MATH 4260: LINEAR ALGEBRA II

Daily Log for Lectures, Readings, and Vocabulary

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CONTENTS

Day 1: Monday, August 12.

Material from *Linear Algebra* by Meckes & Meckes

I expect that you are very familiar with the notation and techniques of Sections 1.1, 1.2, and 1.3. We will revisit Gaussian elimination in a more rapid and abstract context later when we review matrix multiplication and the RREF.

Here are three examples, two totally made up and one not so made up, that all have some common linear algebraic features despite their heavy cosmetic differences.

1.1 Example. For what b_1, b_2, $b_3 \in \mathbb{R}$ can we find $x_1, \ldots, x_5 \in \mathbb{R}$ such that

$$\begin{cases} x_1 + 2x_2 + x_3 + 7x_4 + x_5 = b_1 \\ 2x_1 + 4x_2 + 2x_3 + 14x_4 + 2x_5 = b_2 \\ 2x_3 + 8x_4 = b_3? \end{cases}$$

If we can find such x_k, are they unique?

We proceed with "elementary row operations." Subtract 2 times the first equation from the second to find that the problem is equivalent to

$$\begin{cases} x_1 + 2x_2 + x_3 + 7x_4 + x_5 = b_1 \\ 0 = b_2 - 2b_1 \\ 2x_3 + 8x_4 = b_3. \end{cases}$$

Divide both sides of the third equation by 2 to get a second equivalent problem

$$\begin{cases} x_1 + 2x_2 + x_3 + 7x_4 + x_5 = b_1 \\ 0 = b_2 - 2b_1 \\ x_3 + 4x_4 = b_3/2. \end{cases}$$

For readability, interchange the second and third equations to get another equivalent problem

$$\begin{cases} x_1 + 2x_2 + x_3 + 7x_4 + x_5 = b_1 \\ x_3 + 4x_4 = b_3/2 \\ 0 = b_2 - 2b_1. \end{cases}$$

Last, for no good reason except hope and faith, subtract the (new) second equation from the (original) first equation to end with

$$\begin{cases} x_1 + 2x_2 + 3x_4 + x_5 = b_1 - b_3/2 \\ x_3 + 4x_4 = b_3/2 \\ 0 = b_2 - 2b_1. \end{cases}$$

The upshot of our final problem is that the "data" involving the variables is rather simpler, as there are fewer of them in play. We also see a necessary condition for solving the problem: b_1 and b_2 must satisfy $b_2 - 2b_1 = 0$. That is, $b_2 = 2b_1$, and so we will not be able to solve the problem for arbitrary right sides b_1, b_2, and b_3. For example, there will be no solutions when $b_1 = 1$ and $b_2 = 0$. Of course, the final system is the reduced row echelon form (or row reduced echelon form) of the original, and we achieved it via the three elementary row operations: subtracting a multiple of one equation (row) from another, multiplying both sides of an equation (every entry in a row) by the same nonzero number, and interchanging two equations (rows).

Now we solve the final problem: we must have

$$x_1 = (b_1 - b_3/2) - 2x_2 - 3x_4 - x_5 \quad \text{and} \quad x_3 = b_3/2 - 4x_4.$$

This immediately destroys uniqueness of the solution, even if the "solvability condition" $b_2 - 2b_1 = 0$ is met, as we are "free" (pun intended) to pick x_2, x_4, and x_5 to be any values we like, and each choice of these "free variables" will create a different solution. Consequently, the problem has infinitely many solutions when $b_2 - 2b_1 = 0$ and no solution when $b_2 - 2b_1 \neq 0$.

However, saying that a set has infinitely many members is not a very useful measurement in linear algebra; indeed, most of the interesting sets that we will study are infinite. We can get better control over the "size" of the solution set by focusing on the degrees of freedom in the solution: it looks like there are 3, coming from each of the free variables x_2, x_4, and x_5. The language of vectors will make this precise.

Here is another problem that we can phrase using functions and calculus but that really reduces to a linear system. Like the first, this is totally made up.

1.2 Example. For what quadratics q can we find a quadratic p such that

$$(x+1)p'(x) = q(x) \text{ for all } x?$$

Some further notation (really, some *coordinates*) will help: say $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$. We should view b_0, b_1, and b_2 as given, and we want to find a_0, a_1, and a_2. Since $q'(x) = 2a_2x + a_1$, we have

$$\begin{aligned}(x+1)p'(x) = q(x) \iff (x+1)(2a_2x + a_1) &= b_0 + b_1x + b_2x^2 \\ \iff 2a_2x^2 + (a_1 + 2a_2)x + a_1 &= b_0 + b_1x + b_2x^2.\end{aligned}$$

We recall that two polynomials are equal for all x if and only if their corresponding coefficients are equal, so we need

$$2a_2 = b_2, \qquad a_1 + 2a_2 = b_1, \qquad \text{and} \qquad b_0 = a_1.$$

This immediately lets us solve for

$$a_2 = \frac{b_2}{2} \quad \text{and} \quad a_1 = b_0$$

in terms of the given coefficients, but it imposes no restrictions on a_0. So, we have infinitely many solutions

$$p(x) = a_0 + b_0x + \frac{b_2x^2}{2},$$

and there is one "degree of freedom" here, coming from a_0.

However, we also have a solvability condition on the coefficients of q:

$$b_1 = a_1 + 2a_2 = b_0 + b_2.$$

That is, we can only solve this problem if $b_1 = b_0 + b_2$, and we cannot solve the problem uniquely. This is just like our first example.

1.3 Problem. We can rephrase the solution to the previous example more cleanly using derivatives. Show that if p and q are quadratics with $(x+1)p'(x) = q(x)$, then

$$p(x) = a_0 + q(0)x + \frac{q''(0)}{4}x^2$$

for some $a_0 \in \mathbb{R}$. [Hint: *what are the Taylor coefficients of q?*]

Our last example comes from differential equations (no knowledge of which is presumed for this course).

1.4 Example. A popular second-order linear ordinary differential equation is

$$\epsilon^2 f'' + f = g.$$

Here $\epsilon > 0$ and a continuous function g defined on $\mathbb{R}$ are given, and we want to find a function f on $\mathbb{R}$ such that $\epsilon^2 f''(x) + f(x) = g(x)$ for all x. The dreaded method of variation of parameters furnishes us such a solution:

$$f(x) = c_1 \cos\left(\frac{x}{\epsilon}\right) + c_2\epsilon \sin\left(\frac{x}{\epsilon}\right) + \frac{1}{\epsilon}\int_0^x \sin\left(\frac{x-\tau}{\epsilon}\right) g(\tau)\ d\tau.$$

This is not something that I presume you know off the top of your head; just accept it that every solution to the ODE above has this form for some c_1, $c_2 \in \mathbb{R}$. In fact, it should not be hard to see that $c_1 = f(0)$; with a little more work (possibly expanding the sine inside the integral using a trig addition formula), you can show $c_2 = f'(0)$.

The difference between this problem and the previous two examples is that we can *always* solve the ODE; there are no apparent solvability conditions on g. We still lack uniqueness, but we always get a solution. However, as posed this problem is too general and vague for physical reasonableness. Often we want some extra conditions on the "forcing" function g and/or the solution f. Let's impose the following.

1. The problem is inherently symmetric: if f is even, then so is $\epsilon^2 f'' + f$. Symmetries often cut down on the amount of data that we need to manage and the amount of work that we need to do. So, assume that f and g are even: $f(-x) = f(x)$ and $g(-x) = g(x)$.

2. Assume that g is sufficiently well-behaved for large x that the improper integral

$$\int_0^\infty |g(x)|\ dx := \lim_{b\to\infty}\int_0^b |g(x)|\ dx$$

converges. This integral is one way of measuring the "size" of g as a function, and its convergence says that g is "not too large." Incidentally, since g is even, this also implies that the integral $\int_{-\infty}^{0}|g(x)|\,dx$ converges, and so the integral $\int_{-\infty}^{\infty}|g(x)|\,dx$ converges, too.

3. Assume that we want f to vanish at ∞ in the sense that

$$\lim_{x\to\infty} f(x) = 0.$$

If we think of f as the response to the driving force g (maybe with f measuring the displacement of a harmonic oscillator), this says that the response dies out over long scales. By the way, since f is even, this says $\lim_{x\to-\infty} f(x) = 0$, too.

I claim that with some algebra and calculus (nothing too fancy), these conditions specify the coefficients c_1 and c_2 for us. Namely, we are forced to take

$$c_1 = \frac{1}{\epsilon}\int_0^{\infty} \sin\left(\frac{x}{\epsilon}\right) g(x)\,dx \quad \text{and} \quad c_2 = \frac{1}{\epsilon}\int_0^{\infty} \cos\left(\frac{x}{\epsilon}\right) g(x)\,dx.$$

With these choices, some more algebra of integrals gives an explicit formula for f:

$$f(x) = \frac{1}{\epsilon}\int_x^{\infty} \sin\left(\frac{x-\tau}{\epsilon}\right) g(\tau)\,d\tau.$$

But despite the uniqueness of the solution assuming the extra conditions, there is a solvability condition lurking around. Since f is even, $f'(0) = 0$, and since $c_2 = f'(0)$, we really need

$$\int_0^{\infty} \cos\left(\frac{x}{\epsilon}\right) g(x)\,dx = 0.$$

Certainly not all g satisfy this condition.

While these examples look very different, they all have substantial features in common. Every example asks us to solve an equation of the form $\mathcal{T}v = w$, where $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ is a linear operator, $\mathcal{V}$ and $\mathcal{W}$ are vector spaces, $w \in \mathcal{W}$ is given, and $v \in \mathcal{V}$ is unknown. Big picture, we can add elements of $\mathcal{V}$ and multiply them by real numbers (or maybe complex numbers) to get new elements of $\mathcal{V}$, and all the arithmetic works exactly as we think it should. Same for $\mathcal{W}$. And $\mathcal{T}$ respects linearity:

$$\mathcal{T}(v_1 + v_2) = \mathcal{T}v_1 + \mathcal{T}v_2 \quad \text{and} \quad \mathcal{T}(\alpha v) = \alpha(\mathcal{T}v)$$

for all v_1, v_2, $v \in \mathcal{V}$ and $\alpha \in \mathbb{R}$. (There's a sneaky point in here in that $v_1 + v_2$ is addition in $\mathcal{V}$ and $\mathcal{T}v_1 + \mathcal{T}v_2$ is addition in $\mathcal{W}$, but basically no one cases about that.)

The challenge common to these problems is twofold. First, $\mathcal{T}$ has a nontrivial kernel: there exists $v_0 \in \mathcal{V} \setminus \{0\}$ such that $\mathcal{T}v_0 = 0$. This destroys uniqueness: if $\mathcal{T}v = w$, then

$$\mathcal{T}(v + \alpha v_0) = \mathcal{T}v + \alpha\mathcal{T}v_0 = \mathcal{T}v + 0 = \mathcal{T}v = w,$$

too. Here $\alpha \in \mathbb{R}$ is arbitrary, and since $v_0 \neq 0$, we get infinitely many more solutions $v + \alpha v_0$.

Second, the range of $\mathcal{T}$ is not all of $\mathcal{W}$: there exists $w \in \mathcal{W}$ such that $\mathcal{T}v \neq w$ for all $v \in \mathcal{V}$. And so boom, we cannot solve $\mathcal{T}v = w$ for all $w \in \mathcal{W}$. This destroys existence of solutions.

However, it is possible to find subspaces $\mathcal{V}_0$ of $\mathcal{V}$ and $\mathcal{W}_0$ of $\mathcal{W}$ such that $\mathcal{T}v \neq 0$ for all $v \in \mathcal{V}_0 \setminus \{0\}$, and such that for all $w \in \mathcal{W}_0$, there exists $v \in \mathcal{V}_0$ such that $\mathcal{T}v = w$. That is, we can solve $\mathcal{T}v = w$ uniquely for $v \in \mathcal{V}_0$ given $w \in \mathcal{W}_0$.

That being said, the first two examples have more in common with each other than they do with the third. The first two are really finite-dimensional problems, while the third is infinite-dimensional. As much as possible in this course, we will develop ideas and results for arbitrary vector spaces, regardless of dimension—and we will fail quite often. We will see how the assumption of finite-dimensionality leads to quick proofs, how results might fail in infinite dimensions, and how more structure (specifically, the structure of functional analysis) is needed to get things to work in infinite dimensions. And as much as possible, we'll do examples following a "rule of three": see it in $\mathbb{R}^n$, see it in a finite-dimensional space that is not $\mathbb{R}^n$ (but, necessarily, isomorphic to $\mathbb{R}^n$), and see it in infinite-dimensional space (typically a vector space of functions).

If none of that made sense to you, you're probably in good company. The goal of this course is to get it to make sense: to see the common linear algebraic structure underlying these seemingly disparate problems, and to see how the tools of linear algebra make cosmetically complicated problems much simpler.

Day 2: Wednesday, August 14.

Material from *Linear Algebra* by Meckes & Meckes

Pages 378–382 of Appendix A.1 contain effectively all the information on functions that we will need in the course (and some things that we don't need right now). Note that the text does not use the ordered pair definition of function, and after this introductory material we will not, either, in practice (virtually no one does).

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Function (N), domain, codomain, range, image of a set under a function

Functions are foundational to all of mathematics. We will need functions to define vector spaces, the primary setting in which we will work, and linear operators, the primary connection between vector spaces. Moreover, essentially all vector spaces consist of functions; we will see that column vectors and matrices are functions of "discrete" variables, while some of the most interesting infinite-dimensional vector spaces consist of functions.

Here is a first stab at the definition of function.

2.1 Undefinition. *A* FUNCTION *from a set A to a set B is a rule or operation that pairs (or associates, or maps) every element of A with one and only one element of B.*

The problem with this definition (which is why it is an undefinition) is the use of weasel words: "rule," "operation," "pairs," "associates," "maps." What do these words mean? We will make this annoyingly precise, but first we consider some examples to see how broad functions can be.

2.2 Example. The following should all be functions.

(i) The pairing of real numbers x with their doubles $2x$ is a function: every real number is paired with another number, and only one number at that.

(ii) The pairing of people in a room with the date (1 through 31) on which they were born. Everyone has only one birthday.

(iii) The pairing of people in a room with the color of the chair in which they are seated (assuming everyone is sitting in a chair and every chair has a discernible color). This last function does not involve numbers at all!

The better definition of function involves more set-theoretic machinery, specifically, the ordered pair. The idea of an ordered pair (x, y) is that another ordered pair (a, b) equals (x, y) if and only if $x = a$ and $y = b$. That is, ordered pairs are equal if and only if their corresponding components are equal—that encodes the idea of "order." It is not necessary to memorize the following definition, but it is here for completeness.

2.3 Definition. *Let x and y be elements of a set. The* ORDERED PAIR *whose first component is x and whose second component is y is the set*

$$(x, y) := \{\{x\}, \{x, y\}\}.$$

2.4 Problem (Wholly optional). Use this definition of ordered pair to prove that $(x, y) = (a, b)$ if and only if $x = a$ and $y = b$.

Now we are ready to define functions.

2.5 Definition. *Let A and B be sets. A* FUNCTION f FROM A TO B *is a set of ordered pairs with the following properties.*

(i) *If $(x, y) \in f$, then $x \in A$ and $y \in B$.*

(ii) *For each $x \in A$, there is a unique $y \in B$ such that $(x, y) \in f$.*

We often use the notation $f : A \to B$ to mean that f is a function from A to B. If $(x, y) \in f$, then we write $y = f(x)$. The set A is the DOMAIN *of f, and the set B is the*

CODOMAIN *of f. The* **IMAGE** *or* **RANGE** *of f is the set*

$$f(A) := \{f(x) \mid x \in A\}.$$

More generally, if $E \subseteq A$, then the **IMAGE OF E UNDER f** *is*

$$f(E) := \{f(x) \mid x \in E\}.$$

The first condition in this definition encodes the act of pairing: elements of A are paired with elements of B as ordered pairs. The second condition encodes the idea that *every* element of A is pair with *one and only one* element of B. (There is a slicker way of phrasing this definition using Cartesian products, but we can avoid that extra bit of technology for now.)

2.6 Example. Let

$$f = \{(1,-1),(2,1),(3,-1),(4,1)\}.$$

Then f is clearly a set of ordered pairs. We study possible domains and codomains of f.

(i) Let $A = \{1,2,3,4\}$ and $B = \{1,-1\}$. Then for each $x \in A$, there is one and only one $y \in B$ such that $(x,y) \in f$, and so f is a function from A to B. Moreover, $f(A) = B$. It happens that $f(1) = f(3)$, and also $f(2) = f(4)$, but that does not violate any part of the definition of function. (It does mean that f is not one-to-one or injective, a condition that we will discuss later.)

(ii) Let $A = \{1,2,3\}$ and $B = \{1,-1\}$. Since $(4,1) \in f$ but $4 \notin A$, f cannot be a function from A to B; the first condition in the definition of function is violated.

(iii) Let $A = \{1,2,3,4,5\}$ and $B = \{1,-1\}$. Since $5 \in A$ but $(5,y) \notin f$ for all $y \in B$, f cannot be a function from A to B; part of the second condition in the definition of function is violated.

(iv) Let $A = \{1,2,3,4\}$ and $B = \{1,-1,0\}$. Again, for each $x \in A$, there is one and only one $y \in B$ such that $(x,y) \in f$, and so f is a function from A to B. It happens that $f(A) \neq B$, since $0 \notin f(A)$, but that does not violate any part of the definition of function. (It does mean that f is not onto or surjective, a condition that we will discuss later.)

2.7 Problem. (i) Why is $\{(1,-1),(1,1),(2,1),(3,-1),(4,1)\}$ not a function from $\{1,2,3,4\}$ to $\{1,-1\}$?

(ii) Let $f = \{(x,x^2) \mid x \in \mathbb{R}\}$. Let $I = [0,\infty)$. Show that $f(I) = I$.

(iii) Why is $\{(x,y) \mid x,\ y \in \mathbb{R} \text{ and } y^2 = x\}$ not a function from $\mathbb{R}$ to $\mathbb{R}$?

2.8 Problem (Optional but worth at least reading). Let A, B, C, and D be sets and let $f\colon A \to B$ and $g\colon C \to D$ be functions. Prove that $f = g$ if and only if $A = C$ and $f(x) = g(x)$ for all $x \in A$ (equivalently, for all $x \in C$). [Hint: *remember that f and g are sets of ordered pairs. To prove the forward implication, if $f = g$, we want to show $x \in A \iff x \in C$ and $f(x) = g(x)$ for all $x \in A$. So, take some $x \in A$ and obtain $(x, f(x)) \in g$. Why does this force $x \in C$ and $g(x) = f(x)$? To prove the reverse implication and show $f = g$, we want to establish $(x, y) \in f \iff (x, y) \in g$. If $(x, y) \in f$, why do we have $x \in A$ and thus $x \in C$? Since $f(x) = g(x)$, why does this lead to $(x, y) \in g$?*]

Life starts with sets and then we connect them with functions (which are themselves sets). Naturally, we may also want to consider sets of functions. If A and B are sets, we denote by

$$B^A$$

the set of all functions from A to B.

2.9 Example. The set $\{1, 2\}^{\{1\}}$ is the set of all functions from $\{1\}$ to $\{1, 2\}$. Any function from $\{1\}$ to $\{1, 2\}$ must be a set consisting of a single ordered pair whose first coordinate is 1 and whose second coordinate is either 1 or 2. So,

$$\{1, 2\}^{\{1\}} = \big\{\{(1, 1)\}, \{(1, 2)\}\big\}.$$

2.10 Problem. What are all the elements of $\{1, -1\}^{\{1,2,3,4\}}$? [Hint: *there are eight.*]

Now we show how functions give a rigorous definition of column vectors and matrices, the building blocks of elementary linear algebra. Consider the vector

$$\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \in \mathbb{R}^3.$$

This vector must be different from

$$\begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix},$$

even though the same numbers appear in both—the two vectors have different *entries*. The fundamental difference is one of *ordering*: 2, 4, and 8 appear in different entries, slots, or positions between the two vectors. We might say that the first vector should mean the same as the function f from $\{1, 2, 3\}$ to $\mathbb{R}$ such that $f(1) = 2$, $f(2) = 4$, and $f(3) = 8$. Then we might say that $\mathbb{R}^3$ should be the set of all functions from $\{1, 2, 3\}$ to $\mathbb{R}$.

Consider next the matrix

$$\begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix} \in \mathbb{R}^{2\times 3}.$$

This is a "two-dimensional" array of data, and so while there are six numbers in play, it will be more meaningful to describe this matrix with two directions—rows and columns. If we think of the (i,j)-entry of this matrix as the number in row i and column j (rows before columns, always), then this matrix could be the function f with

$$f(1,1) = 2, \quad f(2,1) = 4, \quad f(1,2) = 6, \quad f(2,2) = 8, \quad f(1,3) = 10, \quad \text{and} \quad f(2,3) = 12.$$

Here f is a function from $\{(i,j) \mid i = 1,\ 2,\ j = 1,\ 2,\ 3\}$ to $\mathbb{R}$.

Day 3: Friday, August 16.

Material from *Linear Algebra* by Meckes & Meckes

We are dancing around the idea of a vector space by considering addition and scalar multiplication in the function spaces $\mathbb{R}^n$ (column vectors), $\mathbb{R}^{m \times n}$ ($m \times n$ matrices), $\mathbb{R}^\infty$ (sequences = functions from $\mathbb{N}$ = natural numbers to $\mathbb{R}$), and $\mathbb{R}^X$ (functions from any set X to $\mathbb{R}$). You may want to read the discussions of column vector arithmetic on pp.24–26. The essential arithmetical properties of real numbers that we tacitly and joyfully assume appear as properties of fields on pp.39–43. Throughout this course, the only fields that we will consider are $\mathbb{R}$ and the complex numbers $\mathbb{C}$, and whenever you see the symbol $\mathbb{F}$ you can think of it as meaning either the real or complex numbers. Matrix and sequence arithmetic appear on pp.55–56 in the context of vector spaces. The book uses the notation $M_{m,n}(\mathbb{R}) = \mathbb{R}^{m \times n}$ and $M_n(\mathbb{R}) = \mathbb{R}^{n \times n}$; I will not.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Sequence in a set X

After all the bluster last time about a rigorous definition for functions, we will be content with the original undefinition: if A and B are sets, a function f from A to B (denoted by $f \colon A \to B$) is a rule that pairs each $x \in A$ with a unique $y \in B$, written $y = f(x)$. To do linear algebra, we need algebra, and to do algebra, we need arithmetic. We study arithmetic with functions, which is really arithmetic with vectors.

Recall that $\mathbb{R}^n$ is the set of all functions from the set $\{1, \ldots, n\}$ to $\mathbb{R}$. If $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x}$ is a function on $\{1, \ldots, n\}$ that takes real values. We write $\mathbf{x}(k) = x_k$ and express

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

In other words, the column vector on the right is just a (very convenient!) notational device for the function $\mathbf{x}$ on $\{1, \ldots, n\}$ that takes the values $\mathbf{x}(k) = x_k$ for $k = 1, \ldots, n$ and $x_k \in \mathbb{R}$.

We work with $\mathbb{R}^3$ for some time just for convenience. There is really only one natural way to add functions (...column vectors...) in $\mathbb{R}^3$, and that is componentwise, or entrywise:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

Abbreviate

$$\mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{y} := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{z} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

Then $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all elements of $\mathbb{R}^3$, which is to say, real-valued functions on the set $\{1, 2, 3\}$. We say that $\mathbf{z} = \mathbf{x} + \mathbf{y}$, and what this means is that $\mathbf{z}(k) = \mathbf{x}(k) + \mathbf{y}(k)$ for all k.

In this last sentence, and in the displayed calculation above, the symbol $+$ is doing double duty. We have the familiar addition of real numbers ($x_1 + y_1$, $x_2 + y_2$, $x_3 + y_3$), and we have the addition of elements of $\mathbb{R}^3$. Perhaps we should be pedantic and give a new symbol for the latter—say that $\mathbf{x} +_{\mathbb{R}^3} \mathbf{y}$ is the function that takes the value $\mathbf{x}(k) + \mathbf{y}(k)$ at k. That is,

$$\big(\mathbf{x} +_{\mathbb{R}^3} \mathbf{y}\big)(k) = \mathbf{x}(k) + \mathbf{y}(k),$$

where on the right $+$ is the familiar addition of the real numbers $\mathbf{x}(k)$ and $\mathbf{y}(k)$. What a horrible, burdensome way of living. We will not write like this. The point, to be repeated often, is that addition of column vectors respects our hopefully intuitive addition of functions: do it pointwise, entrywise, componentwise.

We move to matrix addition. Recall that $\mathbb{R}^{m \times n}$ is the set of all functions from

$$\{(i, j) \mid i = 1, \ldots, m, \ j = 1, \ldots, n\}$$

to $\mathbb{R}$. For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2},$$

we expect

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}.$$

How does this respect our notion of A and B as functions from $\{(1,1), (1,2), (2,1), (2,2)\}$ to $\mathbb{R}$? We have $A(1,1) = a_{11}$, $B(1,1) = b_{11}$, and $(A + B)(11) = a_{11} + b_{11}$. That is, $(A + B)(1,1) = A(1,1) + B(1,1)$. This perfectly respects our idea that function addition is performed pointwise.

More generally, let X be any set, and let f, $g \colon X \to \mathbb{R}$ be functions. Probably the most natural way to assign meaning to the symbol $f + g$ is to make it the function defined by $(f + g)(x) := f(x) + g(x)$. Again, we add functions pointwise.

3.1 Problem. With this definition, explain why

$$f + g = \{(x, f(x) + g(x)) \mid x \in X\}.$$

And, again, we could be horribly pedantic and write something like $f +_{\mathbb{R}^X} g$ instead of $f + g$, to emphasize that $+$ is an operation on real numbers, while $+_{\mathbb{R}^X}$ is an operation on the set of all functions from X to $\mathbb{R}$, which we are calling $\mathbb{R}^X$. We are all probably happier not doing that.

Column vectors and matrices are inherently "finite-dimensional" objects. There is a nice kind of function that retains the discrete structure of column vectors and matrices while being infinite: the sequence.

3.2 Definition. *Let $\mathbb{N}$ denote the natural numbers ($\mathbb{N} = \{1, 2, 3, \ldots\}$). A* **SEQUENCE** *in a set X is a function $f : \mathbb{N} \to X$. If $f(k) = x_k$, then we often write $f = (x_k)$.*

3.3 Example. Define
$$f : \mathbb{N} \to \mathbb{R} : k \mapsto k^2.$$
Then f is a sequence in $\mathbb{R}$, and we might write $f = (k^2)$. Strictly speaking, as a set of ordered pairs,
$$(k^2) = \left\{(k, k^2) \;\middle|\; k \in \mathbb{N}\right\}.$$

Sequences are hugely useful in analysis because they allow us to express "continuous concepts" (like limits and continuity) in terms of "discrete" objects (the domain of a sequence is $\mathbb{N}$ which, while infinite, is still tamer—more discrete—than $\mathbb{R}$). For us in linear algebra, sequences will serve as a pleasant source of examples that introduce infinite-dimensional complexities while largely retaining the manipulability of column vectors and matrices. In particular, since all function addition is componentwise, we should have
$$(x_k) + (y_k) = (x_k + y_k).$$
Again, we are overworking the symbol $+$, once for addition of sequences, once for addition of real numbers.

Per our prior notation, we might say that $\mathbb{R}^{\mathbb{N}}$ is the set of all real-valued sequences. Another evocative notation for this set is $\mathbb{R}^\infty$, since a sequence is morally an infinitely long vector. We will use both notations, although eventually we may want to consider "doubly infinite" sequences indexed by negative numbers, too. (Such sequences arise with Fourier coefficients, among other uses.)

We have thus far defined addition in $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, $\mathbb{R}^\infty$, and $\mathbb{R}^X$ for any set X (this last set $\mathbb{R}^X$ subsuming all prior ones). In each case, we used the intuitive notion of addition in $\mathbb{R}$ to define pointwise addition of functions. Function addition inherits many useful properties of real addition, in particular commutativity and associativity:
$$f + g = g + f \quad \text{and} \quad (f + g) + h = f + (g + h),$$
where, above, f, g, and h are real-valued functions defined on a common set. That is, the order in which we add real-valued functions should not matter, nor should the order in which we group them. Hopefully all of this perfectly respects our intuition from calculus.

Among many other useful properties of real addition is the notion of an **IDENTITY ELEMENT** for addition: the number $0 \in \mathbb{R}$ satisfies $x + 0 = x$ for all $x \in \mathbb{R}$. We can get a "zero" for function addition by defining the zero function pointwise. In the case of $\mathbb{R}^n$, we put

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

and of course $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. (Sadly, there is no universal notation for the zero matrix in $\mathbb{R}^{m \times n}$. Maybe O? Z?)

For sequences, the sequence (0) takes the value 0 at each $k \in \mathbb{N}$, which may be hard to see because there is no k-dependence in (0). But, working componentwise,

$$(x_k) + (0) = (x_k + 0) = (x_k),$$

and that is certainly what a zero sequence should do.

Most generally, for any set X, we can define

$$z \colon X \to \mathbb{R} \colon x \mapsto 0,$$

and then for any function $f \colon X \to \mathbb{R}$, we have $f(x) + z(x) = 0$ for all $x \in \mathcal{X}$. That is, z serves as the additive identity for function addition in the set $\mathbb{R}^X$. (Should we write z_X?)

There is another essential algebraic operation for linear algebra: multiplication by real numbers. (It may be surprising that we do not really multiply vectors—dot products and cross products notwithstanding—after all the function multiplication that we do in calculus.) Again, this should work componentwise. For example, in the case of $\mathbb{R}^3$, if $\alpha \in \mathbb{R}$, then

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}.$$

For sequences,

$$\alpha(x_k) = (\alpha x_k).$$

For functions $f \colon X \to \mathbb{R}$, the symbol αf should denote the function satisfying $(\alpha f)(x) = \alpha f(x)$.

3.4 Problem. Explain why $\alpha f = \{(x, \alpha f(x)) \mid x \in X\}$.

Arithmetic with multiplying functions by real numbers works *exactly as it should*, and we will not belabor the obvious. The point of today was to build a stock of examples for what we will call vector spaces in the future, and all of these vector spaces ($\mathbb{R}^n$, $\mathbb{R}^{m \times n}$, $\mathbb{R}^\infty$, $\mathbb{R}^X$) were built on functions and properties of real numbers. We conclude with some picky comments about column and row vectors vs.matrices.

3.5 Remark. *We do not think of* $\mathbb{R}^{m \times 1}$ *as the set of all functions from*

$$\{(i, 1) \mid i = 1, \ldots, m\}$$

to $\mathbb{R}$*, but rather we define* $\mathbb{R}^{m \times 1} := \mathbb{R}^m$*. We do not think of* $\mathbb{R}^1$ *as the set of all functions* *from* $\{1\}$ *to* $\mathbb{R}$*, but rather we define* $\mathbb{R}^1 := \mathbb{R}$*. Thus* $\mathbb{R}^{1\times 1} = \mathbb{R}$*. However, for* $n \geq 2$*, we do not say* $\mathbb{R}^{1\times n} = \mathbb{R}^n$ *but instead continue to think of* $\mathbb{R}^{1\times n}$ *as the set of all functions from*

$$\{(1,j) \mid j = 1,\ldots,n\}$$

to $\mathbb{R}$*. The point is that we want to distinguish column vectors in* $\mathbb{R}^n$ *from row vectors in* $\mathbb{R}^{1\times n}$*. Most broadly, we could say*

$$\mathbb{R}^n = \begin{cases} \mathbb{R}, & n = 1 \\ \mathbb{R}^{\{1,\ldots,n\}}, & n \geq 2 \end{cases}$$

and

$$\mathbb{R}^{m\times n} = \begin{cases} \mathbb{R}, & m = n = 1 \\ \mathbb{R}^m, & n = 1 \\ \mathbb{R}^{\{(i,j) \ \mid \ i=1,\ldots,m, \ j=1,\ldots,n\}}, & n \geq 2. \end{cases}$$

Of course, no one thinks like this on a daily basis.

Day 4: Monday, August 19.

Material from *Linear Algebra* by Meckes & Meckes

The axioms for a vector space appear on p.51. The main point is the "bottom line" box, repeated on p.58. See p.50 and Proposition 1.9 for examples of vector spaces. See pp.57–58 for essential arithmetic in vector spaces that follows from the axioms. Do Quick Exercise #21 on p.52.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Zero vector, additive inverse. I will *not* ask you to define what a vector space is or memorize the axioms.

Be able to explain what the zero vector and the additive inverse *do*. Don't just say that the zero vector is 0 or $\mathbf{0}$ or $0_{\mathcal{V}}$; rather, $0_{\mathcal{V}} + v = v$ for all $v \in \mathcal{V}$. Likewise, the additive inverse isn't $-v$ but rather the unique vector w such that $v + w = 0_{\mathcal{V}}$.

The function sets $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, $\mathbb{R}^\infty$, and $\mathbb{R}^X$ (with X an arbitrary set) that we considered are all, of course, examples of vector spaces—and in some sense they will be almost all of the vector spaces that we study. For functions f and g in any one of these sets, we define $f + g$ and αf (for $\alpha \in \mathbb{R}$) pointwise (or componentwise) and we get a new function in that set. (While we can naturally multiply functions pointwise, and we do so all the time in calculus, we will not typically do so in this course.) All of the arithmetic works exactly as it should, because all of the arithmetic is inherited from arithmetic on $\mathbb{R}$, and all arithmetic on $\mathbb{R}$ works as it should.

We now abstract from these situations the absolute essentials of the structure—the properties without which we cannot do anything worthwhile and from which we can prove everything worthwhile. First, we take the convention that the symbol $\mathbb{F}$ represents either $\mathbb{R}$ or $\mathbb{C}$, with $\mathbb{C} := \{a + ib \mid a,\ b \in \mathbb{R},\ i^2 = -1\}$.

4.1 Definition. *A* **VECTOR SPACE OVER** $\mathbb{F}$ *consists of four objects:* $\mathcal{V}$, $\mathbb{F}$, $+_{\mathcal{V}}$, *and* $\cdot$, *which are described below.*

- $\mathcal{V}$ *is a nonempty set.*
- $\mathbb{F}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$.
- *For each* $v,\ w \in \mathcal{V}$, *there exists* $v +_{\mathcal{V}} w \in \mathcal{V}$, *which satisfies the axioms below.*
- *For each* $\alpha \in \mathbb{F}$ *and* $v \in \mathcal{V}$, *there exists* $\alpha \cdot v \in \mathcal{V}$, *which satisfies the axioms below.*

That is, $+_{\mathcal{V}}$ *is a function from* $\{(v, w) \mid v,\ w \in \mathcal{V}\}$ *to* $\mathcal{V}$ *and* $\cdot$ *is a function from* $\{(\alpha, v) \mid \alpha \in \mathbb{R},\ v \in \mathcal{V}\}$ *to* $\mathcal{V}$. *We call* $+_{\mathcal{V}}$ **VECTOR ADDITION** *and* $\cdot$ **SCALAR MULTIPLICATION**. *Often we abuse terminology and call just* $\mathcal{V}$ *the vector space.*

Vector addition and scalar multiplication satisfy the following axioms.

Axioms for vector addition.

1. *Commutativity:* $v +_{\mathcal{V}} w = w +_{\mathcal{V}} v$ *for all* $v,\ w \in \mathcal{V}$.

2. *Associativity:* $v +_{\mathcal{V}} (w +_{\mathcal{V}} u) = (v +_{\mathcal{V}} w) +_{\mathcal{V}} u$ *for all* $v,\ w,\ u \in \mathcal{V}$.

3. *Identity: there exists* $0_{\mathcal{V}} \in \mathcal{V}$ *such that* $v + 0_{\mathcal{V}} = v$ *for all* $v \in \mathcal{V}$.

4. *Inverse: for each* $v \in \mathcal{V}$, *there exists* $-v \in \mathcal{V}$ *such that* $v +_{\mathcal{V}} (-v) = 0_{\mathcal{V}}$.

Axioms for scalar multiplication.

5. *Identity:* $1 \cdot v = v$ *for all* $v \in \mathcal{V}$.

6. *Associativity:* $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$ *for all* $\alpha,\ \beta \in \mathbb{F}$ *and* $v \in \mathcal{V}$.

Axioms relating vector addition and scalar multiplication.

7. *Distributivity:* $(\alpha + \beta) \cdot v = (\alpha \cdot v) +_{\mathcal{V}} (\beta \cdot v)$ *for all* $\alpha,\ \beta \in \mathbb{F}$ *and* $v \in \mathcal{V}$.

8. *Distributivity again:* $\alpha \cdot (v +_{\mathcal{V}} w) = (\alpha \cdot v) +_{\mathcal{V}} (\beta \cdot w)$ *for all* $\alpha \in \mathbb{F}$ *and* $v,\ w \in \mathcal{V}$.

We discuss at length these foundational, essential axioms.

4.2 Remark. *The grouping of the axioms is taken from Strang's Introduction to Linear Algebra. The phrase that a vector space "consists of four objects" is weasel words; really, we might think of a vector space as an "ordered* 4*-tuple"* $(\mathcal{V}, \mathbb{F}, +_{\mathcal{V}}, \cdot)$*, where* $\mathcal{V}$ *is a nonempty set,* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$*, and* $+_{\mathcal{V}}$ *and* $\cdot$ *are the maps above. (And here an ordered* 4*-tuple* (a, b, c, d) *is the function* $f \colon \{1, 2, 3, 4\} \to \{a, b, c, d\}$ *with* $f(1) = a$*,* $f(2) = b$*,* $f(3) = c$*, and* $f(4) = d$*.) Of course, no one ever does this in public.*

Here are some more focused comments on the axioms.

(i) *Commutativity of vector addition means that the order in which we add vectors is irrelevant. (Mathematicians are typically uncomfortable using the plus symbol for something that does not commute.)*

(ii) *Associativity of vector addition means that the way in which we group vectors is irrelevant for addition.*

(iii) *We will shortly show that the zero vector is unique and therefore merits the definite article "the."*

(iv) *The symbol* $-v$ *for the additive inverse is just that: a symbol. It is possible to prove that given* $v \in \mathcal{V}$*, there is only one vector* $w \in \mathcal{V}$ *such that* $v +_{\mathcal{V}} w = 0_{\mathcal{V}}$*. It is also possible to show that* $-v = (-1) \cdot v$*; that is, there is an intimate, and expected, connection between the additive inverse in* $\mathcal{V}$ *and scalar multiplication by the additive inverse of the multiplicative identity in* $\mathbb{F}$*.*

(v) *For associativity of scalar multiplication, given* α*,* $\beta \in \mathbb{F}$ *and* $v \in \mathcal{V}$*, we obtain* $\beta \cdot v \in \mathcal{V}$ *and thus* $\alpha \cdot (\beta \cdot v) \in \mathcal{V}$*. But we also have* $\alpha\beta \in \mathbb{F}$*, where juxtaposition of* α *and* β *here indicates their product according to arithmetic in* $\mathbb{F}$*, and so we have* $(\alpha\beta) \cdot v \in \mathcal{V}$*. Associativity of scalar multiplication asserts that these two instances of multiplication are really the same, as we would expect.*

(vi) *The first distributive axiom illustrates why we might want to decorate vector addition as* $+_{\mathcal{V}}$*. On the left,* $\alpha + \beta$ *is addition of numbers in* $\mathbb{F}$*, while on the right* $(\alpha \cdot v) +_{\mathcal{V}} (\beta \cdot v)$ *is vector addition of the vectors* $\alpha \cdot v$ *and* $\beta \cdot v$ *in* $\mathcal{V}$*.*

4.3 Example. We prove some consequences of the vector space axioms using *only* these axioms. Below, $\mathcal{V}$ is a vector space over $\mathbb{F}$.

(i) First we show that the zero vector is unique. The axioms tell us that there exists $0_{\mathcal{V}} \in \mathcal{V}$ such that $v +_{\mathcal{V}} 0_{\mathcal{V}} = v$ for all $v \in \mathcal{V}$. Suppose there is another vector that does this. ("What things do defines what things are.") That is, suppose that $w \in \mathcal{V}$ also satisfies $v +_{\mathcal{V}} w = v$ for all $v \in \mathcal{V}$. We can make $0_{\mathcal{V}}$ "talk" to w by taking $v = 0_{\mathcal{V}}$ in the previous equality; after all, it holds for all $v \in \mathcal{V}$. Then we get $0_{\mathcal{V}} +_{\mathcal{V}} w = 0_{\mathcal{V}}$. By the vector space axioms, $0_{\mathcal{V}} +_{\mathcal{V}} w = w$. Thus $w = 0_{\mathcal{V}}$.

(ii) Next we show the useful fact that if $\alpha \in \mathbb{F}$ and $v \in \mathcal{V}$ with $\alpha \cdot v = 0_{\mathcal{V}}$, then either $\alpha = 0$ or $v = 0_{\mathcal{V}}$. By $\alpha = 0$ we mean the number $0 \in \mathbb{F}$. This is a statement of the form

"$P \implies Q$ or R," and such statements are logically equivalent to both "P and not $Q \implies R$" and P and not $R \implies Q$." We work with the first version: assume that $\alpha \cdot v = 0_{\mathcal{V}}$ but $\alpha \neq 0$. Then α has a reciprocal $\alpha^{-1} \in \mathbb{F}$, which satisfies $\alpha^{-1}\alpha = 1$. Here the juxtaposition $\alpha^{-1}\alpha$ means multiplication in $\mathbb{F}$.

So, from $\alpha \cdot v = 0_{\mathcal{V}}$, we have $\alpha^{-1} \cdot (\alpha \cdot v) = \alpha^{-1} \cdot 0_{\mathcal{V}}$. On the left, we use vector space axioms to rewrite

$$\alpha^{-1} \cdot (\alpha \cdot v) = (\alpha^{-1}\alpha) \cdot v = 1 \cdot v = v.$$

On the right, we actually need to do a little more work and prove separately that $\beta \cdot 0_{\mathcal{V}} = 0_{\mathcal{V}}$ for all $\beta \in \mathbb{F}$. (The axioms do not explicitly tell us anything about how the additive identity for vector addition interacts with scalar multiplication.) Assuming this to be true, we get $\alpha^{-1} \cdot 0_{\mathcal{V}} = 0_{\mathcal{V}}$ and thus $v = 0_{\mathcal{V}}$.

4.4 Problem. In the example above, try proving that if $\alpha \cdot v = 0_{\mathcal{V}}$ and $v \neq 0_{\mathcal{V}}$, then $\alpha = 0$. How far do you get? Is this any harder than our approach?

Typically we will not use $+_{\mathcal{V}}$ or $\cdot$ anymore. That is, we write $v +_{\mathcal{V}} w = v + w$ and $\alpha \cdot v = \alpha v$. Also, we will write 0 instead of $0_{\mathcal{V}}$ and $\mathbf{0}$ for the zero vector in $\mathbb{R}^n$. As needed, we may include subscripts for clarity. For example, we have shown that if $\alpha v = 0$ for $\alpha \in \mathbb{F}$ and $v \in \mathcal{V}$, then either $\alpha = 0$ or $v = 0$; context tells us that the 0 appearing in $\alpha v = 0$ and $v = 0$ is the zero vector, while the 0 appearing in $\alpha = 0$ is the scalar 0.

We will adhere to the conventions that letters like u, v, and w denote elements of a vector space, while α, β, c, and d denote elements of $\mathbb{F}$ (with more letters in each case as needed from earlier or later in the alphabet). We will reserve boldface letters for column vectors in $\mathbb{R}^n$ (they deserve this special treatment because they have scalar components, and because they are just so special to linear algebra).

Day 5: Wednesday, August 21.

Material from *Linear Algebra* by Meckes & Meckes

Pages 55–57 have many examples of vector spaces. Our notation in this log and in class uses $\mathbb{C}^k([a,b])$ to mean what the book calls $D^k[a,b]$ on p.57. The book also adopts a more formal/algebraic view of polynomials (p.57 again) than our view of polynomials as functions on $\mathbb{R}$ or $\mathbb{C}$.

We study a number of examples, and nonexamples, of vector spaces. Most really arise as *subspaces* of other vector spaces, and we will discuss that presently. Here is the general situation. One starts with a vector space $\mathcal{W}$ that is often "too large to be interesting." For example, for an arbitrary set X, the set of all functions $\mathbb{F}^X$ is a vector space (as we review momentarily), but rarely in life do we consider *all* functions from X to $\mathbb{F}$; in calculus, for example, we really only care about continuous and differentiable functions. Instead, we find a subset $\mathcal{V} \subseteq \mathcal{W}$ that is also a vector space when vector addition and scalar multiplication are restricted to $\mathcal{V}$, i.e., $v + w \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ for all v, $w \in \mathcal{V}$ and $\alpha \in \mathbb{F}$, and $0 \in \mathcal{V}$. Often

$\mathcal{V}$ is chosen to be "small enough to be interesting."

5.1 Example. As we have said repeatedly, $\mathbb{F}^X$ is a vector space with vector addition and scalar multiplication defined pointwise (or componentwise, or entrywise). Axioms like commutativity, associativity, and distributivity follow from arithmetic in $\mathbb{F}$. We emphasize that the zero vector is the function

$$z \colon X \to \mathbb{F} \colon x \mapsto 0.$$

That is,

$$z = \{(x, 0) \mid x \in X\},$$

and sometimes we just write 0 instead of z (which could lead to unfortunate expressions like $0 = \{(x, 0) \mid x \in X\}$ if we overthink things—better than underthinking them?) The additive inverse of $f \in \mathbb{F}^X$ is the function $-f$ defined pointwise by

$$-f \colon X \to \mathbb{F} \colon x \mapsto -f(x).$$

For practice in picky reading comprehension, we emphasize that the symbol $-f$ denotes one function, while $-f(x)$ is the additive inverse in $\mathbb{F}$. Thus

$$-f = \{(x, -f(x)) \mid x \in X\}.$$

Of course, $-f(x) = -1 \cdot f(x)$ as multiplication in $\mathbb{F}$.

Here are the most important function spaces for calculus and differential equations.

5.2 Example. Let $I \subseteq \mathbb{R}$ be an interval, which may be open or closed, bounded or unbounded. (Most of calculus is done on intervals, after all.)

(i) Let

$$\mathcal{C}(I) := \left\{ f \in \mathbb{R}^I \;\middle|\; f \text{ is continuous on } I \right\}.$$

Much of calculus works because *limits are linear*: if f, $g \in \mathbb{R}^I$ and $x_0 \in I$ and

$$\lim_{x \to x_0} f(x) \quad \text{and} \quad \lim_{x \to x_0} g(x)$$

exist, then

$$\lim_{x \to x_0} \big(f(x) + g(x)\big)$$

exists with

$$\lim_{x \to x_0} \big(f(x) + g(x)\big) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

Likewise, for any $\alpha \in \mathbb{R}$,

$$\lim_{x \to x_0} \alpha f(x) = \alpha \lim_{x \to x_0} f(x).$$

Since $f \in \mathcal{C}(I)$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

for all $x_0 \in I$ (ignoring the possible particular cases of left/right limits at the endpoints of I, if an endpoint even belongs to I), linearity of limits implies $f + g \in \mathcal{C}(I)$ and $\alpha f \in \mathcal{C}(I)$ for f, $g \in \mathcal{C}(I)$ and $\alpha \in \mathbb{R}$. Last, since all constant functions are continuous, the zero function 0 (defined by $0(x) = 0$ for all $x \in I$) is an element of $\mathcal{C}(I)$. Thus $\mathcal{C}(I)$ is a vector space (over $\mathbb{R}$).

(ii) Let $r \geq 0$ be an integer and let

$$\mathcal{C}^r(I) := \left\{ f \in \mathbb{R}^I \mid f \text{ is } r\text{-times differentiable on } I \text{ and } f^{(r)} \text{ is continuous on } I \right\}.$$

Here $f^{(k)}$ is the kth derivative of f; for example, $f^{(3)} = f'''$. We put

$$\mathcal{C}^0(I) := \mathcal{C}(I)$$

since $f^{(0)} = f$. Each $\mathcal{C}^r(I)$ is a vector space because differentiation is linear: if f and g are differentiable, then $(f+g)' = f' + g'$ and $(\alpha f)' = \alpha f'$. We impose the requirement that $f^{(r)}$ be continuous mostly for "mathematical niceness," e.g., in differential equations, one often wants that the solution to an rth order differential equation has a continuous rth derivative.

5.3 Example. Let

$$\mathcal{C}^\infty(I) := \cap_{r=0}^\infty \mathcal{C}^r(I) = \left\{ f \in \mathbb{R}^I \mid f \in \mathcal{C}^r(I) \text{ for all } r \geq 1 \right\}.$$

We call functions in $\mathcal{C}^\infty(I)$ **INFINITELY DIFFERENTIABLE**. Prove that $\mathcal{C}^\infty(I)$ is a vector space.

Here are some "finite-dimensional" (non)examples.

5.4 Example. **(i)** Let

$$\mathcal{V} := \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \;\middle|\; x_1 \in \mathbb{R} \right\}.$$

We claim that $\mathcal{V}$ is a vector space with the usual componentwise vector addition and scalar multiplication. We just check

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ 0 \end{bmatrix} \in \mathcal{V} \quad \text{and} \quad \alpha \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha \cdot 0 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ 0 \end{bmatrix} \in \mathcal{V}.$$

What is critical here is that vector addition and scalar multiplication keep 0 as the second component. Also,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathcal{V},$$

again because, critically, the second component is 0.

(ii) The set

$$\mathcal{W} := \left\{ \begin{bmatrix} x_1 \\ 1 \end{bmatrix} \;\middle|\; x_1 \in \mathbb{R} \right\}$$

is not a vector space with, again, the usual componentwise vector addition and scalar multiplication. We only need to break one of the axioms, but we show that many fail.

We probably expect that $\mathcal{W}$ is not closed under addition because the second component will have us adding $1+1=2$, which destroys the 1 in the second component. To be explicit, we give a concrete example of how this fails:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathcal{W} \quad \text{but} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \notin \mathcal{W}.$$

Next, we probably expect that $\mathcal{W}$ is not closed under scalar multiplication because the second component will have us multiplying $\alpha \cdot 1 = \alpha \neq 1$ when $\alpha \neq 1$. To be explicit, we give a concrete example of how this fails:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{W} \quad \text{but} \quad 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin \mathcal{W}.$$

Finally, $\mathcal{W}$ lacks an additive identity for vector addition. The only possible additive identity is the zero vector in $\mathbb{R}^2$, and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin \mathcal{W},$$

because of that unpleasant second component.

We go "infinite-dimensional" again.

5.5 Example. Denote by ℓ^∞ the set of all **BOUNDED** sequences:

$$\ell^\infty := \{(a_k) \in \mathbb{R}^\infty \mid \exists M > 0 \; \forall k \in \mathbb{N} : |a_k| \leq M\}.$$

For example, if $a_k = 1$ for all k, then $|a_k| \leq 1$ for all k, and so $(a_k) \in \ell^\infty$. Likewise, if $b_k = 1/2^k$ for all k, then $|b_k| \leq 1/2$ for all k, and so $(b_k) \in \ell^\infty$.

We show that ℓ^∞ is a vector space over $\mathbb{R}$. The zero sequence (0) is certainly an element of ℓ^∞, since $|0| < 1$. (Remember that (0) is the map $\mathbb{N} \to \mathbb{R} \colon k \mapsto 0$.)

Next we check scalar multiplication. Let $\alpha \in \mathbb{R}$ and $(a_k) \in \ell^\infty$. We want to show $\alpha(a_k) \in \ell^\infty$, and we know $\alpha(a_k) = (\alpha a_k)$. Our goal, therefore, is to find $M > 0$ such that $|\alpha a_k| \leq M$ for all k. Since $(a_k) \in \ell^\infty$, we know there is $N > 0$ such that $|a_k| \leq N$ for all k. Now we need a property of absolute value:

$$|xy| = |x||y|, \;\; x, \;\; y \in \mathbb{R}.$$

Then $|\alpha a_k| = |\alpha||a_k| \leq |\alpha| N$. Taking $M = |\alpha| N$ is the bound we want.

Last, we check vector addition. Let (a_k), $(b_k) \in \ell^\infty$. We want to show $(a_k) + (b_k) \in \ell^\infty$, and we know $(a_k) + (b_k) = (a_k + b_k)$. Our goal, therefore, is to find $M > 0$ such that

$|a_k + b_k| \le M$ for all k. Since (a_k), $(b_k) \in \ell^\infty$, we know there are M_1, $M_2 > 0$ such that $|a_k| \le M_1$ and $|b_k| \le M_2$. Now we need another property of absolute value (the **TRIANGLE INEQUALITY**):

$$|x + y| \le |x| + |y|, \ x, \ y \in \mathbb{R}.$$

Then $|a_k + b_k| \le |a_k| + |b_k| \le M_1 + M_2$. Taking $M = M_1 + M_2$ is the bound we want.

