Preliminaries. If S is a set, we write $x \in S$ to mean that x is an element of S. For example, $1 \in \{0, 1, 2\}$. The set of all real numbers is $\mathbb{R}$; the set of all column vectors with n rows is $\mathbb{R}^n$; and the set of all $m \times n$ matrices is $\mathbb{R}^{m \times n}$. We call elements of $\mathbb{R}^{1 \times n}$ row vectors. For example,

$$0 \in \mathbb{R}, \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2, \qquad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \in \mathbb{R}^{1 \times 3}, \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \qquad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \in \mathbb{R}^{3 \times 2},$$

$$\text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2 \times 3}.$$

We typically identify $\mathbb{R} = \mathbb{R}^1 = \mathbb{R}^{1 \times 1}$ and $\mathbb{R}^n = \mathbb{R}^{n \times 1}$. That is, real numbers, column vectors with one row, and row vectors with one column are all the same. Likewise, column vectors with n rows and matrices with n rows and one column are the same. However, a column vector with n rows is not the same as a row vector with n columns. That is, $\mathbb{R}^n \neq \mathbb{R}^{1 \times n}$. We can (and will) make these distinctions precise by defining vectors and matrices as *functions* of their indices; this is comforting set-theoretically and more or less useless practically.

Instructions. The following problems are designed to test your mastery of standard material in MATH 3260: Linear Algebra I. I won't collect or grade them, and we'll probably revisit and review many of these topics throughout the term (albeit at a faster pace and in a more abstract framework than in Linear I). If you don't know what a word or concept means, you should look it up in your Linear I textbook or another linear algebra source (or online). You may have met concepts under various synonyms, e.g., "kernel" for "null space" or "linear transformation" for "linear operator." There are many valid ways to express things.

You should not be alarmed if you struggle with some of these problems, especially if it has been some time since your Linear I class. However, any struggles here will indicate things that you need to review. In particular, if you find the more computational problems with matrix-vector arithmetic difficult, you should work very hard at that. Those mechanics will be essential to the new and more advanced that we'll develop. While we will review them in class, we will not spend as much time on them as you did in Linear I, and the new topics in the course will be very difficult without a solid understanding of computational matrix-vector arithmetic and algebra.

1. Compute each of the following vector operations or explain why it is undefined:

(i) $2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

2. Pairs of matrices A and vectors $\mathbf{x}$ are given below. For each pair, decide if the matrix-vector product $A\mathbf{x}$ is defined and determine how many rows it has; do this without doing any actual calculations. Then compute $A\mathbf{x}$, if it is defined.

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(v) $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

(vi) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$

3. Let ℓ, $d \in \mathbb{R}$ with $d \neq 0$. Explain in words what elementary row operations the matrix-vector multiplications

$$\begin{bmatrix} 1 & 0 & 0 \\ -\ell & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

perform. How about

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}?$$

4. What are the transposes of the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 6 & 7 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}?$$

5. Pairs of matrices A and B are given below. For each pair, determine if the matrix product AB is defined and determine how many rows and columns it has; do this without doing any actual calculations. Then compute AB, if it is defined.

(i) $A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 0 & 7 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 4 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$

6. **(i)** If A and B are square matrices, what does it mean for B to be the inverse of A?

(ii) What is the inverse of

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}?$$

(iii) Why is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

not invertible?

7. In the following, $A \in \mathbb{R}^{3\times 3}$.

(i) Let d_1, d_2, $d_3 \in \mathbb{R}$ and let

$$D := \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}.$$

Describe in words the effect of multiplying DA versus multiplying AD.

(ii) Let

$$P = \begin{bmatrix} 0 & 1 & 00 & 0 & 11 & 0 & 0 \end{bmatrix}.$$

Describe in words the effect of multiplying PA versus multiplying AP.

8. Let

$$\mathbf{x} := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{y} := \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{e}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Compute the dot products

$$\mathbf{x} \cdot \mathbf{x}, \qquad \mathbf{x} \cdot \mathbf{y}, \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{e}_2.$$

What do you notice about the nature of your numerical results?

9. Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

10. Let b_1, b_2, $b_3 \in \mathbb{R}$. Suppose that the problem

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution. What does this tell you about b_1, b_2, and b_3?

