Set Theory and Quantifiers

This document outlines essential concepts, vocabulary, and notation that we will use frequently and without much comment.

Set theory.

1 Undefinition. *A* **SET** *is a collection of objects, called* **ELEMENTS**. *If x is an element of the set A, then we write $x \in A$, and if y is not an element of the set A, then we write $y \notin A$.*

This is an undefinition, not a definition, because we have not defined what "collections" or "objects" means. And we will not. If a set A consists of only finitely many elements, then we may denote A by listing those elements between curly braces. For example, the set consisting precisely of the numbers 1, 2, and 3 is $\{1, 2, 3\}$; the set consisting precisely of the number 1 is $\{1\}$, and $1 \in \{1\}$.

2 Example. Let $A = \{1, 2, 3\}$. Then $1 \in A$ but $4 \notin A$.

If A is a set, and if $P(x)$ is a statement that is either true or false for each $x \in A$, then we denote the set of all elements x of A for which $P(x)$ is true by

$$\{x \in A \mid P(x)\}.$$

We read the expression $P(x)$ as "it is the case that $P(x)$" or "it is the case that $P(x)$ is true."

3 Example. If $U = \{1, 2, 3\}$, then

$$\{x \in U \mid x \text{ is even}\} = \{2\}.$$

4 Definition. *A set A is a* **SUBSET** *of a set B if for each $x \in A$, it is the case that $x \in B$. That is, every element of A is an element of B. If A is a subset of B, we write $A \subseteq B$.*

In symbols,

$$A \subseteq B \iff (x \in A \Longrightarrow x \in B).$$

5 Example. $\{1, 2\} \subseteq \{1, 2, 3\}$ and $\{1, 2, 3\} \subseteq \{1, 2, 3\}$.

6 Definition. *Two sets A and B are* **EQUAL**, *written $A = B$, if $A \subseteq B$ and $B \subseteq A$. An element x and the set $\{x\}$ whose sole element is x cannot be equal: $x \neq \{x\}$.*

In symbols,

$$A = B \iff (x \in A \iff x \in B).$$

7 Hypothesis. **(i)** *There exists a set $\varnothing$ that contains no element. That is, if x is an element of any set A, then $x \notin \varnothing$.*

(ii) *An element x of a set A cannot be equal to the set $\{x\}$ whose only element is x. That is, $x \neq \{x\}$.*

8 Definition. *Let A, $B \subseteq U$. The* **UNION** *of A and B is the set*

$$A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\},$$

the **INTERSECTION** *of A and B is the set*

$$A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\},$$

and the **COMPLEMENT** *of A in B is the set*

$$B \setminus A := \{x \in B \mid x \notin A\}.$$

That is, $A \cup B$ is the set of all elements in either A or B (or both), $A \cap B$ is the set of all elements in both A and B, and $B \setminus A$ is the set of all elements in B but not in A.

9 Example. Let

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{2, 4, 6\}.$$

Then

$$A \cup B = \{1, 2, 3, 4, 6\},$$
$$A \cap B = \{2\},$$

and

$$B \setminus A = \{4, 6\}.$$

10 Problem. Let A and B be as in Example 9. Determine the elements of each of the following sets.

(i) $A \setminus B$

(ii) $(A \setminus B) \cup B$

(iii) $(A \cap B) \setminus A$

(iv) $A \setminus \varnothing$

(v) $\varnothing \setminus B$

Quantifiers.

We use the symbols $\forall$, $\exists$, and $\exists!$ to abbreviate three very common phrases. Let A be a set and, for $x \in A$, let $P(x)$ be a property that is either true or false.

• The string of symbols $\forall x \in A : P(x)$ is read as "for all $x \in A$ it is the case that $P(x)$ is true." For $\forall x \in A : P(x)$ to be true, we need to show that picking any $x \in A$ results in $P(x)$ being true.

