(x)$. We will build up to that rather more slowly. Note, though, that our “toy” problem is covered on pp. 27–28 under “Missing solutions”—this is the exact same thing that we did in class, just a bit more briskly.

Day 7: Monday, August 28. We are doing more examples than the book, and going more slowly than the book does, so there is nothing really new for today. Continue studying the reading from Day 6.

Day 8: Wednesday, August 30. We are still separating variables. We ended by complaining that solving the implicit equation for x can be hard; see "Getting Stuck" on p. 28 for a different example of being stuck.

Day 9: Friday, September 1. We finally stopped separating variables! Pages 29–33 in the book discuss two models, a "savings" model from economics and a "mixing" problem from…science…that lead to separable ODE. You might enjoy reading the derivations of these models to see how ODE arise in non-population situations. You might benefit more from reading how these models are solved via separation of variables. Note that both involve an indefinite integral of the form $\int du/u$, which leads to an antiderivative of the form $\ln(|u|)+C$. Removing the absolute value is a tricky step, and it's one of the reasons we solved exponential growth as an IVP with assumptions on the sign of the initial condition. We then started talking about slope fields and briefly touched on the material on pp. 36–39, which we will revisit next week.

Day 10: Wednesday, September 6. All of the material on slope fields on pp. 36–43 is worth reading carefully. If, after reading that, you feel that you need more information about slope fields, you might consider the applications (mixing problems, circuits) on pp. 43–47. I will not ask you to derive or analyze those models, so, if you want, you could just consider those problems as "ODE to be solved" and read how slope fields give you perspective on them as mathematical objects (not physically relevant models).

You may find the slope field plotter available at

$$\texttt{https://www.geogebra.org/m/W7dAdgqc}$$

helpful. Be sure to adjust both the length of the line segments and their density to get the picture that works best for you. Uncheck the boxes for "Solution A," etc., if you don't want to see (numerically generated) solution graphs. Our next task is to develop an algorithm that will generate those graphs.

Day 11: Friday, September 8. The introduction to Section 1.4 on p. 52 sets the stage for numerical methods and offers some helpful general perspectives. Section 1.4 then derives Euler's method using a different approach (tangent lines and local linear approximations). This is pp. 52–54. You are not required to know this, but I will eventually test you on the integral derivation. This is discussed in the book instead on pp. 648–649, which also suggests some improvements to Euler's methods via better integral approximations. Page 647 provides a broader perspective on approximation arguments via Taylor expansions; this requires some multivariable calculus, and you are not required to know it, but, again, it will broaden your perspectives if you do. The rest of Section 1.4 (pp. 55-61) is worth reading to see the *analysis* of the results of Euler's method; don't worry about performing the arithmetic yourself, but do worry about interpreting the results yourself. Look carefully at "Errors" on p. 60 and "The Big Three" on p. 61—those big three are what our course is all about!

Day 12: Monday, September 11. Pages 63–64 gently introduce the topic of existence and uniqueness by reminding you of solving polynomial equations—it is possible to prove that a solution exists even if you can't find a formula for it. And it may be possible to prove that there is only one solution. The book then packages separately its existence and uniqueness theorems on pp. 64 and pp. 66. You are not required to know these statements from the book, as they require some multivariable calculus. However, I claim that our version in the lecture notes for separable problems *does* meet these more general hypotheses. See p. 67 for another example of an ODE that does not have unique solutions, and then see p. 68 for another paraphrase of uniqueness: "if two solutions are ever in the same place at the same time, then they are the same function."

Day 13: Wednesday, September 13. We discussed a comparison principle that arises from the existence and uniqueness theorem. This is discussed on pp. 68–69 under "Role of equilibrium solutions" and "Comparing solutions." We then started considering how the domain of a solution x to an autonomous equation $\dot{x} = f(x)$ might have something to do with f. Another "blow-up in finite time" situation appears under "Extendability" on pp. 64–65, while a good reminder of what autonomous ODE are and why they matter appears on p. 74.

Day 14: Wednesday, September 15. We distilled the observations of Wednesday's examples into the "maximal existence" theorem. Unfortunately, the book doesn't cover that, so you should read the lecture notes very, very carefully and work the problems therein (merely rereading this theorem without doing the problems won't help). Then we started a new study of the logistic equation to confirm the predictions from slope fields and Euler's method. The book does this very rapidly on p. 76 and somewhat less rapidly on pp. 10–12; we will go more leisurely than either of these treatments. For context about what we are building toward with phase lines, read pp. 74–75—the "metaphor of the rope" is very valuable.