By the way, everything above works if we replace $\mathbb{R}$ with $\mathbb{C}$, since the modulus on $\mathbb{C}$ is multiplicative and enjoys the triangle inequality.

5.6 Problem. Limits of sequences behave exactly as we expect. First, if $(a_k) \in \mathbb{R}^\infty$ and $L \in \mathbb{R}$, we say that $\lim_{k\to\infty} a_k = L$ if we can make a_k arbitrarily close to L by taking k sufficiently large. It follows that if $\lim_{k\to\infty} a_k = L_1$ and $\lim_{k\to\infty} b_k = L_2$, then $\lim_{k\to\infty}(a_k + b_k) = L_1 + L_2$ and $\lim_{k\to\infty} \alpha a_k = \alpha L_1$.

(i) Prove that

$$\mathcal{V} := \left\{(a_k) \in \mathbb{R}^\infty \ \middle| \ \lim_{k\to\infty} a_k \text{ exists}\right\}$$

is a vector space.

(ii) Prove that

$$c_0 := \left\{(a_k) \in \mathbb{R}^\infty \ \middle| \ \lim_{k\to\infty} a_k = 0\right\}$$

is a vector space (the notation c_0 is unfortunate, as it looks like a coefficient in some sum, but traditional).

(iii) Prove that

$$\mathcal{V}_\alpha := \left\{(a_k) \in \mathbb{R}^\infty \ \middle| \ \lim_{k\to\infty} a_k = \alpha\right\}$$

is not a vector space when $\alpha \neq 0$. Explain *all* of the ways in which $\mathcal{V}_\alpha$ fails to be a vector space.

5.7 Problem. So far, we have not paid too much attention to the field over which we are considering our vector spaces. Explain why $\mathbb{R}$ is a vector space over the field $\mathbb{R}$, $\mathbb{C}$ is a vector space over both $\mathbb{R}$ and $\mathbb{C}$, but $\mathbb{R}$ is not a vector space over $\mathbb{C}$.

Here is a finite-dimensional example in disguise. We begin with a slightly atypical definition (caution: your experience in other texts and algebra classes may radically differ).

5.8 Definition. *A* **POLYNOMIAL ON** $\mathbb{F}$ *is a function of the form*

$$p \colon \mathbb{F} \to \mathbb{F} \colon x \mapsto \sum_{k=0}^{n} a_k x^k$$

for some integer $n \ge 0$ *and some coefficients* $a_0, \ldots, a_n \in \mathbb{F}$. *If* $a_n \neq 0$, *the* **DEGREE** *of* p *is* $\deg(p) := n$.

Informally, a polynomial is just a sum of multiples of nonnegative integer powers of x.

5.9 Example. **(i)** Denote (again, atypically) by $\mathbb{P}(\mathbb{F})$ the set of all polynomial functions on $\mathbb{F}$. Here is the informal proof that $\mathbb{P}(\mathbb{F})$ is a vector space: adding polynomials results in polynomials, and multiplying polynomials by constants results in polynomials. To be a little more precise about the zero vector (function), note that $p(x) = 0$ for all x is a polynomial with $n = 0$ and $a_0 = 0$.

(ii) Let

$$\mathcal{V} := \{p \in \mathbb{P}(\mathbb{F}) \mid \deg(p) = 2\}.$$

Then $\mathcal{V}$ is not a vector space. First, it has no zero vector, since $\deg(0) = 0$. Next, $\mathcal{V}$ is not closed under vector addition, as we could subtract quadratics and get a linear or constant polynomial. To be concrete, with $p(x) = 3x^2$ and $q(x) = -3x^2 + x$, we have $p, q \in \mathcal{V}$ but $(p+q)(x) = x$, thus $p + q \notin \mathcal{V}$. Finally, $\mathcal{V}$ is not closed under scalar multiplication, as $0p = 0 \notin \mathcal{V}$ for all $p \in \mathcal{V}$. However, $\alpha p \in \mathcal{V}$ for all $\alpha \neq 0$ and $p \in \mathcal{V}$. This shows that sometimes we must be very precise in how we break the vector space axioms.

5.10 Problem. In contrast to the last example,

$$\mathcal{V} := \{p \in \mathbb{P}(\mathbb{F}) \mid \deg(p) \leq 2\},$$

is a vector space. Prove that.

5.11 Problem. We have now met two of the three kinds of vector spaces from our original three motivating problems. Certainly $\mathbb{R}^5$ and $\mathbb{R}^3$ appeared in Example 1.1, while the space of (at most) quadratics from Problem 5.10 above appeared in Example 1.2. Prove that

$$\mathcal{V} := \left\{ f \in \mathcal{C}^2(\mathbb{R}) \;\middle|\; f \text{ is even}, \lim_{x\to\infty} f(x) = 0 \right\}$$

and

$$\mathcal{W} := \left\{ f \in \mathcal{C}(\mathbb{R}) \;\middle|\; f \text{ is even}, \int_0^\infty |f(x)|\, dx < \infty \right\}$$

are vector spaces; these appeared in Example 1.4. [Hint: *for $\mathcal{W}$, use the triangle inequality and the comparison test for improper integrals (look up the comparison test as needed) to establish that if f and g are absolutely integrable on $[0, \infty)$, then so is $f + g$.*]

Day 6: Friday, August 23.

Material from *Linear Algebra* by Meckes & Meckes

Linear combinations are defined on p.53 and subspaces on p.55. See also the discussion of subspaces on pp.58–59. Read the philosophy on p.63 about "recognizing sameness" and pp.64–65 on linear operators. See Examples 1, 2, and 3 of linear operators on pp.86–87.

Do Quick Exercises #22, #23, #24 in Section 1.5 and #1 in Section 2.1.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Subspace (N), linear combination, span, linear operator (N)

We have seen that vector spaces can arise as nonempty subsets of larger vector spaces that are closed under addition and scalar multiplication. Often the original vector space is "too large" to be interesting, like $\mathbb{R}^{\mathbb{R}}$, while the "smaller" space is more restrictive and has nicer features, like $\mathcal{C}(\mathbb{R})$. Depending on our point of view, we may still want to relate the smaller space to the original space via the notion of subspace.

6.1 Definition. *Let $\mathcal{W}$ be a vector space over $\mathbb{F}$. A set $\mathcal{V} \subseteq \mathcal{W}$ is a* SUBSPACE *of $\mathcal{W}$ if the following hold.*

(i) *$\mathcal{V}$ contains the zero vector: $0 \in \mathcal{V}$.*

(ii) *$\mathcal{V}$ is closed under addition: $v + w \in \mathcal{V}$ for all v, $w \in \mathcal{V}$.*

(iii) *$\mathcal{V}$ is closed under scalar multiplication: $\alpha v \in \mathcal{V}$ for all $\alpha \in \mathbb{F}$, $v \in \mathcal{V}$.*

All of our examples of vector spaces so far have been subspaces of vector spaces like $\mathbb{F}^n$, $\mathbb{R}^I$ for some interval $I \subseteq \mathbb{R}$, or $\mathbb{R}^\infty$. However, plenty of subsets of a vector space fail to be subspaces for violating one or more of the subspace axioms above.

6.2 Problem. Let $\mathcal{V}$ be a vector space and let $v_0 \in \mathcal{V} \setminus \{0\}$. Explain all of the ways in which $\{v_0\}$ fails to be a subspace of $\mathcal{V}$.

Nonetheless, every subset of a vector space "generates" a subspace via the following important interaction between vector addition and scalar multiplication.

6.3 Definition. *Let $\mathcal{V}$ be a vector space over the field $\mathbb{F}$.*

(i) *A* **LINEAR COMBINATION** *of the vectors* $v_1, \ldots, v_n \in \mathcal{V}$ *is a vector of the form*

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \sum_{k=1}^{n} \alpha_k v_k$$

for some $\alpha_k \in \mathbb{F}$.

(ii) *Let* $\mathcal{B} \subseteq \mathcal{V}$ *be nonempty. (This set* $\mathcal{B}$ *need not be a subspace of* $\mathcal{V}$*.) The* **SPAN** *of* $\mathcal{B}$ *is the set of all linear combinations of vectors in* $\mathcal{B}$*. That is,*

$$\operatorname{span}(\mathcal{B}) := \left\{ \sum_{k=1}^{n} \alpha_k v_k \;\middle|\; \alpha_1, \ldots, \alpha_n \in \mathbb{F},\ v_1, \ldots, v_n \in \mathcal{V}, n \in \mathbb{N} \right\}.$$

If $\mathcal{B}$ *is a finite set, say,* $\mathcal{B} = \{w_j\}_{j=1}^m$*, then we write*

$$\operatorname{span}(\mathcal{B}) = \operatorname{span}(\{w_j\}_{j=1}^m) = \operatorname{span}(w_1, \ldots, w_m)$$

and omit the curly braces.

By the way, the sigma notation here reads as follows. If $w_1, \ldots, w_n \in \mathcal{V}$, then we define $\sum_{k=1}^{n} w_k$ recursively as

$$\sum_{k=1}^{n} w_k := \begin{cases} w_1, & n = 1 \\ w_n + \sum_{k=1}^{n-1} w_k, & n \geq 2. \end{cases}$$

Sometimes we may start the sum at an index other than 1. If $w_m, \ldots, w_n \in \mathcal{V}$ with $m \leq n$, put

$$\sum_{k=m}^{n} w_k := \sum_{k=1}^{n-(m-1)} w_{k+(m-1)}.$$

Last, we adopt the useful convention that the "empty sum" is the zero vector: if $m > n$, then

$$\sum_{k=m}^{n} w_k := 0.$$

6.4 Theorem. *Let* $\mathcal{V}$ *be a vector space over the field* $\mathbb{F}$ *and let* $\mathcal{B} \subseteq \mathcal{V}$ *be nonempty. Then* $\operatorname{span}(\mathcal{B})$ *is a subspace of* $\mathcal{V}$.

Proof. We prove this in the extremely special case when $\mathcal{B} = \{v_1, v_2\}$ to avoid too much sigma notation. We have

$$\operatorname{span}(\mathcal{B}) = \operatorname{span}(v_1, v_2) = \{\alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1,\ \alpha_2 \in \mathbb{F}\}.$$

First we check that the zero vector is in $\operatorname{span}(\mathcal{B})$. Since we are free to pick α_1 and α_2 to be any numbers in $\mathbb{F}$, we set them to be 0, so $0v_1 + 0v_2 \in \operatorname{span}(\mathcal{B})$. And $0v_1 + 0v_2 = 0 + 0 = 0$. (The first two instances of 0 in the preceding sentence were scalars; the last two were the zero vector in $\mathcal{V}$.)

Next we check closure under scalar multiplication. Given $\alpha_1, \alpha_2 \in \mathbb{F}$, we want to show $\alpha(\alpha_1 v_1 + \alpha_2 v_2) \in \text{span}(\mathcal{B})$. We just distribute the multiplication and rearrange parentheses:

$$\alpha(\alpha_1 v_1 + \alpha_2 v_2) = (\alpha\alpha_1)v_1 + (\alpha\alpha_2)v_2.$$

This is certainly a linear combination of v_1 and v_2, since $\alpha\alpha_1, \alpha\alpha_2 \in \mathbb{F}$. And so $\alpha(\alpha_1 v_1 + \alpha_2 v_2) \in \text{span}(\mathcal{B})$.

Last, we check closure under vector addition. We want to show

$$(\alpha_1 v_1 + \alpha_2 v_2) + (\beta_1 v_1 + \beta_2 v_2) \in \mathbb{F}$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$. We do some arithmetic:

$$(\alpha_1 v_1 + \alpha_2 v_2) + (\beta_1 v_1 + \beta_2 v_2) = (\alpha_1 v_1 + \beta_1 v_1) + (\alpha_2 v_2 + \beta_2 v_2) = (\alpha_1 + \beta_1)v_1 + (\alpha_2 + \beta_2)v_2 \in \text{span}(\mathcal{B}).$$

Here we "factored" not the scalars but the vectors v_1 and v_2, using distributive properties of vector space arithmetic. ■

We "live" in vector spaces in linear algebra; all of our interesting and relevant questions can somehow be posed (and solved?) using the vector space structure, possibly gussied up with additional features (like norms and inner products). However, to state those problems exactly, we need to be able to "move between" vector spaces in a way that "respects" the vector space operations (of vector addition and scalar multiplication). We also need an instrument to determine when two similar-looking vector spaces are really the same. We achieve this via the following.

6.5 Definition. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces (over the same field $\mathbb{F}$). A function $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ is a* **LINEAR OPERATOR** *(or* **LINEAR MAP** *or* **LINEAR TRANSFORMATION***) if $\mathcal{T}$ satisfies the following two properties.*

1. *Additivity. $\mathcal{T}(v + w) = \mathcal{T}(v) + \mathcal{T}(w)$ for all $v, w \in \mathcal{V}$.*

2. *Homogeneity. $\mathcal{T}(\alpha v) = \alpha\mathcal{T}(v)$ for all $\alpha \in \mathbb{F}$ and $v \in \mathcal{V}$.*

Note that the vector space operations on both sides of the two equals signs in the above are different. The addition in $\mathcal{T}(v + w)$ is addition in $\mathcal{V}$, while the addition in $\mathcal{T}(v) + \mathcal{T}(w)$ is addition in $\mathcal{W}$. The scalar multiplication in $\mathcal{T}(\alpha v)$ is scalar multiplication in $\mathcal{V}$, while the scalar multiplication in $\alpha\mathcal{T}(v)$ is scalar multiplication in $\mathcal{W}$.

We start by working through many examples of linear operators.

6.6 Example. The map

$$\mathcal{T}\colon \mathbb{R} \to \mathbb{R}\colon v \mapsto 2v$$

is linear. First we check homogeneity:

$$\mathcal{T}(\alpha v) = 2(\alpha v) = \alpha(2v) = \alpha\mathcal{T}(v)$$

Next we check additivity:

$$\mathcal{T}(v+w) = 2(v+w) = 2v+2w = \mathcal{T}(v) + \mathcal{T}(w).$$

6.7 Problem. Generalize the preceding example vastly by showing that scalar multiplication is a linear operator. More precisely, assume that $\mathcal{V}$ is a vector space over $\mathbb{F}$, and do the following.

(i) Fix $\lambda \in \mathbb{F}$ and show that the map

$$\mathcal{T} \colon \mathcal{V} \to \mathcal{V} \colon v \mapsto \lambda v$$

is a linear operator.

(ii) By choosing λ above appropriately, show that the map

$$I \colon \mathcal{V} \to \mathcal{V} \colon v \mapsto v$$

is linear. This is the **IDENTITY OPERATOR**.

(iii) By choosing λ above appropriately, show that the map

$$\mathcal{T} \colon \mathcal{V} \to \mathcal{V} \colon v \mapsto 0$$

is linear. This is the **ZERO OPERATOR**. (Outside this problem, we will typically call it 0 and thus vastly overwork that one poor symbol by making it denote an element of $\mathbb{F}$, an element of $\mathcal{V}$, and a linear operator simultaneously.)

6.8 Problem. Let $\mathcal{V}$ be a vector space and let $v_0 \in \mathcal{V} \setminus \{0\}$. Prove that the "constant"

$$\mathcal{T} \colon \mathcal{V} \to \mathcal{V} \colon v \mapsto v_0$$

is not linear. Explain why $\mathcal{T}$ fails both additivity and homogeneity. This problem shows that every vector space has a "nonlinear" operator defined on it.

Most of the familiar operations from calculus yield linear operators, since *limits are linear*.

6.9 Example. If $f \in \mathcal{C}^1([0,1])$, then f is differentiable and $f' \in \mathcal{C}([0,1])$, so the map

$$\mathcal{T} \colon \mathcal{C}^1([0,1]) \to \mathcal{C}([0,1]) \colon f \mapsto f'$$

is defined. It is also linear, because the derivative is linear:

$$\mathcal{T}(f+g) = (f+g)' = f' + g' = \mathcal{T}(f) + \mathcal{T}(g) \quad \text{and} \quad \mathcal{T}(\alpha f) = (\alpha f)' = \alpha f' = \alpha \mathcal{T}(f).$$

6.10 Example. Define

$$\mathcal{T} \colon \mathbb{R}^2 \to \mathbb{R}^2 \colon \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

We show that $\mathcal{T}$ is linear (and eventually we will connect it intimately with a matrix). First we check homogeneity:

$$\mathcal{T}\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \mathcal{T}\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} -\alpha x_2 \\ \alpha x_1 \end{bmatrix} = \alpha \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = \alpha \mathcal{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right).$$

Next we check additivity. This gets bulky quickly, so we do two calculations and hope they meet in the middle:

$$\mathcal{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \mathcal{T}\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} -\alpha(x_2 + y_2) \\ \alpha(x_1 + y_1) \end{bmatrix}$$

and

$$\mathcal{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \mathcal{T}\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -\alpha x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} -\alpha y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} -\alpha x_2 - \alpha y_2 \\ x_1 + y_1 \end{bmatrix} = \begin{bmatrix} -\alpha(x_2 + y_2) \\ x_1 + y_1 \end{bmatrix}.$$

They do.

6.11 Example. The fundamental theorem of calculus tells us that if $f \in \mathcal{C}([0,1])$, then the map

$$F \colon [0,1] \to \mathbb{R} \colon x \mapsto \int_0^x f(s)\ ds$$

is differentiable on $[0,1]$ with $F' = f$. Then F' is continuous, so $F \in \mathcal{C}^1([0,1])$. We may therefore define a map

$$\mathcal{T} \colon \mathcal{C}([0,1]) \to \mathcal{C}^1([0,1])$$

by

$$(\mathcal{T}f)(x) := \int_0^x f(s)\ ds.$$

Observe carefully what we did: we started with a function f on $[0,1]$ and we wanted to define a new function $\mathcal{T}f$ on $[0,1]$, which required us to specify the values $(\mathcal{T}f)(x)$ for each $x \in [0,1]$.

We check that $\mathcal{T}$ is linear. First,

$$\big(\mathcal{T}(\alpha f)\big)(x) = \int_0^x \alpha f(s)\ ds = \alpha \int_0^x f(s)\ ds = \alpha(\mathcal{T}f)(x)$$

by the linearity of the definite integral. Next,

$$(\mathcal{T}(f+g))(x) = \int_0^x \big(f(s) + g(s)\big)\ ds = \int_0^x f(s)\ ds + \int_0^s g(s)\ ds = (\mathcal{T}f)(x) + (\mathcal{T}f)(x),$$

again by the linearity of the integral. Since we have these pointwise equalities for all $x \in [0,1]$, the functions $\mathcal{T}(\alpha f)$ and $\alpha \mathcal{T}f$ are equal, as are $\mathcal{T}(f+g)$ and $\mathcal{T}(f) + \mathcal{T}(g)$.

The preceding example shows that antidifferentiation (properly defined via definite integrals) is a linear operator. We expect that differentiation and integration undo each other, and we will see how, through the right lenses, the operators of Examples 6.9 and 6.11 are "inverses" of each other. For now, it is worthwhile to reflect on our mathematical progress: in precalculus, we studied functions probably "in isolation" from each other; in calculus, we studied functions together via the common features of continuity, differentiability, and integrability; in linear algebra, we are studying functions (linear operators) that act on other functions (which are now viewed as vectors).

6.12 Problem. Here is a chance to think about linear operators as functions and thus as sets of ordered pairs. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, both over $\mathbb{F}$. Prove that a function $\mathcal{T}: \mathcal{V} \to \mathcal{W}$ is linear if and only if the following both hold.

1. If (v_1, w_1), $(v_2, w_2) \in \mathcal{T}$, then $(v_1 + v_2, w_1 + w_2) \in \mathcal{T}$.

2. If $(v, w) \in \mathcal{T}$ and $\alpha \in \mathbb{F}$, then $(\alpha v, \alpha w) \in \mathcal{T}$.

As we saw in Examples 1.1, 1.2, and 1.4, many problems (realistic or artificial, interesting or boring) can be written in the form "Solve $\mathcal{T}(v) = w$ for v given w, with $\mathcal{T}: \mathcal{V} \to \mathcal{W}$ a linear operator, $v \in \mathcal{V}$, and $w \in \mathcal{W}$." To become competent at this, we will do a few more examples of linear operators in isolation and then consider how linear operators interact with subspaces of $\mathcal{V}$ and $\mathcal{W}$ and with each other.

6.13 Problem. To be fair, Example 1.1 involves matrix-vector multiplication as the linear operator, and we have not yet discussed that. What was the linear operator in Example 1.2?

Hereafter, we will often remove the parentheses when evaluating a linear operator $\mathcal{T}$. That is, we frequently write

$$\mathcal{T}v := \mathcal{T}(v).$$

Day 7: Monday, August 26.

Material from *Linear Algebra* by Meckes & Meckes

Pages 67–69 introduce matrix-vector multiplication "componentwise"; you will probably recognize this as a dot product formulation. Lemma 2.11 deduces as a consequence of this definition our in-class definition of matrix-vector multiplication. Pages 69–73 discuss eigenvalues. Pages 73–75 review linear systems as matrix-vector equations and give an application of eigenvalues. See Example 3 on p.87 for a connection between a "continuous" integral operator and the "discrete" operator of matrix-vector multiplication. Example 4 on pp.87–88 generalizes our sequence "shift" example.

Do Quick Exercises #3 and #5 in Section 2.1.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

The linear operator induced by a matrix. Eigenvalue of a linear operator (N), eigenvector of a linear operator (N), eigenpair of a linear operator. Also, eigenvalue, eigenvector, and eigenpair for a matrix.

Sequences in $\mathbb{R}^\infty$ are morally "infinitely long column vectors," and, being infinite, their entries can be "shifted" in ways that those of a finite column vector cannot.

7.1 Example. For $(a_k) \in \mathbb{R}^\infty$, let $\mathcal{T}(a_k) = (a_{k+1})$. That is, $\mathcal{T}$ "shifts" all of the entries in (a_k) ahead by 1. For example, if we cartoonishly write

$$(a_k) = (a_1, a_2, a_3, a_4, \ldots),$$

then

$$\mathcal{T}(a_k) = (a_2, a_3, a_4, a_5, \ldots).$$

We show that $\mathcal{T}$ is linear. First, for (a_k), $(b_k) \in \mathbb{R}^\infty$, we have

$$\mathcal{T}\big[(a_k) + (b_k)\big] = \mathcal{T}(a_k + b_k) = (a_{k+1} + b_{k+1}) = (a_{k+1}) + (b_{k+1}) = \mathcal{T}(a_k) + \mathcal{T}(b_k).$$

Next, for $\alpha \in \mathbb{R}$ and $(a_k) \in \mathbb{R}^\infty$, we have

$$\mathcal{T}\big[\alpha(a_k)\big] = \mathcal{T}(\alpha a_k) = (\alpha a_{k+1}) = \alpha(a_{k+1}) = \alpha\mathcal{T}(a_k).$$

Now we take up the study of an essential linear operator whose presence we have delayed for quite a while: matrix-vector multiplication. We motivate its definition by considering what equality in the following linear system means.

7.2 Example. We are probably trained to write something like

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11. \end{cases}$$

as a matrix-vector equation, with matrix-vector multiplication on the left. Here is how the matrix emerges naturally from vector addition and scalar multiplication. This system is equivalent to the vector equality

$$\begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix},$$

and on the left we expand

$$\begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 2x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Of course this should also equal the matrix-vector product

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

That is, we want to define

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Here, then, is the idea: a matrix-vector product should be a linear combination of the columns of the matrix weighted by the entries of the vector. This is perhaps not the most familiar way of defining the product (and indeed an equivalent, and probably easier, way of doing so involves the dot product of the rows of the matrix with the vector), but it is highly useful for *understanding* matrix-vector multiplication.

7.3 Definition. *Let*

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{F}^{m \times n} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n.$$

Then

$$A\mathbf{x} := x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \sum_{j=1}^{n} x_j\mathbf{a}_j \in \mathbb{F}^m.$$

Note carefully that $\mathbf{x} \in \mathbb{F}^n$, $A \in \mathbb{F}^{m \times n}$, and $A\mathbf{x} \in \mathbb{F}^m$. Since $A\mathbf{x}$ is a linear combination of the columns of A, all of which have m rows, this is what we should expect.

7.4 Example. We compute

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+3 \\ 4+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Matrix-vector multiplication is of course linear in the sense that

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} \tag{7.1}$$

and

$$A(\alpha\mathbf{x}) = \alpha A\mathbf{x} \tag{7.2}$$

for all $A \in \mathbb{F}^{m\times n}$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{F}^n$, and $\alpha \in \mathbb{F}$. For example,

$$A(\alpha\mathbf{x}) = \sum_{j=1}^{n} \alpha x_j \mathbf{a}_j = \alpha \sum_{j=1}^{n} x_j \mathbf{a}_j = \alpha A\mathbf{x}.$$

7.5 Problem. Show that $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

The identities in (7.1) and (7.2) show that every $A \in \mathbb{F}^{m\times n}$ "induces" a linear operator from $\mathbb{F}^n$ to $\mathbb{F}^m$.

7.6 Theorem. *Let $A \in \mathbb{F}^{m\times n}$. The map*

$$\mathcal{T}_A \colon \mathbb{F}^n \to \mathbb{F}^m \colon \mathbf{v} \mapsto A\mathbf{v}$$

is linear. We call $\mathcal{T}_A$ the LINEAR OPERATOR INDUCED BY THE MATRIX *A.*

We emphasize that $\mathcal{T}_A$ and A are different objects! Both are functions, but A is a function from $\{(i,j) \mid i = 1,\ldots,m,\ j = 1,\ldots,n\}$, whereas $\mathcal{T}_A$ is a function from $\mathbb{F}^n$ to $\mathbb{F}^m$. Later we will develop the notion of isomorphism to explain how one might reasonably consider $\mathcal{T}_A$ and A to be "the same." Later we will also address the reverse question: if $\mathcal{T} \colon \mathbb{F}^n \to \mathbb{F}^m$ is linear, is there a matrix $A \in \mathbb{F}^{m\times n}$ such that $\mathcal{T}\mathbf{v} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$? (Yes.)

For now we pursue a different question: what makes a linear operator "easy"? Arguably the "easiest" linear operator defined on any vector space is scalar multiplication, as proved in Problem 6.7. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. A linear operator $\mathcal{T} \colon \mathcal{V} \to \mathcal{V}$ acts as scalar multiplication on a vector $v \in \mathcal{V}$ if

$$\mathcal{T}v = \lambda v$$

for some $\lambda \in \mathbb{F}$. This happens for any scalar λ if $v = 0$.

7.7 Problem. Explain that. More precisely, using only Definition 6.5, show that $\mathcal{T}0 = 0$ for any linear operator $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$. (To be clear, the 0 in $\mathcal{T}0$ is the zero vector for $\mathcal{V}$, while on the right it is the zero vector for $\mathcal{W}$. Perhaps subscripts would be nice here: $\mathcal{T}0_{\mathcal{V}} = 0_{\mathcal{W}}$.

Consequently, in considering the problem $\mathcal{T}v = \lambda v$, we restrict to the more interesting case of nonzero v.

7.8 Definition. *Let $\mathcal{V}$ be a vector space over $\mathbb{F}$ and let $\mathcal{T}: \mathcal{V} \to \mathcal{V}$ be linear. A scalar $\lambda \in \mathbb{F}$ is an* EIGENVALUE *of $\mathcal{T}$ if there exists $v \in \mathcal{V} \setminus \{0\}$ such that*

$$\mathcal{T}v = \lambda v.$$

Such a vector $v \neq 0$ is an EIGENVECTOR *of $\mathcal{T}$ corresponding to λ (equivalently, λ is an eigenvalue of $\mathcal{T}$ "corresponding" to v), and the ordered pair (λ, v) is an* EIGENPAIR *of $\mathcal{T}$.*

It is important to be aware that we only define eigenvalues and eigenvectors for linear operators mapping a space $\mathcal{V}$ back into itself. Indeed, if we want $\mathcal{T}v = \lambda v$, then since $v \in \mathcal{V}$, we also have $\lambda v \in \mathcal{V}$, and thus $\mathcal{T}v \in \mathcal{V}$. That is, eigenvalues simply do not make sense for an operator $\mathcal{T}: \mathcal{V} \to \mathcal{W}$ with $\mathcal{V} \neq \mathcal{W}$ (or maybe with $\mathcal{V}$ not a *subspace* of $\mathcal{W}$). We will eventually generalize the notion of eigenvalue to "singular values" when $\mathcal{V} \neq \mathcal{W}$.

We will also see that eigenvalues reveal vastly useful data about linear operators, and we will develop some methods for computing eigenvalues from scratch. It is frustrating that the eigenvalue problem is really overdetermined, for we want to solve the single equation $\mathcal{T}v = \lambda v$ with the two unknowns λ and v. For now, we focus on basic calculations and a variety of examples.

7.9 Example. We claim that 2 is an eigenvalue of the linear operator induced by

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

with corresponding eigenvector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We check this by computing

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Of course, we also talk about eigenvalues and eigenvectors of matrices. The following is mostly a rehash of Definition 7.8.

7.10 Definition. *Let $A \in \mathbb{F}^{n \times n}$. A scalar $\lambda \in \mathbb{F}$ is an* EIGENVALUE *of A if there exists $\mathbf{v} \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ such that*

$$A\mathbf{v} = \lambda \mathbf{v}.$$

Such a vector $\mathbf{v} \neq \mathbf{0}$ is an EIGENVECTOR *of A corresponding to λ (equivalently, λ is an eigenvalue of A "corresponding" to $\mathbf{v}$), and the ordered pair $(\lambda, \mathbf{v})$ is an* EIGENPAIR *of A.*

7.11 Problem. Let $A \in \mathbb{F}^{n \times n}$. Check that Definitions 7.8 and 7.10 are the same in the sense that $(\lambda, \mathbf{v})$ is an eigenpair for A (according to Definition 7.10) if and only if $(\lambda, \mathbf{v})$

is an eigenpair of the the linear operator induced by A (according to Definition 7.10 and Theorem 7.6).

We will see that $A \in \mathbb{F}^{n \times n}$ always has at least one eigenvalue (in $\mathbb{C}$, not necessarily in $\mathbb{R}$) and at most n distinct eigenvalues. However, operators more generally need not have so restrictive—or so generous—an amount of eigenvalues.

7.12 Example. For an interval $I \subseteq \mathbb{R}$, denote by $\mathcal{C}^\infty(I)$ the set of all infinitely differentiable functions on I. That is, $f \in \mathcal{C}^\infty(I)$ if and only if each derivative $f^{(k)}$ exists for all $k \geq 1$. This, unsurprisingly, is a vector space, and the differentiation operator

$$\mathcal{T}\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})\colon f \mapsto f'$$

is defined and linear.

We compute its eigenvalues: $\mathcal{T}f = \lambda f$ with $f \in \mathcal{C}^\infty(\mathbb{R}) \setminus \{0\}$ if and only if

$$f'(x) = \lambda f(x) \text{ for all } x \in \mathbb{R} \quad \text{and} \quad f(x) \neq 0 \text{ for at least one } x \in \mathbb{R}.$$

The first condition is the pointwise equality that defines $\mathcal{T}f = \lambda f$, and the second equality is the pointwise condition that means $f \neq 0$. That first condition means that f is a function whose derivative is a multiple of itself; we know from calculus that such functions are multiples of exponentials. Specifically, $f'(x) = \lambda f(x)$ for all $x \in \mathbb{R}$ if and only if

$$f(x) = f(0)e^{\lambda x},$$

where we are free to choose $f(0) \in \mathbb{R}$ to be any value.

Taking $f(0) = 1$, we see that $f(x) := e^{\lambda x}$ is an eigenvalue (one might say, "eigenfunction") of $\mathcal{T}$, and so every real number is an eigenvalue of $\mathcal{T}$. If we allow our functions to be complex-valued and accept that the calculus of complex-valued functions of a real variable is the same as the real-valued calculus that we know and love, and if we have a definition of e^z for $z \in \mathbb{C}$ that preserves the derivative identity $f'(x) = \lambda e^{\lambda x}$ when $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, then every complex number would be an eigenvalue of $\mathcal{T}$.

7.13 Problem. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. What are the eigenvalues of the zero operator on $\mathcal{V}$? The identity operator on $\mathcal{V}$? (These operators were defined in Problem 6.7.)

Day 8: Wednesday, August 28.

Here is an eigenvalue example that illustrates how the choice of vector space—context!—matters.

8.1 Example. **(i)** Consider the shift operator from Example 7.1:

$$\mathcal{T}\colon \mathbb{R}^\infty \to \mathbb{R}^\infty\colon (a_k) \mapsto (a_{k+1}).$$

To search for eigenvalues and eigenvectors, we study the equation

$$\mathcal{T}(a_k) = \lambda(a_k)$$

with $(a_k) \neq 0$. That is, we want $a_k \neq 0$ for at least one k and

$$(a_{k+1}) = (\lambda a_k).$$

Since sequences are equal if and only if their corresponding terms are equal, we want

$$a_{k+1} = \lambda a_k \tag{8.1}$$

for all integers $k \geq 1$. We see what this means for a few small values of k:

$$\begin{aligned} a_2 &= a_{1+1} = \lambda a_1 \\ a_3 &= a_{2+1} = \lambda a_2 = \lambda(\lambda a_1) = \lambda^2 a_1 \\ a_4 &= a_{3+1} = \lambda a_3 = \lambda(\lambda^2)a_1 = \lambda^3 a_1. \end{aligned}$$

It looks like

$$a_{k+1} = \lambda^k a_1$$

for all k, equivalently,

$$a_k = \lambda^{k-1} a_1 \tag{8.2}$$

for all k. We could prove this by induction on k from the relation (8.2), but we could also just take (8.2) as a *candidate* for an eigenvector with eigenvalue λ and check. We compute

$$\mathcal{T}(\lambda^{k-1}a_1) = (\lambda^{(k-1)+1}a_1) = \lambda(\lambda^{k-1}a_1).$$

Thus $(\lambda^{k-1}a_1)$ is an eigenvector for λ provided that $(\lambda^{k-1}a_1) \neq 0$.

If $a_1 \neq 0$ and $\lambda \neq 0$, then $(\lambda^{k-1}a_1)$ is definitely not the zero sequence, so any $\lambda \in \mathbb{R} \setminus \{0\}$ is an eigenvalue. We might want to be more careful with $\lambda = 0$, as there $\lambda^{k-1}a_1 = 0$ for $k \geq 2$, regardless of the choice of a_1. At $k = 1$, if we interpret $0^0 = 1$, then the sequence (v_k) defined by

$$v_k := \begin{cases} 1, & k \neq 0 \\ 0, & k = 0 \end{cases}$$

is not the zero sequence and

$$v_{k+1} = 0 = 0 \cdot v_k$$

for all $k \geq 1$. Then

$$\mathcal{T}(v_k) = (v_{k+1}) = 0,$$

and so (v_k) is an eigenvector corresponding to the eigenvalue 0.

(ii) Consider $\mathcal{T}$ now as an operator from ℓ^∞ to ℓ^∞, where ℓ^∞ was defined in Example 5.5. That $\mathcal{T}(a_k) \in \ell^\infty$ for any $(a_k) \in \ell^\infty$ is easy: if there is $M > 0$ such that $|a_k| \le M$ for all k, then certainly $|a_{k+1}| \le M$ for all k, too. Above we showed that if $\mathcal{T}(a_k) = \lambda(a_k)$, then $a_k = \lambda^{k-1}a_1$ for some $a_1 \in \mathbb{R}$. For (a_k) to be an eigenvector, we need $a_1 \ne 0$. But now the sequence $(\lambda^{k-1}a_1)$ need not be bounded. Indeed, if $|r| > 1$, then the sequence (r^k) is unbounded—it satisfies $\lim_{k\to\infty} r^k = \infty$ if $r > 0$, while if $r < 0$, the terms of the sequence (r^k) grow unboundedly large as $k \to \infty$, i.e., $\lim_{k\to\infty} r^{2k} = \infty$ and $\lim_{k\to\infty} r^{2k+1} = -\infty$. Consequently, $(\lambda^{k-1}a_1)$ is only bounded when $|\lambda| \le 1$, and so the eigenvalues of $\mathcal{T}$ as a linear operator from ℓ^∞ to ℓ^∞ are not all of $\mathbb{R}$ but only the interval $[-1, 1]$.

Here is an operator that has no eigenvalues.

8.2 Example. Define $\mathcal{T} \colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ by $(\mathcal{T}f)(x) = xf(x)$. That is, $\mathcal{T}$ is the (unimaginatively named) "multiplication by x" operator. Suppose that $\mathcal{T}f = \lambda f$ for some $\lambda \in \mathbb{R}$ and nonzero $f \in \mathcal{C}([0,1])$. By "nonzero" we mean that $f(x) \ne 0$ for at least one $x \in [0,1]$.

Pointwise, we have $\mathcal{T}f = \lambda f$ if and only if $(\mathcal{T}f)(x) = \lambda f(x)$ for all $x \in [0,1]$, thus if and only if

$$xf(x) = \lambda f(x), \ 0 \le x \le 1.$$

This is equivalent to

$$(x - \lambda)f(x) = 0, \ 0 \le x \le 1,$$

and so, for each $x \in [0,1]$, either

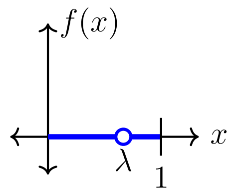
$$x - \lambda = 0 \quad \text{or} \quad f(x) = 0, \tag{8.3}$$

or possibly both.

If $x - \lambda = 0$, that means $x = \lambda$. But this is only possible if $\lambda \in [0,1]$. So, we consider two cases on λ.

1. $\lambda \in \mathbb{R} \setminus [0,1]$. That is, $\lambda < 0$ or $\lambda > 1$. Then in (8.3), it can never be the case that $x - \lambda = 0$ for some $x \in [0,1]$, and so it must be the case that $f(x) = 0$ for all x. But then $f = 0$, which is not allowed for an eigenvector. So, no $\lambda \in \mathbb{R} \setminus [0,1]$ is an eigenvalue.

2. $\lambda \in [0,1]$. Then for $x \in [0,1] \setminus \{\lambda\}$, we have from (8.3) that $f(x) = 0$. That is, f is 0 for all but one point in $[0,1]$. Here is the graph of f when $0 < \lambda < 1$.



Since f is continuous at λ, we have

$$f(\lambda) = \lim_{x\to\lambda} f(x) = \lim_{x\to\lambda} 0 = 0.$$

But then $f(x) = 0$ for all $x \in [0, 1]$, which is not allowed for an eigenvector. A similar argument with left or right limits, when $\lambda = 0$ or $\lambda = 1$, respectively, shows that $f = 0$ in those two cases as well. Thus no point in $[0, 1]$ is an eigenvalue.

Working with continuous complex-valued functions does not change the situation in the previous example, as the same continuity arguments would show that no $\lambda \in \mathbb{C}$ could be an eigenvalue of the "multiply by x" operator. However, here is a situation in which changing the field from $\mathbb{R}$ to $\mathbb{C}$ does improve the eigenvalue situation.

8.3 Example. Let

$$A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(i) Define

$$\mathcal{T} \colon \mathbb{R}^2 \to \mathbb{R}^2 \colon \mathbf{v} \mapsto A\mathbf{v}.$$

Here we consider $\mathbb{R}^2$ as a vector space over $\mathbb{R}$. (It is definitely not a vector space over $\mathbb{C}$, as $i\mathbf{v} \notin \mathbb{R}^2$ for $\mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$.)

We have $\mathcal{T}\mathbf{v} = \lambda\mathbf{v}$ if and only if

$$\begin{cases} -v_2 &= \lambda v_1 \\ v_1 &= \lambda v_2. \end{cases}$$

Like all eigenvalue-eigenvector problems, this is still overdetermined (two equations in the three unknowns λ, v_1, and v_2), but we can substitute the formula for v_1 from the second equation into the first to find

$$-v_2 = \lambda(\lambda v_2) = \lambda^2 v_2,$$

thus

$$(\lambda^2 + 1)v_2 = 0.$$

If $v_2 = 0$, then the second equation implies $v_1 = 0$ and so $\mathbf{v} = \mathbf{0}$, which is not permissible. So, to solve the eigenvalue-eigenvector problem, we need

$$\lambda^2 + 1 = 0,$$

thus $\lambda = \pm i \notin \mathbb{R}$.

Recall from Definition 7.8 that if $\mathcal{T} \colon \mathcal{V} \to \mathcal{V}$ is a linear operator and $\mathcal{V}$ is a vector space over $\mathbb{F}$, then an eigenvalue λ must belong to $\mathbb{F}$. Here $\mathbb{F} = \mathbb{R}$, so this "multiply by A" operator has no eigenvalues.

(ii) Now define

$$\mathcal{T} \colon \mathbb{C}^2 \to \mathbb{C}^2 \colon \mathbf{v} \mapsto A\mathbf{v},$$

where we consider $\mathbb{C}^2$ as a vector space over $\mathbb{C}$. (It is also a vector space over $\mathbb{R}$.) The "action" of this operator is exactly the same as in the previous part (multiply by A), but the domain of this operator is different (and larger). All of the previous work shows that $\mathcal{T}\mathbf{v} = \lambda\mathbf{v}$ only if $\lambda = \pm i$, and now we are considering $\mathbb{C}^2$ as a vector space over $\mathbb{C}$. So, the (putative) eigenvalues do belong to the field.

8.4 Problem. Why "putative" at the end of the example above? We did not show the existence of $\mathbf{v} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ such that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{v} = i\mathbf{v}.$$

Do that. Do the same for $-i$. How, if at all, are the eigenvectors for i and $-i$ related?

The previous example encourages us to reconsider an aspect of our definition of matrix eigenvalues.

8.5 Remark. *Definition 7.10 requires that an eigenvalue of $A \in \mathbb{F}^{n \times n}$ belong to $\mathbb{F}$. This may introduce an ambiguity for $A \in \mathbb{R}^{n \times n}$, as for such A we also have $A \in \mathbb{C}^{n \times n}$. From now on, we say that $\lambda \in \mathbb{C}$ is an* **EIGENVALUE** *of $A \in \mathbb{R}^{n \times n}$ if there is $\mathbf{v} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. While we will be very interested in when a matrix (with real or complex, nonreal entries) has real eigenvalues, we will not discount a complex, nonreal number as an eigenvalue of a matrix with strictly real entries.*

We have claimed that eigenvalues reveal valuable data about linear operators beyond when they act simply as scalar multiplication. To describe and evaluate such data, we need more tools, and so we shift our focus back to more general properties of linear operators.

The recent work on matrix operators and the result of Theorem 7.6 beg a converse question. We know that if $A \in \mathbb{F}^{n \times n}$, then A induces a linear operator $\mathcal{T} \colon \mathbb{F}^n \to \mathbb{F}^n$ given by multiplying by A. What if $\mathcal{T} \colon \mathbb{F}^n \to \mathbb{F}^n$ is *any* linear operator? Is $\mathcal{T}$ really matrix-vector multiplication?

Here is a suggestive example.

8.6 Example. The map

$$\mathcal{T} \colon \mathbb{R}^3 \to \mathbb{R}^3 \colon \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ v_2 - 2v_1 \\ v_3 \end{bmatrix}$$

is linear; this is straightforward but tedious to check from the definition. (The experienced reader might note that $\mathcal{T}$ acts on $\mathbf{v} \in \mathbb{R}^3$ by subtracting twice the first row of $\mathbf{v}$ from the second; this is an elementary row operation, the likes of which we shall see frequently soon.) Some clever "backwards" algebra reveals

$$\begin{bmatrix} v_1 \\ v_2 - 2v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ -2v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We recognize the expression on the right as a linear combination—specifically, the matrix-vector multiplication

$$v_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

That is, $\mathcal{T}$ is the linear operator induced by the matrix above (and so we did not even have to check from the definition that $\mathcal{T}$ was linear).

Day 9: Friday, August 30.

No class.

Day 10: Wednesday, September 4.

No class.

Day 11: Friday, September 6.

Material from *Linear Algebra* by Meckes & Meckes

See the various proof references within today's material below. Read the eigenvalue examples on pp.121–122.

Do Quick Exercises #23 and #24 in Section 2.5

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Eigenspace, range of a linear operator, kernel of a linear operator, null space of a matrix

And we're back.

Now we show that every linear operator from $\mathbb{F}^n$ to $\mathbb{F}^m$ is really matrix-vector multiplication. To do this, we need one new piece of technology that will later become a regular favorite.

11.1 Definition. *Let $n \geq 1$ be an integer and $1 \leq j \leq n$ be an integer. The* jTH STANDARD BASIS VECTOR IN $\mathbb{F}^n$ *is the vector $\mathbf{e}_j \in \mathbb{F}^n$ whose jth entry is 1 and whose other entries are all 0.*

The notation $\mathbf{e}_j$ is fairly standard, but it does not indicate the dimension n; that is usually clear from context.

11.2 Example. **(i)** In $\mathbb{F}^3$, the standard basis vector are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(ii) For

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{F}^3,$$

we have

$$\mathbf{v} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

That is,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$$

(iii) More generally, for $\mathbf{v} \in \mathbb{F}^n$, we have

$$\mathbf{v} = \sum_{j=1}^{n} v_j \mathbf{e}_j,$$

where v_j is, of course, the jth entry of $\mathbf{v}$. This is one of the essential "basis" properties of the vectors $\mathbf{e}_j$ that we will explore at length later.

Now we are ready for our result about linear operators on Euclidean spaces.

11.3 Theorem. *Let $\mathcal{T}\colon \mathbb{F}^n \to \mathbb{F}^m$ be linear. Then there exists $A \in \mathbb{F}^{m \times n}$ such that $\mathcal{T}\mathbf{v} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.*

Proof. Theorem 2.8 in the textbook. ■

11.4 Problem. Represent each of the following linear operators (which are "elementary row operations") by matrix-vector multiplication.

(i) The row interchange

$$\mathbb{F}^2 \to \mathbb{F}^2 \colon \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}.$$

(ii) The scaling

$$\mathbb{F}^2 \to \mathbb{F}^2 \colon \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} d_1 v_1 \\ d_2 v_2 \end{bmatrix},$$

where d_1, $d_2 \in \mathbb{F}$ are given.

This begs a new question. Every matrix $A \in \mathbb{F}^{m\times n}$ gives rise to a linear operator from $\mathbb{F}^n$ to $\mathbb{F}^m$. And every linear operator from $\mathbb{F}^n$ to $\mathbb{F}^m$ is incarnated by matrix-vector multiplication against a very specific matrix. But matrices in $\mathbb{F}^{m\times n}$ and linear operators from $\mathbb{F}^n$ to $\mathbb{F}^m$ are not the same; both, strictly speaking, are functions, but with very different domains and codomains. Yet they are acting in the same way on elements of $\mathbb{F}^n$. Is there a precise mathematical way of expressing this "sameness" while also maintaining this distinction between matrices and linear operators?

There is, and the answer is more linear operators, and more vector spaces. We now build some new machinery to encode "sameness," and this starts with examining more closely how a linear operator $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ interacts with the spaces $\mathcal{V}$ and $\mathcal{W}$. Recall that with $\mathcal{T}$ as a function (Definition 2.5), $\mathcal{V}$ is the **DOMAIN** of $\mathcal{T}$ and $\mathcal{W}$ is the **CODOMAIN**. And as a function, the **RANGE** of $\mathcal{T}$ is the set

$$\mathcal{T}(\mathcal{V}) := \{\mathcal{T}v \mid v \in \mathcal{V}\}.$$

However, the range is more than just a set.

11.5 Theorem. *Let $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ be linear. Then the range $\mathcal{T}(\mathcal{V})$ is a subspace of $\mathcal{W}$.*

Proof. Theorem 2.30 in the book. ■

The range tells us how much of $\mathcal{W}$ a linear operator $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ can "reach." If we want to solve the all-important linear equation $\mathcal{T}v = w$ for $v \in \mathcal{V}$ given $w \in \mathcal{W}$, we want the range to be as large as possible, probably ideally $\mathcal{T}(V) = \mathcal{W}$.

11.6 Problem. Use the fundamental theorem of calculus to prove that the range of

$$\mathcal{T}\colon \mathcal{C}^1([0,1]) \to \mathcal{C}([0,1])\colon f \mapsto f'$$

is $\mathcal{C}([0,1])$.

Another space associated with $\mathcal{T}$ gives equally critical data.

11.7 Definition. *The **KERNEL** of a linear operator $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ is the space*

$$\ker(\mathcal{T}) := \{v \in \mathcal{V} \mid \mathcal{T}v = 0\}.$$

*We might want to emphasize which zero vector is in play here: $v \in \ker(\mathcal{T})$ if and only if $\mathcal{T}v = 0_{\mathcal{W}}$. The kernel is often called the **NULL SPACE** in the context of the matrix-vector multiplication operator, i.e., the **NULL SPACE** of $A \in \mathbb{F}^{m\times n}$ is $\{\mathbf{v} \in \mathbb{F}^n \mid A\mathbf{v} = 0\}$.*

11.8 Theorem. *Let $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ be linear. Then the kernel $\ker(\mathcal{T})$ is a subspace of $\mathcal{V}$.*

Proof. Theorem 2.36 in the book. ■

11.9 Problem. Let

$$\mathcal{V} := \left\{ f \in \mathcal{C}^1([0,1]) \;\middle|\; f(0) = 0 \right\}.$$

Show that the kernel of

$$\mathcal{T} \colon \mathcal{V} \to \mathcal{C}([0,1]) \colon f \mapsto f'$$

is **TRIVIAL** in the sense that $\mathcal{T}f = 0$ if and only if $f = 0$.

11.10 Problem. Show that the kernel controls uniqueness of solutions to the all-important linear equation $\mathcal{T}v = w$ in the following sense. Let $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$ be linear and suppose that $v_0 \in \ker(\mathcal{T}) \setminus \{0\}$. Let $w \in \mathcal{W}$ and suppose that $\mathcal{T}v_\star = w$ for some $v_\star \in \mathcal{V}$. Show that $\mathcal{T}(v_\star + \alpha v_0) = w$ for all $\alpha \in \mathbb{F}$ and so the problem $\mathcal{T}v = w$ has infinitely many solutions.

A third kind of subspace associated to a linear operator arises when $\mathcal{T} \colon \mathcal{V} \to \mathcal{V}$ has an eigenvalue. The eigenvectors from Examples 7.12 and 8.1 were really *multiples* of one particular eigenvector. This might call to mind the notion of span (Definition 6.3), and spans are subspaces (Theorem 6.4).

11.11 Theorem. *Let $\mathcal{T} \colon \mathcal{V} \to \mathcal{V}$ be a linear operator with the eigenvalue λ. Then the* **EIGENSPACE**

$$\mathcal{E}_\lambda(\mathcal{T}) := \{ v \in \mathcal{V} \mid \mathcal{T}v = \lambda v \}$$

is a subspace of $\mathcal{V}$.

Proof. Corollary 2.40 in the book. ■

11.12 Problem. Explain why $\mathcal{E}_\lambda(\mathcal{T})$ is not the set of all eigenvectors of $\mathcal{T}$ corresponding to λ.

The proof of Theorem 11.11 hinged on manipulating the eigenvalue-eigenvector equation $\mathcal{T}v = \lambda v$ into the "kernel"-type equation $(\mathcal{T} - \lambda I)v = 0$. This was a perfectly natural (and naive) symbolic manipulation, and it bears more examination. We can add vectors and multiply them by scalars; linear operators act on vectors, so pointwise (vectorwise?) we should be able to add linear operators and multiply them by scalars.

11.13 Definition. *Let $\mathcal{T}_1$, $\mathcal{T}_2 \colon \mathcal{V} \to \mathcal{W}$ be linear operators and $\alpha \in \mathbb{F}$. We define the operators $\mathcal{T}_1 + \mathcal{T}_2$ and $\alpha\mathcal{T}_1$ "pointwise" by*

$$(\mathcal{T}_1 + \mathcal{T}_2)v := \mathcal{T}_1 v + \mathcal{T}_2 v \quad \text{and} \quad (\alpha\mathcal{T}_1)v := \alpha(\mathcal{T}_1 v) \tag{11.1}$$

for $v \in \mathcal{V}$.