• The string of symbols $\exists x \in A : P(x)$ is read as "there exists $x \in A$ such that $P(x)$ is true." For $\exists x \in A : P(x)$ to be true, we just need to show that $P(x)$ is true for one $x \in A$. Perhaps $P(x)$ is true for all $x \in A$ ($\forall x \in A : P(x) \Longrightarrow \exists x \in A : P(x)$), but determining that is unnecessary.

• The string of symbols $\exists! x \in A : P(x)$ is read as "there exists a unique $x \in A$ such that $P(x)$ is true." For $\exists! x \in A : P(x)$ to be true, we need to show that $P(x)$ is true for one $x \in A$ and no other. Often we do this by assuming that $P(x_1)$ and $P(x_2)$ are true for some x_1, $x_2 \in A$, and then we show that $x_1 = x_2$, whatever "=" means in the context at hand.

In our course, I will often write these symbols on the board, and you should feel free to use them during an exam, but in notes and other formal writing I typically will not.

11 Example. **(i)** The sentence "For all real numbers x it is the case that x^2 is nonnegative" compresses to

$$\forall x \in \mathbb{R} : x^2 \geq 0.$$

Here $\mathbb{R}$ denotes the set of all real numbers. This is a true statement.

(ii) The sentence "There exists a real number x such that $x^2 = 4$" compresses to

$$\exists x \in \mathbb{R} : x^2 = 4.$$

This too is true.

(iii) The sentence "There exists a unique positive real number x such that $x^2 = 4$" compresses to

$$\exists! x \in (0, \infty) : x^2 = 4.$$

And this is also true, although the sentence $\exists! x \in \mathbb{R} : x^2 = 4$ is false. In the first part of the compressed symbolic form, we could also have written

$$\exists! x > 0 : x^2 = 4$$

and used the equivalence of $x \in (0, \infty)$ and $x > 0$ to phrase things differently.

We can chain quantifiers together as much as necessary, and we will not be too picky about saying "such that" every single time we write in English words.

12 Example. The sentence "For all real numbers x, there is a real number y such that there is a unique positive real number z with $z^2 = xy$" compresses to $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \exists! z > 0 : z^2 = xy$. (What is y?)

13 Problem. Let $I \subseteq \mathbb{R}$ be an interval and let f be a real-valued function defined on I. Translate each of the following sentences into symbolic form using $\forall$, $\exists$, and/or $\exists!$ whenever possible.

(i) For all $x \in I$ and all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in I$ with $|x - y| < \delta$, then

$$|f(x) - f(y)| < \epsilon.$$

(ii) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in I$ with $|x - y| < \delta$, then

$$|f(x) - f(y)| < \epsilon.$$

(iii) There exist $x \in I$ and $\delta > 0$ such that if $y \in I$ with $|x - y| < \delta$, then

$$|f(y)| > \frac{f(x)}{2}.$$

(iv) For all $a, b \in I$ and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$, there exists $x \in I$ such that $f(x) = c$.

(v) For all $y \in \mathbb{R}$ there exists a unique $x \in I$ such that $f(x) = y$.

When *negating* quantified statements, it can be helpful to write them out symbolically and then formally flip each $\forall$ to $\exists$ and each $\exists$ to $\forall$. More precisely, the negation of the statement $\forall x : P(x)$ is $\exists x :\sim P(x)$, where $\sim P(x)$ is an abbreviation for the statement "it is not the case that $P(x)$" or "it is not the case that $P(x)$ is true" (or "it is the case that $P(x)$ is false"). Likewise, the negation of $\exists x : P(x)$ is $\forall x :\sim P(x)$. (There is no standard way to flip $\exists!$ in a negation, as the negation of unique existence is either nonunique existence or nonexistence.)

14 Example. We abbreviate the sentence "For all real numbers x, there is a real number y such that the product xy is nonnegative" by "$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : xy \geq 0$." Symbolically, its negation is $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : xy < 0$." In words, this negation reads "There is $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ the product xy is negative." (Which is true, the original sentence or its negation?)

15 Problem. Negate all but the last quantified symbolic statements from Problem 13 and then write out those negations as complete sentences using English words and none of the symbols $\forall$, $\exists$, and/or $\exists!$.