Day 15: Monday, September 18. We finished our exhaustive (and exhausting) qualitative analysis of the logistic equation and distilled all that work into the concept of a phase line. Reread carefully the material from Day 14. Then look at the summary of drawing phase lines on pp. 76–77. We will do more examples.

Day 16: Wednesday, September 20. We drew and interpreted phase lines and then reviewed for the exam. The material on pp. 76–79 is essential. The general behavior of solutions to autonomous ODE is summarized in the three bullet points at the bottom of p. 78.

Day 17: Friday, September 22. You took an exam; the joy in the room was palpable.

Day 18: Monday, September 25. We classified equilibrium points as on pp. 84–85 and talked about linearization as on pp. 85–87. You should also read pp. 80–83, which contain further examples and reiterate some concepts in new (and hopefully helpful) language. Finally, you should read pp. 87–89 on the modified logistic equation. This discussion is a great reference for how to construct a simple mathematical model and analyze it qualitatively.

Day 19: Wednesday, September 27. The textbook does not do harvesting problems, but it does do mixing problems, which are morally the same. Pages 31–33 discuss a mixing problem that leads to a homogeneous linear ODE, while pp. 129–130 set up a different mixing problem that leads to a nonhomogeneous linear ODE. If you want to see where linear ODE come from, these are good examples. Next, see pp. 110–112 for terminology and examples. Finally, pp. 112–114 remind us of how to solve homogeneous linear ODE via separation of variables. Next up is the nonhomogeneous case!

Day 20: Friday, September 29. We started solving nonhomogeneous problems with the method of variation of parameters. This is *not* how the textbook does things in Section 1.9, which presents integrating factors (on pp. 125–128). My opinion is that integrating factors are largely deprecated and archaic and do not generalize well to other problems, unlike variation of parameters. The integrating factor method hinges on rewriting the ODE $\dot{x} = a(t)x+b(t)$, which is a form that arises very naturally in applications, as $\dot{x}-a(t)x = b(t)$, which is a form that does not arise very naturally in applications. Then one notices that $\dot{x} - a(t)x$ looks almost like the product rule and multiplies both sides of $\dot{x} - a(t)x = b(t)$ by a new function to make the left side equal to the derivative of a product. Weird stuff! No weirder than guessing $x(t) = u(t)e^{A(t)}$, as we did with variation of parameters, but my experience is that people (quite naturally!) have difficulty "recognizing" the product rule after multiplying by the integrating factor. Hopefully *using* the product rule to calculate the derivative of $x(t) = u(t)e^{A(t)}$ is easier. So, if you want to learn integrating factors instead of variation of parameters, go right ahead, and I will accept solutions correctly done with integrating factors on quizzes and exams. However, I will not provide any solutions using integrating factors.

Day 21: Monday, October 2. Example 2.7.18 in the lecture notes is treated with integrating factors on pp. 127–128 of the textbook. Try solving the ODE in "Problems with the integration" on p. 128 with variation of parameters and note that you encounter the same antiderivative that stumps the book. Pages 114–115 discuss the linearity principle. Otherwise, you will need to rely on the lecture notes more than the book for material corresponding to today. I will assign some of the problems in Section 2.7.5 for the next problem set and expect that you have read (but definitely not memorized) Theorem 2.7.22. I definitely do not expect that you can get the formula (2.7.19) for the harvesting solution by hand; this is why the year is 2023 and we have computers. I hope you feel confident that with enough computing time and elbow grease, you could eventually get it, and that you appreciate how sensitive the parameter h_0 is to the other data x_0, r, and p.

Day 22: Wednesday, October 4. We developed the method of undetermined coefficients for solving certain constant-coefficient linear ODE. The book develops this on pp. 116–121. You should read every page and every example here very carefully. Section 2.8.4 in the lecture notes is a summary, which we did not discuss in class, and which will give you more perspectives on more complicated constant-coefficient problems.

Day 23: Friday, October 6. We finally moved on from first-order ODE…right back to the start of the book. Pages 12–13 develop the predator prey model. Pages 150–151 find equilibrium solutions to a predator-prey model with specific coefficients.