On the left in (11.1), the sum $\mathcal{T}_1 + \mathcal{T}_2$ is the name for the new operator that pairs $v \in \mathcal{V}$ with the vector addition $\mathcal{T}_1 v + \mathcal{T}_2 v \in \mathcal{W}$, and likewise $\alpha\mathcal{T}_1$ is the new operator that pairs

$v \in \mathcal{V}$ with the scalar multiplication $\alpha(\mathcal{T}_1 v) \in \mathcal{W}$. This is *exactly* how we defined addition and scalar multiplication of functions from an arbitrary set X into $\mathbb{R}$.

With this operator arithmetic, we can define a new vector space of linear operators.

11.14 Theorem. *Let $\mathbf{L}(\mathcal{V}, \mathcal{W})$ denote the set of all linear operators from the vector space $\mathcal{V}$ to the vector space $\mathcal{W}$. Then $\mathbf{L}(\mathcal{V}, \mathcal{W})$ is a vector space with addition and scalar multiplication defined in Definition 11.13.*

Proof. Theorem 2.5 in the book. ■

Marvel at how far we have come: we started with spaces of column vectors and functions, generalized their fundamental properties to vector spaces, built the machinery of linear operators to connect vector spaces, and now we have created a new vector space out of linear operators.

Day 12: Monday, September 9.

Material from *Linear Algebra* by Meckes & Meckes

The composition of linear operators is defined on p.81. See Theorem 2.6 for distribution.

Do Quick Exercises #8 and #10 in Section 2.2.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Composition of linear operators (how do you define it?)

We recall that if $\mathcal{V}$ and $\mathcal{W}$ are vector spaces, then the set $\mathbf{L}(\mathcal{V}, \mathcal{W})$ of linear operators from $\mathcal{V}$ to $\mathcal{W}$ is a vector space with the "pointwise" algebraic operations of

$$\mathcal{T}_1 + \mathcal{T}_2 \colon \mathcal{V} \to \mathcal{W} \colon v \mapsto \mathcal{T}_1 v + \mathcal{T}_2 v \quad \text{and} \quad \alpha\mathcal{T} \colon \mathcal{V} \to \mathcal{W} \colon v \mapsto \alpha(\mathcal{T} v)$$

We consider a number of examples of this space of operators.

12.1 Example. **(i)** We have previously identified each operator $\mathcal{T} \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ with a matrix $A \in \mathbb{F}^{m\times n}$ in the sense that $\mathcal{T}\mathbf{v} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$. While $\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ and $\mathbb{F}^{m\times n}$ are different vector spaces (their elements are different vectors!), one of our lingering goals is to address precisely how they are "the same."

(ii) Possibly the simplest nontrivial operator space is $\mathbf{L}(\mathbb{F}, \mathcal{W})$, where $\mathcal{W}$ is any vector space. This is because the field $\mathbb{F}$, considered as a vector space over itself, is probably the simplest vector space, other than the trivial space $\{0\}$. Let $\mathcal{T} \in \mathbf{L}(\mathbb{F}, \mathcal{W})$. Then for any

$v \in \mathbb{F}$, we have $v = v \cdot 1$, and so

$$\mathcal{T}v = \mathcal{T}(v \cdot 1) = v\mathcal{T}(1),$$

where the second equality is due to the linearity of $\mathcal{T}$ and the assumption $v \in \mathbb{F}$. That is, $\mathcal{T}v$ is just the scalar multiplication of v against the vector $\mathcal{T}(1)$, and so all operators in $\mathbf{L}(\mathbb{F}, \mathcal{W})$ are really scalar multiplication.

(iii) Consider the situation opposite to the one above: $\mathbf{L}(\mathcal{V}, \mathbb{F})$, where $\mathcal{V}$ is any vector space. We call this the **(ALGEBRAIC) DUAL SPACE** of $\mathcal{V}$ and write $\mathcal{V}' := \mathbf{L}(\mathcal{V}, \mathbb{F})$. We may just call this the dual space, but be aware that there is another kind of dual space when $\mathcal{V}$ has a norm, and that space is usually denoted $\mathcal{V}^*$. We call an operator $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathbb{F})$ a **LINEAR FUNCTIONAL** on $\mathcal{V}$. It turns out that many interesting properties of $\mathcal{V}$ can be encoded via the linear functionals on $\mathcal{V}$.

For a concrete example, consider $\mathbf{L}(\mathbb{F}^3, \mathbb{F})$. We know that any $\mathcal{T} \in \mathbf{L}(\mathbb{F}^3, \mathbb{F})$ has the form $\mathcal{T}\mathbf{v} = A\mathbf{v}$ for some $A \in \mathbb{F}^{1 \times 3}$, say, $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$. Then

$$\mathcal{T}\mathbf{v} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

This is, of course, the **DOT PRODUCT** of the column vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

That is, every linear functional on $\mathbb{F}^3$ is given by taking the dot product against some fixed vector, and the same is true for $\mathbb{F}^n$.

(iv) If $\mathcal{V} = \mathcal{W}$, then we write $\mathbf{L}(\mathcal{V}) := \mathbf{L}(\mathcal{V}, \mathcal{V})$, and we say that $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ is a **LINEAR OPERATOR ON** $\mathcal{V}$. The space $\mathbf{L}(\mathcal{C}^\infty(\mathbb{R}))$ is immense and includes all differential and integral operators. The set

$$\mathcal{V} := \{\mathcal{T} \in \mathbf{L}(\mathcal{C}^\infty(\mathbb{R})) \mid \mathcal{T}f = af'' + bf' + cf \text{ for some } a,\ b,\ c \in \mathbb{R}\}$$

is a subspace and is arguably the central object of study in an ODE class.

We live in vector spaces, and we move between them via linear operators; together, linear operators and vector spaces encode the all-important problem of solving $\mathcal{T}v = w$ for an operator $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ with $\mathcal{V}$ and $\mathcal{W}$ vector spaces and $w \in \mathcal{W}$ given. But we can move among multiple vector spaces with successive operators.

12.2 Theorem. *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be vector spaces and let $\mathcal{T}_1 \in \mathbf{L}(\mathcal{U}, \mathcal{V})$ and $\mathcal{T}_2 \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. Fix $u \in \mathcal{U}$. Then $\mathcal{T}_1 u \in \mathcal{V}$, and so $\mathcal{T}_2(\mathcal{T}_1 u) \in \mathcal{W}$. We can therefore define a map*

$$\mathcal{T}_2\mathcal{T}_1 \colon \mathcal{U} \to \mathcal{W} \colon u \mapsto \mathcal{T}_2(\mathcal{T}_1 u).$$

This map $\mathcal{T}_2\mathcal{T}_1$ is the **COMPOSITION** *of $\mathcal{T}_2$ and $\mathcal{T}_1$, and it is linear: $\mathcal{T}_2\mathcal{T}_1 \in \mathbf{L}(\mathcal{U}, \mathcal{W})$. PIC*

Proof. Proposition 2.4 in the book. ■

Given operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$, we probably formally expect distributive laws like

$$\mathcal{T}_3(\mathcal{T}_1 + \mathcal{T}_2) = \mathcal{T}_3\mathcal{T}_1 + \mathcal{T}_3\mathcal{T}_2 \quad \text{and} \quad (\mathcal{T}_1 + \mathcal{T}_2)\mathcal{T}_3 = \mathcal{T}_1\mathcal{T}_3 + \mathcal{T}_2\mathcal{T}_3.$$

What do these mean? Since we want to add $\mathcal{T}_1$ and $\mathcal{T}_2$, both should be elements of the same operator space $\mathbf{L}(\mathcal{U}, \mathcal{V})$, so $\mathcal{T}_1 + \mathcal{T}_2 \in \mathbf{L}(\mathcal{U}, \mathcal{V})$. Since we want to compose $\mathcal{T}_3$ with $\mathcal{T}_1 + \mathcal{T}_2$, and since $\mathcal{T}_1 + \mathcal{T}_2$ maps into $\mathcal{V}$, the operator $\mathcal{T}_3$ should have domain $\mathcal{V}$, thus $\mathcal{T}_3 \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. Since operators are functions, we need pointwise (vectorwise?) equalities here:

$$\mathcal{T}_3(\mathcal{T}_1 + \mathcal{T}_2)u = (\mathcal{T}_3\mathcal{T}_1 + \mathcal{T}_3\mathcal{T}_2)u \tag{12.1}$$

for all $u \in \mathcal{U}$. What does each side of this equality mean?

The composition on the left of (12.1) is

$$\mathcal{T}_3(\mathcal{T}_1 + \mathcal{T}_2)u = \mathcal{T}_3[(\mathcal{T}_1 + \mathcal{T}_2)u],$$

and then the sum here is, by definition of the sum of operators,

$$(\mathcal{T}_1 + \mathcal{T}_2)u = \mathcal{T}_1u + \mathcal{T}_2u.$$

By linearity, the composition is

$$\mathcal{T}_3(\mathcal{T}_1u + \mathcal{T}_2u) = \mathcal{T}_3(\mathcal{T}_1u) + \mathcal{T}_3(\mathcal{T}_2u). \tag{12.2}$$

The sum on the right of (12.1) is, by definition of the sum of operators,

$$(\mathcal{T}_3\mathcal{T}_1 + \mathcal{T}_3\mathcal{T}_2)u = \mathcal{T}_3\mathcal{T}_1u + \mathcal{T}_3\mathcal{T}_2u,$$

and by definition of composition, this is

$$\mathcal{T}_3\mathcal{T}_1u + \mathcal{T}_3\mathcal{T}_2u = \mathcal{T}_3(\mathcal{T}_1u) + \mathcal{T}_3(\mathcal{T}_2u).$$

And that is exactly (12.2).

The point of this discussion was not to convince us that operator composition distributes over operator addition—of course it does, or we would not be writing composition via juxtaposition like multiplication or using the symbol $+$. Rather, the point above was to practice definitions: what does equality mean (pointwise/vectorwise), what does composition mean, what does addition mean. *What does it all mean?*

Here is a concrete example of operator composition, which is really matrix multiplication in disguise.

12.3 Example. Define

$$\mathcal{T}_1\colon \mathbb{R}^2 \to \mathbb{R}^3\colon \mathbf{u} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{u} \quad \text{and} \quad \mathcal{T}_2\colon \mathbb{R}^3 \to \mathbb{R}^4\colon \mathbf{v} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{v}.$$

Then

$$\mathcal{T}_1\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and so

$$\mathcal{T}_2(\mathcal{T}_1\mathbf{u}) = \mathcal{T}_2\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Later we will view this more succinctly as a product of matrices.

The space $\mathbf{L}(\mathcal{V})$ has the additional structure of operator composition, in addition to addition of operators and scalar multiplication of operators. That is, if $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathbf{L}(\mathcal{V})$, then the compositions $\mathcal{T}_1\mathcal{T}_2$ and $\mathcal{T}_2\mathcal{T}_1$ are both defined.

12.4 Problem. Why is that not true more generally in $\mathbf{L}(\mathcal{V}, \mathcal{W})$?

However, there is no guarantee that $\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1$ for $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathbf{L}(\mathcal{V})$ in general. That is, operator composition is not necessarily commutative.

12.5 Example. (i) Define

$$\mathcal{T}_1\colon \mathbb{R}^2 \to \mathbb{R}^2\colon \mathbf{v} \mapsto \begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix} \quad \text{and} \quad \mathcal{T}_2\colon \mathbb{R}^2 \to \mathbb{R}^2\colon \mathbf{v} \mapsto \begin{bmatrix} 3v_1 \\ v_2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Of course, $\mathcal{T}_1$ and $\mathcal{T}_2$ encode elementary row operations. We compute

$$\mathcal{T}_2(\mathcal{T}_1\mathbf{v}) = \mathcal{T}_2\begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix} = \begin{bmatrix} 3v_1 \\ v_2 - 2v_1 \end{bmatrix} \quad \text{but} \quad \mathcal{T}_1(\mathcal{T}_2\mathbf{v}) = \mathcal{T}_1\begin{bmatrix} 3v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 \\ v_2 - 6v_1 \end{bmatrix}.$$

We expect $v_2 - 2v_1 \neq v_2 - 6v_1$ in general; indeed, these numbers are equal only when $-2v_1 = -6v_1$, thus only if $v_1 = 0$. So, $\mathcal{T}_1\mathcal{T}_2\mathbf{v} = \mathcal{T}_2\mathcal{T}_1\mathbf{v}$ only for

$$\mathbf{v} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}.$$

(ii) Let $\mathcal{T}_1 \in \mathbf{L}(\mathcal{C}^\infty(\mathbb{R}))$ be the multiplication operator $(\mathcal{T}_1 f)(x) := xf(x)$, and let $\mathcal{T}_2$ be differentiation. Then

$$(\mathcal{T}_2(\mathcal{T}_1 f))(x) = (\mathcal{T}_1 f)'(x) = (xf(x))' = f(x) + xf'(x)$$

but

$$(\mathcal{T}_1(\mathcal{T}_2 f))(x) = (\mathcal{T}_1 f')(x) = xf'(x).$$

We have $f(x) + xf'(x) = xf'(x)$ only when $f(x) = 0$, so $\mathcal{T}_1\mathcal{T}_2 f = \mathcal{T}_2\mathcal{T}_1 f$ only for $f = 0$.

12.6 Problem. Explain why the last result is not surprising by checking that

$$0 \in \{v \in \mathcal{V} \mid \mathcal{T}_1\mathcal{T}_2 v = \mathcal{T}_2\mathcal{T}_1 v\}$$

for any $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathbf{L}(\mathcal{V})$. Is this set a subspace of $\mathcal{V}$?

Day 13: Wednesday, September 11.

Material from *Linear Algebra* by Meckes & Meckes

Pages 78–79 present basic properties of isomorphisms, and Theorem 2.9 proves that $\mathbb{F}^{m \times n}$ and $\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ are isomorphic.

Do Quick Exercise #7 in Section 2.2.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Injective/one-to-one linear operator (N), surjective/onto linear operator (N), isomorphism (N), isomorphic vector spaces (N)

We finally have the tools that we need to encode "sameness" of vector spaces. We start with an illustrative example.

13.1 Example. Of course,

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \middle| \ v_1, \ v_2, \ v_3 \in \mathbb{R} \right\} = \mathbb{R}^{1,2,3},$$

and we let

$$\mathbb{P}_2(\mathbb{R}) := \left\{ f \in \mathbb{R}^{\mathbb{R}} \ \middle| \ f(x) = ax^2 + bx + c \text{ for some } a, \ b, \ c \in \mathbb{R} \right\}$$

be the set of all "at-most quadratic" functions on $\mathbb{R}$. We claim that $\mathbb{R}^3$ and $\mathbb{P}_2(\mathbb{R})$ are "the same" in that their vector space operations behave "the same." Any vector in either space is controlled by exactly three real numbers; we add vectors in $\mathbb{R}^3$ componentwise, while we add functions in $\mathbb{P}_2(\mathbb{R})$ by combining "like terms," i.e., the same powers of x, which is a kind of componentwise addition.

The spaces $\mathbb{R}^3$ and $\mathbb{P}_2(\mathbb{R})$ really appear to be "the same" when viewed through the lens of a special linear operator. Any linear operator $\mathcal{T}\colon \mathbb{R}^3 \to \mathbb{P}_2(\mathbb{R})$ maps column vectors to functions, so to define $\mathcal{T}\mathbf{v}$ for $\mathbf{v} \in \mathbb{R}^3$, we need to specify the values of $(\mathcal{T}\mathbf{v})(x)$ for $x \in \mathbb{R}$. Put

$$(\mathcal{T}\mathbf{v})(x) := v_1x^2 + v_2x + v_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

It is not too hard to show that $\mathcal{T}$ is linear. For example, to show $\mathcal{T}(\mathbf{v}+\mathbf{w}) = \mathcal{T}\mathbf{v}+\mathcal{T}\mathbf{w}$, we need to show $(\mathcal{T}(\mathbf{v}+\mathbf{w}))(x) = (\mathcal{T}\mathbf{v})(x) + (\mathcal{T}\mathbf{w})(x)$, and that amounts to checking

$$(v_1 + w_1)x^2 + (v_2 + w_2)x + (v_3 + w_3) = (v_1x^2 + v_2x + v_3) + (w_1x^2 + w_2x + w_3)$$

for all $x \in \mathbb{R}$. Of course this is true.

What is more interesting is how $\mathcal{T}$ relates elements of $\mathbb{R}^3$ and $\mathbb{P}_2(\mathbb{R})$. First, given any $f \in \mathbb{P}_2(\mathbb{R})$, if we write $f(x) = ax^2 + bx + c$, then

$$f = \mathcal{T}\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

That is, for every $f \in \mathbb{P}_2(\mathbb{R})$, there is $\mathbf{v} \in \mathbb{R}^3$ such that $\mathcal{T}\mathbf{v} = f$. In other words, we can represent any $f \in \mathbb{P}_2(\mathbb{R})$ by some $\mathbf{v} \in \mathbb{R}^3$ "under the lens" of $\mathcal{T}$.

Next, this representation is unique. Suppose $\mathcal{T}\mathbf{v} = \mathcal{T}\mathbf{w}$ for some $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$. Here equality means equality of functions, so $(\mathcal{T}\mathbf{v})(x) = (\mathcal{T}\mathbf{w})(x)$ for all $x \in \mathbb{R}$. By definition of $\mathcal{T}$, this implies

$$v_1x^2 + v_2x + v_3 = w_1x^2 + w_2x + w_3, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

This equality holds if and only if $v_1 = w_1$, $v_2 = w_2$, and $v_3 = w_3$. (Why this is true is actually a linear algebra problem of linear systems; we will just take it as a fact that two polynomials are equal on $\mathbb{R}$ if and only if the coefficients on their corresponding powers are equal.) Thus $\mathbf{v} = \mathbf{w}$.

The operator from the previous example is a special case of a much more general, but still special, kind of operator.

13.2 Definition. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$.*

(i) *The operator $\mathcal{T}$ is* SURJECTIVE *or* ONTO *if for each $w \in \mathcal{W}$ there exists $v \in \mathcal{V}$ such that $\mathcal{T}v = w$.*

(ii) *The operator $\mathcal{T}$ is* INJECTIVE *or* ONE-TO-ONE *if whenever $\mathcal{T}v_1 = \mathcal{T}v_2$ for v_1, $v_2 \in \mathcal{V}$, then $v_1 = v_2$.*

(iii) *The operator $\mathcal{T}$ is an* **ISOMORPHISM**, *and the spaces $\mathcal{V}$ and $\mathcal{W}$ are* **ISOMORPHIC**, *if $\mathcal{T}$ is both surjective and injective.*

13.3 Problem. **(i)** Explain why

$$\mathcal{T}_1 \colon \mathcal{C}^1([0,1]) \to \mathcal{C}([0,1]) \colon f \mapsto f'$$

is surjective but not injective.

(ii) Put

$$\mathcal{V} := \left\{ f \in \mathcal{C}^1([0,1]) \;\middle|\; f(0) = 0 \right\}.$$

Explain why

$$\mathcal{T}_2 \colon \mathcal{V} \to \mathcal{C}([0,1]) \colon f \mapsto f'$$

is both surjective and injective and thus an isomorphism.

13.4 Problem. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are isomorphic vector spaces. Is every operator $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ an isomorphism?

The fundamental problem of linear algebra, that of solving $\mathcal{T}v = w$ uniquely for all $w \in\in \mathcal{W}$ with $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$, is solvable precisely when $\mathcal{T}$ is an isomorphism. There is no glamorous way of checking surjectivity, but there is a shortcut to injectivity. We have

$$\mathcal{T}v_1 = \mathcal{T}v_2 \iff \mathcal{T}v_1 - \mathcal{T}v_2 = 0 \iff \mathcal{T}(v_1 - v_2) = 0 \iff v_1 - v_2 \in \ker(\mathcal{T}).$$

Since injectivity demands $v_1 = v_2$, equivalently, $v_1 - v_2 = 0$, and since $0 \in \ker(\mathcal{T})$ always, this calculation suggests that the following is true, and it is.

13.5 Theorem. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. Then $\mathcal{T}$ is injective if and only if* $\ker(\mathcal{T}) = \{0\}$.

Proof. Theorem 2.37 in the book. ■

The notion of isomorphism helps us qualify how matrices and linear operators on Euclidean spaces are "the same."

13.6 Theorem. *The spaces $\mathbb{F}^{m \times n}$ and $\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ are isomorphic. Specifically, the operator $\mathcal{T} \colon \mathbb{F}^{m \times n} \to \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ such that $(\mathcal{T}A)\mathbf{v} = A\mathbf{v}$ for all $A \in \mathbb{F}^{m \times n}$ and $\mathbf{v} \in \mathbb{F}^n$, is an isomorphism.*

Proof. We have already proved surjectivity in Theorem 11.3. To see this from the definition, let $\mathcal{S} \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$. We need to find $A \in \mathbb{F}^{m \times n}$ such that $\mathcal{T}A = \mathcal{S}$. This equality is equality of operators on $\mathbb{F}^n$, so we need $(\mathcal{T}A)\mathbf{v} = \mathcal{S}\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$. By definition of $\mathcal{T}$, the matrix A needs to satisfy $A\mathbf{v} = \mathcal{S}\mathbf{v}$. The proof of Theorem 11.3 tells us how to construct A: take $A = \begin{bmatrix} \mathcal{S}\mathbf{e}_1 & \cdots & \mathcal{S}\mathbf{e}_n \end{bmatrix}$, where $\mathbf{e}_j$ is the jth standard basis vector for $\mathbb{F}^n$ (Definition 11.1).

For injectivity, we show $\ker(\mathcal{T}) = \{0\}$. Here 0 is the zero matrix in $\mathbb{F}^{m \times n}$, i.e., the $m \times n$ matrix whose entries are all 0 (i.e., the number 0). We already have $0 \in \ker(\mathcal{T})$, so we just need to show that if $\mathcal{T}A = 0$, then $A = 0$. In the equality $\mathcal{T}A = 0$, the symbol 0 represents the zero operator from $\mathbb{F}^n$ to $\mathbb{F}^m$. (The poor symbol 0 is getting quite a workout here.)

So, assume that $A \in \mathbb{F}^{m \times n}$ with $\mathcal{T}A = 0$. Then $(\mathcal{T}A)\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathbb{F}^n$, where now $\mathbf{0}$ is the zero vector in $\mathbb{F}^m$. Since this is true for all $\mathbf{v}$, we can take $\mathbf{v}$ conveniently: let $\mathbf{v} = \mathbf{e}_j$ for $j = 1, \ldots, n$. It is a fact that $A\mathbf{e}_j = \mathbf{a}_j$, where $\mathbf{a}_j$ is the jth column of A. Then $\mathbf{a}_j = \mathbf{0}$ for each j, and therefore each column of A is the zero vector (in $\mathbb{F}^m$). Consequently, A is the zero matrix in $\mathbb{F}^{m \times n}$, as desired. ■

13.7 Problem. Check that fact: $A\mathbf{e}_j = \mathbf{a}_j$ for each j.

As an exercise in baroque notation, the operator $\mathcal{T}$ from the previous theorem is an element of $\mathbf{L}(\mathbb{F}^{m \times n}, \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m))$!

Day 14: Friday, September 13.

We consider some concrete problems for injectivity and surjectivity.

14.1 Example. Let $\mathcal{T} \colon \mathcal{C}([0,1]) \to \mathcal{C}^1([0,1])$ be the "antidifferentiation" operator given by

$$(\mathcal{T}f)(x) := \int_0^x f(s)\ ds, \ \ f \in \mathcal{C}([0,1]).$$

To check injectivity, we assume $\mathcal{T}f = 0$, and we want to show $f = 0$. That is, pointwise, we have $(\mathcal{T}f)(x) = 0$ for all $x \in [0,1]$, and the goal is $f(x) = 0$ for all $x \in [0,1]$. The definition of $\mathcal{T}$ implies

$$\int_0^x f(s)\ ds = 0, \ \ x \in [0,1].$$

One way to extract information about a function f from an integral involving f is to try to differentiate that integral. Since the equality above is true *for all* x, we have

$$\frac{d}{dx}\left[\int_0^x f(s)\ ds\right] = \frac{d}{dx}[0].$$

Using the fundamental theorem of calculus on the left, we get

$$f(x) = 0$$

for all x. Thus $f = 0$, and $\mathcal{T}$ is injective.

For surjectivity, we want to take $g \in \mathcal{C}^1([0,1])$ and find $f \in \mathcal{C}([0,1])$ such that $\mathcal{T}f = g$, i.e., such that $(\mathcal{T}f)(x) = g(x)$ for all x. Can we do this? We need

$$\int_0^x f(s)\ ds = g(x), \ \ 0 \le x \le 1. \qquad (14.1)$$

We could try the trick above of differentiating both sides to get $f(x) = g'(x)$. But if we check our work, we find (by the fundamental theorem of calculus)

$$(\mathcal{T}g')(x) = \int_0^x g'(s)\, ds = g(x) - g(0), \tag{14.2}$$

and this does not equal $g(x)$ unless $g(0) = 0$. Indeed, taking $x = 0$ in (14.1) implies

$$g(0) = \int_0^0 f(s)\, ds = 0,$$

and so we have a "solvability condition": if $\mathcal{T}f = g$, then $g(0) = 0$. There are plenty of functions in $\mathcal{C}^1([0,1])$ that do not meet this; take $g(x) = \cos(x)$ or $g(x) = 1$. And so $\mathcal{T}$ is not surjective: there is no $f \in \mathcal{C}([0,1])$ such that $(\mathcal{T}f)(x) = 1$ for all x.

But this points to at least a characterization of the range of $\mathcal{T}$: we might conjecture

$$\mathcal{T}\big(\mathcal{C}([0,1])\big) = \big\{ g \in \mathcal{C}^1([0,1]) \;\big|\; g(0) = 0 \big\}.$$

We already know that $\mathcal{T}f$ satisfies $(\mathcal{T}f)(0) = 0$, so assume $g \in \mathcal{C}^1([0,1])$ with $g(0) = 0$. The calculation in (14.2) implies $\mathcal{T}f = g$, and so g is in the range of $\mathcal{T}$.

14.2 Example. We study the shift operator

$$\mathcal{T} \colon \mathbb{R}^\infty \to \mathbb{R}^\infty \colon (a_k) \mapsto (a_{k+1}).$$

To check injectivity, we assume $\mathcal{T}(a_k) = 0$, where 0 is the zero sequence. Then $(a_{k+1}) = 0$, so $a_{k+1} = 0$ for all $k \ge 1$ (where the second 0 is the scalar 0). That is, $a_j = 0$ for $j \ge 2$, but this says nothing about a_1. Indeed, if we put

$$a_k = \begin{cases} 1, & k = 1 \\ 0, & k \ge 2, \end{cases}$$

then we have $(a_k) \ne 0$ but $\mathcal{T}(a_k) = 0$, so $\mathcal{T}$ is not injective.

For surjectivity, let $(b_k) \in \mathbb{R}^\infty$. We want $(a_k) \in \mathbb{R}^\infty$ such that $\mathcal{T}(a_k) = (b_k)$, so we want $(a_{k+1}) = (b_k)$. Termwise, this means $a_{k+1} = b_k$, and we could reindex this to $a_j = b_{j-1}$ for $j \ge 2$. This tells us nothing about a_1, however, and so we could simply set

$$a_k = \begin{cases} 0, & k = 1 \\ b_{k-1}, & k \ge 2 \end{cases} \tag{14.3}$$

to conclude $\mathcal{T}(a_k) = (b_k)$.

14.3 Problem. In the context of surjectivity of the previous example, let

$$z_k = \begin{cases} 1, & k = 1 \\ 0, & k \neq 0 \end{cases}$$

and show that $\mathcal{T}\big[(a_k)+\alpha(z_k)\big] = (b_k)$ with (a_k) defined by (14.3). Interpret this calculation in the context of Problem 11.10.

14.4 Problem. Prove that the operator

$$\mathcal{T}\colon \mathbb{R}^\infty \to \mathbb{R}^\infty\colon (a_k) \mapsto (ka_k)$$

is an isomorphism.

Day 15: Monday, September 16.

Material from *Linear Algebra* by Meckes & Meckes

Pages 78–80 discuss invertible linear operators. Proposition 2.2 proves that inverses are linear, and the example on p.80 gives a finite-dimensional inverse calculation. See also pp.380–382 in Appendix A for inverses of functions more generally.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Invertible linear operator (N), inverse of a linear operator

Previously we have checked injectivity and surjectivity of an operator separately. This was a good idea, since our recent examples have failed to be both! However, it can be more efficient to test for both simultaneously.

15.1 Problem. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Prove that $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ is an isomorphism if and only if for all $w \in \mathcal{W}$, there exists a unique $v \in \mathcal{V}$ such that $\mathcal{T}v = w$.

15.2 Example. We check if

$$\mathcal{T}\colon \mathbb{R}^2 \to \mathbb{R}^2\colon \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix}$$

is an isomorphism. For $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^2$, we calculate

$$\mathcal{T}\mathbf{v} = \mathbf{w} \iff \begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\iff \begin{cases} v_1 = w_1 \\ v_2 - 2v_1 = w_2 \end{cases}$$

$$\iff \begin{cases} v_1 = w_1 \\ v_2 = w_2 + 2w_1 \end{cases} .$$

Yes, $\mathcal{T}$ is an isomorphism: for all $\mathbf{w} \in \mathbb{R}^2$, there is a unique vector $\mathbf{v} \in \mathbb{R}^2$ such that $\mathcal{T}\mathbf{v} = \mathbf{w}$, and this vector $\mathbf{v}$ is given by $\mathbf{v} = \mathcal{S}\mathbf{w}$, where $\mathcal{S}$ is the linear operator

$$\mathcal{S} \colon \mathbb{R}^2 \to \mathbb{R}^2 \colon \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ w_2 + 2w_1 \end{bmatrix}.$$

We are not actually going to check that $\mathcal{S}$ is linear, but by now hopefully it is easy to represent $\mathcal{S}$ as matrix-vector multiplication. The calculation above shows that, for $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^2$, we have

$$\mathcal{T}\mathbf{v} = \mathbf{w} \iff \mathbf{v} = \mathcal{S}\mathbf{w}. \tag{15.1}$$

Knowing this alone actually proves that $\mathcal{T}$ is an isomorphism, even without the formula for $\mathcal{S}$. Indeed, for surjectivity, let $\mathbf{w} \in \mathbb{R}^2$ and compute, by the defining property (15.1) of $\mathcal{S}$, that $\mathcal{T}(\mathcal{S}\mathbf{w}) = \mathbf{w}$ (take $\mathbf{v} = \mathcal{S}\mathbf{w}$). For injectivity, suppose $\mathcal{T}\mathbf{v} = \mathbf{0}$. Then the defining property (15.1) of $\mathcal{S}$ implies $\mathbf{v} = \mathcal{S}\mathbf{0} = \mathbf{0}$ since $\mathcal{S}$ is linear (here we take $\mathbf{w} = \mathbf{0}$).

15.3 Problem. Generalize the result at the end of this example as follows. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are vector spaces and $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. If there exists $\mathcal{S} \in \mathbf{L}(\mathcal{W}, \mathcal{V})$ such that $\mathcal{T}v = w$ if and only if $v = \mathcal{S}w$ for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$, show that $\mathcal{T}$ is an isomorphism.

Here is the abstraction of the previous example (which also proves that the result in the problem can be strengthened to an "if and only if" statement). Let $\mathcal{V}$ and $\mathcal{W}$ be isomorphic vector spaces with isomorphism $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$. So, $\mathcal{T}$ is a linear operator that is injective and surjective. Then for each $w \in \mathcal{W}$, there is a unique $v \in \mathcal{V}$ such that $\mathcal{T}v = w$. Now define a map (not necessarily a linear operator!) $\mathcal{S} \colon \mathcal{W} \to \mathcal{V}$ by setting, for $w \in \mathcal{W}$, the vector $\mathcal{S}w \in \mathcal{V}$ to be the unique vector $v \in \mathcal{V}$ such that $\mathcal{T}v = w$. That is,

$$\mathcal{T}v = w \iff \mathcal{S}w = v \quad \text{and so} \quad \mathcal{T}(\mathcal{S}w) = w. \tag{15.2}$$

Both injectivity and surjectivity of $\mathcal{T}$ are critical for $\mathcal{S}$ to be a function from $\mathcal{W}$ to $\mathcal{V}$: surjectivity guarantees the existence of at least one $v \in \mathcal{V}$ such that $\mathcal{T}v = w$, given $w \in \mathcal{W}$, while injectivity guarantees the uniqueness of v.

15.4 Problem. Since $\mathcal{S} \colon \mathcal{W} \to \mathcal{V}$ is a function, $\mathcal{S}$ really is a set of ordered pairs. Prove that

$$\mathcal{S} = \{(w, v) \mid w \in \mathcal{W} \text{ and } v \in \mathcal{V} \text{ satisfy } \mathcal{T}v = w\}.$$

Of course, $\mathcal{S}$ will not be just any function but a linear operator from $\mathcal{W}$ to $\mathcal{V}$. First we check that if $\alpha \in \mathbb{F}$ and $w \in \mathcal{W}$, then $\mathcal{S}(\alpha w) = \alpha \mathcal{S} w$. We know that $\mathcal{S}w$ is the unique vector in $\mathcal{V}$ to satisfy $\mathcal{T}(\mathcal{S}w) = w$, and likewise $\mathcal{S}(\alpha w)$ is the unique vector in $\mathcal{V}$ to satisfy $\mathcal{T}(\mathcal{S}(\alpha w)) = \alpha w$. Then we have $\alpha w = \alpha \mathcal{T}(\mathcal{S}w)$, and because $\mathcal{T}$ is linear this becomes

$$\alpha w = \alpha \mathcal{T}(\mathcal{S}w) = \mathcal{T}(\alpha \mathcal{S}w).$$

Thus

$$\mathcal{T}(\alpha \mathcal{S}w) = \alpha w = \mathcal{T}(\mathcal{S}(\alpha w)),$$

so by the injectivity of $\mathcal{T}$ we have $\alpha \mathcal{S}w = \mathcal{S}(\alpha w)$.

Now let $w_1, w_2 \in \mathcal{W}$. We want to show $\mathcal{S}(w_1+w_2) = \mathcal{S}w_1+\mathcal{S}w_2$. We know that $\mathcal{S}(w_1+w_2)$ is the unique vector in $\mathcal{V}$ to satisfy $\mathcal{T}[\mathcal{S}(w_1+w_2)] = w_1+w_2$, $\mathcal{S}w_1$ is the unique vector in $\mathcal{V}$ to satisfy $\mathcal{T}(\mathcal{S}w_1) = w_1$, and $\mathcal{S}w_2$ is the unique vector in $\mathcal{V}$ to satisfy $\mathcal{T}(\mathcal{S}w_2) = w_2$. The linearity of $\mathcal{T}$ implies

$$w_1 + w_2 = \mathcal{T}(\mathcal{S}w_1) + \mathcal{T}(\mathcal{S}w_2) = \mathcal{T}(\mathcal{S}w_1 + \mathcal{S}w_2),$$

and so

$$\mathcal{T}(\mathcal{S}w_1 + \mathcal{S}w_2) = w_1 + w_2 = \mathcal{T}[\mathcal{S}(w_1 + w_2)].$$

The injectivity of $\mathcal{T}$ implies $\mathcal{S}w_1 + \mathcal{S}w_2 = \mathcal{S}(w_1 + w_2)$.

Of course, we want to call $\mathcal{S}$ the inverse of $\mathcal{T}$ and write $\mathcal{S} = \mathcal{T}^{-1}$. But the definite article "the" needs justification—is $\mathcal{S}$ really the only operator to satisfy (15.2)? Suppose there are two: let $\mathcal{S}_1, \mathcal{S}_2 \in \mathbf{L}(\mathcal{W}, \mathcal{V})$ such that

$$\mathcal{T}v = w \iff \mathcal{S}_1 w = v \quad \text{and} \quad \mathcal{T}v = w \iff \mathcal{S}_2 w = v \tag{15.3}$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. Our goal is $\mathcal{S}_1 = \mathcal{S}_2$, i.e., $\mathcal{S}_1 w = \mathcal{S}_2 w$ for all $w \in \mathcal{W}$.

Fix $w \in \mathcal{W}$. Take $v = \mathcal{S}_1 w$ and then $v = \mathcal{S}_2$ in (15.3) to obtain, respectively, $\mathcal{T}(\mathcal{S}_1 w) = w$ and $\mathcal{T}(\mathcal{S}_2 w) = w$. That is, for all $w \in \mathcal{W}$, we have $\mathcal{T}(\mathcal{S}_1 w) = \mathcal{T}(\mathcal{S}_2 w)$, and so injectivity of $\mathcal{T}$ forces $\mathcal{S}_1 w = \mathcal{S}_2 w$.

The defining property (15.2) of $\mathcal{S}$ means that $\mathcal{T}(\mathcal{S}w) = w$ for all $w \in \mathcal{W}$. That is, $\mathcal{T}\mathcal{S} = I_{\mathcal{W}}$, where

$$I_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W} \colon w \mapsto w$$

is the identity operator on $\mathcal{W}$. We claim that we also have $\mathcal{S}(\mathcal{T}v) = v$ for all $v \in \mathcal{V}$. Indeed, if we fix $v \in \mathcal{V}$ and take $w = \mathcal{T}v$, then (15.2) implies $v = \mathcal{S}w$. Substituting $\mathcal{T}v$ for w gives $v = \mathcal{S}(\mathcal{T}v)$, and so $\mathcal{S}\mathcal{T} = I_{\mathcal{V}}$, where

$$I_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{V} \colon v \mapsto v$$

is the identity operator on $\mathcal{V}$.

15.5 Problem. Do $\mathcal{T}$ and $\mathcal{S}$ commute?

We celebrate with a theorem.

15.6 Theorem. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ be an isomorphism. Then the map $\mathcal{S} : \mathcal{W} \to \mathcal{V}$ defined by taking $\mathcal{S}w$ for $w \in \mathcal{W}$ to be the unique vector $v \in \mathcal{V}$ such that $\mathcal{T}v = w$ is a linear operator from $\mathcal{W}$ to $\mathcal{V}$. Moreover, $\mathcal{S}$ is the only operator from $\mathcal{W}$ to $\mathcal{V}$ to satisfy*

$$\mathcal{T}v = w \iff \mathcal{S}w = v. \tag{15.4}$$

We call $\mathcal{S}$ the INVERSE *of $\mathcal{T}$, say that $\mathcal{T}$ is* INVERTIBLE, *and write $\mathcal{S} = \mathcal{T}^{-1}$. We have*

$$\mathcal{T}(\mathcal{S}w) = w \text{ for all } w \in \mathcal{W} \quad \text{and} \quad \mathcal{S}(\mathcal{T}v) = v \text{ for all } v \in \mathcal{V}.$$

15.7 Problem. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ be an isomorphism. Prove that $\mathcal{T}^{-1}$ is also an isomorphism (from $\mathcal{W}$ to $\mathcal{V}$) and that $(\mathcal{T}^{-1})^{-1} = \mathcal{T}$.

15.8 Example. A variety of previous problems and examples lead us to conclude that the operator

$$\mathcal{T} : \mathcal{C}^1([0,1]) \to \mathcal{C}([0,1]) : f \mapsto f'$$

is not an isomorphism, because $\mathcal{T}$ is not injective ($\mathcal{T}f = 0$ for any constant f). This does not say that $\mathcal{C}^1([0,1])$ and $\mathcal{C}([0,1])$ are not isomorphic, merely that if they are, it will not be through $\mathcal{T}$.

If we define

$$\mathcal{V} := \{f \in \mathcal{C}^1([0,1]) \mid f(0) = 0\} \quad \text{and} \quad \mathcal{T}_0 : \mathcal{V} \to \mathcal{C}([0,1]) : f \mapsto f',$$

then we expect that $\mathcal{T}_0$ is an isomorphism. (Note that $\mathcal{T} \neq \mathcal{T}_0$, since the domains of these operators are different. Rather, $\mathcal{T}_0$ is the "restriction" of $\mathcal{T}$ to $\mathcal{V}$ in the sense that $\mathcal{V} \subseteq \mathcal{C}^1([0,1])$ and $\mathcal{T}_0 f = \mathcal{T}f$ for all $f \in \mathcal{V}$.)

Here is the calculation: for $f \in \mathcal{V}$ and $g \in \mathcal{C}([0,1])$, we have

$$\begin{aligned}
\mathcal{T}_0 f = g &\iff (\mathcal{T}_0 f)(x) = g(x) \text{ for all } x \in [0,1] \\
&\iff f'(x) = g(x) \text{ for all } x \in [0,1] \\
&\iff \int_0^x f'(s)\,ds = \int_0^x g(s)\,ds \text{ for all } x \in [0,1] \\
&\iff f(x) - f(0) = \int_0^x g(s)\,ds \text{ for all } x \in [0,1] \\
&\iff f(x) = \int_0^x g(s)\,ds \text{ for all } x \in [0,1], \text{ since } f(0) = 0.
\end{aligned}$$

The third if and only if merits expansion. Certainly if $f' = g$ on $[0,1]$, then all of the integrals $\int_0^x f'(s)\,ds$ and $\int_0^x g(s)\,ds$ are equal. Conversely, if these integrals are equal for all x, we may differentiate both sides of the equality $\int_0^x f'(s)\,ds = \int_0^x g(s)\,ds$ to obtain $f' = g$.

We conclude that $\mathcal{T}_0$ is an isomorphism with

$$(\mathcal{T}_0^{-1}g)(x) = \int_0^x g(s)\ ds,\ 0 \le x \le 1.$$

15.9 Problem. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. A LEFT INVERSE of $\mathcal{T}$ is a linear operator $\mathcal{L} \in \mathbf{L}(\mathcal{W}, \mathcal{V})$ such that $\mathcal{L}\mathcal{T}v = v$ for all $v \in \mathcal{V}$, and a RIGHT INVERSE of $\mathcal{T}$ is an linear operator $\mathcal{R} \in \mathbf{L}(\mathcal{W}, \mathcal{V})$ such that $\mathcal{T}\mathcal{R}w = w$ for all $w \in \mathcal{W}$.

(i) Prove that if $\mathcal{T}$ has a left inverse, then $\mathcal{T}$ is injective.

(ii) Prove that if $\mathcal{T}$ has a right inverse, then $\mathcal{T}$ is surjective.

(iii) Prove that $\mathcal{T}$ is invertible if and only if $\mathcal{T}$ has both a left inverse $\mathcal{L}$ and a right inverse $\mathcal{R}$, in which case $\mathcal{L} = \mathcal{R} = \mathcal{T}^{-1}$.

Day 16: Wednesday, September 18.

Material from *Linear Algebra* by Meckes & Meckes

Pages 90–91 motivate the definition of matrix-matrix multiplication in terms of operator composition. Equation (2.11) gives an entrywise definition of this product. Lemma 2.11 proves our "columnwise" definition from that entrywise definition. Study the box at the bottom of p.91 carefully. Finally, Lemma 2.14 gives the familiar definition of AB as "the (i, j)-entry of AB is the dot product of the ith row of A with the jth column of B."

Do Quick Exercises #11 and #14 in Section 2.3.

We have devoted significant effort to understanding linear operators between arbitrary vector spaces. Few of our results have specified exactly what the vector spaces were. Now we have the tools to appreciate more operators on Euclidean spaces (spaces of column vectors: $\mathbb{F}^n$) and their connections to matrices.

Theorem 13.6 tells us that all linear operators between Euclidean spaces are given by matrix-vector multiplication. Specifically, let $\mathcal{T} \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$. Then the MATRIX REPRESENTATION OF $\mathcal{T}$ is the matrix

$$[\mathcal{T}] := \begin{bmatrix} \mathcal{T}\mathbf{e}_1 & \cdots & \mathcal{T}\mathbf{e}_n \end{bmatrix} \in \mathbb{F}^{m \times n},$$

where $\mathbf{e}_j$ is the jth standard basis vector for $\mathbb{F}^n$ (Definition 11.1). The matrix $[\mathcal{T}]$ satisfies

$$\mathcal{T}\mathbf{v} = [\mathcal{T}]\mathbf{v}$$

for any $\mathbf{v} \in \mathbb{F}^n$; on the left, we just have the abstract application of $\mathcal{T}$ to $\mathbf{v}$ (i.e., the evaluation of the function $\mathcal{T}$ at $\mathbf{v}$), while on the right, this application is given "concretely" by the matrix-vector product $[\mathcal{T}]\mathbf{v}$.

16.1 Example. For the linear operator

$$\mathcal{T}\colon \mathbb{R}^2 \to \mathbb{R}^2 \colon \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix},$$

we compute

$$\mathcal{T}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 - 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and

$$\mathcal{T}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

thus

$$[\mathcal{T}] = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Indeed,

$$[\mathcal{T}]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix} = \mathcal{T}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Now let $\mathcal{T}_1 \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^p)$ and $\mathcal{T}_2 \in \mathbf{L}(\mathbb{F}^p, \mathbb{F}^m)$. Their composition $\mathcal{T}_2\mathcal{T}_1$, defined by $(\mathcal{T}_2\mathcal{T}_1)\mathbf{u} = \mathcal{T}_2(\mathcal{T}_1\mathbf{u})$, satisfies $\mathcal{T}_2\mathcal{T}_1 \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$. Thus $[\mathcal{T}_1] \in \mathbb{F}^{p \times n}$, $[\mathcal{T}_2] \in \mathbb{F}^{m \times p}$, and $[\mathcal{T}_2\mathcal{T}_1] \in \mathbb{F}^{m \times n}$. Moreover,

$$\mathcal{T}_1\mathbf{u} = [\mathcal{T}_1]\mathbf{u}, \ \mathbf{u} \in \mathbb{F}^n, \quad \text{and} \quad \mathcal{T}_2\mathbf{v} = [\mathcal{T}_2]\mathbf{v}, \ \mathbf{v} \in \mathbb{F}^p,$$

and we compute twice

$$(\mathcal{T}_2\mathcal{T}_1)\mathbf{u} = [\mathcal{T}_2\mathcal{T}_1]\mathbf{u} \quad \text{and also} \quad (\mathcal{T}_2\mathcal{T}_1)\mathbf{u} = \mathcal{T}_2(\mathcal{T}_1\mathbf{u}) = \mathcal{T}_2\big([\mathcal{T}_1]\mathbf{u}\big) = [\mathcal{T}_2]\big([\mathcal{T}_1]\mathbf{u}\big)$$

for all $\mathbf{u} \in \mathbb{F}^n$. That is,

$$[\mathcal{T}_2\mathcal{T}_1]\mathbf{u} = [\mathcal{T}_2]\big([\mathcal{T}_1]\mathbf{u}\big) \tag{16.1}$$

for all $\mathbf{u} \in \mathbb{F}^n$.

Since (16.1) is true for all $\mathbf{u} \in \mathbb{F}^n$, we are free to take $\mathbf{u} = \mathbf{e}_j$ as the jth standard basis vector, so

$$[\mathcal{T}_2\mathcal{T}_1]\mathbf{e}_j = [\mathcal{T}_2]\big([\mathcal{T}_1]\mathbf{e}_j\big) \tag{16.2}$$

for each j. It is a fact that if $A \in \mathbb{F}^{m \times n}$ and $\mathbf{e}_j$ is the jth standard basis vector for $\mathbb{F}^n$, then $A\mathbf{e}_j$ is the jth column of A. That is, if $A = \begin{bmatrix} \mathbf{a}_1 & \cdots \mathbf{a}_n \end{bmatrix}$, then

$$A\mathbf{e}_j = \mathbf{a}_j.$$

16.2 Problem. Prove it.

In words, then, (16.2) says that the jth column of the matrix $[\mathcal{T}_2\mathcal{T}_1]$ is the matrix-vector product of the matrix $[\mathcal{T}_2]$ and the jth column of $[\mathcal{T}_1]$.

16.3 Problem. Stare at (16.2) until you fully believe the sentence above.

This suggests a meaningful way of defining the product of two matrices. Let $A \in \mathbb{F}^{m\times p}$ and $B \in \mathbb{F}^{p\times n}$. Then the operators

$$\mathcal{T}_A \colon \mathbb{F}^p \to \mathbb{F}^m \colon \mathbf{v} \mapsto A\mathbf{v} \quad \text{and} \quad \mathcal{T}_B \colon \mathbb{F}^n \to \mathbb{F}^p \colon \mathbf{u} \mapsto B\mathbf{u}$$

have matrix representations

$$[\mathcal{T}_A] = A \quad \text{and} \quad [\mathcal{T}_B] = B.$$

16.4 Problem. Why?

We would like to define the matrix product AB so that the composition

$$\mathcal{T}_A\mathcal{T}_B \colon \mathbb{F}^n \to \mathbb{F}^m \colon \mathbf{u} \mapsto A(B\mathbf{u})$$

has matrix representation

$$[\mathcal{T}_A\mathcal{T}_B] = AB. \tag{16.3}$$

How should we define this symbol AB? The work before Problem 16.3 suggests that we want the jth column of AB to be the matrix-vector product of A and the jth column of B. That is, if $B = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}$, then we should define

$$AB := \begin{bmatrix}A\mathbf{b}_1 & \cdots & A\mathbf{b}_n\end{bmatrix}. \tag{16.4}$$

16.5 Problem. With this definition of AB and still assuming $A \in \mathbb{F}^{m\times p}$ and $B \in \mathbb{F}^{p\times n}$, explain why $AB \in \mathbb{F}^{m\times n}$. Then check that (16.3) is indeed true, as desired.

Day 17: Friday, September 20.

We took Exam 1.

Day 18: Monday, September 23.

Material from *Linear Algebra* by Meckes & Meckes

Pages 97–100 discuss matrix inverses. We are beginning to discuss an operator-theoretic approach to row reduction and Gaussian elimination. You should already be familiar with all of Section 1.2 on elimination.

Do Quick Exercise #15 in Section 2.3

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Invertible matrix (N), matrix inverse, identity matrix

18.1 Example. We compute the matrix product

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{18.1}$$

using the "columnwise" definition. First, we compute the matrix-vector products

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

There are many ways to do this, including Definition 7.3 and Problem 16.2. Thus

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}.$$

The result is that we have interchanged the columns of the matrix on the left in (18.1); this is no accident, as the matrix on the right is a permutation matrix, which we will study later.

18.2 Problem. Compute

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

and conclude that matrix multiplication need not commute. Describe in words the result of this multiplication and contrast it to the result of the previous example.

Now that we have a way of multiplying matrices that corresponds to operator composition, we might ask what properties of operator composition extend to matrix multiplication. One such property is inverting operators. What is the right way to invert matrices?

Before doing that, we take a detour into inverting operator *compositions*. Suppose that $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are vector spaces and

$$\mathcal{T}_1 \colon \mathcal{U} \to \mathcal{V} \quad \text{and} \quad \mathcal{T}_2 \colon \mathcal{V} \to \mathcal{W}$$

are both invertible. Experience suggests that $\mathcal{T}_2\mathcal{T}_1 \colon \mathcal{U} \to \mathcal{W}$ is also invertible, and that $(\mathcal{T}_2\mathcal{T}_1)^{-1} = \mathcal{T}_1^{-1}\mathcal{T}_2^{-1}$. Is it?

First, note that since $\mathcal{T}_2^{-1} \in \mathbf{L}(\mathcal{W},\mathcal{V})$ and $\mathcal{T}_1^{-1} \in \mathbf{L}(\mathcal{V},\mathcal{U})$, we have $\mathcal{T}_1^{-1}\mathcal{T}_2^{-1} \in \mathbf{L}(\mathcal{W},\mathcal{U})$. A consequence of Problem 15.9 is that we will have $(\mathcal{T}_2\mathcal{T}_1)^{-1} = \mathcal{T}_1^{-1}\mathcal{T}_2^{-1}$ if

$$(\mathcal{T}_2\mathcal{T}_1)(\mathcal{T}_1^{-1}\mathcal{T}_2^{-1}) = I_{\mathcal{W}} \quad \text{and} \quad (\mathcal{T}_1^{-1}\mathcal{T}_2^{-1})(\mathcal{T}_2\mathcal{T}_1) = I_{\mathcal{V}}. \tag{18.2}$$

We work through the first of these equalities "pointwise." Fix $w \in \mathcal{W}$. Then, successively applying the definition of operator composition multiple times, we have

$$\big((\mathcal{T}_2\mathcal{T}_1)(\mathcal{T}_1^{-1}\mathcal{T}_2^{-1})\big)w = \mathcal{T}_2(\mathcal{T}_1\mathcal{T}_1^{-1})\mathcal{T}_2^{-1}w = \mathcal{T}_2 I_{\mathcal{V}}\mathcal{T}_2^{-1}w = \mathcal{T}_2\mathcal{T}_2^{-1}w = I_{\mathcal{W}}w = w.$$

18.3 Problem. Check the second equality in (18.2).

We conclude what we expected.