11. Explain why the list of vectors

$$\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right)$$

spans $\mathbb{R}^2$ but is not a basis for $\mathbb{R}^2$.

12. Explain why the list of vectors

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right)$$

is linearly independent but does not form a basis for $\mathbb{R}^4$.

13. Let

$$\mathcal{V} := \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1,\ x_2 \in \mathbb{R} \right\}.$$

Explain why $\mathcal{V}$ is a subspace of $\mathbb{R}^3$. Find a basis for $\mathcal{V}$. What is the dimension of $\mathcal{V}$?

14. Explain why the set

$$\mathcal{V} := \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1,\ x_2 \in \mathbb{R} \right\}$$

is not a subspace of $\mathbb{R}^3$. Give as many reasons as you can.

15. Let $\mathcal{C}([0,1])$ be the set of all continuous real-valued functions on the interval $[0,1]$. Show that $\mathcal{C}([0,1])$ is an infinite-dimensional vector space.

16. With $\mathcal{C}([0,1])$ as the space of all continuous real-valued functions on the interval $[0,1]$, for which $\alpha \in \mathbb{R}$ is the set

$$\mathcal{V}_\alpha := \{ f \in \mathcal{C}([0,1]) \mid f(0) = \alpha \}$$

a subspace of $\mathcal{C}([0,1])$?

17. Give three different examples of matrices $A \in \mathbb{R}^{3\times 5}$ in row reduced echelon form (sometimes called "reduced row echelon form," and either way hereafter abbreviated as RREF) with pivots in columns 1 and 4 only.

18. Let

$$A := \begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ 2 & 4 & 2 & 14 & 2 \\ 0 & 0 & 2 & 8 & 0 \end{bmatrix}.$$

(i) Show that the RREF of A is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(ii) Find a basis for and the dimension of the column space of A. Call this space $\mathsf{C}(A)$.

(iii) Find a basis for and the dimension of the null space of A. Call this space $\mathsf{N}(A)$.

(iv) Find a basis for and the dimension of the column space of A^{T}.

(v) Find a basis for and the dimension of the null space of A^{T}.

(vi) Compute $\dim[\mathsf{C}(A)] + \dim[\mathsf{N}(A^{\mathsf{T}})]$. Compute $\dim[\mathsf{N}(A)] + \dim[\mathsf{C}(A^{\mathsf{T}})]$. What do you observe relative to the number of rows and columns of A?

(vii) Check that $\mathsf{C}(A)$ and $\mathsf{N}(A^{\mathsf{T}})$ are orthogonal subspaces in the sense that if $\mathbf{x} \in \mathsf{C}(A)$ and $\mathbf{y} \in \mathsf{N}(A^{\mathsf{T}})$, then $\mathbf{x} \cdot \mathbf{y} = 0$. Do the same for $\mathsf{N}(A)$ and $\mathsf{C}(A^{\mathsf{T}})$. [Hint: *just check the orthogonality of the bases.*]

19. For

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2,$$

define

$$\mathcal{T}\mathbf{x} := \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

(i) Show that $\mathcal{T}$ is a linear operator on $\mathbb{R}^2$.

(ii) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $\mathcal{T}\mathbf{x} = A\mathbf{x}$.

(iii) Now let

$$\widetilde{\mathcal{T}}\mathbf{x} := \begin{bmatrix} x_1 \\ 1 \end{bmatrix}.$$

Explain why $\widetilde{\mathcal{T}}$ is not a linear operator.

20. Three maps $\mathcal{T}$ are defined on the space $\mathcal{C}([0,1])$ of continuous real-valued functions on the interval $[0,1]$. Which of these maps are linear operators?

(i) For $f \in \mathcal{C}([0,1])$, $\mathcal{T}f$ is the real number

$$\mathcal{T}f := \int_0^1 f(x)\, dx.$$

(ii) For $f \in \mathcal{C}([0,1])$, $\mathcal{T}f$ is the real number

$$\mathcal{T}f := \int_0^1 |f(x)| \, dx.$$

(iii) For $f \in \mathcal{C}([0,1])$, $\mathcal{T}f$ is the function on $[0,1]$ given by

$$(\mathcal{T}f)(x) := \int_0^x f(s) \, ds, \; 0 \leq x \leq 1.$$