Day 24: Monday, October 9. We developed qualitative and numerical tools for analyzing systems. Geogebra has many phase portrait/plane and vector/direction field apps, including this one:

`https://www.geogebra.org/m/utcMvuUy`.

Pages 150–154 present numerical results and phase portraits for a predator-prey model, and pp. 155–156 do a "modified" model in which the prey grows logistically, not exponentially. Pages 166–168 develop vector/direction fields; the term "direction field" appears at the bottom of p. 169 in the context of a harmonic oscillator model (we will study that model later, and excessively). Skim pp. 170–172 on vector notation, which we will develop later, but read the "metaphor of the parking lot" on p. 172 closely. You can see all of these numerical and qualitative techniques in action on one further population model on pp. 175–178. If you are curious about how to implement numerical methods for systems, you can read Section 2.5, but you are not required to. Figures 2.43 and 2.44 on p. 196 offer another example of how to go from individual x- and y-graphs to the graph of the parametric curve (x, y).

Day 25: Wednesday, October 11. Section 2.7 develops the SIR model. You should read this entire section carefully, although you may omit the "analytic description of the solution curves."

Day 26: Friday, October 13. We started talking about the harmonic oscillator. We only did the undamped oscillator, but here are all the details that we will continue studying next time. Pages 156–161 develop the undamped harmonic oscillator, and pp. 168–170 give its direction field. Section 2.3 develops the damped harmonic oscillator and gives some pictures of direction fields and solution graphs. Page 240 converts the damped harmonic oscillator to a linear system. And pp. 388–389 develop the driven harmonic oscillator.

Day 27: Monday, October 16. We finished deriving the different harmonic oscillator models; see the readings for Day 26 and the direction fields and numerics included in them. Pages 167–168 discuss vector notation for derivatives, and pp. 243–246 discuss matrices.

Day 28: Wednesday, October 18. We converted the first-order system for the harmonic oscillator into the language of vectors and matrices. This is developed (for linear homogeneous systems) on pp. 243–246, and equilibrium solutions are discussed on pp. 246–248. Pages 189–190 give an example of checking that a vector-valued function solves a certain linear system: plug and chug.

Day 29: Friday, October 20. We discussed more properties of equilibrium solutions to linear homogeneous and nonhomogeneous systems. Pages 246–249 introduce the determinant and its applications to linear homogeneous problems. You will have to rely on the lecture notes for the nonhomogeneous case and the matrix inverse.

Day 30: Monday, October 23. We discussed linear independence and totally and partially decoupled linear systems. Page 191 presents a totally decoupled system, and pp. 192–194 do the partially decoupled. Read that first. Then read about linearity on pp. 249–251. After that, read pp. 252–255 on solving initial value problems (which uses linearity and linear independence) and pp. 255–256 to see fundamental solution sets in action (although the book does not use this terminology). Another example of a fundamental solution set appears on pp. 256–258, although we will have to do some more work to see how to construct this set from scratch.

Day 31: Wednesday, October 25. The book motivates the guess $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ for solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ a little differently than we did. Pages 264–266 discuss these solutions as "straight-line solutions" because of how their parametric graphs look; we will do this a bit later. The book concludes the same eigenvalue-eigenvector relationship $A\mathbf{v} = \lambda\mathbf{v}$ on p. 267 as we did. Pages 267–268 compute eigenvalues and eigenvectors for a particular matrix. Then pp. 268–273 give more examples and theory. You should work through every detail on these pages for more practice.

Day 32: Friday, October 27. Exam 2.

Day 33: Monday, October 30. The book "puts it all together" on pp. 273–275: this is where you see eigenvalues and eigenvectors coming together to give straight-line solutions that really form a fundamental solution set. Read these pages carefully. Then see the examples on pp. 275–277.

Day 34: Wednesday, November 1. The material under "Sinks" on pp. 285–289 and "Sources" on pp. 289–290 is essential. You should read this very, very carefully. The Geogebra program

`https://www.geogebra.org/m/utcMvuUy`.

will help you plot phase portraits for systems, and I encourage you to put in the systems from the book and from class to try to replicate their dynamics online.