18.4 Theorem. *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be vector spaces with $\mathcal{T}_1 \in \mathbf{L}(\mathcal{U}, \mathcal{V})$ and $\mathcal{T}_2 \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ invertible. Then $\mathcal{T}_2\mathcal{T}_1 \in \mathbf{L}(\mathcal{U}, \mathcal{W})$ is invertible and*

$$(\mathcal{T}_2\mathcal{T}_1)^{-1} = \mathcal{T}_1^{-1}\mathcal{T}_2^{-1}.$$

Now we focus on matrices. Let $A \in \mathbb{F}^{m \times n}$. What *should* it mean for A to be invertible?

Experience suggests that we want a matrix S so that the products AS and SA are defined and equal identity matrices.

18.5 Problem. Check that $[I_{\mathbb{F}^n}] = I_n$, where I_n is the **IDENTITY MATRIX**, whose jth column is the standard basis vector (Definition 11.1) $\mathbf{e}_j$, i.e.,

$$I_n = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}.$$

But what should the dimensions of S and those identity matrices be? Here it is helpful to remember the slogan that *what things do defines what things are.* Every matrix induces a linear operator. With $A \in \mathbb{F}^{m \times n}$, define

$$\mathcal{T}_A \colon \mathbb{F}^n \to \mathbb{F}^m \colon \mathbf{v} \mapsto A\mathbf{v},$$

so $A = [\mathcal{T}_A]$. We know what it means for $\mathcal{T}_A$ to be invertible: there must exist $\mathcal{S} \in \mathbf{L}(\mathbb{F}^m, \mathbb{F}^n)$ such that

$$\mathcal{T}_A\mathcal{S} = I_{\mathbb{F}^m} \quad \text{and} \quad \mathcal{S}\mathcal{T}_A = I_{\mathbb{F}^n}, \tag{18.3}$$

where $\mathcal{S}\mathbf{w} = [\mathcal{S}]\mathbf{w}$ with $[\mathcal{S}] \in \mathbb{F}^{n \times m}$. Recall that $I_{\mathbb{F}^n}$ is the identity operator on $\mathbb{F}^n$, i.e., $I_{\mathbb{F}^n} \in \mathbf{L}(\mathbb{F}^n)$ with $I_{\mathbb{F}^n}\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.

We connect these equalities to matrices as follows. The equalities (18.3) say

$$[\mathcal{T}_A\mathcal{S}] = [I_{\mathbb{F}^m}] \quad \text{and} \quad [\mathcal{S}\mathcal{T}_A] = [I_{\mathbb{F}^n}],$$

equivalently,

$$[\mathcal{T}_A][\mathcal{S}] = I_m \quad \text{and} \quad \mathcal{S}[\mathcal{T}_A] = I_n,$$

and, last,

$$A[\mathcal{S}] = I_m \quad \text{and} \quad [\mathcal{S}]A = I_n.$$

This suggests that we should define $A \in \mathbb{F}^{m \times n}$ to be invertible if there is $S \in \mathbb{F}^{n \times m}$ such that $AS = I_m$ and $SA = I_n$. And this is what we will do.

There is just one problem: no such S can exist if $m \neq n$. That is, according to the rules of operator inverses, it is impossible for a nonsquare matrix to be invertible. More generally, the following negative result is true, although we do not have the tools to prove it yet.

18.6 Theorem. *Let m, $n \geq 1$ be integers with $m \neq n$ and let $\mathcal{T} \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$. Then $\mathcal{T}$ is not invertible.*

More positively, we have the following definition.

18.7 Definition. *A matrix $A \in \mathbb{F}^{n \times n}$ is* **INVERTIBLE** *if either of the following (equivalent) conditions holds:*

(i) *The operator $\mathcal{T}_A : \mathbb{F}^n \to \mathbb{F}^m : \mathbf{v} \mapsto A\mathbf{v}$ is invertible (in the sense of Theorem 15.6).*

(ii) *There exists $S \in \mathbb{F}^{n \times n}$ such that $AS = I_n$ and $SA = I_n$, where I_n is the $n \times n$ identity matrix (Problem 18.5). This matrix S is the* **INVERSE MATRIX** *of A.*

In fact, the second condition has a redundancy. Around the time that we prove Theorem 18.6, we will show that $S \in \mathbb{F}^{n \times n}$ satisfies $AS = I_n$ if and only if S also satisfies $SA = I_n$. That is, we only need to check one of those equalities for the other to hold. This result, like Theorem 18.6, is a powerful consequence of dimension counting arguments.

18.8 Problem (Optional, maybe annoying, definitely useful). Actually computing a matrix inverse is often computationally expensive and irrelevant; knowing the existence of the inverse is more important. One case that often arises in practice is the inverse of a 2×2 matrix. Show that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is

$$\frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

18.9 Problem. Prove that if $A \in \mathbb{F}^{n \times n}$ is invertible, its inverse is unique. That is, show that if there are matrices B, $C \in \mathbb{F}^{n \times n}$ such that

$$AB = BA = I_n \quad \text{and} \quad AC = CA = I_n,$$

then $B = C$. [Hint: *try doing this two ways. First, use results about the uniqueness of operator inverses. Second, start with $B = BI_n$. How do we get C to show up in an equality with I_n?*]

We now use our wealth of operator-theoretic and matrix-theoretic tools to study in detail linear systems of equations—recall our very first problem in Example 1.1. (This usually comes first in a first course on linear algebra: linear systems, then matrices, then linear operators. But we have a more evolved sensibility in this second course!) Remember as well that the fundamental problem of linear algebra is, arguably (and we should feel free to argue about this), to solve $\mathcal{T}v = w$ with $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$, $\mathcal{V}$ and $\mathcal{W}$ vector spaces, and $w \in \mathcal{W}$ given. Here we address this problem in the form $A\mathbf{x} = \mathbf{b}$, with $A \in \mathbb{F}^{m \times n}$ and $\mathbf{b} \in \mathbb{F}^m$.

We begin with a very simple toy problem:

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11. \end{cases}$$

There are three "elementary row operations" that we can perform on this system to transform it into an "equivalent" system—that is, x_1 and x_2 solve the original system if and only if they solve the new one. First is the "interchange." The order in which we list equations does not matter:

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11 \end{cases} \iff \begin{cases} 3x_1 + 2x_2 = 11 \\ x_1 - 2x_2 = 1. \end{cases}$$

Second is "scaling." If $\alpha \neq 0$, then we can multiply both sides of any equation by α and get an equivalent system. (This is because of the fundamental relationship that $x = y$ if and only if $\alpha x = \alpha y$ for $\alpha \neq 0$.) So, for example,

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11. \end{cases} \iff \begin{cases} -3x_1 + 6x_2 = -3 \\ 3x_1 + 2x_2 = 11. \end{cases}$$

Third, we can add (a multiple of) one equation to another and get an equivalent system. This is possibly the strangest property, but we know something like $x = y$ if and only if $x + z = y + z$. Thus, for example,

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11 \end{cases} \iff \begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 + -3(x_1 - 2x_2) = 11 + (-3). \end{cases}$$

To be really pedantic, in the second equation we took $x = 3x_1 + 2x_2$, $y = 11$, and $z = -3(x_1 - 2x_2)$. But from the first equation we could also say $z = -3$.

Day 19: Wednesday, September 25.

Material from *Linear Algebra* by Meckes & Meckes

Pages 12–14 perform row reduction on a 4×3 system. Pages 102–104 introduce the three kinds of elementary matrices.

Do Quick Exercises #5 and #6 in Section 1.2.

We study the system

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11 \end{cases}$$

exhaustively via row operations under several different lenses. First, we subtract 3 times the first equation (E1) from the second (E2) to find

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11 \end{cases} \xrightarrow{\mathsf{E2} \mapsto \mathsf{E2}-3\times\mathsf{E1}} \begin{cases} x_1 - 2x_2 = 1 \\ 8x_2 = 8. \end{cases}$$

The idiosyncratic "pseudocode" $(\mathsf{E2}) \mapsto (\mathsf{E2}) - 3 \times (\mathsf{E1})$ is meant to suggest that the second equation on the left is replaced by that original second equation minus three times the first equation.

Next, we multiply both sides of the (new) second equation (E2) by 1/8 to find

$$\begin{cases} x_1 - 2x_2 = 1 \\ 8x_2 = 8 \end{cases} \xrightarrow{\mathsf{E2} \mapsto 1/8\times\mathsf{E2}} \begin{cases} x_1 - 2x_2 = 1 \\ x_2 = 1. \end{cases}$$

This tells us immediately the value of x_2: we have $x_2 = 1$. We substitute this into the first equation to find $x_1 - 2 = 1$, thus $x_1 = 3$. We have solved the system.

Here is how we could view this at the level of matrices. Put

$$A := \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} := \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Then the original system is equivalent to $A\mathbf{x} = \mathbf{b}$. We introduce the "augmented matrix"

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right].$$

Then we subtract 3 times the first row (R1) from the second (R2):

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \xrightarrow{\text{R2} \mapsto \text{R2}-3\times\text{R1}} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right].$$

Next we scale the (new) second row (R2) by 1/8:

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right] \xrightarrow{\text{R2} \mapsto 1/8\times\text{R2}} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right].$$

This last matrix is the augmented matrix for the system

$$\begin{cases} x_1 - 2x_2 = 1 \\ x_2 = 1, \end{cases}$$

which we previously solved.

We could simplify this system even further by manipulating the first equation. If we add 2 times the second equation to the first (equivalently, and pedantically, subtract -2 times the second equation from the first), we have

$$\begin{cases} x_1 - 2x_2 = 1 \\ x_2 = 1 \end{cases} \xrightarrow{\text{E1} \mapsto \text{E1}-(-2)\times\text{E2}} \begin{cases} x_1 = 3 \\ x_2 = 1, \end{cases}$$

and there is nothing more to do. At the level of matrices, we have added 2 times the second row to the first (equivalently, and still pedantically, subtracted -2 times the second equation from the first):

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{R1} \mapsto \text{R1}-(-2)\times\text{R2}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

One of the virtues of matrices is that they encode these "elementary row operations" via fairly "elementary" matrices. Put

$$E_{21} := \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \qquad D_{22} := \begin{bmatrix} 1 & 0 \\ 0 & 1/8 \end{bmatrix}, \quad \text{and} \quad E_{12} := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Each of these matrices is a variation on the 2×2 identity matrix; the subscript tells us what entry is somehow changed (although the notation does not tell us what the new value there is). The symbol E indicates that the matrix "eliminates" something, while D is a "diagonal" matrix that "scales" a row. The subscripts on E, read backward, tell us what row is subtracted from what; the matrix E_{21} causes (a multiple of) the first row to be subtracted from the second.

19.1 Problem. Let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Compute $E_{21}\mathbf{v}$, $D_{22}\mathbf{v}$, and $E_{12}\mathbf{v}$ and express the results in terms of v_1 and v_2. Describe in words how each matrix-vector multiplication changes the rows of $\mathbf{v}$.

The result is that

$$E_{21}\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right],$$

$$D_{22}E_{21}\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right],$$

and

$$E_{12}D_{22}E_{21}\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]. \tag{19.1}$$

Abbreviate

$$M := E_{12}D_{22}E_{21}.$$

Since we multiply matrices columnwise, we have

$$E_{12}D_{22}E_{21}\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = M\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} MA & M\mathbf{b} \end{bmatrix}.$$

But by (19.1), this also reads

$$\begin{bmatrix} MA & M\mathbf{b} \end{bmatrix} = \begin{bmatrix} I_2 & \mathbf{c} \end{bmatrix}, \qquad \mathbf{c} := \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

with I_2, as usual, as the 2×2 identity matrix. Since two matrices are equal if and only if their corresponding columns are equal, we get $MA = I_2$. Thus A is invertible and $A^{-1} = M$ (which we could have checked by Problem 18.8).

Here is what we should take away from this systematic analysis of this overly simple problem.

- The elementary row operations of subtracting a multiple of one row from another and scaling a row by a nonzero number can transform a "complicated" system into an equivalent "simpler" one whose solution, if it exists, we can more or less read off from the structure of the "simpler" system.

- These two elementary row operations can be encoded by matrix multiplication, where the "elementary" matrices involved are identity matrices with one entry replaced by the multiplier factor (or the negative thereof).

- If $A \in \mathbb{F}^{n \times n}$ is invertible, then we can write A^{-1} as the product of these "elementary" matrices.

Strictly speaking, we have proved none of these statements. Also, missing is any discussion of interchanging rows/equations. We will see an example later where that is necessary. For now, we do another, larger example for a specific $A \in \mathbb{F}^{3\times 3}$ in which we construct a product M of elementary matrices such that MA is the identity matrix. In principle, this would allow us to solve $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{F}^3$. (Computationally, this is "expensive" and a bad idea.)

19.2 Example. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}.$$

We multiply A by a series of elimination and scaling elementary matrices so that the final product is the identity matrix:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix} \xrightarrow[E_{21}]{\text{R2} \mapsto \text{R2}-2\times\text{R1}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 8 & 7 & 9 \end{bmatrix}, \qquad E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[E_{31}]{\text{R3} \mapsto \text{R3}-4\times\text{R1}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix}, \qquad E_{31} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[E_{32}]{\text{R3} \mapsto \text{R3}-3\times\text{R2}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad E_{32} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow[D_{33}]{\text{R3} \mapsto (1/2)\times\text{R3}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D_{33} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\xrightarrow[E_{23}]{\text{R2} \mapsto \text{R2}-1\times\text{R3}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{23} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[E_{13}]{\text{R1} \mapsto \text{R1}-1\times\text{R3}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{13} := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[E_{12}]{\text{R1} \mapsto \text{R1}-1\times\text{R2}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{12} := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[D_{11}]{\mathsf{R1} \mapsto (1/2)\times\mathsf{R1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D_{11} := \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that

$$D_{11}E_{13}E_{23}D_{33}E_{32}E_{31}E_{21}A = I_3,$$

and so

$$A^{-1} = D_{11}E_{13}E_{23}D_{33}E_{32}E_{31}E_{21}.$$

19.3 Problem (Optional, tedious, maybe worthwhile). Check that each of the elementary matrices in the previous example "does what it should." For example $E_{31}\mathbf{v}$ subtracts 4 times the first row of $\mathbf{v}$ from the third row.

We did not bother multiplying all of those elementary matrices out to find the actual formula for A^{-1}. If a formula is absolutely necessary, a tried-and-true algorithm for hand computation of A^{-1} for invertible $A \in \mathbb{F}^{n\times n}$ is to work on the "identity-augmented matrix" $\begin{bmatrix} A & I_n \end{bmatrix}$ and perform on this entire matrix the row operations that reduce A to I_n. Collecting the elementary matrices that perform this matrix into a single matrix M, we have

$$M \begin{bmatrix} A & I_n \end{bmatrix} = \begin{bmatrix} I_n & B \end{bmatrix},$$

where B is "whatever happens to I_n." More precisely, of course, $MA = I_n$ and $MI_n = B$, thus, again, $M = A^{-1}$.

Day 20: Friday, September 27.

No class due to university closure.

Day 21: Monday, September 30.

We continue trying to solve (and, more importantly, trying to understand) the linear problem $A\mathbf{x} = \mathbf{b}$ by performing elementary row operations on A (and $\mathbf{b}$) to reduce A to the identity matrix. The strategy is that we use the "leading nonzero entry" (the "pivot") in a row to "zero out" the column below that entry and keep doing so, moving left to right, until A is "transformed" into an upper-triangular matrix. Then we "work upwards" and create zeros above the "pivots," along with rescaling the pivots to be 1. This process is called **Gauss–Jordan elimination** (the downwards step is just **Gaussian elimination**).

Here is a situation in which elimination and scaling alone are not enough.

21.1 Example. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 3 \\ 8 & 7 & 9 \end{bmatrix}.$$

As before, we create zeros in the first column below the $(1,1)$-entry by multiplying by elimination matrices:

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 5 \end{bmatrix}.$$

The problem is the new $(2,2)$-entry of 0: that is not allowed on the diagonal of the identity matrix. (The other problem is the new $(3,2)$-entry of 3, which is also not allowed.)

The right idea is to swap the second and third rows. That is, we want to multiply $E_{31}E_{21}A$ by a matrix $P \in \mathbb{R}^3$ such that

$$P \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_3 \\ v_2 \end{bmatrix}.$$

There are several ways of determining such P, including rearranging algebraically the right side of this desired equality or taking the vectors on which P acts to be the standard basis vectors. However we do it, the right P is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is a "permutation" matrix, since its columns (and rows) are rearrangements (permutations) of the columns (and rows) of the identity matrix. We find

$$PE_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix},$$

and from here we can find a product M of "elementary" matrices such that $MPE_{31}E_{21}A = I_3$.

21.2 Problem. Find that product M.

21.3 Problem. Do as we said and not as we did and define

$$\mathcal{T} \colon \mathbb{R}^3 \to \mathbb{R}^3 \colon \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ v_3 \\ v_2 \end{bmatrix}.$$

Find the matrix representation of $\mathcal{T}$ and check that it is P as stated above in the previous example.

We formalize the notation of permutation matrix.

21.4 Definition. *A matrix $P \in \mathbb{R}^{n \times n}$ is a* **PERMUTATION MATRIX** *if the columns of P are the columns of the $n \times n$ identity matrix I_n. That is, the columns of P are the standard basis vectors for $\mathbb{R}^n$.*

21.5 Problem. Let $n \geq 1$ be an integer and let $1 \leq i,\ j \leq n$. Let P_{ij} be the permutation matrix whose ith column is the standard basis vector $\mathbf{e}_j$ and whose jth column is the standard basis vector $\mathbf{e}_i$.

(i) Let $\mathbf{v} \in \mathbb{F}^n$. For $k = 1, \ldots, n$, what is the kth row of $P_{ij}\mathbf{v}$ in terms of the rows of $\mathbf{v}$?

(ii) Let $A \in \mathbb{F}^{n \times n}$. Compare and contrast the effects of multiplying $P_{ij}A$ versus AP_{ij}.

(iii) Why is P_{ij} invertible? What is its inverse?

Now let $P \in \mathbb{F}^{n \times n}$ be *any* permutation matrix. Answer the questions above with P_{ij} replaced by P.

21.6 Problem. Let $A \in \mathbb{F}^{m \times n}$, let $1 \leq p \leq n$, and let $1 \leq j_1 < \cdots < j_p \leq n$. Suppose that we want to "select" columns j_1 through j_p of A and put them in a matrix in the order in which they appeared in A. This will result in an $m \times p$ matrix. Find a matrix $S \in \mathbb{F}^{n \times p}$ such that AS has this structure. [Hint: *revisit Problem 13.7. As needed, work with some small matrices until you see the right pattern in S in general.*]

Here is a situation in which A is not invertible, and so we will fail to solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$. We first discuss how Gauss–Jordan elimination fails, and then we seek to *understand* what this failure says about the extent to which we can solve $A\mathbf{x} = \mathbf{b}$.

21.7 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

We might notice that the second row of A is double the first row, and that should lead us to expect problems. Indeed, they arise via the elimination

$$A \xrightarrow[E_{21}]{\text{R2} \mapsto \text{R2}-2\times\text{R1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The problem is that row of zeros. There is no way to multiply A by any other "elementary" matrices so that a 1 appears in the $(2,2)$-entry of that product.

Here is what elimination says about solving $A\mathbf{x} = \mathbf{b}$. Suppose that this equality is true. Then $E_{21}A\mathbf{x} = E_{21}\mathbf{b}$. That is, if we can solve $A\mathbf{x} = \mathbf{b}$, we must have

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}\mathbf{x} = E_{21}\mathbf{b}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Computing each side, we arrive at (return to?) the linear system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = b_1 \\ 0 = b_2 - 2b_1 \\ 5x_3 = b_3. \end{cases}$$

The second equation is the killer. It says $b_2 - 2b_1 = 0$, so $b_2 = 2b_1$. This is a "solvability condition" for the problem: if there is a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, then the entries of $\mathbf{b}$ must meet $b_2 = 2b_1$. (To be fair, this is only a condition relating the first two entries of $\mathbf{b}$; it says nothing about b_3.) If $b_2 \neq 2b_1$, then no solution can exist, and there is no point in trying to solve the problem.

What if the solvability condition is met, and $b_2 - 2b_1 = 0$? There are still two other equations in play. The third immediately gives $x_3 = b_3/5$, which we substitute into the first to find that x_1 and x_2 must meet

$$x_1 + 2x_2 + \frac{3b_3}{5} = b_1.$$

This is one equation in two unknowns—not a recipe for unique solutions. One approach is to solve for x_1 in terms of x_2:

$$x_1 = b_1 - \frac{3b_3}{5} - 2x_2.$$

Then for every choice of x_2, we get a new x_1. Together, they form a solution to $A\mathbf{x} = \mathbf{b}$.

In the language of vectors, we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 - 3b_3/5 - 2x_2 \\ x_2 \\ b_3/5 \end{bmatrix} = \begin{bmatrix} b_1 - 3b_3/5 \\ 0 \\ b_3/5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

This should remind us of Problem 11.10. (Does it?) On one hand, we have infinitely many solutions to $A\mathbf{x} = \mathbf{b}$ given the solvability condition $b_2 = 2b_1$. On the other, all solutions have a very similar form, and there is only one "degree of freedom" given by that multiplier x_2.

This problem raises (at least) two questions, neither of which we have the tools to answer just yet.

1. Is there a more meaningful way to describe the size of the solution set to $A\mathbf{x} = \mathbf{b}$ than "infinite"? Yes: we need bases.

2. Is there a way to "rig the game" so that we can always solve $A\mathbf{x} = \mathbf{b}$ uniquely if we specify $\mathbf{x}$ and $\mathbf{b}$ correctly? That is, are there subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^3$ such that the operator $\mathcal{T}: \mathcal{V} \to \mathcal{W}: \mathbf{v} \mapsto A\mathbf{v}$ is an isomorphism? Yes: we need the geometry of inner products and orthogonal complements.

21.8 Problem. We could have gone further with elementary row operations in the previous example. Find a matrix $E \in \mathbb{R}^{3\times 3}$ such that

$$E\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 0 & 0 & 5\end{bmatrix} = \begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}.$$

Express E as the product of "elementary" matrices; you do not need to multiply all of that product out. The first factor in that product (on the right) will be E_{21} from the example; a permutation matrix will also be necessary in the mix.

We finally study a nonsquare matrix. Theorem 18.6 should lead us to believe that we will somehow fail to solve $A\mathbf{x} = \mathbf{b}$ in this case—either no $\mathbf{x}$ will exist, or it will not be unique. For now, we just focus on arithmetic and matrix multiplication: we want to make A look as much like the identity matrix as possible.

21.9 Example. Let

$$\begin{bmatrix}1 & 2 & 1 & 7\\ 2 & 4 & 2 & 14\\ 0 & 0 & 2 & 8\end{bmatrix},$$

so $A \in \mathbb{R}^{3\times 4}$. If we want to multiply A by some matrix B and have the product BA defined, we need $B \in \mathbb{R}^{p\times 3}$ for some p. And if we want B to be invertible, we need $p = 3$. So, the elementary matrices that we apply to A here will continue to be 3×3.

Here we go:

$$\begin{bmatrix}1 & 2 & 1 & 7\\ 2 & 4 & 2 & 14\\ 0 & 0 & 2 & 8\end{bmatrix} \xrightarrow[E_{21}]{\text{R2} \mapsto \text{R2}-2\times\text{R1}} \begin{bmatrix}1 & 2 & 1 & 7\\ 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 8\end{bmatrix}, \qquad E_{21} := \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow[D_{33}]{\text{R3} \mapsto (1/2)\times\text{R3}} \begin{bmatrix}1 & 2 & 1 & 7\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 4\end{bmatrix}, \qquad D_{33} := \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}$$

$$\xrightarrow[E_{13}]{\text{R1} \mapsto \text{R1}-\text{R3}} \begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 4\end{bmatrix}, \qquad E_{13} := \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow[P_{23}]{\text{R2} \mapsto \text{R3},\ \text{R3} \mapsto \text{R2}} \begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}, \qquad P_{23} := \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}.$$

We have done all the row operations that worked in the past to convert a square A to the identity matrix. How exactly should we interpret the result here?

Day 22: Wednesday, October 2.

Material from *Linear Algebra* by Meckes & Meckes

All of Section 1.2 is worth reading line by line. Theorem 1.1 provides the algorithm for Gaussian elimination. Pages 102–104 discuss elementary matrices. Theorems 2.22 and 2.24 give factorizations of the RREF.

Do all of the Quick Exercises in Section 1.2. Do Quick Exercises #16 and #17 in Section 2.4

Here is a summary of what we have been calculating. We have worked on matrices A_1, $A_2 \in \mathbb{R}^{3\times 3}$ and $A_3 \in \mathbb{R}^{3\times 4}$ and found matrices E_1, E_2, $E_3 \in \mathbb{R}^{3\times 3}$ such that

$$E_1 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_2 A_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$E_3 A_3 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Such matrices A_1 appeared in both Examples 19.2 and 21.1 with A_2 in Example 21.7 and Problem 21.8 and A_3 in Example 21.9. The matrices E_1, E_2, and E_3 were products of what we called, euphemistically, "elementary matrices." After a lot of experience, we now define this concept precisely.

22.1 Definition. *An* ELEMENTARY MATRIX *is one of the following three kinds of $n \times n$ matrices.*

(i) *An* ELIMINATION MATRIX E_{ji} *(where $i \neq j$) such that $E_{ji}A$ is formed by subtracting α times row i of A from row j of A. The matrix E_{ji} is formed by replacing the (i, j)-entry of the $n \times n$ identity matrix I_n with $-\alpha$.*

(ii) *A* SCALING MATRIX D_{ii} *such that $D_{ii}A$ is formed by multiplying row i of A by α. The matrix D_{ii} is formed by replacing the (i, i)-diagonal entry of I_n by α.*

(iii) *A* PERMUTATION MATRIX P_{ij} *such that $P_{ij}A$ is formed by interchanging rows i and j of A. The matrix P_{ij} is formed by interchanging columns i and j of I_n. (This is a special case of Definition 21.4.)*

22.2 Problem. Prove that any elementary matrix is invertible in at least two ways. First, explain in words what the inverse of each kind of elementary matrix should *do*. Then describe (in words, like the previous definition) how to construct each inverse out of I_n. (You do not have to verify that the product of these putative inverses and the original elementary matrices is I_n.) Optionally, explain why the multiplication operators induced by these elementary matrices (e.g., $\mathcal{T}_{E_{ij}}\mathbf{v} = E_{ij}\mathbf{v}$) is invertible by considering injectivity and surjectivity.

Now we observe the special structures of the matrices E_kA_k above, $k = 1, 2, 3$. These matrices have what we will call the REDUCED ROW ECHELON FORM **(RREF)** structure.

RREF1. Any row whose entries are all 0 is below every row with nonzero entries:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

RREF2. The first nonzero entry of any row is 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We euphemistically call such an entry of 1 a LEADING 1.

RREF3. Any leading 1 is the only nonzero entry in its column.

RREF4. Each of these matrices contains columns of identity matrices. We first highlight (in blue) the columns of I_3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we highlight (still in blue) the columns of I_2 in the second and third matrices:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(Strictly speaking, the columns of I_2 similarly appear in the first matrix, but more interesting will be the case when we do row operations and do not get an identity matrix.) Both kinds of columns appear in the order that they do in I_2 and I_3, although not all columns of I_3 are present.

We can make the matrices E_2A_2 and E_3A_3 a bit nicer by permuting their *columns*. Now is a good time to look at Problem 21.5. We have

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & F \\ 0 & 0 \end{bmatrix}, \qquad F := \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{22.1}$$

and

$$\begin{bmatrix}1&2&0&3\\0&0&1&4\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&2&3\\0&1&0&4\\0&0&0&0\end{bmatrix}=\begin{bmatrix}I_2&F\\0&0\end{bmatrix}, \qquad F:=\begin{bmatrix}2&3\\0&4\end{bmatrix}. \tag{22.2}$$

In the last matrices on the right in each calculation, the bottom rows of 0 represent not the scalar 0 but zero *matrices*. (What are their sizes?)

Up to a permutation matrix, the matrices E_2A_2 and E_3A_3 have a special "block" structure: the top left block is an identity matrix, the bottom two blocks are all 0, and the top left block is "junk." Every matrix (except the zero matrix) can be written in this form.

This leads us to two interpretations of the RREF. First, it is a "canonical form" satisfying the four conditions above. Second, it is a special "factorization" of a matrix. Both are useful interpretations.

22.3 Theorem (RREF: existence and uniqueness). *Let $A \in \mathbb{F}^{m\times n}$. There exists $E \in \mathbb{F}^{m\times m}$ such that E is a product of elementary matrices and $R := EA$ is in* REDUCED ROW ECHELON FORM (RREF) *in the sense that it satisfies the following.*

Row Property 1. *Any nonzero row of R is below any row with nonzero entries.*

Row Property 2. *If a row contains nonzero entries, the first nonzero entry of that row is 1, called the* LEADING *1 or the* PIVOT *for that row.*

Column Property 1. *The other entries of any column containing a leading 1 are 0. That is, a column containing a leading 1 is a column of the $m\times m$ identity matrix I_m, equivalently, a standard basis vector for $\mathbb{F}^m$. Such a column is called a* PIVOT COLUMN.

Column Property 2. *If columns i and j of I_m appear in R and $i < j$, then the first appearance of column i must be to the left of column j.*

The matrix R is unique in the sense that if EA and $\widetilde{E}A$ both satisfy the four properties above with E and $\widetilde{E}$ invertible, then $EA = \widetilde{E}A$. We sometimes write $R = \mathsf{rref}(A)$.

Proof. Both existence and uniqueness are a byproduct of the "proof by algorithm" given in Theorem 1.1 in the book. Existence should be obvious: just do the elimination. For uniqueness, the key point of the algorithm is that we can run it *in the same way every time*. In the language of the book, when using row operation R3, if necessary, to "get a nonzero entry at the top of the column" or "in the second row of [the] column," just select the *first* nonzero entry in the column under consideration. When using row operation R1, work strictly downward (eliminate in row i before row j when $i < j$). When using row operation R1 "to make all the entries in the same column as any pivot 0" work strictly upward (eliminate in row j before row i when $i < j$).

This ensures that we always reach the RREF of A in the same way. (This is nice theoretically but may be problematic numerically; there are good reasons not to "pivot" with just the first nonzero entry of a column all the time.) If we then *define* the RREF of A to be the matrix that we reach via this specific sequence of elementary row operations, then that matrix will have the four properties above.

However, it may be less clear that no matter how we do the elementary row operations, we always reach the same RREF, or that if we can write $EA = R$ with E invertible (if necessary, specifying that E is a product of elementary matrices) and R in RREF, then there is only one choice for R. (There will definitely *not* be only one choice for E, as we certainly *can* do the elementary row operations in different orders.)

We give a proof of uniqueness (which in particular only relies on E being invertible, not the product of elementary matrices) later, after building more intuition. ■

22.4 Problem. Explain *all* of the reasons why

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is not in RREF.

22.5 Problem. By considering the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

explain the importance of the adjective "first" in Column Property 2 of Theorem 22.3.

The RREF factorizations that appeared in (22.1) and (22.2) can be hugely useful.

22.6 Theorem (RREF: factored version). *Let $A \in \mathbb{F}^{m \times n} \setminus \{0\}$. Then there exist an integer r with $1 \le r \le \min\{m, n\}$ and invertible matrices $E \in \mathbb{F}^{m \times m}$ and $P \in \mathbb{F}^{n \times n}$ such that $EA = \mathsf{rref}(A)$ and*

$$EA = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P. \tag{22.3}$$

The integer r is unique.

Proof. Write $R = \mathsf{rref}(A)$, so $R = EA$ for some invertible E. (It is possible that there are multiple such E.) Let r be the number of columns in R that are columns of the $m \times m$ identity matrix I_m. (Equivalently, and importantly, r is the number of rows of R with a leading 1.) Let P be a permutation matrix such that the first r of RP columns are the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_r$ for $\mathbb{F}^m$. (It is possible that there are multiple such P.) Then RP has the form on the right in (22.3).

For the uniqueness of r, if $EA = \mathsf{rref}(A)$ and EA also has the factorization in (22.3), then r is the number of columns of the $m \times m$ identity matrix that appear in $\mathsf{rref}(A)$. This ensures that there is only one possible value for r in that factorization. ∎

22.7 Example. Here is how we interpret the special structure of the block matrix in (22.3), which we hereafter call B, so $\mathsf{rref}(A) = BP$. (It is important to remember that B is not necessarily $\mathsf{rref}(A)$.) First,

$$B := \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{m \times n}.$$

This is necessary for the product on the right of (22.3) to be defined and to have the same dimensions as the product on the left, given the specified sizes of A, E, and P. The zero blocks and F may or may not be present.

(i) We allow the case $r = n < m$, in which case

$$B = \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

The block F is no longer present, as otherwise B would have more than n columns. The matrix zero block at the bottom must be present, as B has m rows, but I_n has $n < m$ rows. Also, $\mathsf{rref}(A) = B$ and here $P = I_n$, as otherwise Column Property 2 would not hold. This shows that not every permutation matrix is allowed in (22.3).

For example, we could have $r = n = 2$ and $m = 3$ and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(ii) We allow the case $r = m < n$, in which case

$$B = \begin{bmatrix} I_m & F \end{bmatrix}.$$

The block F must be present (it could be anything—maybe all 0), but if F is not present, then $B = I_m$, which violates the inequality $m < n$. The zero blocks cannot be present, as I_m has m rows, so with zero blocks B would have more than m rows.

For example, we could have $r = m = 2$ and $n = 3$ and

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

(iii) We allow the case $r = m = n$, in which case all matrices are square and $R = I_n = I_m$. Otherwise, with F present, B would have more than n columns, and with zero blocks, B would have more than m rows.

(iv) If $r < m$ and $r < n$, then B must have both zero blocks and the F block. Without the zero blocks, it would be the case that B would have $r < m$ rows; without the F block, it would be the case that B would have $r < n$ columns.

For example, we could have $r = 2$, $m = 3$, and $n = 4$ and use what will be one of our favorite recurring examples:

$$B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

22.8 Problem. Example 21.9 constructs $E \in \mathbb{R}^{3\times3}$ such that

$$E \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) By revisiting the elementary row operations in that example, explain why E in Theorem 22.6 might not be unique. [Hint: *could* P_{23} *have appeared earlier? Could* D_{33} *have appeared later?*]

(ii) Find a permutation matrix $\widetilde{P}$ such that

$$E \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \widetilde{P}.$$

Contrast this with (22.2) and explain how this shows that P and F from Theorem 22.6 may not be unique.

(iii) Explain why there cannot exist a matrix $A \in \mathbb{R}^{3\times4}$ such that

$$\mathsf{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Conclude (as noted in part (i) of Example 22.7) that not every permutation matrix P can appear in (22.3).

22.9 Problem. Find a matrix $A \in \mathbb{R}^{3\times4}$ whose entries are all nonzero such that

$$\mathsf{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Provide a matrix $E \in \mathbb{R}^{3\times3}$ such that $EA = \mathsf{rref}(A)$; you may express E as a product of elementary matrices, and you do not have to multiply that product out.

22.10 Example. We find all $R \in \mathbb{F}^{3\times 5}$ that are in RREF with a leading 1 in columns 2 and 4 only. Throughout, we denote by $*$ an entry whose value may be an arbitrary number in $\mathbb{F}$. We break our reasoning into the following steps.

1. The first column of R must be $\mathbf{0}$. Otherwise, that column would contain a nonzero entry, and that nonzero entry would be the leading nonzero entry in its row. Then that entry would have to be 1, but the first column of R does not contain a leading 1.

2. Since the second column of R must contain a leading 1, we therefore have three possibilities for R currently:

$$\begin{bmatrix} 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \end{bmatrix}, \qquad \text{or} \qquad \begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \end{bmatrix}. \tag{22.4}$$

3. We claim that the second and third cases are impossible. In the second case, the first row cannot be all 0, as then we would have a row with entries all 0 (row 1) above a row with some nonzero entries (row 2). Any nonzero entry in the first row would be the leading nonzero entry in that row and therefore 1, and so the rest of that column would be 0. But then the standard basis vector $\mathbf{e}_2$ for $\mathbb{F}^3$ would appear for the first time (in column 2) before $\mathbf{e}_1$ (in columns 3, 4, or 5).

4. The same contradiction results in the third case: either the first row is 0 or $\mathbf{e}_1$ appears for the first time only after $\mathbf{e}_3$.

5. So, R must have the form of the first matrix in (22.4). The RREF properties in Theorem 22.3 say nothing about the $(1,3)$-entry of this matrix: it is not a leading nonzero entry in row 1 (and so it does not have to be 1), and column 3 does not contain a leading 1 (and so this entry does not have to be 0). So, we leave it arbitrary. However, the other entries of column 3 must be 0, as otherwise they would be leading nonzero entries in rows 2 and 3. So, R has the form

$$\begin{bmatrix} 0 & 1 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}.$$

6. Since column 4 contains a leading 1, and since there is already a leading 1 in row 1, this leading 1 in column 4 must appear in rows 2 or 3. So, there are two possibilities for R now:

$$\begin{bmatrix} 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}.$$

7. We claim that the third case is impossible. For in this case the $(2,5)$-entry of row 2 must be 0, as otherwise there would be a leading nonzero entry in column 5. Then the entries of row 2 are all 0, but row 3 has a nonzero entry.

8. We are down to

$$\begin{bmatrix} 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

as the only possible form of R. The RREF properties say nothing about the $(1,5)$-entry, as it is not a leading nonzero entry in row 1, and there is no leading 1 in column 5. The same holds for the $(2,5)$-entry. But the $(3,5)$-entry must be 0, as otherwise there would be a leading nonzero entry in column 5, which would have to be a leading 1. So, R has the form

$$\begin{bmatrix} 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

9. That is, all matrices $R \in \mathbb{F}^{3\times5}$ that are in RREF with a leading 1 in columns 2 and 4 only have the form

$$R = \begin{bmatrix} 0 & 1 & a & 0 & b \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for some a, b, $c \in \mathbb{F}$.

22.11 Problem. Let $\mathcal{V}$ be the set of all matrices R described at the end of Example 22.10. Is $\mathcal{V}$ a subspace of $\mathbb{F}^{3\times5}$?

22.12 Problem. Generalize one of the arguments in Example 22.10 as follows. Let $A \in \mathbb{F}^{m\times n}$ and $EA = R$, where $E \in \mathbb{F}^{m\times m}$ is invertible and R is in RREF. Prove that the first column of R is either the zero vector $\mathbf{0} \in \mathbb{F}^m$ or the first standard basis vector $\mathbf{e}_1 \in \mathbb{F}^m$. Go further and argue that the first *nonzero* column of R is $\mathbf{e}_1$.

The following example is designed to be used in the proof of uniqueness for the RREF.

22.13 Example. Here another properties of the RREF that we might tease out of the experience of building Example 22.10. Let $R \in \mathbb{F}^{m\times n}$ be in RREF.

(i) *The (j,j)-entry of R is either 0 or 1 for $j = 1, \ldots, \min\{m,n\}$. That is, on the "diagonal" R is either 0 or 1. Moreover, the entries of column j are 0 in rows $j+1$ and below.*

Here is why. By Problem 22.12, the first column of R is either $\mathbf{0}$ or $\mathbf{e}_1$; that takes care of the $(1,1)$-entry. Suppose the $(2,2)$-entry is nonzero; then because the $(2,1)$-entry is 0 (it is the second entry of $\mathbf{e}_1$), the $(2,2)$-entry must be the leading nonzero entry row 2 and thus 1. Then every entry in column 2 that is not in row 2 must be 0; in particular, the $(3,2)$-entry is 0. And also the $(3,1)$-entry is 0 since the first column is either $\mathbf{0}$ or $\mathbf{e}_1$. Do it again: if the $(3,3)$-entry is nonzero, then since the $(3,1)$- and $(3,2)$-entries are 0, the $(3,3)$-entry is the leading nonzero entry in row 3 and thus 1.

Now turn the crank: if we know that the entries of column j, $1 \le j \le k$, are 0 in rows

$j+1$ and below, then the $(k+1, j)$-entries are 0 for $j = 1, \ldots, k$. So, if the $(k+1, k+1)$-entry is nonzero, it must be the leading nonzero entry in row $k+1$ and thus 1.

(ii) *If $\mathbf{e}_j$ is a column of R and $i < j$, then $\mathbf{e}_i$ must appear as a column of R at least once before $\mathbf{e}_j$.* Here is why—suppose $\mathbf{e}_i$ does not appear at all. Then the entries of row i of R are all 0; otherwise, some nonzero entry in row i would be the leading nonzero entry in that row, and then $\mathbf{e}_i$ would appear. So, $\mathbf{e}_i$ appears as *some* column of R, and so it must appear *before* $\mathbf{e}_j$.

(iii) *Any column of R that appears before the first appearance of $\mathbf{e}_i$ has zeros in rows i and below.* Let $\mathbf{r}$ be such a column and suppose that an entry of $\mathbf{r}$ in row i or below is nonzero. Since $\mathbf{e}_i$ has not yet appeared, all entries in row i in the columns before $\mathbf{r}$ must be 0; otherwise, such an entry would be the leading nonzero entry in row i, which would force a prior appearance of $\mathbf{e}_i$. So, if $\mathbf{r}$ has a nonzero entry in row i, that is the leading nonzero entry in row i, and therefore $\mathbf{r} = \mathbf{e}_i$. But $\mathbf{e}_i$ has not yet appeared.

This nonzero entry of $\mathbf{r}$ must therefore be in row $i+1$ or below. If this entry is the leading nonzero entry of row $i+1$, then $\mathbf{r} = \mathbf{e}_{i+1}$. But then $\mathbf{e}_{i+1}$ appears for the first time before $\mathbf{e}_i$. If this entry is not the leading nonzero entry of row $i+1$, then row $i+1$ has its leading nonzero entry in some column before $\mathbf{r}$. And that column is then $\mathbf{e}_{i+1}$, which again appears before $\mathbf{e}_i$.

22.14 Problem. **(i)** Determine all possible forms of a matrix $R \in \mathbb{F}^{1\times n}$ in RREF.

(ii) Determine all possible forms of a matrix $R \in \mathbb{F}^{m\times 1}$ $(= \mathbb{F}^m)$ in RREF.

22.15 Problem. Let $A \in \mathbb{F}^{m\times n}$ and $EA = R$, where $E \in \mathbb{F}^{m\times m}$ is invertible and R is in RREF. Prove that the jth column of A is $\mathbf{0}$ if and only if the jth column of R is $\mathbf{0}$. [Hint: *think about the matrix products $EA = R$ and $A = E^{-1}R$ using the definition* (16.4) *of matrix multiplication—what happens when one column in the factor on the right is $\mathbf{0}$?*]

Here is the proof of uniqueness of the RREF of $A \in \mathbb{F}^{m\times n}$, as claimed in Theorem 22.3. This is optional reading. The argument is based on a method of Holzmann, available here:

`https : //www.cs.uleth.ca/ holzmann/notes/reduceduniq.pdf.`

Assume that $A \neq 0$, as otherwise $R = \widetilde{R} = 0$. Let $EA = R$ and $\widetilde{E}A = \widetilde{R}$, where E, $\widetilde{E} \in \mathbb{F}^{m\times m}$ are invertible and R, $\widetilde{R} \in \mathbb{F}^{m\times n}$ are in RREF. By Problem 22.12, the first nonzero column of R is $\mathbf{e}_1$. By Problem 22.15, any zero column of R corresponds to a zero column of A and thus to a zero column of $\widetilde{R}$. We conclude that R and $\widetilde{R}$ have the same number of leading zero columns (if any), and that the first nonzero column of each is $\mathbf{e}_1$, and $\mathbf{e}_1$ occurs for the first time in the same column in both R and $\widetilde{R}$.

Suppose that the first p columns of R and $\widetilde{R}$ starting with this first occurrence of $\mathbf{e}_1$ agree (so $p \geq 1$). Denote by $\mathbf{r}$ and $\widetilde{\mathbf{r}}$ the $(p+1)$st column after this first occurrence of $\mathbf{e}_1$. Suppose that the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_j$, $j \geq 1$, appear within these first p columns of R and $\widetilde{R}$ starting with the first occurrence of $\mathbf{e}_1$ (since these p columns agree, the same standard basis vectors must appear among them).

Let $S \in \mathbb{F}^{n\times(j+1)}$ be the selection matrix (Problem 21.6) that selects $\mathbf{e}_1, \ldots, \mathbf{e}_j$ from within these first p columns and also the $(p+1)$st column. Then

$$EAS = RS = \left[\begin{array}{c|c} I_j & \mathbf{r} \\ 0 & \end{array}\right] \quad \text{and} \quad \widetilde{E}AS = \widetilde{R}S = \left[\begin{array}{c|c} I_j & \widetilde{\mathbf{r}} \\ 0 & \end{array}\right]. \tag{22.5}$$

That is, the first p columns of EAS and $\widetilde{E}AS$ are the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_p$ for $\mathbb{F}^m$, while the last column is $\mathbf{r}$ and $\widetilde{\mathbf{r}}$, respectively.

For example, if

$$R = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \widetilde{R} = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

then $p = 4$ and $j = 2$ and we would choose S to select columns 2, 3, and 4 from R and $\widetilde{R}$, thus

$$EAS = RS = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{E}AS = \widetilde{R}S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we examine the structure of $\mathbf{r}$ and $\widetilde{\mathbf{r}}$. Since the only standard basis vectors to the left of $\mathbf{r}$ in R are $\mathbf{e}_1, \ldots, \mathbf{e}_j$, either $\mathbf{r} = \mathbf{e}_{j+1}$ or $\mathbf{r}$ is 0 in rows $j+1$ and below. This is a result from Example 22.13. Consequently, we can refine the structure in (22.5) to read

$$\underbrace{EAS = \left[\begin{array}{c|c} I_j & \mathbf{r}_0 \\ 0 & 0 \end{array}\right]}_{I} \quad \text{or} \quad \underbrace{EAS = \left[\begin{array}{c|c} I_j & \mathbf{e}_{j+1} \\ 0 & \end{array}\right]}_{II}$$

and

$$\underbrace{\widetilde{E}AS = \left[\begin{array}{c|c} I_j & \widetilde{\mathbf{r}}_0 \\ 0 & \end{array}\right]}_{\widetilde{I}} \quad \text{or} \quad \underbrace{\widetilde{E}AS = \left[\begin{array}{c|c} I_j & \mathbf{e}_{j+1} \\ 0 & \end{array}\right]}_{\widetilde{II}},$$

where $\mathbf{r}_0$ and $\widetilde{\mathbf{r}}_0$ contain the first j rows of $\mathbf{r}$ and $\widetilde{\mathbf{r}}$.

If both cases II and $\widetilde{II}$ hold, then $\mathbf{r} = \mathbf{e}_{j+1} = \widetilde{\mathbf{r}}$. Suppose that cases I and $\widetilde{I}$ hold. The idea is now to view EAS and $\widetilde{E}AS$ as augmented matrices. Specifically, let $\widehat{S}$ be the selection matrix that selects just the columns of EA and $\widetilde{E}A$ containing $\mathbf{e}_1, \ldots, \mathbf{e}_j$ up to column p, so

$$EA\widehat{S} = \begin{bmatrix} I_j \\ 0 \end{bmatrix} = \widetilde{E}A\widehat{S}.$$

Consider the linear system $A\widehat{S}\mathbf{x} = AS\mathbf{e}_{j+1}$; we have selected this right side because $\mathbf{r}$ is the $(j+1)$st column of EAS. Applying E, this system is equivalent to

$$A\widehat{S}\mathbf{x} = AS\mathbf{e}_{j+1} \iff EA\widehat{S}\mathbf{x} = EAS\mathbf{e}_{j+1} \iff \begin{bmatrix} I_j \\ 0 \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{r}_0 \\ 0 \end{bmatrix} \iff \mathbf{x} = \mathbf{r}_0.$$

Applying $\widetilde{E}$ gives $\mathbf{x} = \widetilde{\mathbf{r}}_0$ as well, thus $\mathbf{r} = \widetilde{\mathbf{r}}$.

Last, suppose that cases I and $\widetilde{II}$ hold (the same argument will work if cases II and $\widetilde{I}$ hold). As above, we can take $\mathbf{x} = \mathbf{r}_0$ to solve $A\widehat{S}\mathbf{x} = AS\mathbf{e}_{j+1}$. But then

$$A\widehat{S}\mathbf{r}_0 = AS\mathbf{e}_{j+1} \Longrightarrow \widetilde{E}A\widehat{S}\mathbf{r}_0 = \widetilde{E}AS\mathbf{e}_{j+1} \Longrightarrow \begin{bmatrix} I_j \\ 0 \end{bmatrix} \mathbf{r}_0 = \mathbf{e}_{j+1}.$$

But the entries of

$$\begin{bmatrix} I_j \\ 0 \end{bmatrix} \mathbf{x}$$

are 0 in rows $j+1$ and below, and so no such product can equal $\mathbf{e}_{j+1}$. This shows that cases I and $\widetilde{II}$ cannot simultaneously hold.

Day 23: Friday, October 4.

Today was really just a retread of Day 22, so read the (copious) notes there.

Day 24: Monday, October 7.

Material from *Linear Algebra* by Meckes & Meckes

The column space is defined on p.116.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Column space

We have touted the RREF as an important factorization of a matrix. What good does it actually do?

Suppose that $A \in \mathbb{F}^{m \times n}$ has the RREF

$$EA = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P$$

with $1 \le r < n$. Since $r < n$, Example 22.7 reminds us that both the block of F is genuinely present in the RREF (although the blocks of 0 may not be present). Of course, E and P are invertible.

We claim that A has a nontrivial kernel (null space), and we can "parametrize" it as follows. We have $A\mathbf{x} = \mathbf{0}_n$ (it will be useful to emphasize the size of the zero vector here and there) if and only if

$$E^{-1} \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P\mathbf{x} = \mathbf{0}_n.$$

Multiply both sides by E and abbreviate $\mathbf{y} = P\mathbf{x}$, so $A\mathbf{x} = \mathbf{0}$ if and only if

$$\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} \mathbf{y} = \mathbf{0}_n. \tag{24.1}$$

It might be easier to understand the structure here if we work with a very small matrix, say $m = n = 3$ and $r = 2$. Then the situation is

$$\begin{bmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y} = \mathbf{0}_3.$$

This is the same as the linear system

$$\begin{cases} y_1 \qquad + f_1 y_3 = 0 \\ \qquad y_2 + f_2 y_3 = 0 \\ \qquad\qquad 0 = 0, \end{cases}$$

and so

$$y_1 = -f_1 y_3 \quad \text{and} \quad y_2 = -f_2 y_3.$$

Thus

$$\mathbf{y} = \begin{bmatrix} -f_1 y_3 \\ -f_2 y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} -f_1 \\ -f_2 \\ 1 \end{bmatrix}.$$

Here is how this structure shows up in (24.1). Denote by $\mathbf{y}^{(r)}$ the first r rows of $\mathbf{y}$ and by $\mathbf{y}^{(n-r)}$ the last $n-r$ rows of $\mathbf{y}$. Then (24.1) is equivalent to

$$\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}^{(r)} \\ \mathbf{y}^{(n-r)} \end{bmatrix} = \mathbf{0}_n, \tag{24.2}$$

and that in turn is equivalent to

$$I_r \mathbf{y}^{(r)} + F\mathbf{y}^{(n-r)} = \mathbf{0}_r. \tag{24.3}$$

24.1 Problem. By considering carefully the sizes of the 0 blocks in the RREF above (if they are even present), explain why (24.2) and (24.4) really are equivalent. (Doing the block multiplication shows that (24.2) implies (24.4), but why does (24.4) imply (24.2)?)

Then (24.4) is equivalent to

$$\mathbf{y}^{(r)} = -F\mathbf{y}^{(n-r)}, \tag{24.4}$$

and from that we have

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}^{(r)} \\ \mathbf{y}^{(n-r)} \end{bmatrix} = \begin{bmatrix} -F\mathbf{y}^{(n-r)} \\ \mathbf{y}^{(n-r)} \end{bmatrix} = \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} \mathbf{y}^{(n-r)}.$$

Now recall that $\mathbf{y} = P\mathbf{x}$ and rewrite $\mathbf{z} = \mathbf{y}^{(n-r)}$. So here is what we have shown: $A\mathbf{x} = \mathbf{0}_n$ if and only if

$$\mathbf{x} = P^{-1} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} \mathbf{z}$$

for some $\mathbf{z} \in \mathbb{F}^{n-r}$. This formula for $\mathbf{x}$ is the key to controlling $\ker(A)$.

For now, suppose $\mathbf{z} \neq \mathbf{0}_{n-r}$. Then

$$\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} \mathbf{z} = \begin{bmatrix} -F\mathbf{z} \\ \mathbf{z} \end{bmatrix} \neq \mathbf{0}_n,$$

since the bottom $n-r$ rows are $\mathbf{z} \neq \mathbf{0}_{n-r}$. Since P is invertible,

$$P^{-1} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} \mathbf{z} \neq \mathbf{0}.$$

Here is what we have proved.

24.2 Theorem. *Let $A \in \mathbb{F}^{m \times n}$ and suppose that*

$$\mathsf{rref}(A) = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P,$$

where $1 \leq r < n$ and $P \in \mathbb{F}^{n \times n}$ is invertible (the 0 blocks may or may not be present). Then

$$\ker(A) = \left\{ P^{-1} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} \mathbf{z} \;\middle|\; \mathbf{z} \in \mathbb{F}^{n-r} \right\}.$$

In particular, $\ker(A)$ *is nontrivial.*

This number r is, of course, the rank of A, and it will be hugely important in controlling the behavior of solutions to $A\mathbf{x} = \mathbf{b}$. For now, having $r < n$ destroys uniqueness of solutions if they exist.

24.3 Problem. Let $n > m \geq 1$. Prove that no linear operator $\mathcal{T} : \mathbb{F}^n \to \mathbb{F}^m$ is injective. [Hint: *consider the RREF of the matrix representation of $\mathcal{T}$ and recall $r \leq \min\{m, n\}$. What is $\min\{m, n\}$ here?*]

Theorem 24.2 is particularly helpful in understanding more about invertible matrices.

24.4 Theorem. *A matrix $A \in \mathbb{F}^{n \times n}$ is invertible if and only if $\mathsf{rref}(A) = I_n$.*

Proof. ($\Longleftarrow$) This direction is (slightly) easier, so we do it first. If $\mathsf{rref}(A) = I_n$, then there is $E \in \mathbb{F}^{n \times n}$ invertible such that $EA = I_n$, thus $A = E^{-1}$. Since E^{-1} is also invertible, so is A.

($\Longrightarrow$) If A is invertible, then there is $B \in \mathbb{F}^{n \times n}$ invertible such that $BA = I_n$. The matrix I_n is already in RREF, so since B is invertible, the uniqueness of the RREF forces $\mathsf{rref}(A) = I_n$.

Here is a different proof that does not rely on the uniqueness of the RREF (which was not easy to establish). Suppose $\mathsf{rref}(A) \neq I_n$. Then

$$\mathsf{rref}(A) = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P$$

with $1 \leq r < n$ and P invertible. (Since $\mathsf{rref}(A)$ has n rows as well as n columns, those 0 blocks are genuinely present—otherwise I_r would have to have n rows and then $r = n$. However, the presence or absence of the 0 blocks is not really important here.) Theorem 24.2 implies that $\ker(A) \neq \{\mathbf{0}\}$, and so there is $\mathbf{x} \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{x} = I_n$. But then A is not invertible. ■

Our last immediate consequence of the RREF is a relaxation of what needs to be checked to ensure that a matrix is invertible. Our current definition of an invertible matrix $A \in \mathbb{F}^{n \times n}$ requires the existence of $B \in \mathbb{F}^{n \times n}$ such that both $AB = I_n$ and $BA = I_n$. We claim that only one such equality needs to hold.

24.5 Theorem. *Let $A \in \mathbb{F}^{n \times n}$ and suppose that there exists $B \in \mathbb{F}^{n \times n}$ such that either $AB = I_n$ or $BA = I_n$. Then A is invertible.*

Proof. Be very careful in that we are not assuming that B is invertible, as otherwise we would be done.

Suppose first that $BA = I_n$. By Theorem 24.4, we will be done if we can show that $\mathsf{rref}(A) = I_n$. Motivated by what might go wrong if A is not invertible, we study $\ker(A)$. Let $A\mathbf{x} = \mathbf{0}$, so $BA\mathbf{x} = \mathbf{0}$ as well. But also $BA\mathbf{x} = \mathbf{x}$, so $\mathbf{x} = \mathbf{0}$. Thus $\ker(A)$ is trivial. If $\mathsf{rref}(A) \neq I_n$, then Theorem 24.2 implies that $\ker(A)$ is nontrivial. So $\mathsf{rref}(A) = I_n$, as desired.

Now suppose that $AB = I_n$. The previous paragraph shows that if $CD = I_n$ for some C, $D \in \mathbb{F}^{n \times n}$, then D is invertible; thus (with $C = A$ and $D = B$) B is invertible. Then $ABB^{-1} = I_nB^{-1}$, so $A = B^{-1}$, which is invertible. ■

We now turn to counting problems. Theorem 24.2 shows us how $\mathbb{F}^{n-r}$ effectively "parametrizes" $\ker(A)$ for $A \in \mathbb{F}^{m \times n}$ when the block I_r shows up in $\mathsf{rref}(A)$. What does this number $n - r$ say about the "size" of the kernel? What, more generally, can the RREF possibly say about solving $A\mathbf{x} = \mathbf{b}$?

Here is what it does not say.

24.6 Problem. Let $\mathcal{V}$ be a nontrivial vector space, i.e., $\mathcal{V} \neq \{0\}$ (where 0 is the zero vector for $\mathcal{V}$). Show that $\mathcal{V}$ contains infinitely many elements.

This result destroys the possibility that counting the number of elements in a vector space will ever yield meaningful information, outside of the trivial space $\{0\}$. It turns out that "counting" the "size" of the range of a linear operator will be more effective than starting with the kernel.

24.7 Definition. *Let $A \in \mathbb{F}^{m \times n}$. The* COLUMN SPACE *of A is*

$$\operatorname{col}(A) := \{A\mathbf{v} \mid \mathbf{v} \in \mathbb{F}^n\}.$$

24.8 Problem. Check that the column space of $A \in \mathbb{F}^{m \times n}$ is the range of $\mathcal{T}_A \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ given by $\mathcal{T}_A\mathbf{v} = A\mathbf{v}$, and so $\operatorname{col}(A)$ is a subspace of $\mathbb{F}^n$.

For a tractable example, let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}.$$

We have $\mathbf{w} \in \operatorname{col}(A)$ if and only if $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^4$, thus

$$\mathbf{w} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3 + v_4\mathbf{a}_4.$$

Now, observe that

$$\mathbf{a}_2 = 2\mathbf{a}_1 \quad \text{and} \quad \mathbf{a}_4 = 3\mathbf{a}_1 + 4\mathbf{a}_3.$$

Thus any $\mathbf{w} \in \operatorname{col}(A)$ has the form

$$\mathbf{w} = v_1\mathbf{a}_1 + v_2(2\mathbf{a}_1) + v_3\mathbf{a}_3 + v_4(3\mathbf{a}_1 + 4\mathbf{a}_3) = (v_1 + 2v_2 + 3v_4)\mathbf{a}_1 + (v_3 + 4v_4)\mathbf{a}_3.$$

That is, the vectors $\mathbf{a}_2$ and $\mathbf{a}_4$ are redundant in describing $\operatorname{col}(A)$, and that is because they are linear combinations of the other columns of A.

This toy example illustrates two general principles that we will develop much further: many interesting subspaces are given by spans (we have already seen this with some concrete eigenspaces), and sometimes some of the vectors in a span are redundant. How do we quantify and qualify redundancy to be as efficient as possible in working with spans?

Day 25: Wednesday, October 9.

Material from *Linear Algebra* by Meckes & Meckes

Pages 140–141 discuss redundancy and linear combinations, and pp.141–142 give two equivalent definitions of linear (in)dependence. Proposition 3.1 relates linear (in)dependence of column vectors to matrix kernels.

Do Quick Exercises #1, #2, and #3 in Section 3.1

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

List/finite sequence of length n in a set X, linearly dependent list in a vector space $\mathcal{V}$, linearly independent list in a vector space $\mathcal{V}$

We return to the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}.$$

We can consider the redundancies among its columns at three levels.

1. *One column is a linear combination of the others.* Here it is the case that

$$\mathbf{a}_2 = 2\mathbf{a}_1 + 0\mathbf{a}_3 + 0\mathbf{a}_4 \quad \text{and} \quad \mathbf{a}_4 = 3\mathbf{a}_1 + 0\mathbf{a}_2 + 4\mathbf{a}_3.$$

This "singles out one column for blame."

2. *A nontrivial linear combination of the columns is the zero vector.* We rewrite the equalities above as

$$2\mathbf{a}_1 + (-1)\mathbf{a}_2 + 0\mathbf{a}_3 + 0\mathbf{a}_4 = \mathbf{0}_3 \quad \text{and} \quad 3\mathbf{a}_1 + 0\mathbf{a}_2 + 4\mathbf{a}_3 + (-1)\mathbf{a}_4 = \mathbf{0}_3.$$

These linear combinations are "nontrivial" because some of the scalar coefficients in them are nonzero. Here no vector is "guiltier" than another.

3. *The kernel is nontrivial.* The equalities above show

$$A \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}_4 \quad \text{and} \quad A \begin{bmatrix} 3 \\ 0 \\ 4 \\ -1 \end{bmatrix} = \mathbf{0}_4.$$

So, there are nonzero vectors in $\ker(A)$.

The first and second properties are equivalent in any vector space, not just $\mathbb{F}^3$. First, we need a slight variation on sigma notation for finite sums. Let $\mathcal{V}$ be a vector space and $v_1, \ldots, v_n \in \mathcal{V}$ and $1 \le j \le n$. Then

$$\sum_{\substack{k=1 \\ k \ne j}}^{n} v_k := \begin{cases} \sum_{k=2}^{n} v_k, & j = 1 \\ \sum_{k=1}^{j-1} v_k + \sum_{k=j+1}^{n} v_k, & 2 \le j \le n-1 \\ \sum_{k=1}^{n-1} v_k, & j = n. \end{cases}$$

25.1 Lemma. *Let $\mathcal{V}$ be a vector space and $v_1, \ldots, v_n \in \mathcal{V}$. The following are equivalent:*

(i) *One of the v_k is a linear combination of the others: there exists j such that*

$$v_j = \sum_{\substack{k=1 \\ k \ne j}}^{n} \alpha_k v_k$$

for some $\alpha_k \in \mathbb{F}$.

(ii) *A nontrivial linear combination of the v_k is 0: there exist $\beta_1, \ldots, \beta_n \in \mathbb{F}$ such that*

$$\sum_{k=1}^{n} \beta_k v_k = 0,$$

and at least one of the β_k is nonzero.

Proof. (i) $\Longrightarrow$ (ii) Just rewrite

$$-v_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} \alpha_k v_k = 0$$

and define

$$\beta_k := \begin{cases} \alpha_k, & k \neq j \\ -1, & k = j. \end{cases}$$

Then $\sum_{k=1}^{n} \beta_k v_k = 0$ and at the very least $\beta_j = -1 \neq 0$.

(ii) $\Longrightarrow$ (i) Say $\beta_j \neq 0$. Then

$$0 = \sum_{k=1}^{n} \beta_k v_k = \beta_j v_j + \sum_{\substack{k=1 \\ k \neq j}} \beta_k v_k,$$

and so

$$v_j = \sum_{\substack{k=1 \\ k \neq j}} \left(-\frac{\beta_k}{\beta_j}\right) v_k.$$

Here it is important that $\beta_j \neq 0$ so we can divide. ■

We probably want to say that either of the conditions in the previous lemma means linear dependence. But *what object* should we call linearly dependent?

The columns of the matrix A above certainly should be linearly dependent, from this lemma. We also probably want to say that the columns of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are linearly dependent, since the nontrivial linear combination

$$(1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}_3$$

is the zero vector. But we should *not* say that the *set* of columns of this matrix is linearly dependent, for that set is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The problem here is that set conventions ignore repetition of elements, while linear dependence relations are very much affected by repetition.

The answer is in our statement of the previous lemma: when we take vectors $v_1, \ldots, v_n \in \mathcal{V}$, we are allowing repetition (maybe $v_1 = v_n$, which certainly happens if $n = 1$), but in addition, we are also encoding a sense of order. We formalize this as follows.

25.2 Definition. *Let X be a set and $n \geq 1$ be an integer. A* **LIST OF LENGTH** n **IN** X *or a* **FINITE SEQUENCE IN** X **OF LENGTH** n *is a function from $\{1, \ldots, n\}$ to X. If $f \colon \{1, \ldots, n\} \to X$ is such a function with $x_k := f(k)$, then we often write $f = (x_1, \ldots, x_n)$. If $n = 1$, then we interpret $(x_1, \ldots, x_1) = (x_1)$.*

A list of length n in X is sometimes called an **ORDERED** n**-TUPLE WITH ENTRIES IN** X. Note that even though we say "in X" in the definition above, a list *in* X is not an element *of* X.

25.3 Example. The lists $(1, 2, 1)$ and $(1, 1, 2)$ in $\mathbb{R}$ are not the same. The first list is the function

$$(1, 2, 1) = \{(1, 1), (2, 2), (3, 1)\},$$

while the second list is the function

$$(1, 1, 2) = \{(1, 1), (2, 1), (3, 2)\}.$$

As functions, their domains and ranges are the same: both domains are $\{1, 2, 3\}$ and both ranges are $\{1, 2\}$. But pointwise these functions are different.

25.4 Problem. Explain why $\mathbb{F}^n$ is the set of all lists of length n in $\mathbb{F}$, and so it is correct to say

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = (v_1, \ldots, v_n).$$

If a matrix $A \in \mathbb{F}^{m \times n}$ is a function from $\{(i, j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$, and if the columns of A are $\mathbf{a}_1, \ldots, \mathbf{a}_n$, is it really correct to say $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$?

Embiggened with lists, we now rigorously define linear dependence and independence.

25.5 Definition. *A list $(v_1, \ldots, v_n)$ in a vector space $\mathcal{V}$ is* **LINEARLY DEPENDENT** *if either condition in Lemma 25.1 holds and* **LINEARLY INDEPENDENT** *if it is not linearly dependent.*

So, linear dependence is an *existential* condition: we must show that either there exists a nontrivial linear combination of the vectors in the list that adds up to the zero vector, or that there exists a vector in the list that is a linear combination of the others. But linear independence is a *universal* condition: we must show that no linear combination of the vectors in the list is zero except the trivial combination ($\sum_{k=1}^{n}\alpha_k v_k \Longrightarrow \forall k : \alpha_k = 0$) or that no vector in the list is a linear combination of the others in the list.

25.6 Example. **(i)** Consider a list of length 1 in the vector space $\mathcal{V}$: this has the form (v) for some $v \in \mathcal{V}$. This list is linearly dependent if and only if there is $\alpha \in \mathbb{F} \setminus \{0\}$ such that $\alpha v = 0$. If $v \neq 0$, $\alpha v = 0$ forces $\alpha = 0$. So, every list of length 1 is linearly independent, except for (0).

(ii) Consider a list of length 2 in the vector space $\mathcal{V}$: this has the form (v_1, v_2) for some v_1, $v_2 \in \mathcal{V}$. This list is linearly dependent if and only if there are α_1, $\alpha_2 \in \mathbb{F}$, not both 0, such that $\alpha_1 v_1 + \alpha_2 v_2 = 0$. Say $\alpha_1 \neq 0$. Then $v_1 = -(\alpha_2/\alpha_1)v_2$. So, a list of length 2 is linearly dependent if and only if one of the vectors in the list is a scalar multiple of the other.

(iii) Let $(v_1, \ldots, v_n)$ be a list in $\mathcal{V}$ and suppose that one vector in the list is 0, say, $v_j = 0$. Put

$$\alpha_k := \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases}$$

Then

$$\sum_{k=1}^{n} \alpha_k v_k = (1)v_j + \sum_{\substack{k=1 \\ k \neq j}} 0v_k = 0 + 0,$$

so this list is linearly dependent. So, any list containing the zero vector is linearly dependent (we saw this above with the case $n = 1$).

(iv) Let $(v_1, \ldots, v_n)$ be a list in $\mathcal{V}$ of length $n \geq 2$. Suppose that $v_{j_1} = v_{j_2}$ where $1 \leq j_1 < j_2 \leq n$. Put

$$\alpha_k := \begin{cases} 1, & k = j_1 \\ -1, & k = j_2 \\ 0, & k \neq j_1,\ j_2. \end{cases}$$

Then

$$\sum_{k=1}^{n} \alpha_k v_k = (1)v_{j_1} + (-1)v_{j_2} + \sum_{\substack{k=1 \\ k \neq j_1,\ j_2}}^{n} 0v_k = v_{j_1} - v_{j_1} + 0 = 0.$$

so this list is linearly dependent. So, any list of two or more vectors with a repeated vector is linearly dependent.

(v) A list $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ in $\mathbb{F}^m$ is linearly dependent if and only if there are $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ not all 0 such that $\sum_{k=1}^{n}\alpha_k \mathbf{v}_k = \mathbf{0}_m$; this is equivalent to

$$\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{0}_m,$$

which in turn is equivalent to a nontrivial kernel for $\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$. So, a list in $\mathbb{F}^m$ is linearly dependent if and only if the matrix whose columns are the entries of that list (with no repetitions of entries) has a nontrivial kernel. (We will eventually develop the tools to isolate the "guilty" columns of that matrix and see exactly how they are linear combinations of the other columns.)

(vi) Recall the notion of algebraic dual space from Example 12.1 and define linear functionals on $\mathcal{C}(\mathbb{R})$ as follows. For $x \in \mathbb{R}$, let φ_x be the "evaluate at x" functional such that $\varphi_x(f) := f(x)$ for $f \in \mathcal{C}(\mathbb{R})$. For $x_1, \ldots, x_n \in \mathbb{R}$, let $(\varphi_{x_1}, \ldots, \varphi_{x_n})$ be a list of such functionals in $(\mathcal{C}(\mathbb{R}))'$. This list is automatically linearly dependent if any terms are the same, which here is equivalent to $x_j = x_k$ for some $j \neq k$. So, assume that all of the x_k are distinct. We claim this list is linearly independent.

Here is why. Suppose that $\sum_{k=1}^n \alpha_k \varphi_{x_k} = 0$ for some $\alpha_k \in \mathbb{F}$. This means that $\sum_{k=1}^n \alpha_k \varphi_{x_k}(f) = 0$ for all $f \in \mathcal{C}(\mathbb{R})$, and so $\sum_{k=1}^n \alpha_k f(x_k) = 0$ for all $f \in \mathcal{C}(\mathbb{R})$. Since this is true for *all* f, we can pick f to be any function that we like. In particular, we could "interpolate" and choose f to be 0 at each x_k except for one value of k. Say that $f_j \in \mathcal{C}(\mathbb{R})$ with

$$f_j(x_k) = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

Then $0 = \sum_{k=1}^n \alpha_k f_j(x_k) = \alpha_j$, and so each α_k is 0.

Does such a function f_j really exist? Here is how to construct it for small n, say, $n = 3$. Just put

$$f_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \qquad f_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)},$$

$$\text{and} \qquad f_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}.$$

There are no problems with division by zero because all of the x_k are distinct.

25.7 Problem. Check that the "evaluate at x" functional really is a linear functional on $\mathcal{C}(\mathbb{R})$.

25.8 Problem. Use the definition of linear independence that the list $(\mathbf{e}_1, \ldots, \mathbf{e}_m)$ of standard basis vectors in $\mathbb{F}^m$ is linearly independent.

25.9 Problem. Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{F}^{n \times n}$ be invertible. Prove that $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is linearly independent. (This is understood to be the list of columns of A with no repetitions among the entries of the list.)

Day 26: Friday, October 11.

Material from *Linear Algebra* by Meckes & Meckes

The "spanning lemma" is Theorem 3.6 (and Corollary 3.7), which the Meckeses call the "linear dependence lemma" and use frequently. Our lemma adds a little detail on linear independence. The eigenvector result is Theorem 3.8. Results on linear (in)dependence in $\mathbb{F}^m$ appear in Corollaries 3.2, 3.3, and 3.5 and Algorithm 3.4.

Do Quick Exercises #4 and #5 in Section 3.1.

It follows from the definition of linear independence that linear independence prevents redundancy: if a list $(v_1, \ldots, v_n)$ in $\mathcal{V}$ is linearly independent, then no vector in that list is a linear combination of the others. This also removes ambiguity: there is only one way to write a vector $v \in \operatorname{span}(v_1, \ldots, v_n)$ as a linear combination of the v_k.

26.1 Lemma (Unique representation). *Let $\mathcal{V}$ be a vector space and let $(v_1, \ldots, v_n)$ be a linearly independent list in $\mathcal{V}$. Suppose that $\sum_{k=1}^n \alpha_k v_k = \sum_{k=1}^n \beta_k v_k$ for some α_k, $\beta_k \in \mathbb{F}$. Then $\alpha_k = \beta_k$ for all k.*

Proof. Subtract to find

$$\sum_{k=1}^n (\alpha_k - \beta_k) v_k = 0,$$

so $\alpha_k - \beta_k = 0$ for all k. ■

So, if $v \in \operatorname{span}(v_1, \ldots, v_n)$, then by definition of span we can write v as a linear combination of the v_k, and by the previous result, there is only one way to do this.

Here is another useful result about spans and linear (in)dependence. It says that any linearly dependent list whose first term is not the zero vector can be pared down to a linearly independent list that respects certain "order" properties of the list. (Shortly we will see a similar result that chops up the order a bit.)

26.2 Lemma (Spanning). **(i)** *Let $(v_1, \ldots, v_n)$ be a linearly dependent list in the vector space $\mathcal{V}$ with $n \geq 2$. Suppose $v_1 \neq 0$. Then there exists $j \geq 2$ such that $v_j \in (v_1, \ldots, v_{j-1})$. Moreover, j can be chosen so that $(v_1, \ldots, v_{j-1})$ is linearly independent.*

(ii) *Let $(v_1, \ldots, v_n)$ be a list in $\mathcal{V}$ with $n \geq 2$ such that $v_j \notin \operatorname{span}(v_1, \ldots, v_{j-1})$ for $j = 1, \ldots, n$. Then $(v_1, \ldots, v_n)$ is linearly independent.*

Proof. **(i)** Since $(v_1, \ldots, v_n)$ is linearly dependent, there exist $\alpha_k \in \mathbb{F}$ not all 0 such that $\sum_{k=1}^n \alpha_k v_k = 0$. Let $m \geq 1$ be the smallest index such that $\alpha_m \neq 0$. At worst $m = n$, but at least $m \geq 2$, for if $\alpha_k = 0$ for $k \geq 2$ but $\alpha_1 \neq 0$, then $\alpha_1 v_1 = 0$, and then $v_1 = 0$. So, we have

$$0 = \sum_{k=1}^n \alpha_k v_k = \alpha_m v_m + \sum_{k=1}^{m-1} \alpha_k v_k,$$

and therefore, since $\alpha_m \neq 0$,

$$v_m = \sum_{k=1}^{m-1} \left(-\frac{\alpha_k}{\alpha_m}\right) v_k \in \operatorname{span}(v_1, \ldots, v_{m-1}).$$

Now let j be the smallest of these m, i.e., the smallest index of the integers $m \in \{2, \ldots, n\}$ such that $v_m \in (v_1, \ldots, v_{m-1})$. The work above shows that at least one such m exists. We claim that $(v_1, \ldots, v_{j-1})$ is linearly independent. Certainly this is true if $j = 2$, since $v_1 \neq 0$, so (v_1) is linearly independent. Otherwise, if $j \geq 3$, suppose $\sum_{k=1}^{j-1} \alpha_k v_k = 0$, and let ℓ be the smallest index such that $\alpha_\ell \neq 0$. As before, we must have $\ell \geq 2$. Then

$$0 = \sum_{j=1}^{j-1} \alpha_k v_k = \sum_{k=1}^{\ell} \alpha_k v_k,$$

so, just as before,

$$v_\ell = \sum_{k=1}^{\ell-1} \left(-\frac{\alpha_k}{\alpha_p}\right) v_k \in \operatorname{span}(v_1, \ldots, v_{\ell-1}).$$

But $p \leq j - 1 < j$, which contradicts the minimality of j.

(ii) This is the contrapositive of the first result above:

$$\big(\text{Linear dependence} \Longrightarrow \exists j : v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})\big)$$
$$\Longleftrightarrow \quad \big(\forall j : v_j \notin \operatorname{span}(v_1, \ldots, v_{j-1}) \Longrightarrow \text{Linear independence}\big). \quad \blacksquare$$

26.3 Problem. Use the spanning lemma to give another proof that the list of standard basis vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ in $\mathbb{F}^n$ is linearly independent.

Experience might suggest that eigenvectors corresponding to distinct eigenvalues are linearly independent (this is hopefully obvious for a diagonal matrix, where the standard basis vectors show up as eigenvectors). Here is how we see this with only a handful of eigenvectors. Let $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ have the distinct eigenvalues λ_1, $\lambda_2 \in \mathbb{F}$, so $\lambda_1 \neq \lambda_2$, and there are v_1, $v_2 \in \mathcal{V} \setminus \{0\}$ such that

$$\mathcal{T} v_1 = \lambda_1 v_1 \quad \text{and} \quad \mathcal{T} v_2 = \lambda_2 v_2.$$

Suppose $\alpha_1 v_1 + \alpha_2 v_2 = 0$. We want to show $\alpha_1 = \alpha_2 = 0$. We can get the eigenvalues to show up in that linear combination by applying $\mathcal{T}$ to both sides:

$$0 = \mathcal{T}(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \mathcal{T} v_1 + \alpha_2 \mathcal{T} v_2 = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2.$$

It now looks like we have a system of linear equations for α_1 and α_2, except the "coefficients" are not scalars in $\mathbb{F}$ but vectors:

$$\begin{cases} \alpha_1 v_1 + \alpha_2 v_2 = 0 \\ \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 = 0. \end{cases}$$

We can make λ_1 and λ_2 interact via elementary row operations: multiply both sides of $\alpha_1 v_1 + \alpha_2 v_2 = 0$ by $-\lambda_2$ and add to the result above to get

$$0 = \big(\alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2\big) + \big(-\lambda_2\alpha_1 v_1 - \lambda_2\alpha_2 v_2\big) = \alpha_1(\lambda_1 - \lambda_2)v_1.$$

Since $v_1 \neq 0$, we have $\alpha_1(\lambda_1 - \lambda_2) = 0$, and since $\lambda_1 \neq \lambda_2$, we have $\alpha_1 = 0$. Back to the original linear combination, we reduce to $\alpha_2 v_2 = 0$, so $\alpha_2 = 0$ since $v_2 \neq 0$.

Here is how this works in general: let $\mathcal{T}v_k = \lambda_k v_k$, with each $v_k \neq 0$ and the λ_k distinct. Consider the list $(v_1, \ldots, v_n)$ of eigenvectors: $v_1 \neq 0$, so if the list is linearly dependent, the spanning lemma gives j such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$ and $(v_1, \ldots, v_{j-1})$ is linearly independent.

26.4 Problem. Explain why, in the $n = 2$ case, we really had $j = 2$.

Write $v_j = \sum_{k=1}^{j-1} \alpha_k v_k$. Apply $\mathcal{T}$ to find $\mathcal{T}v_j = \sum_{k=1}^{j-1} \alpha_k \mathcal{T}v_k$. Use the definition of eigenvalue(vector) to find $\lambda_j v_j = \sum_{k=1}^{j-1} \alpha_k \lambda_k v_k$. We now have the "linear system"

$$\begin{cases} v_j = \sum_{k=1}^{j-1} \alpha_k v_k \\ \lambda_j v_j = \sum_{k=1}^{j-1} \alpha_k \lambda_k v_k. \end{cases}$$

Multiply both sides of $v_j = \sum_{k=1}^{j-1} \alpha_k v_k$ by $-\lambda_j$ and add:

$$\lambda_j v_j - \lambda_j v_j = \sum_{k=1}^{j-1} \alpha_k \lambda_k v_k - \sum_{k=1}^{j-1} \alpha_k \lambda_j v_k,$$

thus

$$\sum_{k=1}^{j-1} \alpha_k (\lambda_k - \lambda_j) v_k = 0.$$

By linear independence of $(v_1, \ldots, v_{j-1})$, $\alpha_k(\lambda_k - \lambda_j) = 0$, and since the eigenvalues are distinct and $\lambda_k - \lambda_j \neq 0$, we must have $\alpha_k = 0$.

At this point we may not be sure where we are. We are trying to derive a contradiction from the linear dependence of $(v_1, \ldots, v_n)$. Yet nowhere have we considered a linear combination of the form $\sum_{k=1}^{n} \beta_k v_k$ that involves all of the vectors in this list—surely that should be part of the argument?

Not necessarily. Our work above shows $\alpha_k = 0$ for $k = 1, \ldots, j-1$, where $v_j = \sum_{k=1}^{j-1} \alpha_k v_k$. But then $v_j = \sum_{k=1}^{j-1} 0 v_k = 0$, and the zero vector is not an eigenvector. That is the contradiction.

26.5 Theorem. *Eigenvectors of a linear operator corresponding to distinct eigenvalues are linearly independent. More precisely, if (λ_k, v_k) is an eigenpair of $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ for $k = 1, \ldots, n$ and $\lambda_j \neq \lambda_k$ for $j \neq k$, then $(v_1, \ldots, v_n)$ is linearly independent.*

A good exercise for the ardent apprentice linear algebraist is to redo the argument above for arbitrary n until it feels completely natural. Starting with $n = 3$ concretely may make things more transparent.

Day 27: Monday, October 14.

Material from *Linear Algebra* by Meckes & Meckes

Algorithm 3.4 describes how to show that a list in $\mathbb{F}^m$ is linearly independent.

Do Quick Exercise #4 in Section 3.1.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Pivot column of a matrix (N)

Here is the other side of the spanning lemma: we can "reduce" or "winnow down" any linearly dependent list into a linearly independent one with the same span. This relies on the following fact.

27.1 Problem. Let $(v_1, \ldots, v_n)$ be a list in the vector space $\mathcal{V}$ with $n \geq 2$ and suppose that some entry v_j of the list is a linear combination of the other vectors in the list. Show that $\operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$. (If $j = 1$, we interpret $(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) = (v_2, \ldots, v_n)$, and if $j = n$, this is $(v_1, \ldots, v_{n-1})$.

Consider the following list in $\mathbb{R}^4$:

$$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6) = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right).$$

The zero vector contributes nothing to the span, so we can ignore that (and we should remove it from the list to protect linear independence anyway). Since $\mathbf{v}_2 \neq \mathbf{0}_4$, we may as well try to include $\mathbf{v}_2$ in the span. Since $\mathbf{v}_3 = 2\mathbf{v}_2$, $\mathbf{v}_3$ contributes nothing new to a span of vectors already containing $\mathbf{v}_2$. So, right now, $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \operatorname{span}(\mathbf{v}_2)$. Next, $\mathbf{v}_4$ is definitely not a scalar multiple of $\mathbf{v}_2$, so we include it. If we think a bit we can find α_2 and α_4 such that $\alpha_2\mathbf{v}_2 + \alpha_4\mathbf{v}_4 = \mathbf{v}_5$, so we should exclude $\mathbf{v}_5$ from the span. Right now, $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_5) = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_4)$.

27.2 Problem. Think a bit and find them.

Last, $\mathbf{v}_6$ is not in the span of $\mathbf{v}_2$ and $\mathbf{v}_4$, since no linear combination of these three vectors adds to $\mathbf{0}_4$ except the trivial one.

27.3 Problem. Check that: if $\alpha_2\mathbf{v}_2 + \alpha_4\mathbf{v}_4 + \alpha_6\mathbf{v}_6 = \mathbf{0}_4$, then $\alpha_2 = \alpha_4 = \alpha_6 = 0$. [Hint: *look at rows 2 and 4 first.*]

We conclude $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_6) = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6)$. We probably want to call $(\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6)$ a "sublist" of $(\mathbf{v}_1, \ldots, \mathbf{v}_6)$, since the entries of the former appear in the latter and in the same order. This is somewhat annoying to define precisely.

27.4 Definition. *Let X be a set and let $(x_1, \ldots, x_n)$ be a list of length n in X. Let $r \le n$. A list $(y_1, \ldots, y_r)$ is a* SUBLIST *of $(x_1, \ldots, x_n)$ if there is a map $\sigma \colon \{1, \ldots, r\} \to \{1, \ldots, n\}$ that is strictly increasing (in the sense that $\sigma(j) < \sigma(k)$ for $j < k$) with $y_k = x_{\sigma(k)}$ for each k.*

In the concrete example above, the sublist of $(\mathbf{v}_1, \ldots, \mathbf{v}_6)$ of interest was $(\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6)$. We could call this sublist $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and put $\sigma(1) = 2$, $\sigma(2) = 4$, and $\sigma(3) = 6$.

We generalize this "reduction" procedure as follows.

27.5 Lemma (Reduction). *Let $(v_1, \ldots, v_n)$ be a linearly dependent list in the vector space $\mathcal{V}$ with at least one of the v_k nonzero. There exists a linearly independent sublist $(v_{k_1}, \ldots, v_{k_r})$ of $(v_1, \ldots, v_n)$ such that* $\operatorname{span}(v_{k_1}, \ldots, v_{k_r}) = \operatorname{span}(v_1, \ldots, v_n)$.

Proof. First, we require at least one entry in the list to be nonzero, as otherwise the span is just $\{0\}$, and there is nothing interesting here. Next, we require the list to be linearly dependent, as otherwise the list is linearly independent, and that is the best kind of list.

We reduce the list as follows. Let v_{k_1} be the first nonzero vector in the list. (At least one exists.) So $\operatorname{span}(v_1, \ldots, v_{k_1}) = \operatorname{span}(v_{k_1})$. Also, (v_{k_1}) is linearly independent because $v_{k_1} \ne 0$.

Let v_{k_2} be the first vector in the list that is not a scalar multiple of v_{k_1}. So $\operatorname{span}(v_1, \ldots, v_{k_2}) = \operatorname{span}(v_{k_1}, v_{k_2})$. Also, (v_{k_1}, v_{k_2}) is linearly independent because neither entry is a scalar multiple of the other (or by the spanning lemma, since $v_{k_2} \notin \operatorname{span}(v_{k_1})$).

Let v_{k_3} be the first vector in the list that is not in $\operatorname{span}(v_{k_1}, v_{k_2})$. So $\operatorname{span}(v_1, \ldots, v_{k_3}) = \operatorname{span}(v_{k_1}, v_{k_2}, v_{k_3})$. Also, $(v_{k_1}, v_{k_2}, v_{k_3})$ is linearly independent by the spanning lemma.

Now turn the crank and keep going: eventually we run out of vectors in the list. ∎

The reduction lemma is an existential result, and the algorithm within requires an annoying entry-by-entry examination of the list. In the very important case of lists of column vectors, there are much easier, and more meaningful, methods of determining which vectors in a list are linearly independent and preserve the span. These hinge on the RREF.

Here is an illustrative example of a much more general phenomenon. We have previously shown the existence of an invertible matrix $E \in \mathbb{R}^{3\times3}$ such that

$$EA = \mathsf{rref}(A) =: R, \qquad A := \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}, \qquad R := \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we introduce a new piece of terminology.

27.6 Definition. **(i)** *Let $R \in \mathbb{F}^{m \times n}$ be in RREF. Column j of R is a* PIVOT COLUMN *of R if column j contains a leading 1 (i.e., if column j is the first appearance in R of a standard basis vector for $\mathbb{F}^m$).*

(ii) *Let $A \in \mathbb{F}^{m \times n}$. Column j of A is a* PIVOT COLUMN *of A if the jth column of* $\mathsf{rref}(A)$ *is a pivot column.*

So, with A and R as above, columns 1 and 3 are the pivot columns. With $\mathbf{r}_j$ as the jth column of R, we have also previously seen that

$$\mathbf{r}_2 = 2\mathbf{r}_1 \quad \text{and} \quad \mathbf{r}_4 = 3\mathbf{r}_1 + 4\mathbf{r}_3.$$

That is, the nonpivot columns of R_0 are linear combinations of the pivot columns. The same is true of the nonpivot columns of A, although this may be less obvious: $\mathbf{a}_2 = 2\mathbf{a}_1$ (well, that should be obvious) and also $\mathbf{a}_4 = 3\mathbf{a}_4 + 4\mathbf{a}_3$.

It should be obvious that $(\mathbf{r}_1, \mathbf{r}_3)$ is linearly independent, since the entries of this list are (nonrepeated) standard basis vectors. It may be less obvious that $(\mathbf{a}_1, \mathbf{a}_3)$ is linearly independent.

27.7 Problem. Check that.

The linear (in)dependence relations among the columns of A appear to coincide with those among the columns of R. This is no accident: $A\mathbf{x} = \mathbf{0}_3$ if and only if $EA\mathbf{x} = \mathbf{0}_3$, since E is invertible, and so this holds if and only if $R_0\mathbf{x} = \mathbf{0}_3$.

Here is the more general (but also more precise) situation.

27.8 Theorem. *Let $A \in \mathbb{F}^{m \times n} \setminus \{0\}$.*

(i) *The pivot columns of A are linearly independent. More precisely, if $\mathbf{a}_{j_k}$ is a pivot column of A for $k = 1, \ldots, r$ with $1 \le j_k < j_{k+1} \le r \le \min\{m, n\}$, then $(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_r})$ is linearly independent.*

(ii) *Any nonpivot column of A is a linear combination of the pivot columns of A.*

(iii) $\mathrm{col}(A)$ *is the span of the pivot columns.*

Proof. **(i)** Say that columns $j_1, \ldots, j_r$ are the pivot columns of A, where $r \le \min\{m, n\}$, and $1 \le j_1 < \cdots < j_r \le n$. (Since $A \ne 0$, A has at least one pivot column, and A cannot have more than $\min\{m, n\}$ pivot columns because (1) A has at most n columns and (2) at most m columns in $\mathsf{rref}(A)$ can contain a leading 1.) Suppose $\sum_{k=1}^r \alpha_k \mathbf{a}_{j_k} = \mathbf{0}_m$ for some $\alpha_k \in \mathbb{F}$.

Let $E \in \mathbb{F}^{m \times m}$ be invertible with $EA = \mathsf{rref}(A)$. Then $E\big(\sum_{k=1}^r \alpha_k \mathbf{a}_{j_k}\big) = \mathbf{0}_m$. Since $E\mathbf{a}_{j_k}$ is a pivot column of $\mathsf{rref}(A)$, $E\mathbf{a}_{j_k}$ must be one of the standard basis vectors for $\mathbb{F}^m$. Specifically, $E\mathbf{a}_{j_k} = \mathbf{e}_k$, since the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_r$ must appear for the first time "in order" in $\mathsf{rref}(A)$. Thus $\sum_{k=1}^r \alpha_k \mathbf{e}_k = \mathbf{0}_m$, so $\alpha_k = 0$ for all k.

(ii) Let $\mathbf{a}_j$ be a nonpivot column of A. So $E\mathbf{a}_j$ is a nonpivot column of $\mathsf{rref}(A)$. Our idea is that the nonpivot columns of $\mathsf{rref}(A)$ are linear combinations of the pivot columns, which are $\mathbf{e}_1, \ldots, \mathbf{e}_r$. Thus $E\mathbf{a}_j = \sum_{k=1}^{r} \alpha_k \mathbf{e}_k$ for some α_k, equivalently, $\mathbf{a}_j = \sum_{k=1}^{r} \alpha_k E^{-1}\mathbf{e}_k$. And each $E^{-1}\mathbf{e}_k$ is a pivot column of A.

Here we give slightly more detail on that "idea." If A has a nonpivot column, it must be the case that

$$\mathsf{rref}(A) = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P$$

with $r < n$ and $F \in \mathbb{F}^{r \times (n-r)}$. (As usual, the zero blocks may or may not be present, and P is a permutation matrix.) Any column of F is a linear combination of the columns of I_r, so any nonpivot column of $\mathsf{rref}(A)$ is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_r$.

(iii) Apply Problem 27.1 repeatedly to all of the nonpivot columns. ■

This corollary provides an explicit recipe for finding the linearly independent columns of a matrix and controlling its column space: just look at the RREF. It also provides a quick opinion on linear dependence of a list.

27.9 Corollary. *Any list of vectors in $\mathbb{F}^m$ of length $n > m$ is linearly dependent.*

27.10 Problem. Prove this corollary. [Hint: *how many pivot columns, at the most, can the matrix whose columns are the vectors in that list have?*]

That is,

$$\text{More columns than rows} \Longrightarrow \text{Linearly dependent.}$$

Day 28: Wednesday, October 16.

Material from *Linear Algebra* by Meckes & Meckes

All of the material on pp.150–153 up to and including Theorem 3.10 is essential. Lemma 3.17 on p.163 contains the full proof of the "counting" lemma. Page 164 discusses dimension. Read Algorithm 3.25 on p.166 and Theorem 3.26 on p.167.

Do Quick Exercise #6 on p.150, #7 on p.152, #11 on p.163, #12 on p.165, #14 on p.166, and #15 on p.168.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Finite-dimensional vector space, basis for a vector space

The RREF tells us exactly *which* columns of A are linearly independent, but in the factorization $EA = \mathsf{rref}(A)$, those columns do not appear explicitly. However, if we write $A = E^{-1}\mathsf{rref}(A)$, then we can extract the linearly independent columns; this is a new idea due to Strang. Consider first, and once again,

$$A := \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix} = E^{-1}R_0P, \qquad R_0 := \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

for some invertible $E \in \mathbb{R}^{3\times3}$ and a permutation matrix $P \in \mathbb{R}^{4\times4}$. The matrix R_0 is just $\mathsf{rref}(A)$ with the columns shuffled into the "right" order so the standard basis vectors come first.

For lack of better notation, write $E^{-1} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. Then

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & 2\mathbf{v}_1 & (3\mathbf{v}_1 + 4\mathbf{v}_2) \end{bmatrix} P.$$

Look at the product on the right: the permutation matrix P just reorders the columns of the first factor $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & 2\mathbf{v}_1 & (3\mathbf{v}_1 + 4\mathbf{v}_2) \end{bmatrix}$, so the columns of A are $\mathbf{v}_1$, $\mathbf{v}_2$, $2\mathbf{v}_1$, and $3\mathbf{v}_1 + 4\mathbf{v}_2$ (not in that order, though). In particular, $(\mathbf{v}_1, \mathbf{v}_2)$ are linearly independent, since this is a list of (nonrepeated) columns of an invertible matrix.

Now factor again:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & 2\mathbf{v}_1 & (3\mathbf{v}_1 + 4\mathbf{v}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

Call the second factor on the right $\widehat{R}$ (we have deleted the subscript since the zero rows/blocks are gone). Thus

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \widehat{R}P.$$

The first factor contains the linearly independent columns of A; the second factor "combines" them into *all* of the columns of A (this is the "recipe" factor, hence the hat on the R—all chefs wear hats); and the third factor reorders them into the right order.

Such a factorization is true in general.

28.1 Theorem (CR-factorization—Strang). *Let $A \in \mathbb{F}^{m\times n} \setminus \{0\}$ and suppose that A has r pivot columns. There exists $R \in \mathbb{F}^{r\times n}$ such that $A = CR$, where the columns of $C \in \mathbb{F}^{m\times r}$ are exactly the pivot columns of A. If $r = n$ then $R = I_n$, while if $r < n$, then R has the form $R = \begin{bmatrix} I_r & F \end{bmatrix} P$, where $P \in \mathbb{F}^{n\times n}$ is a permutation matrix and $F \in \mathbb{F}^{r\times(n-r)}$.*

28.2 Problem. Prove this theorem. [Hint: *write $A = E^{-1}\mathsf{rref}(A)$, and then write*

$$E^{-1}\mathsf{rref}(A) = \begin{bmatrix} V_r & V_{n-r} \end{bmatrix} \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P \quad \text{or} \quad E^{-1}\mathsf{rref}(A) = V_n \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

where in the first case $r < n$ and in the second case $r = n$. The columns of V_r and V_n will *be linearly independent in either case.*]

28.3 Problem. Find a CR factorization for the matrix A that you constructed in Problem 22.9.

Now we return to abstract vector spaces. We develop a third result in the spirit of the spanning and reduction lemmas that relates spans and linearly independent lists. Informally, the next results says that once a span is defined by n linearly independent vectors, we cannot do better than that—we can never describe the same span with fewer than n vectors.

28.4 Lemma (Counting). *Let $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be lists in the vector space $\mathcal{V}$ such that $(v_1, \ldots, v_n)$ is linearly independent and $\operatorname{span}(v_1, \ldots, v_n) \subseteq \operatorname{span}(w_1, \ldots, w_m)$. Then $m \geq n$.*

Proof. This is essentially Lemma 3.17 in the book. We give the proof in the illustrative cases of a few small values for m and n.

Suppose instead that $m < n$.

1. $m = 1$, $n = 2$. Then $\operatorname{span}(v_1, v_2) = \operatorname{span}(w_1)$, so both v_1 and v_2 are scalar multiples of w_1 and thus of each other. This contradicts the linear independence of v_1 and v_2.

2. $m = 2$, $n = 3$. Then $\operatorname{span}(v_1, v_2, v_3) = \operatorname{span}(w_1, w_2)$. Write

$$\begin{cases} v_1 = a_{11}w_1 + a_{21}w_2 \\ v_2 = a_{12}w_1 + a_{22}w_2 \\ v_3 = a_{13}w_1 + a_{23}w_2, \end{cases}$$

so the columns of

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in \mathbb{F}^{2\times 3}$$

are linearly dependent. Let $\boldsymbol{\alpha} \in \mathbb{F}^3 \setminus \{\mathbf{0}_3\}$ such that $A\boldsymbol{\alpha} = \mathbf{0}_2$. Then

$$\begin{aligned} \alpha_1 v_1 + \alpha_2 v_3 + \alpha_3 v_3 &= \alpha_1(a_{11}w_1 + a_{21}w_2) + \alpha_2(a_{12}w_1 + a_{22}w_2) + \alpha_3(a_{13}w_1 + a_{23}w_2) \\ &= (\alpha_1 a_{11} + \alpha_2 a_{12} + \alpha_3 a_{13})w_1 + (\alpha_1 a_{21} + \alpha_2 a_{22} + \alpha_3 a_{23})w_2 = 0. \end{aligned}$$

This contradicts the linear independence of (v_1, v_2, v_3). ■

28.5 Problem. Try doing the proof of the $m = 2$, $n = 3$ case above by writing w_1 and w_2 as linear combinations of v_1, v_2, and v_3. Can you still find a contradiction with the analogue of A? If not, where do you get stuck/what goes wrong?

The following result is central to a well-defined notion of dimension for vector spaces.

28.6 Corollary. *Let* $(v_1, \ldots, v_n)$ *and* $(w_1, \ldots, w_m)$ *be linearly independent lists in* $\mathcal{V}$ *with the same span:* $\text{span}(v_1, \ldots, v_n) = \text{span}(w_1, \ldots, w_m)$. *Then* $m = n$.

Proof. The counting lemma implies both $m \le n$ and $n \le m$, thus $m = n$. (We have $=$ for real numbers if and only if both $\le$ and $\ge$, just like we have $=$ for sets if and only if both $\subseteq$ and $\supseteq$. Both kinds of if and only if statements show up here: the numerical equality of m and n, the set-theoretic equality of $\text{span}(v_1, \ldots, v_n)$ and $\text{span}(w_1, \ldots, w_m)$.) ■

28.7 Problem. If (v_1, v_2) and (w_1, w_2) are linearly independent lists with the same span in a vector space $\mathcal{V}$, must we have $\{v_1, v_2\} = \{w_1, w_2\}$?

At last we are ready for dimension and basis.

28.8 Definition. *A vector space* $\mathcal{V} \neq \{0\}$ *is* **FINITE-DIMENSIONAL** *if there exists a linearly independent list* $(v_1, \ldots, v_n)$ *in* $\mathcal{V}$ *such that* $\mathcal{V} = \text{span}(v_1, \ldots, v_n)$. *The* **DIMENSION** *of* $\mathcal{V}$ *is the integer* $\dim[\mathcal{V}] := n$. *We define* $\dim[\{0\}] := 0$.

The definition of dimension certainly does not depend on the choice of spanning linearly independent list, for if $\mathcal{V} = \text{span}(v_1, \ldots, v_n)$ and $\mathcal{V} = \text{span}(w_1, \ldots, w_m)$ with the two lists here linearly independent, then the (corollary to the) counting lemma implies $m = n$.

28.9 Problem. Explain why the following paraphrase of the above is true: if $\mathcal{V}$ is a finite-dimensional vector space, then any linearly independent list that spans $\mathcal{V}$ has length $\dim[\mathcal{V}]$.

Thus the dimension of $\mathcal{V}$ tells us the minimal amount of data that we need to describe $\mathcal{V}$ completely: $\dim[\mathcal{V}]$ vectors. Of course, we could be inefficient and use more vectors than necessary.

28.10 Problem. We can weaken the definition of finite-dimensional to say that $\mathcal{V}$ is finite-dimensional if and only if $\mathcal{V} = \text{span}(v_1, \ldots, v_m)$ for some list $(v_1, \ldots, v_m)$ in $\mathcal{V}$. Prove that this is true, but also show that any list in $\mathcal{V}$ of length greater than $\dim[\mathcal{V}]$ is linearly dependent. [Hint: *use the reduction and counting lemmas.*]

But the most efficient way to describe a finite-dimensional space is via a basis.

28.11 Definition. *A* **BASIS** *for a finite-dimensional vector space* $\mathcal{V} \neq \{0\}$ *is a linearly independent spanning list: a list* $(v_1, \ldots, v_n)$ *that is linearly independent with* $\mathcal{V} = \text{span}(v_1, \ldots, v_n)$. *Necessarily* $n = \dim[\mathcal{V}]$.

28.12 Example. **(i)** The most famous finite-dimensional space is $\mathbb{F}^n$: we have $\mathbb{F}^n = \text{span}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$, and certainly this list is linearly independent, thus $\dim[\mathbb{F}^n] = n$.

(ii) Let E_{ij} be the $m \times n$ matrix whose (i, j)-entry is 1 and that has 0 in all other entries. (This is an unfortunate clash of notation with elimination matrices.) Then the list of these E_{ij} is linearly independent in $\mathbb{F}^{m \times n}$ and spans $\mathbb{F}^{m \times n}$. (This is easier to say in words than in mathspeak, which might require a doubly indexed finite sum over i and j.) There are mn such matrices, so $\dim[\mathbb{F}^{m \times n}] = mn$.

(iii) Let $A \in \mathbb{F}^{m \times n}$. Then $\text{col}(A)$ is the span of the pivot columns of A, and the pivot columns of A are linearly independent. (Or, more precisely, and annoyingly, any list in $\mathbb{F}^m$ whose entries are exactly the pivot columns of A with no repetitions in the list is linearly independent.) Thus the list of pivot columns of A form a basis for $\text{col}(A)$, and so the number of pivot columns of A is the dimension of $\text{col}(A)$.

(iv) Let $\mathbb{P}_n(\mathbb{F})$ denote the set of all polynomials of degree at most $n \geq 0$ with coefficients in $\mathbb{F}$. Any such polynomial has the form $p(x) = \sum_{k=0}^{n} a_k x^k$, $a_k \in \mathbb{F}$, so $\mathbb{P}_n(\mathbb{F}) = \text{span}(f_0, \ldots, f_n)$ with $f_k(x) := x^k$ for $k \geq 0$.

There are several proofs of the linear independence of $(f_0, \ldots, f_n)$; here is one. For $k \geq 1$, we have $f_k'(x) = kx^{k-1}$, and that almost looks like a scalar multiple of f_k. Indeed, $x^k = xx^{k-1} = xf_k'(x)/k$, so $xf_k'(x) = kx^k = kf_k(x)$. Now let $\mathcal{T}$ be the linear operator on $\mathcal{C}^\infty(\mathbb{R})$ such that $(\mathcal{T}f)(x) := xf'(x)$. Then $\mathcal{T}f_k = kf_k$, and so each f_k is an eigenvector of $\mathcal{T}$ with eigenvalue k. Furthermore, $(\mathcal{T}f_0)(x) = xf_0'(x) = 0$, so f_0 is still an eigenvector of $\mathcal{T}$ with eigenvalue 0. Thus $(f_0, \ldots, f_n)$ is a list of eigenvectors of $\mathcal{T}$ corresponding to distinct eigenvalues, and so it is linearly independent.