Day 35: Friday, November 3. Pages 281–284 present saddles. Then look at pp. 290–291 to contrast sinks, sources, and saddles from the point of view of stability, similar to what we saw with the phase line. Pages 291–293 do an economics model, based on the set-up on p. 241; this might be helpful if you are wondering about useful applications of systems beyond the harmonic oscillator. Pages 325–326 discuss systems with 0 as an eigenvalue along with a positive or negative eigenvalue.

Day 36: Monday, November 6. Pages 324–325 discuss the very special case of a repeated eigenvalue with linearly independent eigenvectors. Pages 315–318 discuss the more complicated case of a repeated eigenvalue with only one linearly independent eigenvector. The example here is essentially the one from class, except the eigenvalue is -2, not -1. The form of the general solution at the bottom of p. 317 is a little different from our fundamental solution set in class, but both give useful information.

Day 37: Wednesday, November 8. The book constructs all solutions to linear systems with repeated eigenvalues and one linearly independent eigenvector on pp. 318–319. This is different from our formulation in class and in the notes. Problem 4.5.28 in the lecture notes fills in the gaps between the two approaches. Then revisit the system from pp. 315–317 at the bottom of p. 318. Look at the "twists" in the phase portraits on p. 320. Finally, look at the harmonic oscillator example on p. 321.

We then moved to the thrilling case of complex eigenvalues. See pp. 750–751 for the only things you need to know about complex numbers (skip "Geometry of complex numbers" on p. 751) and then read p. 752 on Euler's formula. Read carefully the calculations on pp. 296–302. This is essentially the calculation that we did for the very simple ODE $\ddot{x} + x = 0$ in class today. We will make this a little more abstract, more general, and more systematic in our next meeting.

Day 38: Friday, November 10. Reread everything from Wednesday on pp. 296–302 in light of the general calculation that we did today. Do you see how α, β, $\mathbf{v_r}$, and $\mathbf{v_i}$ from the lecture notes match up with the particular calculation in the book?

Day 39: Monday, November 13. Pages 303–310 cover phase portraits for systems with complex eigenvalues. These chiefly offer some additional concrete examples of the different cases. The material on variation of parameters for systems is not in the book, so you will have to rely on the lecture notes (but take heart and be of good cheer that while you need to understand how to derive that formula, you will never, ever have to calculate anything concrete with it in this class).

Day 40: Wednesday, November 15. Section 3.6 imports techniques from linear systems to study second-order linear ODE. You should read every example in this section very carefully. The book uses the language of "overdamped," "critically damped," and "underdamped" oscillators, which we did not use explicitly in class. You might also want to look at Figure 3.48 on p. 351 to see a broad smattering of phase portraits; think of T as $-p$ and D as the coefficient q in the ODE $\ddot{x} + p\dot{x} + qx = 0$. Then look at Figure 3.49 on p. 352 to see the restriction of this smattering to harmonic oscillators with $p \geq 0$ (so $T \leq 0$) and $q > 0$.

Day 41: Friday, November 17. Pages 388–389 review the structure of forced, or driven, harmonic oscillators/nonhomogeneous second-order linear ODE. See the first paragraph on p. 389 for a discussion of some of the most physically meaningful forcing terms. Pages 390–391 prove our (briefly sketched) claim that all solutions to a nonhomogeneous second-order linear ODE are the sum of a solution to the homogeneous problem and one "particular" solution to the nonhomogeneous problem. We studied only sinusoidal forcing today. Page 403 discusses the utility of such forcing functions in applications; see also "Physical interpretation" on p. 407. The book uses complex exponentials instead of our guess with sines and cosines; see pp. 404–405 for an overview of this alternative, equally valid method for a damped harmonic oscillator. Pages 415–416 use similar exponential techniques to solve an undamped harmonic oscillator—note what happens, ominously, when ω is close to $\pm\sqrt{2}$. Then see pp. 421–423 for the solution when $\omega = \sqrt{2}$, and in particular note the graphs in Figure 4.20 and 4.21.

Day 42: Monday, November 27. No class.

Day 43: Wednesday, November 29. We covered verbatim the material on pp. 567–574, omitting the verification of how the transform interacts with the derivative (see the bottom of p. 569). Another example of solving an ODE using Laplace transforms appears on pp. 574–576. Page 566 gives some philosophy and (attempted) motivation for the strange definition of the transform,