Strictly speaking, this is linear independence in $\mathcal{C}^\infty(\mathbb{R})$, but $\mathbb{P}_n(\mathbb{F})$ is a subspace of $\mathcal{C}^\infty(\mathbb{R})$. Considering a list $(v_1, \ldots, v_n)$ as a list in a vector space $\mathcal{V}$ or in a subspace $\mathcal{V}_0$ of $\mathcal{V}$ does not affect its linear (in)dependence—just look at the definition (also, the linear combination $\sum_{k=1}^{n} \alpha_k v_k$ is an element of both $\mathcal{V}$ and $\mathcal{V}_0$).

28.13 Problem. Check explicitly that $(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$ as defined in part (ii) of Example 28.12 is a basis for $\mathbb{F}^{2 \times 3}$.

28.14 Problem. Convince yourself that the following expressions are all euphemisms for a basis.

(i) A linearly independent spanning list

(ii) The longest possible linearly independent list

(iii) The shortest possible spanning list

Day 29: Friday, October 18.

Material from *Linear Algebra* by Meckes & Meckes

Rank and nullity are defined on pp.173 and 175. Read Algorithm 3.33 on p.174 on rank. Theorem 3.35 on p.175 is rank-nullity, and a version of the matrix proof appears on p.176.

Do Quick Exercise #16 on p.173.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Rank of a linear operator or matrix, nullity of a linear operator or matrix

The other side of the "reduction" lemma is the following "extension" lemma.

29.1 Lemma (Extension). *Let $\mathcal{V}$ be a finite-dimensional vector space and let $(v_1, \ldots, v_d)$ be a linearly independent list in $\mathcal{V}$ with $d < \dim[\mathcal{V}] =: n$. Then there exist $v_{d+1}, \ldots, v_n \in \mathcal{V}$ such that $(v_1, \ldots, v_d, v_{d+1}, \ldots, v_n)$ is a basis for $\mathcal{V}$.*

Proof. Since $d < \dim[\mathcal{V}]$, the original list $(v_1, \ldots, v_d)$ cannot be a basis for $\mathcal{V}$ (since all bases have the same length n). Since $(v_1, \ldots, v_d)$ is linearly independent, the only way that it can fail to be a basis for $\mathcal{V}$ is if $\mathcal{V} \neq \operatorname{span}(v_1, \ldots, v_d)$. Thus there is $v_{d+1} \in \mathcal{V} \setminus \operatorname{span}(v_1, \ldots, v_d)$. The spanning lemma then implies that $(v_1, \ldots, v_{d+1})$ is linearly independent. If $d + 1 = n$, then we are done.

Otherwise, $(v_1, \ldots, v_{d+1})$ still cannot be a basis for $\mathcal{V}$; since it is already linearly independent, it cannot span $\mathcal{V}$, and so there is $v_{d+2} \in \mathcal{V} \setminus (v_1, \ldots, v_d, v_{d+1})$. The spanning lemma again implies that $(v_1, \ldots, v_{d+1}, v_{d+2})$ is linearly independent. If $d+2 = n$, then we are done; otherwise, turn the crank. Eventually we will construct $v_{d+1}, \ldots, v_{d+(n-d)} \in \mathcal{V}$ such that $(v_1, \ldots, v_d, v_{d+1}, \ldots, v_n)$ is linearly independent; since this list will have length $n = \dim[\mathcal{V}]$, it must be a basis for $\mathcal{V}$. ∎

29.2 Problem. Let $\mathcal{V}$ be a finite-dimensional vector space with $n := \dim[\mathcal{V}]$.

(i) Prove that any list in $\mathcal{V}$ of length greater than n is linearly dependent.

(ii) Can any list in $\mathcal{V}$ of length greater than or equal to n be reduced to a basis for $\mathcal{V}$?

(iii) Can any list in $\mathcal{V}$ of length less than or equal to n be extended to a basis for $\mathcal{V}$?

29.3 Problem. Let $\mathcal{V}$ be a finite-dimensional vector space and let $\mathcal{U}$ be a subspace of $\mathcal{V}$.

(i) Prove that $\mathcal{U}$ is also finite-dimensional with $\dim[\mathcal{U}] \le \dim[\mathcal{V}]$.

(ii) Prove that $\dim[\mathcal{U}] = \dim[\mathcal{V}]$ if and only if $\mathcal{U} = \mathcal{V}$.

Finite-dimensional vector spaces are, from a certain point of view, discrete objects. They are the spans of finite lists of vectors—of bases—and that finitude makes them tractable. There is only so much data to manage!

The goal of linear algebra is not merely to stay within vector spaces but to move between them via linear operators, and to understand the fundamental problem $\mathcal{T}v = w$. We can understand a lot about this problem when $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$ is linear with one or both of $\mathcal{V}$, $\mathcal{W}$ finite-dimensional. First we focus on the case when $\mathcal{V}$ is finite-dimensional.

The two most important subspaces associated with $\mathcal{T}$ are its kernel and range:

$$\ker(\mathcal{T}) = \{v \in \mathcal{V} \mid \mathcal{T}v = 0\} \quad \text{and} \quad \mathcal{T}(\mathcal{V}) = \{\mathcal{T}v \mid v \in \mathcal{V}\}.$$

To solve $\mathcal{T}v = w$ "as uniquely as possible" for "as many w as possible," we would like $\ker(\mathcal{T})$ to be small and $\mathcal{T}(\mathcal{V})$ to be large. Since size of vector spaces is really measured via dimension, not counting elements, we would like the dimension of $\ker(\mathcal{T})$ to be small and the dimension of $\mathcal{T}(\mathcal{V})$ to be large. Remarkably, these dimensions are closely related.

29.4 Theorem (Rank–nullity for linear operators). *Let $\mathcal{V}$ be a finite-dimensional vector space and $\mathcal{W}$ be any vector space. Let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. Then*

$$\dim[\ker(\mathcal{T})] + \dim[\mathcal{T}(\mathcal{V})] = \dim[\mathcal{V}].$$

The dimension $\dim[\mathcal{T}(\mathcal{V})]$ is sometimes called the RANK of $\mathcal{T}$, and the dimension $\dim[\ker(\mathcal{T})]$ is sometimes called the NULLITY of $\mathcal{T}$, although that may be a little more old-fashioned than rank (and rank may be more commonly used for matrices than for arbitrary linear operators).

29.5 Problem. Assuming the rank–nullity theorem for linear operators to be true, show that the image of any finite-dimensional vector space under a linear operator is finite-dimensional with dimension at most that of the domain space.

29.6 Problem. Here is a slightly sharper version of the previous problem that does not rely on rank–nullity. *In these parts you should not use the rank–nullity theorem at all.*

(i) Let $\mathcal{V}$ be a finite-dimensional vector space and le $(v_1, \ldots, v_n)$ be a basis for $\mathcal{V}$. Let $\mathcal{W}$ be any vector space and let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W}) \setminus \{0\}$. Prove that a sublist of $(\mathcal{T}v_1, \ldots, \mathcal{T}v_n)$ is a basis for $\mathcal{T}(\mathcal{V})$, and conclude that $\mathcal{T}(\mathcal{V})$ is finite-dimensional with $1 \le \dim[\mathcal{T}(\mathcal{V})] \le \dim[\mathcal{V}]$.

(ii) Maintain the assumptions of the previous part but now suppose as well that $\mathcal{T}$ is injective. Prove that $\dim[\mathcal{T}(\mathcal{V})] = \dim[\mathcal{V}]$. What is a basis for $\mathcal{T}(\mathcal{V})$?

There is also a version of the rank–nullity theorem for matrices, which follows from the theorem by considering matrix-vector multiplication as a linear operator, but which can be proved independently using the RREF (conversely, the operator version can be proved from the matrix version by considering matrix representations of linear operators on arbitrary finite-dimensional spaces—a topic that we will take up later).

We do that first.

29.7 Theorem (Rank–nullity for matrices). *Let $A \in \mathbb{F}^{m \times n}$. Then*

$$\dim[\ker(A)] + \dim[\text{col}(A)] = n.$$

In particular, if the number of pivot columns of A is r, then $\dim[\text{col}(A)] = r$, *and we call* $\dim[\text{col}(A)]$ *the* **RANK** *of A.*

Most of the proof of this theorem was done in the challenging calculation that led to Theorem 24.2. We have already argued in part (iii) of Example 28.12 that $\dim[\text{col}(A)]$ is the number of pivot columns of A. First suppose that A has n pivot columns, i.e., all of the columns of A are pivot columns. (By the way, since there are at most $\min\{m, n\}$ pivot columns, this presumes $n \le m$.) Then part (v) of Example 25.6 tells us that $\ker(A)$ is trivial, i.e., $\ker(A) = \{\mathbf{0}_n\}$, thus $\dim[\ker(A)] = 0$, and so

$$\dim[\ker(A)] + \dim[\text{col}(A)] = 0 + n = n.$$

Now suppose that A has $r < n$ pivot columns. Part (iii) of Example 28.12 still shows that $\dim[\text{col}(A)] = r$. As for the kernel, Theorem 24.2 implies that

$$\ker(A) = \text{col}\left(P^{-1}\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}\right)$$

for some invertible $P \in \mathbb{F}^{n \times n}$ and some $F \in \mathbb{F}^{r \times (n-r)}$. We claim that the factor of P^{-1} is irrelevant for dimension counting.

29.8 Problem. Let $B \in \mathbb{F}^{m \times m}$ be invertible and let $C \in \mathbb{F}^{m \times n}$. Prove that $\dim[\text{col}(BC)] = \dim[\text{col}(C)]$.

Thus

$$\dim[\ker(A)] = \dim\left[\text{col}\left(\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}\right)\right].$$

We claim that the list of columns of the matrix on the right is linearly independent; since there are $n - r$ columns, the dimension of the column space on the right will be $n - r$. That will show $\dim[\ker(A)] = n - r$, and then we will have

$$\dim[\ker(A)] + \dim[\text{col}(A)] = (n - r) + r = n.$$

We check linear independence: suppose

$$\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}\mathbf{z} = \mathbf{0}_n$$

for some $\mathbf{z} \in \mathbb{F}^{n-r}$, then

$$\begin{bmatrix} -F\mathbf{z} \\ \mathbf{z} \end{bmatrix} = \mathbf{0}_n,$$

and comparing the bottom $n - r$ rows of each side shows $\mathbf{z} = \mathbf{0}_{n-r}$. This is linear independence.

Day 30: Monday, October 21.

Material from *Linear Algebra* by Meckes & Meckes

The operator-theoretic version of rank–nullity is proved on pp.178–179. Work through the proof of Corollary 3.37 for extra practice. Read the remarks on pp.180–181 on linear systems and geometry (this relates to some questions in class).

Do Quick Exercise #21 on p.179.

30.1 Example. We study the column and null spaces of

$$A := \begin{bmatrix} 1 & 2 & 1 & 7 & 0 \\ 2 & 4 & 2 & 14 & 0 \\ 0 & 0 & 2 & 8 & 0 \end{bmatrix}$$

to see the rank–nullity theorem for matrices in action. First, it should not surprise anyone that

$$\mathsf{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The fifth column is now present to make life interesting.

First we work with the kernel. We have $A\mathbf{x} = \mathbf{0}_3$ if and only if $\mathsf{rref}(A)\mathbf{x} = \mathbf{0}_3$ (why?), and $\mathsf{rref}(A)\mathbf{x} = \mathbf{0}_3$ is equivalent to

$$\left\{ \begin{array}{ccccc} x_1 & +2x_2 & & +3x_4 & = 0 \\ & & x_3 & +4x_4 & = 0 \\ & & & 0 & = 0. \end{array} \right\}$$

We follow tradition and express the "pivot variables" x_1 and x_3 as linear combinations of the "free variables" x_2, x_3, and x_5 (the last not actually appearing). Thus $A\mathbf{x} = \mathbf{0}_3$ if and only if

$$x_1 = -2x_2 - 3x_4 \quad \text{and} \quad x_3 = -4x_4,$$

so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

That is,

$$\ker(A) = \operatorname{span}\left(\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right) = \operatorname{col}\left(\begin{bmatrix}-2 & -3 & 0\\1 & 0 & 0\\0 & -4 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right).$$

The three vectors (columns) above are linearly independent; one can check this directly (do it), or we could write

$$P\begin{bmatrix}-2 & -3 & 0\\1 & 0 & 0\\0 & -4 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\-2 & -3 & 0\\0 & -4 & 0\end{bmatrix}$$

for some permutation matrix $P \in \mathbb{R}^{5\times 5}$. (Be careful: this P will involve more than two row swaps, and so $P \neq P_{ij}$ for any pair of i and j.) Then perhaps it is even easier to see that the columns of the matrix on the right are linearly independent; multiplying by the invertible P does not change that. Then $\dim[\ker(A)] = 3$, also known as $5 - 2$, where 2 is the number of pivot columns of A.

Moreover, we have

$$\mathsf{rref}(A) = \begin{bmatrix}I_2 & F\\0 & 0\end{bmatrix}P, \qquad F := \begin{bmatrix}2 & 3 & 0\\0 & 4 & 0\end{bmatrix}. \tag{30.1}$$

Thus

$$\ker(A) = \operatorname{col}\left(\begin{bmatrix}-F\\I_{5-2}\end{bmatrix}\right).$$

This is a concrete illustration of the very abstract Theorem 24.2 (which maybe we should have illustrated concretely much earlier).

Onwards to the column space. We first claim that $\operatorname{col}(A) \neq \operatorname{col}(\mathsf{rref}(A))$ here and in general. The first and third columns of A are the pivot columns, so

$$\operatorname{col}(A) = \operatorname{span}\left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right),$$

and so $\dim[\operatorname{col}(A)] = 2$. We conclude

$$\dim[\ker(A)] + \dim[\operatorname{col}(A)] = 3 + 2 = 5,$$

which is the number of columns of A. All is as it should be.

30.2 Problem. Fill in some of the details from the previous example.

(i) What is P?

(ii) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ be injective. If $(v_1, \ldots, v_n)$ is linearly independent in $\mathcal{V}$, prove that $(\mathcal{T}v_1, \ldots, \mathcal{T}v_n)$ is linearly independent in $\mathcal{W}$. How does this explain the claim "multiplying by the invertible P does not change that"?

(iii) Check the equality (30.1).

(iv) With A as above, explain why $\text{col}(A) \neq \text{col}(\mathsf{rref}(A))$.

Now we prove the rank–nullity theorem for linear operators. We have $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$ linear with $\mathcal{V}$ finite-dimensional.

30.3 Problem. What happens if $\mathcal{V} = \{0\}$?

Assume that $n := \dim[\mathcal{V}] \geq 1$ and let $d := \dim[\ker(\mathcal{T})]$. By Problem 29.3, $0 \leq d \leq n$.

30.4 Problem. **(i)** If $d = 0$, use part (ii) of Problem 30.2 to show that $\dim[\mathcal{T}(\mathcal{V})] = n$, and so we are done.

(ii) If $d = n$, explain why $\mathcal{T}$ is the zero operator and so $\dim[\mathcal{T}(V)] = 0$, and so we are done.

From now on we assume $1 \leq d < n$. Let $(v_1, \ldots, v_d)$ be a basis for $\ker(\mathcal{T})$.

Use the extension lemma to extend $(v_1, \ldots, v_d)$ to a basis $(v_1, \ldots, v_d, v_{d+1}, \ldots, v_n)$ for $\mathcal{V}$. We will show that $(\mathcal{T}v_{d+1}, \ldots, \mathcal{T}v_n)$ is a basis for $\mathcal{T}(\mathcal{V})$; there are $n - d$ vectors in this list, and so that will give $\dim[\mathcal{T}(V)] = n - d$, thus

$$\dim[\ker(\mathcal{T})] + \dim[\mathcal{T}(\mathcal{V})] = d + (n - d) = d.$$

First we show $\text{span}(\mathcal{T}v_{d+1}, \ldots, \mathcal{T}v_n) = \mathcal{T}(\mathcal{V})$. Certainly the span is contained in the image, so let $w \in \mathcal{T}(\mathcal{V})$. Then $w = \mathcal{T}v$ for some $v \in \mathcal{V}$; write

$$v = \sum_{k=1}^{d} \alpha_k v_k + \sum_{k=d+1}^{n} \beta_k v_k$$

for some α_k, $\beta_k \in \mathbb{F}$, so

$$w = \mathcal{T}v = \sum_{k=1}^{d} \alpha_k \mathcal{T}v_k + \sum_{k=d+1}^{n} \beta_k \mathcal{T}v_k = \sum_{k=d+1}^{n} \beta_k \mathcal{T}v_k,$$

since $\mathcal{T}v_k = 0$ as $v_k \in \ker(\mathcal{T})$ for each k. Thus we have the desired equality.

Now we check linear independence of $(\mathcal{T}v_{d+1}, \ldots, \mathcal{T}v_n)$. Suppose $\sum_{k=d+1}^{n} \gamma_k \mathcal{T}v_k = 0$ for some $\gamma_k \in \mathbb{F}$. We want to show $\gamma_k = 0$ for all k.

Linearity of $\mathcal{T}$ implies $\mathcal{T}\sum_{k=d+1}^{n} \gamma_k v_k = 0$, so $\sum_{k=d+1}^{n} \gamma_k v_k \in \ker(\mathcal{T})$. Now write

$$\sum_{k=d+1}^{n} \gamma_k v_k = \sum_{k=1}^{d} \delta_k v_k$$

for some $\delta_k \in \mathbb{F}$; this is possible because $(v_1, \ldots, v_d)$ is a basis for $\ker(\mathcal{T})$. Rearrange this to

$$\sum_{k=1}^{d} \delta_k v_k + \sum_{k=d+1}^{n} (-\gamma_k) v_k = 0,$$

so $\delta_k = \gamma_k = 0$ for all k by the linear independence of $(v_1, \ldots, v_d, v_{d+1}, \ldots, v_n)$. This completes the proof of operator rank–nullity.

30.5 Problem. All of this might feel a bit backward from the proof of rank-nullity for matrices. There, everything hinged on the number r, which was the rank, and thus, effectively, the dimension of the range. Here, everything hinges on the dimension of the kernel.

Try the following approach to proving operator rank–nullity. Let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W}) \setminus \{0\}$, with $\mathcal{V}$ finite-dimensional. Problem 29.6 implies that $\mathcal{T}(\mathcal{V})$ is finite-dimensional with $1 \le \dim[\mathcal{T}(\mathcal{V})] \le \dim[\mathcal{V}]$; recall that this problem can be done without rank–nullity. Let $r = \dim[\mathcal{T}(\mathcal{V})]$ and let $(w_1, \ldots, w_r)$ be a basis for $\mathcal{T}(\mathcal{V})$. Write $w_k = \mathcal{T} v_k$ for some $v_k \in \mathcal{V}$.

(i) Show that $(v_1, \ldots, v_r)$ is linearly independent.

(ii) Extend $(v_1, \ldots, v_r)$ to a basis $(v_1, \ldots, v_r, v_{r+1}, \ldots, v_n)$ for $\mathcal{V}$, where $\dim[\mathcal{V}] = n$. Try to show that $(v_{r+1}, \ldots, v_n)$ is a basis for $\ker(\mathcal{T})$. Explain precisely where you get stuck.

Rank–nullity is the key to some (possibly) surprising connections between operator behavior between finite-dimensional spaces and the dimensions of the domains and codomains. First, when the domain and codomain have the same dimension, injectivity and surjectivity are equivalent—contrary to all of our earlier work (especially our earlier work on vector spaces that were not finite-dimensional, which we will soon imaginatively term infinite-dimensional), we do not have to check both conditions to see if the spaces are isomorphic.

30.6 Theorem. *Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces with* $\dim[\mathcal{V}] = \dim[\mathcal{W}]$. *Let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. The following are equivalent:*

(i) *$\mathcal{T}$ is injective.*

(ii) *$\mathcal{T}$ is surjective.*

(iii) *$\mathcal{T}$ is an isomorphism.*

Proof. Corollary 3.36 in the book. ■

When the domain and codomain are still both finite-dimensional but perhaps not with the same dimension, then some more operator behaviors are possible (but still these behaviors are restricted).

30.7 Problem. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces. Prove the following.

(i) If there exists an injective operator $\mathcal{T}: \mathcal{V} \to \mathcal{W}$, then $\dim[\mathcal{V}] \le \dim[\mathcal{W}]$. Conversely, if $\dim[\mathcal{W}] < \dim[\mathcal{V}]$, then no operator from $\mathcal{V}$ to $\mathcal{W}$ is injective.

(ii) If there exists a surjective operator $\mathcal{T}: \mathcal{V} \to \mathcal{W}$, then $\dim[\mathcal{W}] \le \dim[\mathcal{V}]$. Conversely, if $\dim[\mathcal{V}] < \dim[\mathcal{W}]$, then no operator from $\mathcal{V}$ to $\mathcal{W}$ is surjective.

(iii) If $\mathcal{V}$ and $\mathcal{W}$ are isomorphic, then $\dim[\mathcal{V}] = \dim[\mathcal{W}]$.

The last point of this problem merits further exploration. If $\dim[\mathcal{V}] = \dim[\mathcal{W}]$, are $\mathcal{V}$ and $\mathcal{W}$ isomorphic, and can we construct an *explicit* isomorphism? Yes and yes.

Day 31: Wednesday, October 23.

Material from *Linear Algebra* by Meckes & Meckes

Theorem 3.14 on p.155 constructs a linear operator by extension from a basis. Work through the proof of Theorem 3.15 for extra practice and read Corollary 3.16.

Do Quick Exercise #9 on p.156 and #10 on p.158.

The following is an example of how bases contain all the necessary data for finite-dimensional vector spaces—not merely for building vectors within a space but also for operators mapping between them.

31.1 Theorem. *Let $\mathcal{V}$ be a finite-dimensional vector space with basis $(v_1, \ldots, v_n)$, and let $\mathcal{W}$ be a vector space with $w_1, \ldots, w_n \in \mathcal{W}$. There exists a unique linear operator $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ such that $\mathcal{T} v_k = w_k$.*

Proof. Let $v \in \mathcal{V}$; we need to figure out how to define $\mathcal{T} v$. All we know is that $(v_1, \ldots, v_n)$ is a basis for $\mathcal{V}$ and how $\mathcal{T}$ acts on this basis. If we expand $v = \sum_{k=1}^n \alpha_k v_k$ for some $\alpha_k \in \mathbb{F}$, we would want $\mathcal{T} v = \sum_{k=1}^n \alpha_k \mathcal{T} v_k$ by linearity. But $\mathcal{T} v_k = w_k$, so this suggests $\mathcal{T} v = \sum_{k=1}^n \alpha_k w_k$.

Is this really a good definition of $\mathcal{T}$? First, there is no ambiguity: we have reduced v to its coefficient data α_k, and there is only one way to select those coefficients for v. That is, $\mathcal{T}$ is "well-defined" as a map from $\mathcal{V}$ to $\mathcal{W}$.

Next, is $\mathcal{T}$ linear? We need to check that $\mathcal{T}(u+v) = \mathcal{T} u + \mathcal{T} v$ and $\mathcal{T}(\alpha v) = \alpha \mathcal{T} v$ for all u, $v \in \mathcal{V}$ and $\alpha \in \mathbb{F}$. (Writing u and v instead of v and w might keep us from overworking w here.) If $v = \sum_{k=1}^n \beta_k v_k$ (where now we are writing β to avoid overworking α), then $\alpha v = \sum_{k=1}^n \alpha \beta_k v_k$, and so

$$\mathcal{T}(\alpha v) = \sum_{k=1}^n \alpha \beta_k w_k = \alpha \sum_{k=1}^n \beta_k w_k = \alpha \mathcal{T} v.$$

Last, is $\mathcal{T}$ unique? Suppose there is an operator $\mathcal{S}: \mathcal{V} \to \mathcal{W}$ such that $\mathcal{S} v_k = w_k$ for all k. We want to show $\mathcal{T} = \mathcal{S}$, which means we need to show $\mathcal{T} v = \mathcal{S} v$ for all $v \in \mathcal{V}$. Expand, once more, $v = \sum_{k=1}^{n} \alpha_k v_k$, so

$$\mathcal{T}v = \sum_{k=1}^{n} \alpha_k w_k = \sum_{k=1}^{n} \alpha_k \mathcal{S} v_k = \mathcal{S} \sum_{k=1}^{n} \alpha_k v_k = \mathcal{S}v.$$

The first equality is the definition of $\mathcal{T}$; the second is the hypothesis on $\mathcal{S}$; and the third is the linearity of $\mathcal{S}$. ■

31.2 Problem. Finish the proof above by checking that $\mathcal{T}(u+v) = \mathcal{T}u + \mathcal{T}v$.

31.3 Problem. Make precise and then prove the following statements.

(i) If two linear operators defined on the same finite-dimensional vector space agree on a basis for that space, then those operators are the same.

(ii) If an operator is 0 on every vector in a basis for a finite-dimensional space, then that operator is the zero operator.

Now here is the result that we actually wanted.

31.4 Problem. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces with $\dim[\mathcal{V}] = \dim[\mathcal{W}]$. Show that $\mathcal{V}$ and $\mathcal{W}$ are isomorphic. [Hint: *if* $(v_1, \ldots, v_n)$ *and* $(w_1, \ldots, w_n)$ *are bases for* $\mathcal{V}$ *and* $\mathcal{W}$*, respectively, what is the only natural choice for an isomorphism from* $\mathcal{V}$ *to* $\mathcal{W}$ *that talks nicely to the bases?*]

31.5 Problem. Let $\mathcal{V}$ be a finite-dimensional vector space with basis $(v_1, \ldots, v_n)$. Recall that $\mathcal{V}' := \mathbf{L}(\mathcal{V}, \mathbb{F})$ is the **(ALGEBRAIC) DUAL SPACE** of $\mathcal{V}$, and that its elements are called **LINEAR FUNCTIONALS** on $\mathcal{V}$. Prove that there exists a so-called **DUAL BASIS** for $\mathcal{V}'$ of the form $(\varphi_1, \ldots, \varphi_n)$ satisfying

$$\varphi_j(v_k) = \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases}$$

[Hint: *use Theorem 31.1 to construct each* φ_j *first; then show that* $(\varphi_1, \ldots, \varphi_n)$ *is linearly independent and spans* $\mathcal{V}'$.]

Day 32: Friday, October 25.

You took Exam 2.

Day 33: Monday, October 28.

Material from *Linear Algebra* by Meckes & Meckes

See the example on pp.216–217 for an operator that has no real eigenvalues. Read pp.217–218 on operator polynomials and the example on p.219, which offers another concrete version of the "algorithm for eigenvalue existence."

Do Quick Exercise #32 on p.219.

We have been studying the interaction of operators and dimension. Since the fundamental problem of linear algebra is understanding the equation $\mathcal{T}v = w$ with $v \in \mathcal{V}$, $w \in \mathcal{W}$, and $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ for some vector spaces $\mathcal{V}$ and $\mathcal{W}$, and since dimension is *the* key structural feature of finite-dimensional vector spaces, it is natural to ask what further information dimension counting provides about that fundamental problem. We have all the answers:

- If $\dim[\mathcal{V}] = \dim[\mathcal{W}]$, then $\mathcal{T}$ is injective if and only if $\mathcal{T}$ is surjective. That is, existence of a solution to $\mathcal{T}v = w$ is equivalent to uniqueness of solutions.

- If $\dim[\mathcal{V}] < \dim[\mathcal{W}]$, then $\mathcal{T}$ cannot be surjective (and $\mathcal{T}$ may or may not be injective). That is, we definitely cannot solve $\mathcal{T}v = w$ for all $w \in \mathcal{W}$.

- If $\dim[\mathcal{V}] > \dim[\mathcal{W}]$, then $\mathcal{T}$ cannot be injective (and $\mathcal{T}$ may or may not be surjective). That is, we definitely cannot solve $\mathcal{T}v = w$ uniquely no matter the choice of $w \in \mathcal{W}$.

Another natural question is how many ways we can pose the problem $\mathcal{T}v = w$ from the point of view of $\mathcal{T}$: if $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional, is $\mathbf{L}(\mathcal{V}, \mathcal{W})$ finite-dimensional and, if so, can we compute its dimension in terms of $\dim[\mathcal{V}]$ and $\dim[\mathcal{W}]$? We have almost all the tools that we need. First, put $n := \dim[\mathcal{V}]$ and $m := \dim[\mathcal{W}]$. We know that $\mathcal{V}$ and $\mathbb{F}^n$ are isomorphic (Problem 31.4), as are $\mathcal{W}$ and $\mathbb{F}^m$. We also know that $\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ and $\mathbb{F}^{m \times n}$ are isomorphic (Theorem 13.6) and that $\dim[\mathbb{F}^{m \times n}] = mn$. Thus $\dim[\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)]$ are isomorphic. We are done if $\mathbf{L}(\mathcal{V}, \mathcal{W})$ and $\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ are isomorphic, which would give $\dim[\mathbf{L}(\mathcal{V}, \mathcal{W})] = mn$.

Here is the deeper truth, which does not rely on any dimension counting at all.

33.1 Theorem. *Suppose that $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ are isomorphic vector spaces, and that $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are also isomorphic vector spaces. Then $\mathbf{L}(\mathcal{V}, \mathcal{W})$ and $\mathbf{L}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}})$ are isomorphic.*

Proof. Let $\mathcal{T} \colon \mathcal{V} \to \widetilde{\mathcal{V}}$ and $\mathcal{S} \colon \mathcal{W} \to \widetilde{\mathcal{W}}$ be isomorphisms. Fix $\mathcal{A} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. How might we pair $\mathcal{A}$ with an operator in $\mathbf{L}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}})$ in a "meaningful" way? First, how do we connect the domain and codomain of $\mathcal{A}$ to the domain and codomain of an operator in $\mathbf{L}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}})$. The following cartoon tells the story:

$$\widetilde{\mathcal{V}} \xrightarrow{\mathcal{T}^{-1}} \mathcal{V} \xrightarrow{\mathcal{A}} \mathcal{W} \xrightarrow{\mathcal{S}} \widetilde{\mathcal{W}}. \tag{33.1}$$

We therefore try defining a map

$$\Phi\colon \mathbf{L}(\mathcal{V},\mathcal{W}) \to \mathbf{L}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}})\colon \mathcal{A} \mapsto \mathcal{S}\mathcal{A}\mathcal{T}^{-1}$$

and we will check if this is an isomorphism.

First, Φ is linear thanks to composition properties of linear operators:

$$\Phi(\alpha\mathcal{A}) = \mathcal{S}(\alpha\mathcal{A})\mathcal{T}^{-1} = \alpha(\mathcal{S}\mathcal{A}\mathcal{T}^{-1}) = \alpha\Phi\mathcal{A}$$

and

$$\Phi(\mathcal{A}_1 + \mathcal{A}_2) = \mathcal{S}(\mathcal{A}_1 + \mathcal{A}_2)\mathcal{T}^{-1} = \mathcal{S}\mathcal{A}_1\mathcal{T}^{-1} + \mathcal{S}\mathcal{A}_2\mathcal{T}^{-1} = \Phi\mathcal{A}_1 + \Phi\mathcal{A}_2.$$

Next, we check injectivity: if $\mathcal{S}\mathcal{A}\mathcal{T}^{-1} = 0$, where here 0 is the zero operator in $\mathbf{L}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}})$, then

$$\mathcal{S}^{-1}\mathcal{S}\mathcal{A}\mathcal{T}^{-1}\mathcal{T} = \mathcal{S}^{-1}0\mathcal{T},$$

and so

$$\mathcal{A} = 0,$$

since composition with the zero operator always returns the zero operator.

Finally, we check surjectivity. Let $\widetilde{\mathcal{A}} \in \mathbf{L}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}})$. We want to find $\mathcal{A} \in \mathbf{L}(\mathcal{V},\mathcal{W})$ such that $\mathcal{S}\mathcal{A}\mathcal{T}^{-1} = \widetilde{\mathcal{A}}$. This is only possible if $\mathcal{A} = \mathcal{S}^{-1}\widetilde{\mathcal{A}}\mathcal{T}$. ■

33.2 Problem. Using the notation above, check that $\mathcal{S}^{-1}\widetilde{\mathcal{A}}\mathcal{T} \in \mathbf{L}(\mathcal{V},\mathcal{W})$. [Hint: *a cartoon like* (33.1) *might help.*]

So far we have not really assumed $\mathcal{V} = \mathcal{W}$ in connecting operator theory to finite-dimensionality. If we do this, we get a powerful result about eigenvalues. Recall that (1) eigenvalues tell us where operators act as simply as possible (as scalar multiplication) and (2) an operator may not have any eigenvalues (Example 8.2). The latter does not occur over finite-dimensional vector spaces *when the underlying field is* $\mathbb{C}$—this is one of the rare times that we have to specify $\mathbb{F} = \mathbb{C}$.

To prove this, we need a notational trick. Let $\mathcal{V}$ be a vector space (not necessarily finite-dimensional right now) over $\mathbb{F}$ and $\mathcal{T} \in \mathbf{L}(\mathcal{V})$. We define nonnegative integer powers of $\mathcal{T}$ as

$$\mathcal{T}^k := \begin{cases} \mathbb{1}_{\mathcal{V}}, & k = 0 \\ \mathcal{T}, & k = 1 \\ \mathcal{T}(\mathcal{T}^{k-1}), & k \geq 2. \end{cases}$$

Then for a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ with coefficients $a_k \in \mathbb{F}$, we define

$$p(T) := \sum_{k=0}^{n} a_k \mathcal{T}^k.$$

That is, $p(T) \in \mathbf{L}(\mathcal{V})$ is the "operator polynomial" such that

$$p(T)v = \sum_{k=0}^{n} a_k \mathcal{T}^k v.$$

For example, if $p(z) = z^2 + 1$, then $p(T) = T^2 + I$, with $Iv = v$ as the identity operator on $\mathcal{V}$. In the particular case of $\mathcal{V} = \mathcal{C}^\infty(\mathbb{R})$ and $\mathcal{T}f := f'$, we have $p(\mathcal{T})f = f'' + f$, one of the central players in an ODE class.

33.3 Problem. Let $\mathbb{P}(\mathbb{F})$ denote the vector space of all polynomials with coefficients in $\mathbb{F}$. Let $\mathcal{V}$ be any vector space over $\mathbb{F}$ and fix $T \in \mathcal{L}(V)$. Show that the map

$$P_T \colon \mathbb{P}(\mathbb{F}) \to \mathcal{L}(\mathcal{V}) \colon p \mapsto p(T)$$

is linear.

There is another useful way to express polynomials, and that nicely carries over to operator polynomials. Here we need product notation: if $w_1, \ldots, w_n \in \mathbb{C}$, then

$$\prod_{j=1}^n w_j := \begin{cases} w_1, & n = 1 \\ \left(\prod_{j=1}^{n-1} w_j\right) w_n, & n \geq 2. \end{cases}$$

With this notation, we state the fundamental theorem of algebra: every polynomial with complex coefficients factors into a product of linear factors with complex coefficients.

33.4 Theorem (Fundamental theorem of algebra). *Let $p(z) = \sum_{k=0}^n a_k z^k$ be a polynomial of degree n with coefficients in $\mathbb{C}$: $a_k \in \mathbb{C}$, $a_n \neq 0$. Then there exist (not necessarily distinct) $z_1, \ldots, z_n \in \mathbb{C}$ such that*

$$p(z) = a_n \prod_{j=1}^n (z - z_j).$$

For example, $z^2+1 = (z+i)(z-i)$. Thus every polynomial p has (at least) two expressions: the Taylor expansion $p(z) = \sum_{k=0}^n a_k z^k$ and the factored form above. *The key difference is that even though all of the coefficients a_k may be real, some or all of the roots z_j may be complex.* Just consider $p(z) = z^2 + 1$.

Now, the operator product $a_n \prod_{j=1}^n (\mathcal{T} - z_j I)$ certainly makes sense: if $\mathcal{S}_1, \ldots, \mathcal{S}_n \in \mathbf{L}(\mathcal{V})$, then

$$\prod_{j=1}^n \mathcal{S}_j := \begin{cases} \mathcal{S}_1, & n = 1 \\ \left(\prod_{j=1}^{n-1} \mathcal{S}_j\right), & n \geq 2. \end{cases}$$

So, if $p(z) = \sum_{k=0}^n a_k z^k$ factors as $p(z) = a_n \prod_{j=1}^n (z - z_j)$, do we have

$$p(T) = a_n \prod_{j=1}^n (\mathcal{T} - z_j I)$$

as well?

33.5 Theorem. *Yes.*

Proof. We induct on n. There is really nothing to do in the $n = 1$ case, as there $p(z) = a_1 z + a_0$ with $a_1 \neq 0$, thus $p(z) = a_1(z - (-a_0/a_1))$ as well. The same algebra shows

$$a_1 \mathcal{T} + a_0 I = a_1 \left(\mathcal{T} - \left(\frac{a_0}{a_1} \right) I \right).$$

Suppose the result is true for some $n \geq 1$, and let p be a polynomial of degree $n+1$:

$$p(z) = \sum_{k=0}^{n+1} a_k z^k = a_{n+1} \prod_{j=1}^{n} (z - z_j).$$

Let

$$q(z) := a_{n+1} \prod_{j=1}^{n} (z - z_j),$$

so q is a polynomial of degree n, and therefore we can write

$$q(z) = \sum_{k=0}^{n} b_k z^k.$$

for some $b_k \in \mathbb{C}$. Then

$$p(z) = q(z)(z - z_{n+1}) = \sum_{k=0}^{n} b_k z^k (z - z_{n+1}) = \sum_{k=0}^{n} (b_k z^{k+1} - b_k z_{n+1} z^k) = \sum_{k=0}^{n} b_k z^{k+1} - \sum_{k=0}^{n} b_k z_{n+1} z^k$$

$$= \sum_{k=1}^{n+1} b_{k-1} z^k - \sum_{k=0}^{n} b_k z_{n+1} z^k = -b_0 z_{n+1} + \sum_{k=1}^{n} (b_{k-1} - b_k z_{n+1}) z^k + b_n z^{n+1}, \quad (33.2)$$

and so, by uniqueness of a polynomial's coefficients,

$$a_k = \begin{cases} -b_0 z_{n+1}, & k = 0 \\ b_{k-1} - b_k z_{n+1}, & 1 \leq k \leq n \\ b_n, & k = n+1. \end{cases} \quad (33.3)$$

The induction hypothesis implies

$$\sum_{k=0}^{n} b_k \mathcal{T}^k = a_{n+1} \prod_{j=1}^{n} (\mathcal{T} - z_j I),$$

and so

$$a_{n+1} \prod_{j=1}^{n+1} (\mathcal{T} - z_j I) = \sum_{k=0}^{n} b_k \mathcal{T}^k (\mathcal{T} - z_j I).$$

The same algebra from (33.2) shows

$$\sum_{k=0}^{n} b_k \mathcal{T}^k (\mathcal{T} - z_j I) = -b_0 z_{n+1} I + \sum_{k=1}^{n} (b_{k-1} - b_k z_{n+1}) \mathcal{T}^k + b_n \mathcal{T}^{n+1} = \sum_{k=0}^{n+1} a_k \mathcal{T}^k,$$

where the last equality is (33.3). ■

33.6 Problem. **(i)** Let $\mathcal{V}$ be a finite-dimensional vector space and let $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of $\mathcal{T}$ if and only if $\mathcal{T} - \lambda I$ is *not* invertible.

(ii) Recall from Example 8.2 that the multiplication operator $\mathcal{T} \colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ has no eigenvalues. (Strictly speaking, we showed in that example that $\mathcal{T}$ has no *real* eigenvalues, but if we broaden our notions of calculus and the space $\mathcal{C}([0,1])$ to allow continuous complex-valued functions, the same argument still shows that no $\lambda \in \mathbb{C}$ satisfies $\mathcal{T}f = \lambda f$ for some nonzero f. This is worth pointing out because Example 8.3 shows that an operator may have only complex, nonreal eigenvalues.) Show that $\mathcal{T} - \lambda I$ is not invertible for $0 \le \lambda \le 1$. [Hint: *if* $(T - \lambda I)f = g$*, what is* $g(\lambda)$*?*] This suggests a generalization of eigenvalue: a "spectral value" for $\mathcal{T}$ is a scalar λ such that $\mathcal{T} - \lambda I$ is not invertible (or, in the context of a normed space, that is invertible with a "badly behaved" inverse).

33.7 Problem (Optional, a little long). **(i)** Let $\mathcal{V}$ be a vector space and $\mathcal{S}, \mathcal{T} \in \mathbf{L}(\mathcal{V})$. Suppose that $\mathcal{S}$ and $\mathcal{T}$ commute: $\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S}$. Prove that $\mathcal{S}\mathcal{T}$ is invertible if and only if both $\mathcal{S}$ and $\mathcal{T}$ are invertible. [Hint: *for any* $\mathcal{A} \in \mathbf{L}(\mathcal{V})$*, we have* $\mathcal{A}\mathcal{S}\mathcal{T} = \mathcal{A}\mathcal{T}\mathcal{S}$ *and* $\mathcal{S}\mathcal{T}\mathcal{A} = \mathcal{T}\mathcal{S}\mathcal{A}$.]

(ii) Let p be a polynomial, $\mathcal{V}$ be a finite-dimensional vector space, and $\mathcal{T} \in \mathbf{L}(\mathcal{V})$. Prove the **POLYNOMIAL SPECTRAL MAPPING THEOREM**: $\lambda \in \mathbb{F}$ is an eigenvalue of $\mathcal{T}$ if and only if $p(\lambda)$ is an eigenvalue of $p(\mathcal{T})$. [Hint: *use Problem 33.6 to show that it suffices to prove that* $\mathcal{T} - \lambda I$ *is invertible if and only if* $p(\mathcal{T}) - p(\lambda)I$ *is invertible. Factor* $p(z) - p(\lambda) = a\prod_{j=1}^{n}(z - z_j)$*, where* $n = \deg(p)$*. Explain why* $z_j = \lambda$ *for at least one* j*. Then explain why if one of the following operators is invertible, the others all are:* $p(\mathcal{T}) - \lambda I$, $\prod_{j=1}^{n}(\mathcal{T} - z_j I)$, $\mathcal{T} - z_j I$ *for any* $1 \le j \le n$.]

Now here is why we care about operator polynomials: they are the key to proving that any linear operator on a finite-dimensional space has an eigenvalue. We have seen examples of operators on infinite-dimensional spaces that do not have eigenvalues (Example 8.2), but this cannot happen on a finite-dimensional space over $\mathbb{C}$. The proof of this fact is an abstraction of the following concrete situation.

33.8 Example. Define

$$\mathcal{T} \colon \mathbb{C}^2 \to \mathbb{C}^2 \colon \mathbf{v} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{v}.$$

We show that $\mathcal{T}$ has eigenvalues (and we do so without determinants).

Here is the trick. The list $(\mathbf{v}, \mathcal{T}\mathbf{v}, \mathcal{T}^2\mathbf{v})$ is linearly dependent in $\mathbb{C}^2$ for any $\mathbf{v} \in \mathbb{C}^2$, since the list has three entries, but $\dim[\mathbb{C}^2] = 2$, of course. For simplicity, we pick $\mathbf{v} = \mathbf{e}_1$, and we compute

$$\mathcal{T}\mathbf{e}_1 = \mathbf{e}_2 \quad \text{and} \quad \mathcal{T}^2\mathbf{e}_1 = \mathcal{T}\mathbf{e}_2 = -\mathbf{e}_1.$$

Then the list is $(\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1)$, and the (hopefully obvious) linear dependence relationship is

$$1\mathbf{e}_1 + 0\mathbf{e}_2 + 1(-\mathbf{e}_1) = \mathbf{0}_2.$$

That is,

$$\mathcal{T}^2\mathbf{e}_1 + I\mathbf{e}_1 = \mathbf{0}_2.$$

Put $p(z) = z^2 + 1$. Then $p(\mathcal{T})\mathbf{e}_1 = \mathbf{0}_2$, and since p factors as $p(z) = (z+i)(z-i)$, this also says that

$$(\mathcal{T} + iI)(\mathcal{T} - iI)\mathbf{e}_1 = \mathbf{0}_2. \tag{33.4}$$

Now we consider cases.

First, if $(\mathcal{T} - iI)\mathbf{e}_1 = \mathbf{0}_2$, then $\mathcal{T}\mathbf{e}_1 = i\mathbf{e}_1$, so $\mathbf{e}_1$ would be an eigenvector of $\mathcal{T}$ corresponding to the eigenvalue i. Second, if $\mathbf{w} := (\mathcal{T} - iI)\mathbf{e}_1 \neq \mathbf{0}_2$, then (33.4) forces $(\mathcal{T} + iI)[(\mathcal{T} - iI)\mathbf{e}_1] = \mathbf{0}_2$. That is, $(\mathcal{T} + iI)\mathbf{w} = \mathbf{0}_2$ and $\mathbf{w} \neq \mathbf{0}_2$, thus $\mathbf{w}$ is an eigenvector of $\mathcal{T}$ corresponding to the eigenvalue $-i$.

33.9 Problem. **(i)** Which is which? Compute $(\mathcal{T} - iI)\mathbf{e}_1$ and decide if it $\mathbf{e}_1$ is an eigenvector corresponding to i, or if $(\mathcal{T} - iI)\mathbf{e}_1$ is an eigenvector corresponding to $-i$.

(ii) Use the approach above to find the other eigenvalue. [Hint: *try* $\mathbf{v} = \mathbf{e}_2$.]

The trick of Example 33.8 generalizes substantially.

33.10 Theorem. *Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{C}$ and $\mathcal{T} \in \mathbf{L}(\mathcal{V})$. Then $\mathcal{T}$ has an eigenvalue: there exist $\lambda \in \mathbb{C}$ and $v \in \mathcal{V} \setminus \{0\}$ such that $\mathcal{T}v = \lambda v$.*

Proof. Proposition 3.66 in the book. ■

33.11 Problem. In the preceding theorem, how many distinct eigenvalues can $\mathcal{T}$ possibly have? We know $\mathcal{T}$ has at least one; here we want an upper bound.

33.12 Problem. Let $(v_1, \ldots, v_n)$ be a basis for the vector space $\mathcal{V}$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ be distinct (i.e., $\lambda_j \neq \lambda_k$ for $j \neq k$). Let $\mathcal{W}$ be a vector space and fix $w_1, \ldots, w_n \in \mathcal{W}$. Define a linear operator $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$ by setting $\mathcal{T}v_k = \lambda_k v_k$ and extending $\mathcal{T}$ to $\mathcal{V}$ by linearity. Prove that the eigenvalues of $\mathcal{T}$ are the scalars $\lambda_1, \ldots, \lambda_n$.

Day 34: Wednesday, October 30.

Material from *Linear Algebra* by Meckes & Meckes

The remark on p.169 offers a very brief perspective on infinite-dimensional spaces. For practice with eigenvalue calculations (without determinants!) read pp.215–216.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Infinite-dimensional vector space (N)—be able to give an example of both a finite-dimensional vector space (of requested dimension n) and an infinite-dimensional vector space

We still have not given any general algorithms for finding eigenvalues; outside of matrix problems, there really are none.

34.1 Example. We claim that the eigenvalues of a diagonal matrix are precisely its diagonal entries. A diagonal matrix D has 0 for all of its off-diagonal entries, i.e., for the (i,j)-entries with $i \neq j$. Such a matrix $D \in \mathbb{F}^{n \times n}$ has the form $D = \begin{bmatrix} d_1\mathbf{e}_1 & \cdots & d_n\mathbf{e}_n \end{bmatrix}$ for some $d_k \in \mathbb{F}$, and so D acts via $D\mathbf{v} = \sum_{k=1}^n v_k d_k \mathbf{e}_k$ for $\mathbf{v} \in \mathbb{F}^n$.

First we show that each d_j is an eigenvalue; we just have to exhibit an eigenvector. Trying $\mathbf{e}_j$ is probably natural, and, indeed, $D\mathbf{e}_j = d_j\mathbf{e}_j$.

Now we show that the *only* eigenvalues are d_j. Suppose that $\lambda \in \mathbb{F}$ is an eigenvalue with eigenvector $\mathbf{v} \in \mathbb{F}^n \setminus \{\mathbf{0}_n\}$. Then $D\mathbf{v} = \lambda\mathbf{v}$, and we expand each side as

$$\sum_{k=1}^n v_k d_k \mathbf{e}_k = \sum_{k=1}^n \lambda v_k \mathbf{e}_k.$$

Thus

$$\sum_{k=1}^n (v_k d_k - \lambda v_k)\mathbf{e}_k = \mathbf{0}_n,$$

so by linear independence of $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$, we have

$$v_k d_k - \lambda v_k = 0$$

for all k. Thus for a given k, either $v_k = 0$ or $d_k - \lambda = 0$. Since $\mathbf{v} \neq \mathbf{0}_n$, we have $v_k \neq 0$ for at least one k, so in at least that case $\lambda = d_k$.

34.2 Problem. Suppose that $A \in \mathbb{F}^{n \times n}$ is UPPER-TRIANGULAR: its (i,j)-entries are 0 when $i > j$, i.e., the entries below the diagonal are all 0.

(i) Prove that A is invertible if and only if no diagonal element is 0. Do not use determinants. [Hint: *to see patterns, first consider a "small" A, say*

$$A = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & d_3 & * \\ 0 & 0 & 0 & d_4 \end{bmatrix},$$

where the precise value of an entry denoted $*$ is irrelevant. Try to solve $A\mathbf{x} = \mathbf{b}$ by "back-substitution": if $d_4 \neq 0$, solve for x_4 in terms of b_4, and then if $d_3 \neq 0$, solve for x_3 in terms of b_3 and b_4, and so on. Explain how this breaks down if one of the diagonal entries is 0. Generalize this to $A \in \mathbb{F}^{n \times n}$ and show that if no diagonal element is 0, then we can always solve $A\mathbf{x} = \mathbf{b}$ uniquely for $\mathbf{x}$ given $\mathbf{b}$, and that is invertibility.

Now suppose a diagonal entry is 0. If $d_1 = 0$, explain why A has a nontrivial kernel element; if $d_4 = 0$, explain why $A\mathbf{x} = \mathbf{b}$ only if $b_4 = 0$. If $d_4 \neq 0$ and $d_3 = 0$ and if we can solve $A\mathbf{x} = \mathbf{b}$, explain how b_3 depends on b_4, and so we cannot solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$. Generalize this to $A \in \mathbb{F}^{n \times n}$: if the $(1,1)$-entry is 0, then A has a nontrivial kernel; if the (n,n)-entry is 0, then $b_n = 0$; and otherwise let j be the largest integer such that the (j,j)-entry of A is 0. Explain how back-substitution implies that b_j depends on $b_{j+1}, \ldots, b_n$.]

(ii) Prove that the eigenvalues of A are precisely the diagonal elements. [Hint: *λ is an eigenvalue if and only if $A - \lambda I_n$ is not invertible.*]

Some of the most interesting and frequently used vector spaces in applications are not finite-dimensional but *infinite*-dimensional.

34.3 Definition. *A vector space $\mathcal{V}$ is* INFINITE-DIMENSIONAL *if it is not finite-dimensional: if $\mathcal{V} \neq \{0\}$ and $\mathcal{V} \neq \operatorname{span}(v_1, \ldots, v_n)$ for any list $(v_1, \ldots, v_n)$ in $\mathcal{V}$.*

Infinite-dimensional vector spaces arise most often as function spaces.

34.4 Example. Our gut feeling is probably that the space of all polynomials with coefficients in $\mathbb{F}$, which we denote by $\mathbb{P}(\mathbb{F})$, is infinite-dimensional. There are just too many degrees possible for it to be finite-dimensional! Here is how this works with Definition 34.3.

First we do something particular. Consider the list $(1, z, 2z, z^2, 3z + 4z^2)$, which we sloppily list just using formulas. It is easy to find a polynomial not in the span of this list: just go one degree higher than any polynomial in the list and use z^3.

Now here is the general situation. Suppose that $(p_1, \ldots, p_d)$ is a list in $\mathbb{P}(\mathbb{F})$. Let $n := \max_{1 \le k \le d} \deg(p_k)$. Put $p(z) := z^{n+1}$. Then $p \notin \operatorname{span}(p_1, \ldots, p_d)$ because any polynomial that is a linear combination of polynomials of degree at most n is itself of degree at most n.

While Definition 34.3 is the natural place to start ("infinite" should mean "not finite"), a negative definition is not necessarily the most helpful. Here is a more positive definition.

34.5 Problem. Prove that a vector space $\mathcal{V}$ is infinite-dimensional if and only if for each integer $n \ge 1$, there is a linearly independent list in $\mathcal{V}$ of length n. [Hint: *contrapositive, maybe contradiction?*]

34.6 Problem. **(i)** Prove that every subspace $\mathcal{W}$ of a finite-dimensional vector space $\mathcal{V}$ is finite-dimensional, with $\dim[\mathcal{W}] \le \dim[\mathcal{V}]$. Show also that equality $\dim[\mathcal{W}] = \dim[\mathcal{V}]$

holds if and only if $\mathcal{W} = \mathcal{V}$.

(ii) Prove that if a vector space $\mathcal{V}$ has an infinite-dimensional subspace $\mathcal{W}$, then $\mathcal{V}$ is infinite-dimensional.

(iii) Let $n \geq 1$ be an integer. Prove that every infinite-dimensional vector space has a subspace of dimension n.

34.7 Example. Here is a different take on Example 34.4 that uses some of the tools just developed. Denote by $\mathbb{P}(\mathbb{F})$ the space of all polynomials with coefficients in $\mathbb{F}$. Put $f_k(x) := x^k$. Part (iv) of Example 28.12 shows that $(f_0, \ldots, f_n)$ is linearly independent for any $n \geq 0$, and so $\mathbb{P}(\mathbb{F})$ is infinite-dimensional. Since $\mathbb{P}(\mathbb{R})$ is a subspace of $\mathcal{C}^\infty(\mathbb{R})$, Problem 34.6 shows that $\mathcal{C}^\infty(\mathbb{R})$ is also infinite-dimensional. Since $\mathcal{C}^\infty(\mathbb{R})$ is a subspace of $\mathcal{C}^r(\mathbb{R})$ for any $r \geq 0$, this shows that each $\mathcal{C}^r(\mathbb{R})$ is also infinite-dimensional. (Restricting polynomials to any subinterval I of $\mathbb{R}$ shows that $\mathcal{C}^r(I)$ is infinite-dimensional for $0 \leq r \leq \infty$.)

34.8 Problem. Generalize this result (recall that part (iv) of Example 28.12 was all about eigenstuff) as follows. Let $\mathcal{V}$ be a vector space and suppose there is a linear operator $\mathcal{T} \colon \mathcal{V} \to \mathcal{V}$ that has infinitely many distinct eigenvalues. Prove that $\mathcal{V}$ is infinite-dimensional.

34.9 Problem. Use part (vi) of Example 25.6 to prove that the algebraic dual space of $\mathcal{C}^\infty(\mathbb{R})$, i.e., the space $\mathbf{L}(\mathcal{C}^\infty(\mathbb{R}), \mathbb{R})$, is infinite-dimensional.

What should the notion of a basis for an infinite-dimensional space *mean*? At the finite-dimensional level, a basis is a spanning, linearly independent list—and lists are inherently finite objects. However, we can still talk about spans of sets, not lists—that was the original Definition 6.3 of spans, after all. And the real goal of linear independence is to ensure unique representations of vectors in spans.

Here, then, is one correct generalization of basis for an infinite-dimensional space.

34.10 Definition. *Let $\mathcal{V}$ be a vector space. A* BASIS *for $\mathcal{V}$ is a set $\mathcal{B} \subseteq \mathcal{V}$ such that* $\operatorname{span}(\mathcal{B}) = \mathcal{V}$ *and every vector $v \in \mathcal{V}$ has a unique representation in* $\operatorname{span}(\mathcal{B})$*, i.e., given $v \in \mathcal{V}$, there are unique $v_1, \ldots, v_n \in \mathcal{V}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $v = \sum_{k=1}^n \alpha_k v_k$.*

Since unique representation really is linear independence, we might try another definition of basis.

34.11 Problem. Let $\mathcal{V}$ be a vector space. Prove that $\mathcal{B} \subseteq \mathcal{V}$ is a basis for $\mathcal{V}$ if and only if $\operatorname{span}(\mathcal{B}) = \mathcal{V}$ and $\mathcal{B}$ is linearly independent in the sense that if $(v_1, \ldots, v_n)$ is any list in $\mathcal{B}$ of any length with no repeated entries (i.e., $v_j \neq v_k$ for $j \neq k$), then $(v_1, \ldots, v_n)$ is linearly independent.

34.12 Problem. Neither Definition 34.10 nor Problem 34.11 assumed that $\mathcal{V}$ was not finite-dimensional. Prove that if $\mathcal{V}$ is finite-dimensional, then both definitions of basis given in Definition 34.10 and Problem 34.11 are equivalent to the original (Definition 28.11).

Now the question is if all (nontrivial) vector spaces have bases.

34.13 Example. With $f_k(x) := x^k$, we claim that $\mathcal{B} := \{f_k\}_{k=0}^{\infty}$ is a basis for $\mathbb{P}(\mathbb{F})$. Indeed, if $p \in \mathbb{P}(\mathbb{F})$, then $p \in \operatorname{span}(f_0, \ldots, f_n)$ with $n = \deg(p)$, so $\mathbb{P}(\mathbb{F}) = \operatorname{span}(\mathcal{B})$.

Next, $\mathcal{B}$ is linearly independent in the sense of Problem 34.11. To see how the general argument works, say that we want to show that the list (f_2, f_4, f_1) is linearly independent. This is a rearrangement or a permutation of the list (f_1, f_2, f_4), and this is a sublist of the linearly independent list $(f_0, f_1, f_2, f_3, f_4)$. Since rearranging or permuting a list does not change its linear (in)dependence, and since any sublist of a linearly independent list is linearly independent, so is (f_2, f_4, f_1).

Here is how this works more generally. Let $(f_{j_1}, \ldots, f_{j_n})$ be a list in $\mathcal{B}$ with no repeated entries. This means that $j_k \neq j_\ell$ for $k \neq \ell$. First, let $\sigma \colon \{1, \ldots, n\} \to \{j_1, \ldots, j_n\}$ be a strictly increasing bijection, so $\sigma(k) < \sigma(k+1)$ for $k = 1, \ldots, n-1$ and for each j_ℓ there exists a unique k such that $\sigma(k) = j_\ell$. (Above, $j_1 = 1$, $j_2 = 2$, and $j_3 = 4$, and $n = 3$. So take $\sigma(1) = 1$, $\sigma(2) = 2$, and $\sigma(3) = 4$.) Then the list $(f_{j_1}, \ldots, f_{j_n})$ is a rearrangement of $(f_{\sigma(1)}, \ldots, f_{\sigma(n)})$. And this list is a sublist of the linearly independent list $(f_0, \ldots, f_{\sigma(n)})$.

34.14 Problem. For k, $n \geq 1$, let

$$e_k^{(n)} := \begin{cases} 1, & k = n \\ 0, & k \neq n. \end{cases}$$

Cartoonishly, $(e_k^{(1)}) = (1, 0, 0, 0, 0 \ldots)$ and $(e_k^{(3)}) = (0, 0, 1, 0, 0, \ldots)$. Use the set $\{(e_k^{(n)})\}_{n=1}^{\infty}$ to show that $\mathbb{R}^\infty$ is infinite-dimensional, but explain why $\{(e_k^{(n)})\}_{n=1}^{\infty}$ is *not* a basis for $\mathbb{R}^\infty$.

It can be shown that *any* vector space has a basis—this is true by definition of a finite-dimensional space, and it follows from some nontrivial set-theoretic arguments for infinite-dimensional spaces. Two follow-up questions are about size and practicality.

First, if an infinite-dimensional vector space has a basis $\mathcal{B}$ (which necessarily cannot be a finite set), can we use the cardinality of $\mathcal{B}$ to give a notion of dimension for the space? (Yes.) Second, and maybe more importantly, who cares? (We really do not.) In most applications involving infinite-dimensional spaces, the space has extra structure—an inner product, a norm—and there are better ways of representing elements in the space relative to those structures. We now turn to that.

Day 35: Friday, November 1.

Material from *Linear Algebra* by Meckes & Meckes

Pages 225–228 introduce inner product spaces and give some standard examples. More examples appear on pp.233–235. The issue with $\int_0^1 [f(x)]^2 \, dx = 0$ but possibly $f \neq 0$ is discussed in #3 on p.268. All of Appendix A.2 on complex numbers will be particularly useful here.

We often reiterate how the goal of the course is to understand the problem $\mathcal{T}v = w$ with $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$ linear, $\mathcal{V}$ and $\mathcal{W}$ vector spaces, and $w \in \mathcal{W}$. The concepts of injectivity, surjectivity, and isomorphism provided us with concrete vocabulary for success and failure: injectivity for unique solutions, surjectivity for existence of solutions, isomorphism for a unique solution to any way of posing the problem.

The subspaces $\ker(\mathcal{T})$ and $\mathcal{T}(\mathcal{V})$ refined and quantified these notions: the problem is not solvable if $w \notin \mathcal{T}(\mathcal{V})$, and it is not uniquely solvable if $\ker(\mathcal{T}) \neq \{0\}$. When $\mathcal{V}$ is finite-dimensional, we can exploit properties of dimensionality to understand $\ker(\mathcal{T})$ and $\mathcal{T}(\mathcal{V})$ further: both are finite-dimensional (by rank–nullity, among other things). The dimension of $\ker(\mathcal{T})$ tells us how many "degrees of freedom" we have in a solution; the larger $\dim[\ker(\mathcal{T})]$ is, the "more" solutions we have to $\mathcal{T}v = w$ when at least one exists. The dimension of $\mathcal{T}(\mathcal{V})$ tells us how "many" kinds of w lead to a solvable problem; the larger $\dim[\mathcal{T}(\mathcal{V})]$ is, the "more" problems $\mathcal{T}v = w$ we can successfully pose.

What else is going on in $\mathcal{V}$ and $\mathcal{W}$? If $\mathcal{T}(\mathcal{V}) \neq \mathcal{W}$ and the problem cannot always be solved, what can we say about vectors $w \in \mathcal{W}$ with $w \notin \mathcal{T}(\mathcal{V})$? Can we "characterize" $\mathcal{T}(\mathcal{V})$ in a simpler way than just the definition? Can we say $w \in \mathcal{T}(\mathcal{V})$ if and only if…"something"? How does $\mathcal{T}(\mathcal{V})$ interact with the "rest" of the structure of $\mathcal{W}$?

We can do all of this, and more, if we add more structure to $\mathcal{V}$ and $\mathcal{W}$: the geometry of *inner products*. While this structure may seem contrived at first glance, it is in fact perfectly natural, and most of the "meaningful" vector spaces in existence inherently come with inner products.

We think about two in particular and then give the general definition.

35.1 Example. For $\mathbf{v}$, $\mathbf{w} \in \mathbb{F}^n$, define

$$\langle \mathbf{v}, \mathbf{w} \rangle := \sum_{k=1}^{n} v_k \overline{w_k}.$$

Here, for $x + iy \in \mathbb{F}$ with x, $y \in \mathbb{R}$, the scalar $\overline{x+iy} := x - iy$ is the **CONJUGATE** of $x + iy$, and we review some of its properties along the way in the context of essential properties of $\langle \cdot, \cdot \rangle$.

1. Because of the way scalar addition and multiplication interact, we have

$$\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle + \langle \mathbf{v}_2, \mathbf{w} \rangle$$

for all $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w} \in \mathbb{F}^n$.

2. Again because of the way scalar addition and multiplication interact, we have

$$\langle \alpha\mathbf{v}, \mathbf{w}\rangle = \alpha \langle \mathbf{v}, \mathbf{w}\rangle$$

for all $\alpha \in \mathbb{F}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{F}^n$.

3. If we reverse the order of things, we have

$$\langle \mathbf{w}, \mathbf{v}\rangle = \sum_{k=1}^{n} w_k\overline{v_k} = \sum_{k=1}^{n} \overline{\overline{w_k}v_k} = \overline{\sum_{k=1}^{n} \overline{w_k}v_k} = \overline{\langle \mathbf{v}, \mathbf{w}\rangle}.$$

Here we have used the properties $\overline{\overline{\alpha}} = \alpha$, $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$, and $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.

4. If we make both slots the same, we have

$$\langle \mathbf{v}, \mathbf{v}\rangle = \sum_{k=1}^{n} v_k\overline{v_k} = \sum_{k=1}^{n} |v_k|^2 \geq 0.$$

Here we are using the property that $\alpha\overline{\alpha} \geq 0$ for all $\alpha \in \mathbb{F}$; indeed, if $\alpha = x + iy$ with x, $y \in \mathbb{R}$, then $\alpha\overline{\alpha} = x^2 + y^2 \geq 0$.

5. What if equality is achieved above and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$? Then $\sum_{k=1}^{n}|v_k|^2 = 0$. Let $1 \leq j \leq n$. Then

$$0 \leq |v_j|^2 \leq \sum_{k=1}^{n} |v_k|^2 = 0,$$

so $|v_j|^2 = 0$, thus $|v_j| = 0$, and therefore $v_j = 0$. Then $\mathbf{v} = \mathbf{0}_n$. Here we are using the definition $|\alpha| := \sqrt{\alpha\overline{\alpha}}$ and the consequent property that $|\alpha| = 0$ if and only if $\alpha = 0$ for any $\alpha \in \mathbb{F}$.

35.2 Example. Let $\mathcal{V} = \mathcal{C}([0, 1])$; recall that functions in $\mathcal{V}$ are real-valued (we could develop the following for complex-valued functions, but that requires a little too much calculus for complex-valued functions of a real variable than we care to pursue). For f, $g \in \mathcal{V}$, put

$$\langle f, g\rangle := \int_0^1 f(x)g(x)\ dx.$$

This integral is defined because the product of continuous functions is continuous and therefore integrable. We compare the properties of $\langle \cdot, \cdot\rangle$ here to the previous example. Nothing in the following would change if we worked on an arbitrary interval $[a, b]$, although more care would be needed if we wanted to use improper integrals (which we do not right now).

1. Linearity of the integral implies

$$\langle f+g,h\rangle=\int_0^1\big(f(x)+g(x)\big)h(x)\ dx=\int_0^1 f(x)h(x)\ dx+\int_0^1 g(x)h(x)\ dx=\langle f,h\rangle+\langle g,h\rangle$$

for all f, g, $h\in\mathcal{V}$.

2. More linearity of the integral implies

$$\langle \alpha f,g\rangle=\int_0^1 \alpha f(x)g(x)\ dx=\alpha\int_0^1 f(x)g(x)\ dx=\alpha\langle f,g\rangle.$$

3. Since f, $g\in\mathcal{V}$ are real-valued, $\langle f,g\rangle\in\mathbb{R}$, and so $\overline{\langle f,g\rangle}=\langle f,g\rangle$. But also

$$\langle g,f\rangle=\int_0^1 g(x)f(x)\ dx=\int_0^1 f(x)g(x)\ dx=\langle f,g\rangle=\overline{\langle f,g\rangle}.$$

4. We compute

$$\langle f,f\rangle=\int_0^1 f(x)f(x)\ dx=\int_0^1[f(x)]^2\ dx\geq 0,$$

since $[f(x)]^2\geq 0$ for all $x\in[0,1]$. Here we are using the monotonicity of the integral: if g, $h\in\mathcal{V}$ with $g(x)\leq h(x)$ for all $x\in[0,1]$, then $\int_0^1 g(x)\ dx\leq\int_0^1 h(x)\ dx$.

5. Suppose $\langle f,f\rangle=0$, so $\int_0^1[f(x)]^2\ dx=0$. What if $f(x_0)\neq 0$ for some $x_0\in[0,1]$? Continuity implies the existence of $\delta>0$ such that $|f(x)|>|f(x_0)|/2>0$ for $x\in(x_0-\delta,x_0+\delta)\cap[0,1]$. And so monotonicty of the integral implies

$$0<\int_{x_0-\delta}^{x_0+\delta}[f(x)]^2\ dx\leq\int_0^1[f(x)]^2\ dx=0.$$

Strictly speaking, this is valid only if $x_0\in[0,1]$; if $x_0=0$, replace $x_0-\delta$ with 0. The second inequality is a different kind of integral monotonicity: if $g\in\mathcal{C}([a,b])$ and $g(x)\geq 0$ for $x\in[a,b]$ and $a\leq c\leq d\leq b$, then $\int_c^d g(x)\ dx\leq\int_a^b g(x)\ dx$. Regardless, this is a contradiction: $0<0$ is false, and so we cannot have $x_0\in[0,1]$ such that $f(x_0)=0$. Thus $f=0$.

We codify the properties of the structures $\langle\cdot,\cdot\rangle$ above into a definition. Here it is important to note explicitly what the underlying field is.

35.3 Definition. *Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. An* **INNER PRODUCT** *on $\mathcal{V}$ is a function*

$$\langle\cdot,\cdot\rangle:\{(v,w)\mid v,\ w\in\mathcal{V}\}\to\mathbb{F}$$

such that the following hold.

1. [Linearity in the first slot I] $\langle v_1+v_2,w\rangle=\langle v_1,w\rangle+\langle v_2,w\rangle$ *for all* v_1, v_2, $w\in\mathcal{V}$.

2. [Linearity in the first slot II] $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ *for all* $\alpha \in \mathbb{F}$ *and* v, $w \in \mathcal{V}$.

3. [Conjugacy] $\overline{\langle v, w \rangle} = \langle w, v \rangle$ *for all* v, $w \in \mathcal{V}$. *(This is trivially true if* $\mathbb{F} = \mathbb{R}$.*)*

4. [Nonnegavitivty] $\langle v, v \rangle \geq 0$ *for all* $v \in \mathcal{V}$.

5. [Definiteness] *If* $\langle v, v \rangle = 0$, *then* $v = 0$.

An **INNER PRODUCT SPACE** *is a vector space on which an inner product is defined; strictly speaking (recall Remark 4.2), we might declare an inner product space to be a 5-tuple* $(\mathcal{V}, \mathbb{F}, +, \cdot, \langle \cdot, \cdot \rangle)$, *where* $(\mathcal{V}, \mathbb{F}, +, \cdot)$ *is a vector space over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $\langle \cdot, \cdot \rangle$ *is an inner product on* $\mathcal{V}$. *While a vector space can be defined over more general fields than* $\mathbb{R}$ *or* $\mathbb{C}$ *(and we are not doing that in this course), inner product spaces require real or complex fields.*

35.4 Example. **(i)** The map from Example 35.1 is an inner product on $\mathbb{F}^n$, and of course we usually call it the **DOT PRODUCT** and write it as $\mathbf{v} \cdot \mathbf{w}$, not as $\langle \mathbf{v}, \mathbf{w} \rangle$.

(ii) The map from Example 35.2 is an inner product on $\mathcal{C}([0,1])$, which we often call (for various historical and cultural reasons) the L^2-inner product on $\mathcal{C}([0,1])$.

35.5 Problem. Let $\mathcal{V}$ be an inner product space.

(i) Fix $w \in \mathcal{V}$. Prove that the map

$$\varphi \colon \mathcal{V} \to \mathbb{F} \colon v \mapsto \langle v, w \rangle$$

is linear. (Recall that since this linear map is scalar-valued, we call it a linear functional on $\mathcal{V}$.)

(ii) Fix w_1, $w_2 \in \mathcal{V}$. Prove that the map

$$\mathcal{T} \colon \mathcal{V} \to \mathcal{V} \colon \langle v, w_1 \rangle w_2$$

is linear.

(iii) Prove that the inner product is **ANTILINEAR** in the second slot in the sense that

$$\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle \qquad \text{and} \qquad \langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$$

for all v, w, w_1, $w_2 \in \mathcal{V}$ and $\alpha \in \mathbb{F}$.

35.6 Problem. Why does defining

$$\langle f, g \rangle := \int_0^1 f'(x) g(x) \, dx$$

not give an inner product on $\mathcal{C}([0,1])$?

35.7 Problem. Let $\mathcal{V}$ be an inner product space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$. What conditions on $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ guarantee that

$$\langle v, w \rangle_{\mathcal{T}} := \langle \mathcal{T}v, w \rangle_{\mathcal{V}}$$

defines an inner product on $\mathcal{V}$?

Day 36: Monday, November 4.

Material from *Linear Algebra* by Meckes & Meckes

Proposition 4.2 on p.228 gives essential properties of inner products that we will use without further comment. Orthonormal bases are discussed on p.239; see the examples on pp.239–240.

Do Quick Exercises #3 on p.230, #6 on p.239, and #7 on p.242.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Orthogonal subset of an inner product space (N), orthogonal list in an inner product space (N), orthonormal subset of an inner product space (N), orthonormal list in an inner product space (N)

We begin to study the valuable *data* about vectors and linear operators that inner products can extract and represent. Our first observation—which may appear unintuitive and unhelpful, but which will be vindicated in later work—is that

36.1 Theorem. *Let $\mathcal{V}$ be an inner product space and $v \in \mathcal{V}$. The following are equivalent.*

(i) $v = 0$.

(ii) $\langle v, w \rangle = 0$ *for all* $w \in \mathcal{V}$.

(iii) $\langle w, v \rangle = 0$ *for all* $w \in \mathcal{V}$.

Proof. The second and third parts are equivalent because $\langle w, v \rangle = \overline{\langle v, w \rangle}$, and, for $z \in \mathbb{F}$, we have $z = 0$ if and only if $\overline{z} = 0$.

We work on the equivalence of the first two parts. If $v = 0$, then

$$\langle v, w \rangle = \langle 0, w \rangle = \langle 0(0), w \rangle = 0 \langle 0, w \rangle = 0.$$

Here we are using the identity $0 = 0(0)$, where the first 0 on the right is the scalar $0 \in \mathbb{F}$ and the second 0 on the right is $0 \in \mathcal{V}$, and the property that $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $\alpha \in \mathbb{F}$ and v, $w \in \mathcal{V}$.

Conversely, suppose that $\langle v, w \rangle = 0$ for all $w \in \mathcal{V}$. We know a special property of inner products and 0 when both inputs to the inner product are the same, and we are allowed to pick any $w \in \mathcal{V}$ here, so we set $w = v$ and compute $0 = \langle v, v \rangle$. The axioms for an inner product then imply $v = 0$. ■

36.2 Problem. Let $\mathcal{V}$ be a vector space and $\mathcal{W}$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Suppose that both $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional with bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$, respectively. Let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. Prove that $\mathcal{T} = 0$ if and only if $\langle \mathcal{T} v_j, w_k \rangle = 0$ for $j = 1, \ldots, n$ and $k = 1, \ldots, m$. [Hint: *recall from Problem 31.3 that it suffices to know the values of $\mathcal{T} v_j$; use Theorem 36.1 and algebraic properties of the inner product to reduce this to knowledge of $\langle \mathcal{T} v_j, w_k \rangle$.*] Explain why checking that $\mathcal{T}_1 = \mathcal{T}_2$ for $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ amounts to doing only mn calculations involving scalars (although, to be fair, the same could be done involving the n calculations $(\mathcal{T}_1 - \mathcal{T}_2) v_j = 0$ with vectors).

36.3 Example. **(i)** Consider the inner product space $\mathbb{F}^n$ with inner product given by the dot product, as usual. Let $\mathbf{v} \in \mathbb{F}^n$. The kth component of $\mathbf{v}$ is $\langle \mathbf{v}, \mathbf{e}_k \rangle$ and so

$$\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{e}_k = \sum_{k=1}^{n} \langle \mathbf{v}, \mathbf{e}_k \rangle \mathbf{e}_k. \tag{36.1}$$

The inner product thus provides a convenient way of expressing $\mathbf{v}$ as a linear combination of the standard basis vectors. Consequently, we can also calculate inner products via inner products:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{k=1}^{n} v_k \overline{w_k} = \sum_{k=1}^{n} \langle \mathbf{v}, \mathbf{e}_k \rangle \overline{\langle \mathbf{w}, \mathbf{e}_k \rangle}. \tag{36.2}$$

(ii) Now let both $\mathbb{F}^n$ and $\mathbb{F}^m$ have the dot product(s) as their inner products and let $A \in \mathbb{F}^{m \times n}$. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be the standard basis for $\mathbb{F}^n$ and $(\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_m)$ be the standard basis for $\mathbb{F}^m$. For example, if $n = 3$ and $m = 2$, then

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{but} \quad \widetilde{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then the jth column of A is $A\mathbf{e}_j$, and the kth component of this vector is $\langle A\mathbf{e}_j, \widetilde{\mathbf{e}}_k \rangle$. Changing one letter, more colloquially this says that the (i, j)-entry of A is $\langle A\mathbf{e}_j, \widetilde{\mathbf{e}}_i \rangle$.

What makes the standard basis vectors so special with respect to the dot product is not really their componentwise formulas (although that *is* what everything ultimately relies on) but rather their "orthonormality":

$$\langle \mathbf{e}_j, \mathbf{e}_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases} \tag{36.3}$$

We abstract and exploit this calculation in much more general contexts.

36.4 Definition. *Let $\mathcal{V}$ be an inner product space.*

(i) *A subset $\mathcal{U} \subseteq \mathcal{V}$ is* **ORTHOGONAL** *if $\langle v, w \rangle = 0$ for all v, $w \in \mathcal{U}$ with $v \neq w$.*

(ii) *A list $(u_1, \ldots, u_n)$ in $\mathcal{V}$ is* **ORTHOGONAL** *if $\langle u_j, u_k \rangle = 0$ for all $j \neq k$.*

36.5 Example. **(i)** Certainly the list of standard basis vectors in $\mathbb{F}^n$ is orthogonal.

(ii) The list

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

is orthogonal in $\mathbb{F}^3$, since

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 1(-3) + 2(0) + 3(1) = 0.$$

(iii) Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$. The list (f, g) is orthogonal in $\mathcal{C}([-\pi, \pi])$, since

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\ dx = \int_{-\pi}^{\pi} \sin(x)\cos(x)\ dx = 0.$$

36.6 Problem. Let x, $y \in \mathbb{R}$. Show that

$$\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ -x \end{bmatrix} \right)$$

is orthogonal in $\mathbb{F}^2$. Draw a picture that illustrates how this corresponds to the usual geometric notion of orthogonality (= perpendicularity).

Of course the list of standard basis vectors in $\mathbb{F}^n$ is linearly independent; we now show that this is an easy consequence of orthogonality.

36.7 Theorem. *Let $(u_1, \ldots, u_n)$ be an orthogonal list in the inner product space $\mathcal{V}$ with $u_j \neq 0$ for all j. Then $(u_1, \ldots, u_n)$ is linearly independent.*

Proof. Theorem 4.3 in the book.

Assume that $\sum_{k=1}^{n}\alpha_k u_k = 0$ for some $\alpha_k \in \mathbb{F}$; we show $\alpha_k = 0$ for all k. We know that $\langle u_k, u_j\rangle = 0$ for $k \neq j$, so to make this identity show up in our assumption, we fix j, take the inner product of the linear combination against u_j, and use algebraic properties of the inner product:

$$0 = \langle 0, u_j\rangle = \left\langle \sum_{k=1}^{n} \alpha_k u_k, u_j \right\rangle = \sum_{k=1}^{n} \langle \alpha_k u_k, u_j\rangle = \sum_{k=1}^{n} \alpha_k \langle u_k, u_j\rangle = \alpha_j \langle u_j, u_j\rangle. \qquad (36.4)$$

The last equality is the identity $\langle u_k, u_j\rangle = 0$ for $k \neq j$. Since $u_j \neq 0$ by our hypotheses on this list, the inner product axioms imply $\langle u_j, u_j\rangle > 0$, thus $\alpha_j = 0$. ■

36.8 Remark. *The calculation in* (36.4) *is an important example of good mathematical grammar: we are using k as the index of summation, and so we should not overwork it by using k in the second slot. That is why we wrote $\langle \sum_{k=1}^{n}\alpha_k u_k, u_j\rangle$, not $\langle \sum_{k=1}^{n}\alpha_k u_k, u_k\rangle$. Indeed, the latter would have us calculate*

$$\left\langle \sum_{k=1}^{n} \alpha_k u_k, u_k \right\rangle = \sum_{k=1}^{n} \alpha_k \langle u_k, u_k\rangle,$$

which is useless, because it does not "extract" any particular coefficient from the sum.

36.9 Problem. Give an example of a linearly independent list in an inner product space that is not an orthogonal list.

36.10 Problem. Prove that no entry in an orthogonal list of nonzero vectors can be repeated in that list. Try to do this without invoking linear independence and using just the definition of an orthogonal list.

Orthogonality by itself does not account for the excellent behavior of the standard basis vectors in (36.3). In particular, orthogonality says nothing about the behavior of the inner products $\langle u, u\rangle$ when both vectors are the same.

36.11 Definition. *Let $\mathcal{V}$ be an inner product space.*

(i) *A subset $\mathcal{U} \subseteq \mathcal{V}$ is* ORTHONORMAL *if*

$$\langle v, w\rangle = \begin{cases} 1, & v = w \\ 0, & v \neq w. \end{cases}$$

(ii) *A list $(u_1, \ldots, u_n)$ in $\mathcal{V}$ is* ORTHONORMAL *if*

$$\langle u_j, u_k\rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

We can adapt the calculation in (36.4) to generalize the expansion (36.1). We can paraphrase this via the slogan "Inner products extract coefficients from linear combinations."

36.12 Theorem. *Let $(u_1, \ldots, u_n)$ be an orthogonal list in the inner product space $\mathcal{V}$, and let $v \in \operatorname{span}(u_1, \ldots, u_n)$. Then*

$$v = \sum_{k=1}^{n} \langle v, u_k \rangle \, u_k.$$

Proof. Theorem 4.9 in the book.

By definition of span, we know that $v = \sum_{k=1}^{n} \alpha_k u_k$ for some $\alpha_k \in \mathbb{F}$. We compute

$$\langle v, u_j \rangle = \left\langle \sum_{k=1}^{n} \alpha_k u_k, u_j \right\rangle = \sum_{k=1}^{n} \alpha_k \langle u_k, u_j \rangle = \alpha_j.$$

The third equality are orthonormality with $\langle u_k, u_j \rangle = 1$ for $k = j$ and 0 otherwise. ■

Now we generalize (36.2), mostly as an exercise in manipulating sums and indices.

36.13 Theorem. *Let $(u_1, \ldots, u_n)$ be an orthogonal list in the inner product space $\mathcal{V}$, and let v, $w \in \operatorname{span}(u_1, \ldots, u_n)$. Then*

$$\langle v, w \rangle = \sum_{k=1}^{n} \langle v, u_k \rangle \, \overline{\langle w, u_k \rangle}.$$

Proof. Theorem 4.10 in the book.

We use Theorem 36.12 to expand

$$\langle v, w \rangle = \left\langle \sum_{k=1}^{n} \langle v, u_k \rangle, \sum_{j=1}^{n} \langle w, u_j \rangle \right\rangle. \tag{36.5}$$

Here we are continuing with the good grammar of Remark 36.8 by not overworking the indices of summation and using k and j separately. Linearity of the inner product in the first slot gives

$$\begin{aligned}\left\langle \sum_{k=1}^{n} \langle v, u_k \rangle u_k, \sum_{j=1}^{n} \langle w, u_j \rangle u_j \right\rangle &= \sum_{k=1}^{n} \left\langle \langle v, u_k \rangle u_k, \sum_{j=1}^{n} \langle w, u_j \rangle u_j \right\rangle \\ &= \sum_{k=1}^{n} \langle v, u_k \rangle \left\langle u_k, \sum_{j=1}^{n} \langle w, u_j \rangle u_j \right\rangle. \end{aligned} \tag{36.6}$$

Then we use antilinearity in the second slot to compute, for each k,

$$\left\langle u_k, \sum_{j=1}^{n} \langle w, u_j \rangle u_j \right\rangle = \sum_{j=1}^{n} \langle u_k, \langle w, u_j \rangle u_j \rangle = \sum_{j=1}^{n} \overline{\langle w, u_j \rangle} \langle u_k, u_j \rangle = \langle w, u_k \rangle, \tag{36.7}$$

where the last equality is orthonormality with $\langle u_k, u_j\rangle = 1$ for $j = k$ and 0 otherwise. Combining (36.5), (36.6), and (36.7) yields the desired identity for $\langle v, w\rangle$. ■

The two theorems above indicate the utility of spanning by orthonormal lists. It would be nice if we could always summon up a *basis* for a finite-dimensional inner product space that is orthonormal, and we always can.

Day 37: Wednesday, November 6.

Material from *Linear Algebra* by Meckes & Meckes

The norm induced by an inner product is defined on p.229. Pages 244–247 present the Gram–Schmidt algorithm in full detail. In particular Algorithm 4.11 gives the full algorithm for a list of arbitrary length. Read the examples on pp.245–246.

Do Quick Exercises #8 and #9 on p.246.

Before proceeding, we define a very useful quantity that will have profound ramifications for our subsequent analysis of vector spaces and linear operators. To do so, recall that $\langle v, v\rangle \geq 0$ for any inner product $\langle\cdot,\cdot\rangle$ on a space $\mathcal{V}$ and any $v \in \mathcal{V}$, so $\sqrt{\langle v, v\rangle}$ is always defined.

37.1 Definition. *Let $\mathcal{V}$ be an inner product space with inner product $\langle\cdot,\cdot\rangle$. The* **NORM INDUCED BY** $\langle\cdot,\cdot\rangle$ *is the map*

$$\|\cdot\| : \mathcal{V} \to [0,\infty) \colon v \mapsto \sqrt{\langle v, v\rangle}.$$

We will develop a staggering number of properties of the norm as a means of measuring "sizes" of vectors in inner product spaces. Here are just two for now.

37.2 Problem. Let $\mathcal{V}$ be an inner product space.

(i) Let $v \in \mathcal{V}$. Prove that $v = 0$ if and only if $\|v\| = 0$.

(ii) Let $(u_1, \ldots, u_n)$ be an orthonormal list and let $v \in \operatorname{span}(u_1, \ldots, u_n)$. Prove that

$$\|v\|^2 = \sum_{k=1}^{n} |\langle v, u_k\rangle|^2.$$

37.3 Theorem (Gram–Schmidt). *Let $\mathcal{V}$ be an inner product space and let $(v_1, \ldots, v_n)$ be a linearly independent list in $\mathcal{V}$. Then there exists an orthonormal list $(u_1, \ldots, u_n)$ in $\mathcal{V}$*

such that $\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(u_1, \ldots, u_j)$ *for* $j = 1, \ldots, n$. *Specifically,*

$$u_k := \begin{cases} v_1/\|v_1\|, & k = 1 \\ \widetilde{u}_k/\|\widetilde{u}_k\|, & 2 \le k \le n, \end{cases} \qquad \widetilde{u}_k := v_k - \sum_{\ell=1}^{k-1} \langle v_\ell, u_\ell \rangle u_\ell.$$

Proof. Algorithm 4.11 in the book. We prove the $n = 1$ and $n = 2$ cases explicitly to show how the recursive formula above arises.

1. $n = 1$. We assume that (v_1) is linearly independent, which just means that $v_1 \neq 0$. We want to find (u_1) such that $\operatorname{span}(v_1) = \operatorname{span}(u_1)$ and $\langle u_1, u_1 \rangle = 1$. For the span, we want $u_1 = \alpha v_1$ for some $\alpha \in \mathbb{F}$. The orthonormality condition demands

$$1 = \langle \alpha v_1, \alpha v_1 \rangle = \alpha\overline{\alpha} \langle v_1, v_1 \rangle = |\alpha|^2 \|v_1\|^2.$$

Thus we need $|\alpha| \|v_1\| = 1$, and so, since $v_1 \neq 0$,

$$|\alpha| = \frac{1}{\|v_1\|}.$$

There is nothing more that we can say. If $\mathbb{F} = \mathbb{R}$, then we can take $\alpha = \pm 1/\|v_1\|$, and putting $u_1 = \pm v_1/\|v_1\|$ works. If $\mathbb{F} = \mathbb{C}$, then we can take $\alpha = e^{i\theta}/\|v_1\|$ for any $\theta \in \mathbb{R}$. Possibly the path of least resistance is just to take $u_1 = v_1/\|v_1\|$.

2. $n = 2$. We assume that (v_1, v_2) is linearly independent, so neither v_1 nor v_2 is a scalar multiple of the other. We want to find (u_1, u_2) such that $\operatorname{span}(u_1) = \operatorname{span}(v_1)$, $\operatorname{span}(u_1, u_2) = \operatorname{span}(v_1, v_2)$, and

$$\langle u_j, u_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

The previous step suggests that we try $u_1 = v_1/\|v_1\|$. Then we want $u_2 = \alpha_1 v_1 + \alpha_2 v_2$ for some α_1, $\alpha_2 \in \mathbb{F}$; $\langle u_1, u_2 \rangle = 0$; and $\langle u_2, u_2 \rangle = 1$. Building on our intuition that inner products extract coefficients from linear combinations (and fearing the conjugates from keeping u_2 in the second slot), we compute that we want

$$0 = \langle u_2, u_1 \rangle = \left\langle \alpha_1 v_1 + \alpha_2 v_2, \frac{v_1}{\|v_1\|} \right\rangle = \frac{1}{\|v_1\|}\big(\alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \langle v_1, v_2 \rangle\big).$$

A little algebra reveals that we want

$$\alpha_1 = -\alpha_2 \frac{\langle v_1, v_2 \rangle}{\|v_1\|^2},$$

and so we should have

$$u_2 = \alpha_2 \left(v_2 - \frac{\langle v_1, v_2 \rangle}{\|v_1\|^2} \right).$$

If we take $\alpha_2 = 1/\left\| v_2 - \langle v_1, v_2 \rangle / \|v_1\|^2 \right\|$, then we should have $\langle u_2, u_2 \rangle = 1$. Of course, this presumes that the norm is nonzero. ■

37.4 Problem. Check that the ideas from the $n = 2$ case pan out. Specifically, let (v_1, v_2) be a linearly independent list in the inner product space $\mathcal{V}$ and define

$$u_1 := \frac{v_1}{\|v_1\|}, \qquad u_2 := \frac{\widetilde{u}_2}{\|\widetilde{u}_2\|}, \qquad \widetilde{u}_2 := v_2 - \langle v_2, u_1 \rangle u_1.$$

First check that $\|\widetilde{u}_2\| \neq 0$. Then show that $\operatorname{span}(u_1, u_2) = \operatorname{span}(v_1, v_2)$. Finally, show that (u_1, u_2) is orthonormal.

37.5 Problem. The Gram–Schmidt process introduced an important linear operator. Let $\mathcal{V}$ be an inner product space and $w \in \mathcal{V} \setminus \{0\}$. Define

$$\mathcal{P}_w \colon \mathcal{V} \to \mathcal{V} \colon v \mapsto \frac{\langle v, w \rangle}{\|w\|^2} w.$$

Prove the following.

(i) $\mathcal{P}_w \in \mathbf{L}(\mathcal{V})$.

(ii) $\mathcal{P}_w^2 = \mathcal{P}_w$.

(iii) $\langle \mathcal{P}_w v, v - \mathcal{P}_w v \rangle = 0$ for all $v \in \mathcal{V}$.

We call $\mathcal{P}_w$ the **ORTHOGONAL PROJECTION ONTO** w. Draw a picture illustrating this projection phenomenon for $\mathcal{V} = \mathbb{R}^2$ and $\mathbf{w} = \mathbf{e}_1$ in the case that $\mathbf{v} \in \mathbb{R}^2$ has positive entries—specifically, draw $\mathbf{v}$, $\mathbf{e}_1$, and $\mathcal{P}_{\mathbf{e}_1}\mathbf{v}$ on the same graph.

We now have enough knowledge of the structure of inner product spaces to study operators mapping between them. Remember, the goal is to characterize the range of a linear operator in more elementary and accessible terms than its definition. We start with a lengthy example in which the trivial structure of the operator highlights deep properties.

Let $\mathcal{V} = \mathcal{W} = \mathbb{R}^3$ and define

$$\mathcal{T} \colon \mathcal{V} \to \mathcal{W} \colon \mathbf{v} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}.$$

The range of this operator is just

$$\mathcal{T}(\mathcal{V}) = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; w_1,\ w_2 \in \mathbb{R} \right\} = \left\{ \mathbf{w} \in \mathbb{R}^3 \;\middle|\; w_3 = 0 \right\}.$$

It is fairly easy to see what is "missing" from $\mathcal{T}(\mathcal{V})$: vectors with a nonzero third component. Put

$$\widetilde{\mathcal{W}} := \left\{ \begin{bmatrix} 0 \\ 0 \\ w_3 \end{bmatrix} \;\middle|\; w_3 \in \mathbb{R}^3 \right\} = \left\{ \mathbf{w} \in \mathbb{R}^3 \;\middle|\; w_1 = w_2 = 0 \right\} = \operatorname{span}(\mathbf{e}_3).$$

Since any $\mathbf{w} \in \mathbb{R}^3$ can be written as

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ w_3 \end{bmatrix},$$

we see that any $\mathbf{w} \in \mathbb{R}^3$ can be written as

$$\mathbf{w} = \mathbf{u} + \widetilde{\mathbf{u}}, \qquad \mathbf{u} \in \mathcal{T}(\mathcal{V}), \ \widetilde{\mathbf{u}} \in \widetilde{\mathcal{W}}.$$

There is rather more to say about this decomposition.

37.6 Problem. **(i)** Prove that this decomposition is unique in the sense that if $\mathbf{w} = \mathbf{u}_1 + \widetilde{\mathbf{u}}_1$ and $\mathbf{w} = \mathbf{u}_2 + \widetilde{\mathbf{u}}_2$ for $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathcal{T}(\mathcal{V})$ and $\widetilde{\mathbf{u}}_1$, $\widetilde{\mathbf{u}}_2 \in \widetilde{\mathcal{W}}$, then $\mathbf{u}_1 = \mathbf{u}_2$ and $\widetilde{\mathbf{u}}_1 = \widetilde{\mathbf{u}}_2$.

(ii) Is it true that if $\mathbf{u} \in \mathbb{R}^3$, then there are unique $\mathbf{v} \in \mathcal{V}$ and $\widetilde{\mathbf{u}} \in \widetilde{\mathcal{W}}$ such that $\mathbf{u} = \mathcal{T}\mathbf{v} + \widetilde{\mathbf{u}}$?

(iii) Prove that this decomposition is "orthogonal" in the sense that if $\mathbf{u} \in \mathcal{T}(\mathcal{V})$ and $\widetilde{\mathbf{u}} \in \widetilde{\mathcal{W}}$, then $\langle \mathbf{u}, \widetilde{\mathbf{u}} \rangle = 0$, where the inner product is, of course, the dot product.

The last result above says that every vector in $\mathcal{T}(\mathcal{V})$ is orthogonal to every vector in $\widetilde{\mathcal{W}}$. The reverse is also true: suppose that $\langle \mathbf{u}, \widetilde{\mathbf{u}} \rangle = 0$ for all $\widetilde{\mathbf{u}} \in \widetilde{\mathcal{W}}$. Take $\widetilde{\mathbf{u}} = \mathbf{e}_3$ to find $\langle \mathbf{u}, \mathbf{e}_3 \rangle = 0$. Then

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} = \mathcal{T} \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \in \mathcal{T}(\mathcal{V}).$$

This gives us a new characterization of $\mathcal{T}(\mathcal{V})$:

$$\mathcal{T}(\mathcal{V}) = \left\{ \mathbf{w} \in \mathbb{R}^3 \ \middle| \ \langle \mathbf{w}, \widetilde{\mathbf{u}} \rangle = 0 \text{ for all } \widetilde{\mathbf{u}} \in \widetilde{\mathcal{W}} \right\}.$$

This is good but not great. Is it possible to characterize $\widetilde{\mathcal{W}}$ somehow in terms of $\mathcal{T}$? Yes: as noted above $\widetilde{\mathcal{W}} = \operatorname{span}(\mathbf{e}_3)$, and it turns out that $\operatorname{span}(\mathbf{e}_3) = \ker(\mathcal{T})$.

37.7 Problem. Check that.

So there we are:

$$\mathcal{T}(\mathcal{V}) = \left\{ \mathbf{w} \in \mathbb{R}^3 \ \middle| \ \langle \mathbf{w}, \widetilde{\mathbf{u}} \rangle = 0 \text{ for all } \widetilde{\mathbf{u}} \in \ker(\mathcal{T}) \right\}. \tag{37.1}$$

This is far too nice to be true in general. One obstacle to generalization is that we want to study linear operators between *different* inner product spaces, and there is no reason to expect that the kernel (which is a subset of the domain) will also be a subset of the codomain (so that we can take the inner product of an element $\mathbf{w}$ of the codomain with an element $\widetilde{\mathbf{u}}$ of the kernel).

Another obstacle is that we spent a long time thinking about the very explicit and trivial structure of this operator $\mathcal{T}$ from above. Everything hinged on computing inner products of the form $\langle \mathbf{w}, \widetilde{\mathbf{u}} \rangle$ with, ideally, $\mathbf{w}$ in the range of $\mathcal{T}$. What if we just started out with such inner products (and dropped the tilde for convenience)? We compute

$$\langle \mathcal{T}\mathbf{v}, \mathbf{u} \rangle = \left\langle \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right\rangle = v_1u_1 + v_2u_2 = v_1u_1 + v_2u_2 + v_3(0) = \left\langle \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \right\rangle = \langle \mathbf{v}, \mathcal{T}\mathbf{u} \rangle .$$

This is something special: we have "popped" the operator $\mathcal{T}$ from one slot in the inner product to the other.

Suppose we knew this from the start, that $\langle \mathcal{T}\mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathcal{T}\mathbf{u} \rangle$ for all $\mathbf{v}$ and $\mathbf{u}$. Now let $\widetilde{\mathbf{u}} \in \ker(\mathcal{T})$. Then

$$\langle \mathcal{T}\mathbf{v}, \widetilde{\mathbf{u}} \rangle = \langle \mathbf{v}, \mathcal{T}\widetilde{\mathbf{u}} \rangle = \langle \mathbf{v}, \mathbf{0}_3 \rangle = 0$$

for all $\mathbf{v} \in \mathbb{R}^3$. Thus $\mathcal{T}(\mathcal{V}) \subseteq \left\{ \mathbf{w} \in \mathbb{R}^3 \mid \langle \mathbf{w}, \widetilde{\mathbf{u}} \rangle = 0 \text{ for all } \widetilde{\mathbf{u}} \in \ker(\mathcal{T}) \right\}$. Proving this only required knowing that $\langle \mathcal{T}\mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathcal{T}\mathbf{u} \rangle$ for all $\mathbf{v}$ and $\mathbf{u}$, with no knowledge of the exact structure of $\mathcal{T}$, or $\ker(\mathcal{T})$, necessary. It would be nice to obtain the reverse inclusion in a similarly easy fashion, without needing to know the precise formula for $\mathcal{T}$ above. If $\mathbf{w} \in \mathbb{R}^3$ such that $\langle \mathbf{w}, \widetilde{\mathbf{u}} \rangle = 0$ for all $\widetilde{\mathbf{u}} \in \ker(\mathcal{T})$, why do we have $\mathbf{w} \in \mathcal{T}(\mathcal{V})$? Why must there exist $\mathbf{v} \in \mathcal{V}$ such that $\mathcal{T}\mathbf{v} = \mathbf{w}$?

It turns out that we *can* prove this, but we need to generalize the two important concepts introduced here via this overwrought example. First, given an inner product space $\mathcal{V}$ and a subspace $\mathcal{U}$, we need a better understanding of the set (subspace?) of the form

$$\{v \in \mathcal{V} \mid \langle v, u \rangle = 0 \text{ for all } u \in \mathcal{U}\} . \tag{37.2}$$

Second, given inner product spaces $\mathcal{V}$ and $\mathcal{W}$, with inner products, respectively, $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, and $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$, we need a better understanding of the operator(s) $\mathcal{S} \in \mathbf{L}(\mathcal{W}, \mathcal{V})$ such that

$$\langle \mathcal{T}v, w \rangle_{\mathcal{W}} = \langle v, \mathcal{S}w \rangle_{\mathcal{V}} \tag{37.3}$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$.

37.8 Problem. Here is an infinite-dimensional version of the overwrought example above. Let $\mathcal{V} = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid f(x + 2\pi) = f(x),\ x \in \mathbb{R}\}$. That is, $\mathcal{V}$ is the space of all infinitely differentiable, 2π-periodic functions on $\mathbb{R}$; you do not have to prove that $\mathcal{V}$ is a vector space.

For f, $g \in \mathcal{V}$, put

$$\langle f, g \rangle := \int_0^{2\pi} f(x)g(x)\, dx.$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{V}$; you do not need to prove this. Define

$$\mathcal{T} \colon \mathcal{V} \to \mathcal{V} \colon f \mapsto f'.$$

Then $\mathcal{T} \in \mathbf{L}(\mathcal{V})$; you do not need to prove this (it should be obvious by now that $\mathcal{T}$ is linear—what is special here is that if f is 2π-periodic and differentiable, then f' is also 2π-periodic).

(i) Prove that $\langle \mathcal{T}f, g \rangle = \langle f, -\mathcal{T}g \rangle$ for all f, $g \in \mathcal{V}$. [Hint: *integrate by parts.*]

(ii) Let $u_0(x) := 1$ for all $x \in \mathbb{R}$. Prove that $\ker(\mathcal{T}) = \operatorname{span}(u_0)$.

(iii) Prove that if $h \in \mathcal{T}(\mathcal{V})$, then $\langle h, g \rangle = 0$ for all $g \in \ker(\mathcal{T})$.

(iv) Suppose that $h \in \mathcal{V}$ with $\langle h, g \rangle = 0$ for all $g \in \ker(\mathcal{T})$. Prove that $h \in \mathcal{T}(\mathcal{V})$. [Hint: *the natural idea is to set $H(x) := \int_0^x h(s)\, ds$, so $H' = h$. However, do we have $H \in \mathcal{V}$? We need $H(x + 2\pi) = H(x)$ for all $x \in \mathbb{R}$. Use properties of integrals to obtain $H(x + 2\pi) = H(x) + \int_x^{x+2\pi} h(s)\, ds$. It is a subtler property of integrals, which you do not have to prove, that when h is 2π-periodic, we have $\int_x^{x+2\pi} h(s)\, ds = \int_0^{2\pi} h(s)\, ds$ for all $x \in \mathbb{R}$. Use this fact along with the condition $\langle h, g \rangle = 0$ for all $g \in \ker(\mathcal{T})$ to obtain $H(x + 2\pi) = H(x)$.*]

Day 38: Friday, November 8.

Material from *Linear Algebra* by Meckes & Meckes

Page 252 defines orthogonal complements. See the examples there. Page 254 defines orthogonal direct sums; see the examples there, too.

Do Quick Exercise #10 on p.252 and #11 on p.255.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Orthogonal complement of a subset or subspace of an inner product space, orthogonal direct sum

Now we extract and generalize the essential ideas from that overworked, overwrought example. First we discuss sets of the form (37.2).

38.1 Definition. *Let $\mathcal{V}$ be an inner product space and $\mathcal{U} \subseteq \mathcal{V}$. The* **ORTHOGONAL COMPLEMENT** *of $\mathcal{U}$ in $\mathcal{V}$ is*

$$\mathcal{U}^\perp := \{v \in \mathcal{V} \mid \langle v, u \rangle = 0 \text{ for all } u \in \mathcal{U}\}.$$

We typically pronounce the symbol $\mathcal{U}^\perp$ as "U-perp."

That is, the orthogonal complement of $\mathcal{U}$ is the set of all vectors in $\mathcal{V}$ that are orthogonal to *all* vectors in $\mathcal{U}$.

38.2 Problem. Let $\mathcal{V}$ be an inner product space and $\mathcal{U} \subseteq \mathcal{V}$. Prove that $\mathcal{U}^\perp$ is a subspace of $\mathcal{V}$, even if $\mathcal{U}$ is just a subset, not a subspace, of $\mathcal{V}$.

38.3 Problem. Let $\mathcal{V}$ be an inner product space and $\mathcal{U} \subseteq \mathcal{V}$. What is $\mathcal{U} \cap \mathcal{U}^{\perp}$?

38.4 Example. **(i)** Continuing that overworked example, let

$$\mathcal{U} := \left\{ \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; u_1,\ u_2 \in \mathbb{R} \right\}. \tag{38.1}$$

Working with the definition of $\mathcal{U}^{\perp}$ may be intimidating, as $\mathbf{v} \in \mathcal{U}^{\perp}$ if and only if $\mathbf{v} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in \mathcal{U}$. But that quantifier "for all" offers us freedom: we can pick $\mathbf{u}$ to be something convenient, and few vectors are more convenient than the standard basis vectors. Since $\mathbf{e}_1$, $\mathbf{e}_2 \in \mathcal{U}$, if $\mathbf{v} \in \mathcal{U}^{\perp}$, then

$$v_1 = \mathbf{v} \cdot \mathbf{e}_1 = 0 \quad \text{and} \quad v_2 = \mathbf{v} \cdot \mathbf{e}_2 = 0.$$

Thus $\mathbf{v} \in \operatorname{span}(\mathbf{e}_3)$, and a direct calculation shows that $\mathbf{u} \cdot \mathbf{e}_3 = 0$ for all $\mathbf{u} \in \mathcal{U}$. We conclude $\mathcal{U}^{\perp} = \operatorname{span}(\mathbf{e}_3)$.

(ii) Let $\mathcal{V} = \{f \in \mathcal{C}^{\infty}(\mathbb{R}) \mid f(x + 2\pi) = f(x) \text{ for all } x \in \mathcal{V}\}$ with inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\ dx$, as in Example 37.8. Let $u_0(x) = 1$. We have $f \in \{u_0\}^{\perp}$ if and only if $\langle f, u_0 \rangle = 0$, thus if and only if $\int_0^{2\pi} f(x)\ dx = 0$. That is,

$$\{u_0\}^{\perp} = \left\{ f \in \mathcal{V} \;\middle|\; \int_0^{2\pi} f(x)\ dx = 0 \right\}.$$

38.5 Problem. Let $\mathcal{V}$ be an inner product space. What are $\mathcal{V}^{\perp}$ and $\{0\}^{\perp}$?

With $\mathcal{U}$ defined in (38.1), we have seen that each $\mathbf{v} \in \mathbb{R}^3$ can be written uniquely in the form $\mathbf{v} = \mathbf{u} + \mathbf{u}^{\perp}$ for some $\mathbf{u} \in \mathcal{U}$ and $\mathbf{u}^{\perp} \in \mathcal{U}^{\perp}$; by "uniquely," we mean that there is only one choice of $\mathbf{u}$ and $\mathbf{u}^{\perp}$ that makes this equality work. This is true in a much more general context for an inner product space $\mathcal{V}$ and a finite-dimensional subspace $\mathcal{U}$.

For motivation, we first consider the case of an inner product space $\mathcal{V}$ with $\mathcal{U} = \operatorname{span}(u_1)$ for a single vector $u_0 \in \mathcal{V}$ with $\|u_1\| = 1$. The list (u_1) is an orthonormal basis for $\mathcal{U}$, and we may as well work with orthonormal bases whenever we can. The following should feel very similar to the $n = 1$ case for the proof of Gram–Schmidt.

We want to write any $v \in \mathcal{V}$ uniquely in the form $v = u + u^{\perp}$, with $u \in \operatorname{span}(u_1)$ and $u^{\perp} \in \operatorname{span}(u_1)^{\perp}$. What are good candidates for u and $u^{\perp}$? We will focus on *existence* of this decomposition first and then worry about uniqueness. If we know u, then $u^{\perp} = v - u$. And since $u \in \operatorname{span}(u_1)$, we need $u = \alpha u_1$ for some $\alpha \in \mathbb{F}$. Then $u^{\perp} = v - \alpha u_1$, and since we need $\langle u, u^{\perp} \rangle = 0$, this means we need

$$0 = \langle \alpha u_1, v - \alpha u_1 \rangle = \langle \alpha u_1, v \rangle - \langle \alpha u_1, \alpha u_1 \rangle = \alpha \langle u_1, v \rangle - \alpha\overline{\alpha} \|u_1\|^2 = \alpha \langle u_1, v \rangle - \alpha\overline{\alpha}$$

since $\|u_1\|^2 = 1$.

Thus

$$\alpha\big(\langle u_1, v\rangle - \overline{\alpha}\big) = 0,$$

so either $\alpha = 0$ or $\alpha = \langle v, u_1\rangle$. The case $\alpha = 0$ means $v = u^\perp$, and so we really had $v \in \operatorname{span}(u_1)^\perp$ from the beginning. (This is not wholly enlightening and does not address how to figure out what $\operatorname{span}(u_1)^\perp$ really is, beyond the definition.) The case $\alpha = \langle v, u_1\rangle$ is more interesting, as it means $u = \langle v, u_1\rangle u_1$.

38.6 Problem. In the context and notation of Problem 37.5, prove that $\langle v, u_1\rangle u_1 = \mathcal{P}_{u_1} v$.

Now we are ready to work with subspaces $\mathcal{U}$ of arbitrary finite dimension.

38.7 Theorem. *Let $\mathcal{V}$ be an inner product space and let $\mathcal{U}$ be a finite-dimensional subspace of $\mathcal{V}$. Then for each $v \in \mathcal{V}$, there exist unique $u \in \mathcal{U}$ and $u^\perp \in \mathcal{U}^\perp$ such that $v = u + u^\perp$.*

Proof. Theorem 4.14 in the book.

First we prove existence of the decomposition. Since $\mathcal{U}$ is a finite-dimensional subspace of $\mathcal{V}$, it has (by the Gram–Schmidt process) an orthonormal basis $(u_1, \ldots, u_n)$. This is one reason that we require $\mathcal{U}$ to be a finite-dimensional subspace of $\mathcal{V}$ and do not allow it to be a general subset or an infinite-dimensional subspace. Drawing on the one-dimensional motivation above, we put

$$\mathcal{P}_{\mathcal{U}} \colon \mathcal{V} \to \mathcal{U} \colon v \mapsto \sum_{k=1}^{n} \langle v, u_k\rangle u_k.$$

We claim that taking $u = \mathcal{P}_{\mathcal{U}} v$ and $u^\perp = v - \mathcal{P}_{\mathcal{U}} v = (I - \mathcal{P}_{\mathcal{U}})v$ works. Certainly this gives $u \in \mathcal{U}$ and $u + u^\perp = v$; we have to check $\langle u, u^\perp\rangle = 0$.

All we can do is compute:

$$\langle \mathcal{P}_{\mathcal{U}} v, v - \mathcal{P}_{\mathcal{U}} v\rangle = \langle \mathcal{P}_{\mathcal{U}} v, v\rangle - \langle \mathcal{P}_{\mathcal{U}} v, \mathcal{P}_{\mathcal{U}} v\rangle.$$

The first inner product is

$$\langle \mathcal{P}_{\mathcal{U}} v, v\rangle = \left\langle \sum_{k=1}^{n} \langle v, u_k\rangle u_k, v \right\rangle = \sum_{k=1}^{n} \langle v, u_k\rangle \langle u_k, v\rangle = \sum_{k=1}^{n} \langle v, u_k\rangle \overline{\langle v, u_k\rangle} = \sum_{j=1}^{n} |\langle v, u_k\rangle|^2.$$

The second inner product is just $\langle \mathcal{P}_{\mathcal{U}} v, \mathcal{P}_{\mathcal{U}} v\rangle = \|\mathcal{P}_{\mathcal{U}} v\|^2$, and since $\mathcal{P}_{\mathcal{U}} v \in \operatorname{span}(u_1, \ldots, u_n)$ and $(u_1, \ldots, u_n)$ is orthonormal, we have

$$\|\mathcal{P}_{\mathcal{U}} v\|^2 = \sum_{k=1}^{n} |\langle v, u_k\rangle|^2 = \langle \mathcal{P}_{\mathcal{U}} v, v\rangle$$

with the last equality following from the work above. Thus $\langle \mathcal{P}_{\mathcal{U}} v, v - \mathcal{P}_{\mathcal{U}} v\rangle = 0$, as desired.

Now we check uniqueness. Suppose we have two such decompositions:

$$v = u + u^\perp \quad \text{and} \quad v = \widetilde{u} + \widetilde{u}^\perp, \qquad u,\ \widetilde{u} \in \mathcal{U}, \qquad u^\perp,\ \widetilde{u}^\perp \in \mathcal{U}^\perp.$$

One way to get all of these components in dialogue with each other is subtraction:

$$0 = (u + u^\perp) - (\widetilde{u} + \widetilde{u}^\perp) = (u - \widetilde{u}) + (u^\perp - \widetilde{u}^\perp).$$

Put

$$w := u - \widetilde{u} \quad \text{and} \quad w^\perp := u^\perp - \widetilde{u}^\perp.$$

We have $w + w^\perp = 0$, and we want to show $w = w^\perp = 0$.

Since $\mathcal{U}$ is a subspace, $w \in \mathcal{U}$, and since $\mathcal{U}^\perp$ is a subspace (which, recall, is true even if $\mathcal{U}$ is not a subspace), $w^\perp \in \mathcal{U}^\perp$. Thus $\langle w, w^\perp \rangle = 0$. We may as well try to us this in conjunction with the identity $w + w^\perp = 0$, so we compute

$$0 = \langle w + w^\perp, w \rangle = \langle w, w \rangle + \langle w^\perp, w \rangle = \|w\|^2. \tag{38.2}$$

Thus $\|w\| = 0$, and so $w = 0$. But then $0 = 0 + w^\perp = w^\perp$, as desired. ■

38.8 Problem. Convince yourself that the uniqueness argument did not rely on dimension, but it did require $\mathcal{U}$ to be a subspace. That is, if $\mathcal{U}$ is any subspace of an inner product space $\mathcal{V}$, then there is *at most* one way (possible no way) to write $v \in \mathcal{V}$ in the form $v = u + u^\perp$ for $u \in \mathcal{U}$ and $u^\perp \in \mathcal{U}^\perp$.

The proof of the previous theorem was *constructive*: it gave us formulas for u and $u^\perp$. We record them here, separately, for future reference.

38.9 Corollary. *Let $\mathcal{V}$ be an inner product space and let $\mathcal{U}$ be a finite-dimensional subspace of $\mathcal{V}$ with orthonormal basis $(u_1, \ldots, u_n)$. Then every $v \in \mathcal{V}$ can be written uniquely as $v = u + u^\perp$, where*

$$u = \mathcal{P}_{\mathcal{U}} v, \qquad u^\perp = v - \mathcal{P}_{\mathcal{U}} v, \qquad \mathcal{P}_{\mathcal{U}} v := \sum_{k=1}^n \langle v, u_k \rangle u_k.$$

38.10 Problem. **(i)** Let $\mathcal{V} = \mathbb{R}^3$ with inner product given by the dot product, and let

$$\mathcal{U} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right).$$

Given $\mathbf{v} \in \mathbb{R}^3$, find $\mathbf{u} \in \mathcal{U}$ and $\mathbf{u}^\perp \in \mathcal{U}^\perp$ such that $\mathbf{v} = \mathbf{u} + \mathbf{u}^\perp$.

(ii) Let $\mathcal{V} = \mathcal{C}([-\pi, \pi])$ with inner product given by $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$, and let $f_1(x) = 1$ and $f_2(x) = \sin(x)$. Put $\mathcal{U} = \operatorname{span}(f_1, f_2)$. Given $f \in \mathcal{V}$, find $g \in \mathcal{U}$ and $g^\perp \in \mathcal{U}^\perp$ such that $f = g + g^\perp$. [Hint: $\int_{-\pi}^{\pi} \sin^2(x)\, dx = \pi$.]

We can generalize the situation in the previous excellent theorem to the case when $\mathcal{U}$ is not necessarily finite-dimensional.

38.11 Definition. *Let $\mathcal{V}$ be an inner product space and let $\mathcal{U}$ be a subspace of $\mathcal{V}$. Suppose that for each $v \in \mathcal{V}$, there are $u \in \mathcal{U}$ and $u^\perp \in \mathcal{U}^\perp$ such that $v = u + u^\perp$. (Such u and $u^\perp$ are necessarily unique by the previous problem.) Then say that $\mathcal{V}$ is the* **ORTHOGONAL DIRECT SUM** *of $\mathcal{U}$ and $\mathcal{U}^\perp$, and we abbreviate this by $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$.*

Given $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$, it is natural to wonder if when $v \in \mathcal{V}$ but $v \notin \mathcal{U}$, do we have $v \in \mathcal{U}^\perp$? Not really.

38.12 Problem. Let $\mathcal{V}$ be an inner product space and let $\mathcal{U}$ be a nontrivial proper (i.e., $\mathcal{U} \neq \{0\}$ and $\mathcal{U} \neq \mathcal{V}$) subspace of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$.

(i) Prove that $\mathcal{U}^\perp \neq \{0\}$. [Hint: *Problem 38.5.*]

(ii) Why should we expect $\mathcal{V} \neq \mathcal{U} \cup \mathcal{U}^\perp$? [Hint: *if $u \in \mathcal{U} \setminus \{0\}$ and $u^\perp \in \mathcal{U}^\perp \setminus \{0\}$, do we have $u + u^\perp$ in that union?*]

(iii) With $\mathcal{V} = \mathbb{R}^2$ (and the inner product being, as always, the dot product) and $\mathcal{U} = \operatorname{span}(\mathbf{e}_1)$, draw a picture illustrating why $\mathbb{R}^2 \neq \operatorname{span}(\mathbf{e}_1) \cup \operatorname{span}(\mathbf{e}_1)^\perp$.

(iv) Let $v \in \mathcal{V}$ with $v \notin \mathcal{U}$. Explain why there exists $u^\perp \in \mathcal{U}^\perp$ such that $\langle v, u^\perp \rangle \neq 0$.

(v) Suppose that $\mathcal{V}$ is finite-dimensional, so both $\mathcal{U}$ and $\mathcal{U}^\perp$ are finite-dimensional, too. Let $(u_1, \ldots, u_n)$ be a basis for $\mathcal{U}$ and $(u_1^\perp, \ldots, u_m^\perp)$ be a basis for $\mathcal{U}^\perp$. Prove that $(u_1, \ldots, u_n, u_1^\perp, \ldots, u_m^\perp)$ is a basis for $\mathcal{V}$ and thus $\dim[\mathcal{V}] = \dim[\mathcal{U}] + \dim[\mathcal{U}^\perp]$. [Hint: *that this lists spans $\mathcal{V}$ should be a quick consequence of $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$. For linear independence, how does Problem 38.3 suggest what happens when $\sum_{k=1}^n \alpha_k u_k = \sum_{k=1}^m \beta_k u_k^\perp$ for α_k, $\beta_k \in \mathbb{F}$?*]

A natural question, then, is if $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$ for any subspace $\mathcal{U}$ of $\mathcal{V}$, not just finite-dimensional ones. The answer is decidedly no, and we will give an example to that effect. Before doing so, we consider another feature of orthogonal complements.

If we think about right angles in two or three dimensions from real life, we may be led to conjecture that $(\mathcal{U}^\perp)^\perp = \mathcal{U}$. Consider, hopefully for the last time, the subspace $\mathcal{U}$ of $\mathbb{R}^3$ defined in (38.1) with $\mathcal{U}^\perp = \operatorname{span}(\mathbf{e}_3)$. We have $(\mathcal{U}^\perp)^\perp = \operatorname{span}(\mathbf{e}_3)^\perp$. If $\langle \mathbf{v}, \mathbf{e}_3 \rangle = 0$, then $v_3 = 0$, and so $\mathbf{v} \in \mathcal{U}$; conversely, if $\mathbf{v} \in \mathcal{U}$, then $\langle \mathbf{v}, \alpha\mathbf{e}_3 \rangle = 0$ for all $\alpha \in \mathbb{R}$. Thus $\operatorname{span}(\mathbf{e}_3)^\perp = \mathcal{U}$, and our conjecture works out here. This is true in general for finite-dimensional $\mathcal{U}$.

38.13 Theorem. *Let $\mathcal{U}$ be a finite-dimensional subspace of the inner product space $\mathcal{V}$. Then $(\mathcal{U}^\perp)^\perp = \mathcal{U}$.*

Proof. Proposition 4.15 in the book.

First suppose $u \in \mathcal{U}$. We need to show that $\langle u, v \rangle = 0$ for every $v \in \mathcal{U}^\perp$. So, take some $v \in \mathcal{U}^\perp$. Then $\langle v, w \rangle = 0$ for all $w \in \mathcal{U}$. We are free to choose $w = u$ to conclude $\langle v, u \rangle = 0$, as desired.

Now let $v \in (\mathcal{U}^\perp)^\perp$. We want to show $v \in \mathcal{U}$. Since $\mathcal{U}$ is a finite-dimensional subspace of $\mathcal{V}$, we have the decomposition $v = u + u^{\perp}$ for some $u \in \mathcal{U}$ and $u^{\perp} \in \mathcal{U}^{\perp}$. If we can show $u^{\perp} = 0$, then we will have $v = u \in \mathcal{U}$. We now know three things: $v = u + u^{\perp}$, $\langle u, u^{\perp} \rangle = 0$, and $\langle v, w \rangle = 0$ for all $w \in \mathcal{U}^{\perp}$. Taking $w = u^{\perp}$,

$$0 = \langle v, w \rangle = \langle u + u^{\perp}, u^{\perp} \rangle = \langle u, u^{\perp} \rangle + \langle u^{\perp}, u^{\perp} \rangle = \|u^{\perp}\|^2,$$

thus $u^{\perp} = 0$. This should smack of the calculation (38.2). ■

38.14 Problem. Convince yourself that proving the containment $\mathcal{U} \subseteq (\mathcal{U}^{\perp})^{\perp}$ did not require $\mathcal{U}$ to be finite-dimensional, or even a subspace at all.

Day 39: Monday, November 11.

Material from *Linear Algebra* by Meckes & Meckes

Page 311 defines the adjoint of a linear operator, and various examples and properties appear on pp.312–314.

Do Quick Exercise #9 on p.312.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Adjoint of a linear operator

Our proofs that $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^{\perp}$ and $(\mathcal{U}^{\perp})^{\perp}$ for an inner product space $\mathcal{V}$ and a subspace $\mathcal{U}$ of $\mathcal{V}$ used, at various key points, the hypothesis that $\mathcal{U}$ was finite-dimensional. What happens if we remove this hypothesis?

Nothing good. We illustrate this with a concrete example. Let $\mathcal{V} = \ell^2$, which we recall is the space of all real-valued "square-summable" sequences:

$$\ell^2 = \left\{ (a_k) \in \mathbb{R}^{\infty} \;\middle|\; \sum_{k=1}^{\infty} |a_k|^2 \text{ converges} \right\}.$$

The inner product on ℓ^2 is

$$\langle (a_k), (b_k) \rangle = \sum_{k=1}^{\infty} a_k b_k.$$

Strictly speaking, we have not proved that ℓ^2 is a vector space or that the series defining $\langle (a_k), (b_k) \rangle$ converges when (a_k), $(b_k) \in \ell^2$. We will. For now, we think of ℓ^2 as a natural, infinite-dimensional generalization of $\mathbb{R}^n$.

39.1 Problem. Recall that $\sum_{k=1}^{\infty} k^{-p}$ converges for any $p > 1$. Prove that $(k^{-p/2}) \in \ell^2$ for any $p > 1$. In particular, $(1/k) \in \ell^2$, notwithstanding our natural squeamishness about the harmonic series.

It will ease our notational burdens somewhat to think of elements of ℓ^2 as what they really are: functions from $\mathbb{N}$ to $\mathbb{R}$. So, we will (mostly) write elements of ℓ^2 as f and g, not (a_k) and (b_k), with the understanding that $f = (f(k))$ in "sequence" notation. In particular, for $j \in \mathbb{N}$, we define

$$e_j \colon \mathbb{N} \to \mathbb{R} \colon k \mapsto \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases}$$

For example, we might write $e_1 = (1, 0, 0, 0, \ldots)$ and $e_3 = (0, 0, 1, 0, \ldots)$. We put

$$\mathcal{U} := \{ f \in \ell^2 \mid f(k) = 0 \text{ for all but finitely many } k. \}$$

So, $e_j \in \mathcal{U}$ for all j, but $(1/k) \notin \mathcal{U}$.

39.2 Problem. Prove that $f \in \mathcal{U}$ if and only if there is $M > 0$ such that $f(k) = 0$ for $k \geq M$.

Now we are ready to break things.

39.3 Example. **(i)** With $\mathcal{U}$ defined above, let $g \in \mathcal{U}^{\perp}$. Then $\langle g, f \rangle = 0$ for all $f \in \mathcal{U}$, so in particular $\langle g, e_j \rangle = 0$ for all j. But then $0 = \langle g, e_j \rangle = g(j)$ for all j, and so $g = 0$. That is, $\mathcal{U}^{\perp} = \{0\}$.

We claim that we cannot write every $f \in \ell^2$ uniquely in the form $f = g + g^{\perp}$ for $g \in \mathcal{U}$ and $g^{\perp} \in \mathcal{U}^{\perp}$. Indeed, we would have $g^{\perp} = 0$, and so $f = g + 0 = g \in \mathcal{U}$. But then $\ell^2 = \mathcal{U}$, and we know $(1/k) \in \ell^2$ and $(1/k) \notin \mathcal{U}$. This shows that we need not have $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^{\perp}$ for an arbitrary inner product space $\mathcal{V}$ and an arbitrary subspace $\mathcal{U}$.

(ii) We showed above that $\mathcal{U}^{\perp} = \{0\}$. Since $\{0\}^{\perp} = \ell^2$, we have $(\mathcal{U}^{\perp})^{\perp} = \{0\}^{\perp} = \ell^2$. But $\ell^2 \neq \mathcal{U}$, so $(\mathcal{U}^{\perp})^{\perp} \neq \mathcal{U}$.

If we want results like $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^{\perp}$ and $(\mathcal{U}^{\perp})^{\perp} = \mathcal{U}$ for an infinite-dimensional subspace of $\mathcal{V}$, then we need *something else*. We have to assume that $\mathcal{U}$ enjoys some other properties. While we will not get into this here, such properties can be made explicit by introducing more of an analytic and topological structure into $\mathcal{V}$ via the norm induced by the inner product.

Instead, we return to study operators that satisfy the relation (37.3). That is, we consider two inner product spaces $\mathcal{V}$ and $\mathcal{W}$, with inner products $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, respectively, and a linear operator $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$, and we try to understand the existence (if possible) and properties (given existence) of an operator $\mathcal{S} \colon \mathcal{W} \to \mathcal{V}$ such that

$$\langle \mathcal{T} v, w \rangle_{\mathcal{W}} = \langle v, \mathcal{S} w \rangle_{\mathcal{V}} \tag{39.1}$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. We will find that existence is not guaranteed but uniqueness is.

39.4 Lemma. *There exists at most one such $\mathcal{S}$.*

Proof. Lemma 5.11 in the book. Suppose that $\mathcal{S}_1, \mathcal{S}_2 \colon \mathcal{W} \to \mathcal{V}$ are linear operators such that

$$\langle \mathcal{T}v, w\rangle_{\mathcal{W}} = \langle v, \mathcal{S}_1 w\rangle_{\mathcal{V}} \quad \text{and} \quad \langle \mathcal{T}v, w\rangle_{\mathcal{W}} = \langle v, \mathcal{S}_2 w\rangle_{\mathcal{V}}$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. We want to show that $\mathcal{S}_1 = \mathcal{S}_2$, equivalently, $\mathcal{S}_1 w = \mathcal{S}_2 w$ for all $w \in \mathcal{W}$. And by linearity, that is equivalent to $(\mathcal{S}_1 - \mathcal{S}_2)w = 0$ for all $w \in \mathcal{W}$. Since $(\mathcal{S}_1 - \mathcal{S}_2)w \in \mathcal{V}$, we have $(\mathcal{S}_1 - \mathcal{S}_2)w = 0$ if and only if $\langle v, (\mathcal{S}_1 - \mathcal{S}_2)w\rangle_{\mathcal{V}} = 0$ for all $v \in \mathcal{V}$.

Thus we want to show that $\langle v, (\mathcal{S}_1 - \mathcal{S}_2)w\rangle_{\mathcal{V}} = 0$ for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. We compute

$$\langle v, (\mathcal{S}_1 - \mathcal{S}_2)w\rangle_{\mathcal{V}} = \langle v, \mathcal{S}_1 w\rangle_{\mathcal{V}} - \langle v, \mathcal{S}_2 w\rangle_{\mathcal{V}} = \langle \mathcal{T}v, w\rangle_{\mathcal{W}} - \langle \mathcal{T}v, w\rangle_{\mathcal{W}} = 0,$$

as desired. ■

Since there is at most one operator in $\mathbf{L}(\mathcal{W}, \mathcal{V})$ satisfying (39.1), it is fair to give this operator a special name and notation, if it exists.

39.5 Definition. *Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces with inner products $\langle \cdot, \cdot\rangle_{\mathcal{V}}$ and $\langle \cdot, \cdot\rangle_{\mathcal{W}}$, respectively, and let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. An operator $\mathcal{S} \in \mathbf{L}(\mathcal{W}, \mathcal{V})$ is the* ADJOINT *of $\mathcal{T}$ if*

$$\langle \mathcal{T}v, w\rangle_{\mathcal{W}} = \langle v, \mathcal{S}w\rangle_{\mathcal{V}}, \ \text{for all } v \in \mathcal{V},\ w \in \mathcal{W}, \tag{39.2}$$

and we write $\mathcal{S} = \mathcal{T}^*$.

39.6 Remark. *Strictly speaking, an adjoint depends on the underlying inner products. Sometimes we can define multiple meaningful inner products on the same vector space, and then we would have to be careful from context about the inner products with which an operator's adjoint interacts.*

We have had the most success working on finite-dimensional vector spaces, so it is natural to wonder what advantage finite-dimensionality might give in understanding adjoints. It turns out that if the domain is finite-dimensional, then every operator has an adjoint.

39.7 Example. Theorem 5.12 in the book. Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces with inner products $\langle \cdot, \cdot\rangle_{\mathcal{V}}$ and $\langle \cdot, \cdot\rangle_{\mathcal{W}}$, respectively, and let $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. Suppose that $\mathcal{V}$ is finite-dimensional, and let $(u_1, \ldots, u_n)$ be an orthonormal basis for $\mathcal{V}$. How should we define $\mathcal{T}^* \in \mathbf{L}(\mathcal{W}, \mathcal{V})$? If the adjoint exists, it must satisfy $\langle \mathcal{T}v, w\rangle_{\mathcal{W}} = \langle v, \mathcal{T}^* w\rangle_{\mathcal{V}}$ for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$.

More fundamentally, for each $w \in \mathcal{W}$ we will have $\mathcal{T}^* w \in \mathcal{V}$, and so we can write

$$\mathcal{T}^* w = \sum_{k=1}^n \langle \mathcal{T}^* w, u_k\rangle_{\mathcal{V}} u_k.$$

Can we convert this into an expression in terms of $\mathcal{T}$? We are used to seeing $\mathcal{T}^*$ in the other slot of the inner product, so we rewrite

$$\langle \mathcal{T}^* w, u_k\rangle_{\mathcal{V}} = \overline{\langle u_k, \mathcal{T}^* w\rangle_{\mathcal{V}}} = \overline{\langle \mathcal{T}u_k, w\rangle_{\mathcal{W}}} = \langle w, \mathcal{T}u_k\rangle_{\mathcal{W}}.$$

This suggests that *if* $\mathcal{T}^*$ is defined, *then* the only reasonable formula for $\mathcal{T}^*$ is

$$\mathcal{T}^* w = \sum_{k=1}^{n} \langle w, \mathcal{T} u_k \rangle_{\mathcal{W}} u_k. \tag{39.3}$$

39.8 Problem. Under the hypotheses and notation of the example above, prove that defining $\mathcal{T}^*$ via (39.3) really does give a linear operator from $\mathcal{W}$ to $\mathcal{V}$ that satisfies the fundamental property (39.2) of the adjoint.

Day 40: Wednesday, November 13.

Material from *Linear Algebra* by Meckes & Meckes

The conjugate transpose of a matrix is defined on p.97.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Conjugate transpose of a matrix, self-adjoint linear operator (N), skew-adjoint linear operator

40.1 Example. Let $A \in \mathbb{F}^{m \times n}$, and let $\mathcal{T}_A \colon \mathbb{F}^n \to \mathbb{F}^m \colon \mathbf{v} \mapsto A\mathbf{v}$. What is $\mathcal{T}_A^*$? We know what this adjoint *does*:

$$\langle \mathcal{T}_A \mathbf{v}, \mathbf{w} \rangle_m = \langle \mathbf{v}, \mathcal{T}_A^* \mathbf{w} \rangle_n,$$

where $\langle \cdot, \cdot \rangle_p$ is the dot product on $\mathbb{F}^p$. And we know what it *is*:

$$\mathcal{T}_A^* \mathbf{w} = \sum_{k=1}^{n} \langle \mathbf{w}, \mathcal{T}_A \mathbf{e}_k \rangle_m \mathbf{e}_k = \sum_{k=1}^{n} \langle \mathbf{w}, A\mathbf{e}_k \rangle_m \mathbf{e}_k$$

with $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ as the standard basis for $\mathbb{F}^n$.

It is natural to desire the matrix representation of $\mathcal{T}_A^*$. Since $\mathcal{T}_A^* \colon \mathbb{F}^m \to \mathbb{F}^n$ is linear, there must be $[\mathcal{T}_A^*] \in \mathbb{F}^{n \times m}$ such that $\mathcal{T}_A^* \mathbf{w} = [\mathcal{T}_A^*]\mathbf{w}$. Of course, the jth column of $[\mathcal{T}_A^*]$ is $\mathcal{T}_A^* \widetilde{\mathbf{e}}_j = \sum_{k=1}^{n} \langle \widetilde{\mathbf{e}}_j, A\mathbf{e}_k \rangle_m \mathbf{e}_k$, where $(\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_m)$ is the standard basis for $\mathbb{F}^m$. Just to be clear, if, say, $n = 3$ and $m = 2$, then

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{but} \quad \widetilde{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The kth entry of this column is therefore $\langle \widetilde{\mathbf{e}}_j, A\mathbf{e}_k \rangle_m$, and that is the (k, j)-entry of $[\mathcal{T}_A^*]$. Another way to see this is that the (i, j)-entry of $[\mathcal{T}_A^*]$ is $\langle [\mathcal{T}_A^*]\widetilde{\mathbf{e}}_j, \mathbf{e}_i \rangle_n$, and that is

$$\langle [\mathcal{T}_A^*]\widetilde{\mathbf{e}}_j, \mathbf{e}_i \rangle_n = \langle \mathcal{T}_A^*\widetilde{\mathbf{e}}_j, \mathbf{e}_i \rangle_n = \left\langle \sum_{k=1}^n \langle \widetilde{\mathbf{e}}_j, A\mathbf{e}_k \rangle_m, \mathbf{e}_i \right\rangle_n = \sum_{k=1}^n \langle \widetilde{\mathbf{e}}_j, A\mathbf{e}_k \rangle_m \langle \mathbf{e}_k, \mathbf{e}_i \rangle_n$$
$$= \langle \widetilde{\mathbf{e}}_j, A\mathbf{e}_i \rangle_m = \overline{\langle A\mathbf{e}_i, \widetilde{\mathbf{e}}_j \rangle_m}$$

since $\langle \mathbf{e}_k, \mathbf{e}_i \rangle_n = 1$ for $k = i$ and 0 otherwise.

What does this mean in terms of A? Recall that the (i,j)-entry of A is $\langle A\mathbf{e}_j, \widetilde{\mathbf{e}}_i \rangle_m$, and so the (j,i)-entry of A is $\langle A\mathbf{e}_i, \widetilde{\mathbf{e}}_j \rangle_m$. *Thus the (i,j)-entry of $[\mathcal{T}_A^*]$ is the conjugate of the (j,i)-entry of A.*

We name this special kind of matrix.

40.2 Definition. *Let $A \in \mathbb{F}^{m \times n}$. The* CONJUGATE TRANSPOSE *of A is the matrix $A^* \in \mathbb{F}^{n \times m}$ such that the (i,j)-entry of A^* is the conjugate of the (j,i)-entry of A.*

The point of the preceding definition and example is that if $A \in \mathbb{F}^{m \times n}$ and $\mathcal{T}_A$ is multiplication by A, so $[\mathcal{T}_A] = A$, then (with respect to the dot products)

$$[\mathcal{T}_A^*] = A^*.$$

Informally, "the columns of A^* are the rows of A, and the rows of A^* are the columns of A."

40.3 Example. If

$$A := \begin{bmatrix} 1 & i \\ -2i & 3 \\ 4 & 5 \end{bmatrix},$$

then

$$A^* = \begin{bmatrix} \overline{1} & \overline{-2i} & \overline{4} \\ \overline{i} & \overline{3} & \overline{5} \end{bmatrix} = \begin{bmatrix} 1 & 2i & 4 \\ -i & 3 & 5 \end{bmatrix}.$$

We give some more examples of adjoints on infinite-dimensional spaces.

40.4 Example. Let $\mathcal{V} = \mathcal{C}([0,1])$, both with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$. Let $m \in \mathcal{C}([0,1])$ and let $\mathcal{T}\colon \mathcal{V} \to \mathcal{V}$ be the multiplication operator $(\mathcal{T}f)(x) = m(x)f(x)$. We want to find $\mathcal{T}^*\colon \mathcal{V} \to \mathcal{V}$ such that $\langle \mathcal{T}f, g \rangle = \langle f, \mathcal{T}^*g \rangle$, where

$$\langle \mathcal{T}f, g \rangle = \int_0^1 (\mathcal{T}f)(x)g(x)\,dx = \int_0^1 m(x)f(x)g(x)\,dx.$$

Since

$$\langle f, \mathcal{T}^*g \rangle = \int_0^1 f(x)(\mathcal{T}^*g)(x)\,dx, \quad \text{and} \quad \langle \mathcal{T}f, g \rangle = \int_0^1 f(x)\big[m(x)g(x)\big]\,dx,$$

this suggests that we just take $(\mathcal{T}^*g)(x) = m(x)g(x)$. That is, here $\mathcal{T}^* = \mathcal{T}$.

Operators that are their own adjoint are particularly nice (in the sense that they have many clear and useful properties) and deserve a special name.

40.5 Definition. *Let $\mathcal{V}$ be an inner product space. An operator $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ is* SELF-ADJOINT *if $\mathcal{T}^* = \mathcal{T}$.*

40.6 Problem. Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces with $\mathcal{V} \neq \mathcal{W}$. Why does it not really make sense to talk about a self-adjoint operator $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$?

40.7 Example. Let

$$\mathcal{V} = \{f \in \mathcal{C}^\infty([0,1]) \mid f(0) = f(1) = f'(0) = f'(1) = 0\}$$

and

$$\mathcal{W} = \{g \in \mathcal{C}^\infty([0,1]) \mid g(0) = g(1) = 0\},$$

both with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\ dx$. Let $\mathcal{T} \colon \mathcal{V} \to \mathcal{V}$ be differentiation: $\mathcal{T}f = f'$. Finding the adjoint of $\mathcal{T}$ involves manipulating integrals of the form $\int_0^1 f'(x)g(x)\ dx$, and that is best done via integration by parts:

$$\langle \mathcal{T}f, g\rangle = \int_0^1 f'(x)g(x)\ dx = f(1)g(1) - f(0)g(0) - \int_0^1 f(x)g'(x)\ dx = \int_0^1 f(x)[-g'(x)]\ dx.$$

The third equality holds because $f(0) = f(1) = 0$ for any $f \in \mathcal{V}$. Thus $\mathcal{T}^*g = -g'$, and so $\mathcal{T}^* = -\mathcal{T}$.

40.8 Problem. Suppose that in the previous example we try to change $\mathcal{V}$ and $\mathcal{W}$ to the following pairs. Explain precisely where you get stuck when you try to replicate the work above.

(i) $\mathcal{V} = \mathcal{W} = \mathcal{C}^\infty([0,1])$.

(ii) $\mathcal{V} = \{f \in \mathcal{C}^\infty([0,1]) \mid f(0) = f(1) = 0\}$, $\mathcal{W} = \{g \in \mathcal{C}^\infty([0,1]) \mid g(0) = g(1) = 0\}$.

(iii) $\mathcal{V} = \{f \in \mathcal{C}^1([0,1]) \mid f(0) = f(1) = f'(0) = f'(1) = 0\}$,
$\mathcal{W} = \{g \in \mathcal{C}([0,1]) \mid g(0) = g(1) = 0\}$.

Operators $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ with $\mathcal{T}^* = -\mathcal{T}$ also have a special name.

40.9 Definition. *Let $\mathcal{V}$ be an inner product space. An operator $\mathcal{T} \in \mathbf{L}(\mathcal{V})$ is* SKEW-ADJOINT *if $\mathcal{T}^* = -\mathcal{T}$.*

40.10 Example. Let

$$\mathcal{V} = \mathcal{W} = \{f \in \mathcal{C}^\infty([-\pi,\pi]) \mid f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R}\},$$

both with inner product $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Let $\mathcal{T}\colon \mathcal{V} \to \mathcal{W}$ be the shift operator $(\mathcal{T}f)(x) := f(x+1)$. To find the adjoint of $\mathcal{T}$, we manipulate

$$\langle \mathcal{T}f, g\rangle = \int_{-\pi}^{\pi} (\mathcal{T}f)(x)g(x)\,dx = \int_{-\pi}^{\pi} f(x+1)g(x)\,dx.$$

We would like to turn this into an integral involving only a factor of $f(x)$ and "something" involving g, and the way to do that is to "remove" the $x+1$ by substituting $s = x+1$, $ds = dx$, $s(-\pi) = -\pi+1$, and $s(\pi) = \pi+1$ to find

$$\int_{-\pi}^{\pi} f(x+1)g(x)\,dx = \int_{1-\pi}^{1+\pi} f(s)g(s-1)\,ds.$$

The problem now is that this integral is not over $[-\pi,\pi]$, and so it is not really the original inner product.

But it is: if $h \in \mathcal{C}(\mathbb{R})$ is 2π-periodic, then the fundamental theorem of calculus gives

$$\partial_x\left[\int_{x-\pi}^{x+\pi} h(s)\,ds\right] = h(x+\pi)-h(x-\pi) = h((x-\pi)+2\pi)-h(x-\pi) = h(x-\pi)-h(x-\pi) = 0.$$

Thus the map $x \mapsto \int_{x-\pi}^{x+\pi} h(s)\,ds$ is constant. In the particular case above, $h(s) = f(s)g(s-1)$ and so

$$\int_{1-\pi}^{1+\pi} f(s)g(s-1)\,ds = \int_{-\pi}^{\pi} f(s)g(s-1)\,ds.$$

Thus with $(\mathcal{T}^*g)(s) := g(s-1)$, we have $\langle \mathcal{T}f, g\rangle = \langle f, \mathcal{T}^*g\rangle$.

40.11 Problem. Recall from multivariable calculus that if h is continuous on the unit square $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1\}$, then

$$\int_0^1 \int_0^x h(x,y)\,dy\,dx = \int_0^1 \int_y^1 h(x,y)\,dx\,dy.$$

Use this fact to find the adjoint of the antiderivative operator $\mathcal{T}\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ given by $(\mathcal{T}f)(x) = \int_0^x f(y)\,dy$, where, as usual, the inner product is $\langle f,g\rangle = \int_0^1 f(s)g(s)\,ds$.

Day 41: Friday, November 15.

Material from *Linear Algebra* by Meckes & Meckes

Proposition 5.16 on p.315 gives "range equals kernel perp." Theorem 4.4 on p.230 is the Pythagorean identity; Theorem 4.6 on p.232 is the Cauchy–Schwarz inequality (see the hilarious footnote); and Theorem 4.7 below that is the triangle inequality for inner product spaces.

Do Quick Exercise #4 on p.233.

We have built up two distinct structures from inner products: the orthogonal complement, which involves subspaces, and the adjoint, which involves linear operators. Now we put them together.

Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces with inner products $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{W}}$, respectively. Let $\mathcal{T} \in \mathbf{L}(\mathcal{V},\mathcal{W})$, and suppose that $\mathcal{T}$ has an adjoint $\mathcal{T}^* \in \mathbf{L}(\mathcal{W},\mathcal{V})$, so $\langle \mathcal{T}v, w\rangle_{\mathcal{W}} = \langle v, \mathcal{T}^*w\rangle_{\mathcal{V}}$ for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. Suitably translated into our current notation, our overwrought guiding example suggested in (37.1) that $\mathcal{T}(\mathcal{V}) = \ker(\mathcal{T}^*)^{\perp}$. Is this always true?

41.1 Problem. (i) Prove that $\mathcal{T}(\mathcal{V}) \subseteq \ker(\mathcal{T}^*)^{\perp}$.

(ii) Try to prove that $\ker(\mathcal{T}^*)^{\perp} \subseteq \mathcal{T}(\mathcal{V})$ with a direct proof, i.e., assume $w \in \ker(\mathcal{T}^*)^{\perp}$ and try to find $v \in \mathcal{V}$ such that $\mathcal{T}v = w$. Where do you get stuck?

The second part of the problem above suggests that a direct proof that $\mathcal{T}(\mathcal{V}) = \ker(\mathcal{T}^*)^{\perp}$ will be, at best, challenging. However, sometimes the orthogonal complement of the orthogonal complement of a subspace is that subspace. What if we tried showing $\mathcal{T}(\mathcal{V})^{\perp} = \ker(\mathcal{T}^*)$ and then hoped that $[\mathcal{T}(\mathcal{V})^{\perp}]^{\perp} = \mathcal{T}(\mathcal{V})$?

We have

$$\begin{aligned} w \in \mathcal{T}(\mathcal{V})^{\perp} &\iff \langle w, u\rangle_{\mathcal{W}} = 0 \text{ for all } u \in \mathcal{T}(\mathcal{V}) \\ &\iff \langle u, w\rangle_{\mathcal{W}} = 0 \text{ for all } u \in \mathcal{T}(\mathcal{V}), \text{ just rewritten for convenience} \\ &\iff \langle \mathcal{T}v, w\rangle_{\mathcal{W}} = 0 \text{ for all } v \in \mathcal{V}, \text{ by definition of } \mathcal{T}(\mathcal{V}) \\ &\iff \langle v, \mathcal{T}^*w\rangle_{\mathcal{V}} = 0 \text{ for all } v \in \mathcal{V}, \text{ by definition of } \mathcal{T}^* \\ &\iff \mathcal{T}^*w = 0 \text{ by properties of inner products} \\ &\iff w \in \ker(\mathcal{T}^*) \text{ by definition of the kernel.} \end{aligned}$$

Here is our result.

41.2 Lemma. *Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces. Suppose that $\mathcal{T} \in \mathbf{L}(\mathcal{V},\mathcal{W})$ has an adjoint $\mathcal{T}^*$. Then*

$$\mathcal{T}(\mathcal{V})^{\perp} = \ker(\mathcal{T}^*).$$

In the event that $[\mathcal{T}(\mathcal{V})^{\perp}]^{\perp} = \mathcal{T}(\mathcal{V})$, which certainly happens if $\mathcal{T}(\mathcal{V})$ is finite-dimensional, then

$$\mathcal{T}(\mathcal{V}) = \ker(\mathcal{T}^*)^{\perp}. \tag{41.1}$$

This is it. This is the kind of characterization of the range that we have been seeking all along. Given $w \in \mathcal{W}$, what feels easier? To summon up $v \in \mathcal{V}$ such that $\mathcal{T}v = w$? Or to compute $\ker(\mathcal{T}^*)$, which amounts to solving $\mathcal{T}^*z = 0$ (something that we probably know how to do, since "homogeneous" problems are so ubiquitous), and then checking $\langle w, z\rangle_{\mathcal{W}} = 0$ for all $z \in \ker(\mathcal{T}^*)$, which amounts to doing a bunch of inner product computations? Unless we are so lucky as to find immediately that $v \in \mathcal{V}$ with $\mathcal{T}v = w$, we might well want to go the route of "range equals kernel perp."

41.3 Problem. Here is a different proof of (41.1), which still ultimately relies on $[\mathcal{T}(\mathcal{V})^{\perp}]^{\perp} = \mathcal{T}(\mathcal{V})$. Part (i) of Problem 41.1 ensures that $\mathcal{T}(\mathcal{V}) \subseteq \ker(\mathcal{T}^*)^{\perp}$, so we just need to show $\ker(\mathcal{T}^*)^{\perp} \subseteq \mathcal{T}(\mathcal{V})$. We prove the contrapositive here: let $w \notin \mathcal{T}(\mathcal{V})$. The goal is to show $w \notin \ker(\mathcal{T}^*)^{\perp}$. Assume in the following that $\mathcal{W} = \mathcal{T}(\mathcal{V}) \oplus \mathcal{T}(\mathcal{V})^{\perp}$.

(i) Explain why it suffices to find $z \in \ker(\mathcal{T}^*)$ such that $\langle w, z\rangle_{\mathcal{W}} \neq 0$.

(ii) Use part (iv) of Problem 38.12 to produce $z \in \mathcal{T}(\mathcal{V})^{\perp}$ such that $\langle w, z\rangle_{\mathcal{W}} \neq 0$.

(iii) Show that $\|\mathcal{T}^*z\|_{\mathcal{W}} = 0$. [Hint: *it has been some time since we worked with norms, so this is Definition 37.1.*] Why does this complete the proof?

It would be nice to have some deeper understanding of when $[\mathcal{T}(\mathcal{V})^{\perp}]^{\perp} = \mathcal{T}(\mathcal{V})$, beyond having $\mathcal{T}(\mathcal{V})$ finite-dimensional. That understanding can be a consequence another property of inner products that we have not used all that much so far. Perhaps our best uses of inner products have been to *represent* vectors in orthonormal spans, i.e., if $(u_1, \ldots, u_n)$ is orthonormal and $v \in \operatorname{span}(u_1, \ldots, u_n)$, then $v = \sum_{k=1}^n \langle v, u_k\rangle\, u_k$, and to *test* vectors, i.e., if $\langle v, w\rangle = 0$ for all $w \in \mathcal{V}$, then $v = 0$. We can also use inner products to *measure* vectors.

Recall in the following that the norm induced by the inner product $\langle \cdot, \cdot\rangle$ on $\mathcal{V}$ is

$$\|v\| := \sqrt{\langle v, v\rangle}.$$

The fundamental notion of a norm is that it measures length.

41.4 Problem. Convince yourself that the norm induced by the dot product on $\mathbb{R}^2$ is a good measurement of length in $\mathbb{R}^2$.

We already know (part (i) of Problem 37.2) that the norm correctly measures the length of the zero vector: $\|v\| = 0$ if and only if $v = 0$. It would be nice if measurements of length also respect scalings: if we scale a vector v by $\alpha \in \mathbb{F}$, hopefully the length of αv is the length of v scaled by α.

41.5 Problem. Check that $\|\alpha v\| = |\alpha|\,\|v\|$ for all $\alpha \in \mathbb{F}$ and $v \in \mathcal{V}$.

Experience teaches us that going from point A to point B, and then (on the weekends) from point B to point C, should take longer than just going from point A to point C. This is the **TRIANGLE INEQUALITY**.

41.6 Problem. Draw a picture illustrating the triangle inequality in $\mathbb{R}^2$. [Hint: *think about "tip-to-tail" addition of vectors.*]

We would like to show that

$$\|v + w\| \leq \|v\| + \|w\| .$$

Square roots are challenging, so we might try proving this inequality by showing instead

$$\|v + w\|^2 \leq \big(\|v\| + \|w\| \big)^2.$$

41.7 Problem. Show that this is equivalent to

$$\mathrm{Re}[\langle v, w\rangle] \leq \|v\| \, \|w\| .$$

[Hint: *use* $\|u\|^2 = \langle u, u\rangle$*, algebraic properties of the norm, and the identity* $z + \overline{z} = 2\,\mathrm{Re}(z)$*, valid for all* $z \in \mathbb{C}$.]

So, do we have this peculiar inequality? Yes. In fact, we get something a bit stronger.

41.8 Problem. To motivate the following more general result, consider the dot product on $\mathbb{R}^2$. Compute

$$(\mathbf{v} \cdot \mathbf{w})^2 = v_1^2 w_1^2 + 2(v_1 w_2)(w_1 v_2) + v_2^2 w_2^2.$$

Use the inequality $2ab \leq a^2 + b^2$, valid for all a, $b \in \mathbb{R}$ (since $0 \leq (a - b)^2$), to find

$$(\mathbf{v} \cdot \mathbf{w})^2 \leq v_1^2 w_1^2 + v_1^2 w_2^2 + w_1^2 v_2^2 + v_2^2 w_2^2.$$

Factor the right side as

$$(v_1^2 + v_2^2)(w_1^2 + w_2^2) = \|\mathbf{v}\|^2 \, \|\mathbf{w}\|^2 .$$

41.9 Problem. Here are some steps to prepare for the following general result.

(i) Prove that if $\mathcal{V}$ is an inner product space and v, $w \in \mathcal{V}$ with $\langle v, w\rangle = 0$, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2 . \tag{41.2}$$

[Hint: *just expand everything.*]

(ii) Draw a picture illustrating this in $\mathbb{R}^2$ and explain why it makes you think about right triangles and why we might call the equality (41.2) the **PYTHAGOREAN** identity.

(iii) Why does the Pythagorean identity not imply a triangle "equality" for orthogonal vectors? [Hint: $\sqrt{a^2+b^2} \neq |a|+|b|$ *in general.*]

41.10 Theorem (Cauchy–Schwarz inequality). *Let $\mathcal{V}$ be an inner product space and v, $w \in \mathcal{V}$. Then*

$$|\langle v, w\rangle| \leq \|v\|\,\|w\|.$$

Proof. There are many proofs of this inequality, none of them quite obvious. Perhaps the easiest place to start is the case $\langle v, w\rangle = 0$; then since $\|v\| \geq 0$ and $\|w\| \geq 0$, we have $0 \leq \|v\|\,\|w\|$, and that is the inequality. If v and w are not orthogonal, then experience teaches us that we can separate them into "orthogonal components."

Suppose $w \neq 0$. (The case $w = 0$ is handled by the previous remarks about $\langle v, w\rangle = 0$.) Recalling our development of Gram–Schmidt and in particular Problem 37.5, we put

$$\mathcal{P}_w v := \frac{\langle v, w\rangle}{\|w\|^2} w.$$

Then

$$v = \mathcal{P}_w v + (v - \mathcal{P}_w v) \quad \text{and} \quad \langle \mathcal{P}_w v, v - \mathcal{P}_w v\rangle = 0,$$

so

$$\|v\|^2 = \|\mathcal{P}_w v + (v - \mathcal{P}_w v)\|^2 = \|\mathcal{P}_w v\|^2 + \|v - \mathcal{P}_w v\|^2.$$

The vector $\mathcal{P}_w v$ contains factors that resemble what we want out of the Cauchy–Schwarz inequality, whereas $v - \mathcal{P}_w v$ may be more opaque. Fortunately, the latter is irrelevant here: since $\|v - P_w v\|^2 \geq 0$, we have

$$\|v\|^2 \geq \|\mathcal{P}_w v\|^2.$$

Rewritten, this says

$$\left\|\frac{\langle v, w\rangle}{\|w\|^2} w\right\| \leq \|v\|,$$

and the left side of this inequality is

$$\left\|\frac{\langle v, w\rangle}{\|w\|^2} w\right\| = \frac{|\langle v, w\rangle|}{\|w\|^2}\|w\| = \frac{|\langle v, w\rangle|}{\|w\|}.$$

Thus

$$\frac{|\langle v, w\rangle|}{\|w\|} \leq \|v\|,$$

and this turns into the Cauchy–Schwarz inequality. ■

41.11 Problem. Reread the proof of the Cauchy–Schwarz inequality and find exactly where an inequality appears for the first (and really only) time. What would make that inequality an equality? Determine a condition on v and w that is equivalent to *equality* in the Cauchy–Schwarz *in*equality. [Hint: $\mathcal{P}_w v = v$ *if and only if* $v \in \operatorname{span}(w)$.]

Now we can return to proving the triangle inequality. We want to show $\|v+w\| \leq \|v\| + \|w\|$, and we will have this if we can show

$$\operatorname{Re}[\langle v, w\rangle] \leq \|v\| \, \|w\| \, .$$

Recall that for any $z \in \mathbb{C}$, we have $\operatorname{Re}(z) \leq |\operatorname{Re}(z)| \leq |z|$. Thus

$$\operatorname{Re}[\langle v, w\rangle] \leq |\, \langle v, w\rangle \,| \leq \|v\| \, \|w\|$$

by the Cauchy–Schwarz inequality. This completes the proof of the triangle inequality for the norm induced by the inner product.

41.12 Problem. Let $\mathcal{V} = \mathcal{C}([0,1])$ with the usual inner product $\langle f, g\rangle = \int_0^1 f(x)g(x) \, dx$. Let $f(x) = 1$ and $g(x) = x$. Compute $\|f\|$, $\|g\|$, $\|f+g\|$, and $\langle f, g\rangle$. Check that the Cauchy–Schwarz and triangle inequalities hold. Are there any equalities?

41.13 Problem. Reread the proof of the triangle inequality above, which was interrupted by several digressions involving the Cauchy–Schwarz inequality. (It might help to rewrite the proof of the triangle inequality without those digressions.) Determine a condition on v and w that is equivalent to equality in the triangle inequality. [Hint: *think about Problem 41.7 and when equality holds in the Cauchy–Schwarz inequality.*]