MATH 4310: PARTIAL DIFFERENTIAL EQUATIONS

Daily Log for Lectures and Readings

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CONTENTS

How to Use This Daily Log

Day 1: Monday, January 6.

Material from *Basic Partial Differential Equations* by Bleecker & Csordas

Section 1.2 has a broad overview of the subject and some important terms (like linear PDE and superposition). You definitely don't have to understand everything in here, but it gives a good vision of the subject and some important examples. We will revisit some of this material throughout the term.

Broadly, we care about PDE (which we use as both a singular and a plural noun, depending on context) because many interesting quantities in life depend on more than one variable and because the study of PDE employs and motivates many interesting concepts in mathematics. In this course, we will usually study functions of two variables, typically one for space and one for time (or sometimes both for two-dimensional space), and usually the unknown function in our equations will be u; we will write $u = u(x,t)$ to emphasize that u depends on x and t, and x will denote space and t time. Major challenge of PDE involve *data* and *geometry*: when the domain of our unknown function is two-dimensional (or higher-dimensional), we must keep track of much more data from the inputs, and there are many more options available for the domain's geometry as a subset of $\mathbb{R}^2$ (or $\mathbb{R}^n$). We will avoid these challenges by taking fairly banal domains when working with one spatial variable and one temporal variable; there, each variable will belong to some subinterval of $\mathbb{R}$, possibly infinite, possibly closed and bounded.

This course will provide many opportunities to revisit topics from single and multivariable calculus; we will become stronger students of familiar calculus because of these opportunities, and we will develop new appreciation for things that we previously learned, most especially the integral. We will also have many opportunities to ask, but not fully answer, questions that connect PDE to other courses—in particular real and complex analysis, linear algebra, and topology. Questions from PDE motivate many of the rigorous results from those courses that we will not prove, fully or even partially, here. But we will prove many results here; after all, a proof is just an argument that we are correct about something.

We will devote significant attention to the following four canonical linear PDE:

$$\begin{aligned} &u_t + u_x = 0 && \text{Transport equation} \\ &u_t - u_{xx} = 0 && \text{Heat equation} \\ &u_{tt} - u_{xx} = 0 && \text{Wave equation} \\ &u_{tt} + u_{xx} = 0 && \text{Laplace's equation.} \end{aligned}$$

It turns out that we can represent all solutions to the transport equation very explicitly and compactly, and so that PDE will be a great "lab rat" as we develop new techniques—we can always see how something new compares to what we know about transport. In fact, once we know how to solve the transport equation as written above, a versatile "rescaling" technique will allow us to solve

$$au_t + bu_x = 0$$

for any choice of a, $b \in \mathbb{R}$. And, thanks to a clever "factoring" technique, we will be able to import many ideas from the transport equation to the wave equation, and so transport and wave morally belong to the same "class" of PDE. But heat and Laplace are totally different, both from each other and from the transport/wave class. In particular, the difference between wave and Laplace, which is just the choice of $\pm$, is remarkable—a banal change in the algebraic structure of the PDE produces a profound change in the behavior of solutions and in the mathematical techniques and tools necessary for their analysis.

All four PDE are **LINEAR AND HOMOGENEOUS** in the sense that if u and v are solutions and c_1, $c_2 \in \mathbb{R}$, then $c_1u + c_2v$ is also a solution.

1.1 Problem (!). Prove that.

This phenomenon is sometimes gussied up with the term **SUPERPOSITION**, which fails for nonlinear problems. Here are two nonlinear equations that we will eventually study:

$u_t + uu_x = 0$	Burgers's equation
$u_t + u_{xxx} + uu_x = 0$	Korteweg–de Vries (KdV) equation.

1.2 Problem (!). If u and v solve Burgers's equation, what goes wrong if you try to show that $c_1u + c_2v$ is also a solution for c_1, $c_2 \in \mathbb{R}$?

Here are some things that we will *not* do. Lawrence C. Evans, in his magisterial graduate-level text *Partial Differential Equations*, captures the challenge and the orientation of PDE study quite evocatively:

> *"There is no general theory concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations that are important for applications within and outside of mathematics, with the hope that insight from the origins of these PDE can give clues as to their solutions."*

Peter Olver's book *Introduction to Partial Differential Equations* gives the following as a mission statement for a first undergraduate course in PDE, and I agree with it fully:

> *"[T]he primary purpose of a course in partial differential equations is to learn the principal solution techniques and to understand the underlying mathematical analysis."*

We will focus rather less on deriving PDE from models and physical principles and rather more on the solution techniques and mathematical analysis.

Two of our major tools in this course will be integrals (definite and improper) and fundamental results from ODE. We will start by reviewing essential properties of the definite integral and then applying them to redevelop familiar results from ODE at a more abstract level (and more rapid pace). Throughout the course, we will see that integrals fundamentally *measure and/or extract useful data about functions* (and all the cool kids want to be data scientists these days) and also *represent functions in convenient and/or meaningful ways*.

We know this from calculus: the number

$$\frac{1}{b-a}\int_a^b f(x)\ dx$$

gives a good measure of the "average value" that the function f takes on the interval $[a, b]$, while the function

$$F(x) := \int_a^x f(t)\ dt$$

is an antiderivative of f in the sense that $F'(x) = f(x)$. Eventually we will see that integrals like

$$\int_a^b |f(x)|\ dx \quad \text{and} \quad \left(\int_a^b |f(x)|^2\ dx\right)^{1/2}$$

are good measures of "size" for f (that is, they are integral **NORMS**). We will also find representing functions via (inverse) Fourier transforms, which are defined via improper integrals, particularly convenient.

But to get anywhere, we need to be comfortable with how integrals work. We only need four properties of integrals in order to get the fundamental theorem of calculus (FTC), and all of those properties have geometric motivations (there are other motivations, too, but geometry/area is probably the most universally accessible). For simplicity (and to annoy the calculus professors), we will write $\int_a^b f$ most of the time, and we will agree that the dummy variable of integration does not matter:

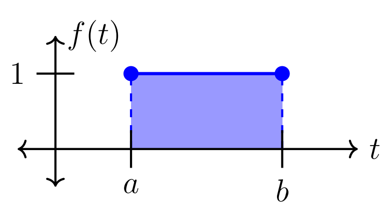
$$\int_a^b f = \int_a^b f(x)\ dx = \int_a^b f(u)\ du = \int_a^b f(s)\ ds = \int_a^b f(t)\ dt = \int_a^b f(\tau)\ d\tau.$$

That last dummy variable τ is the Greek letter "tau."

Here are those properties.

($\int$1) First, the integral of a function $f \colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ should somehow measure the net area of the region between the graph of f and the interval $[a, b]$. Since the most fundamental area is the area of a rectangle, we should expect

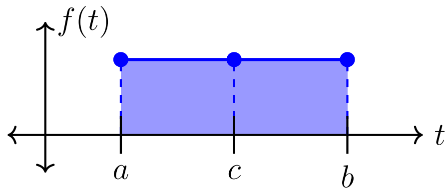
$$\int_a^b 1\ dt = b - a.$$



($\int$2) If we divide the region between the graph of f and the interval $[a, b]$ into multiple components, measure the net area of those components, and add those net areas together, we should get the total net area of the region between the graph of f and the interval

$[a, b]$. There are many such ways to accomplish this division, but perhaps one of the most straightforward is to split $[a, b]$ up into two or more subintervals and consider the net areas of the regions between the graph of f and those subintervals. So, we expect that if $a \le c \le b$, then

$$\int_a^c f(t)\ dt + \int_c^b f(t)\ dt = \int_a^b f(t)\ dt.$$

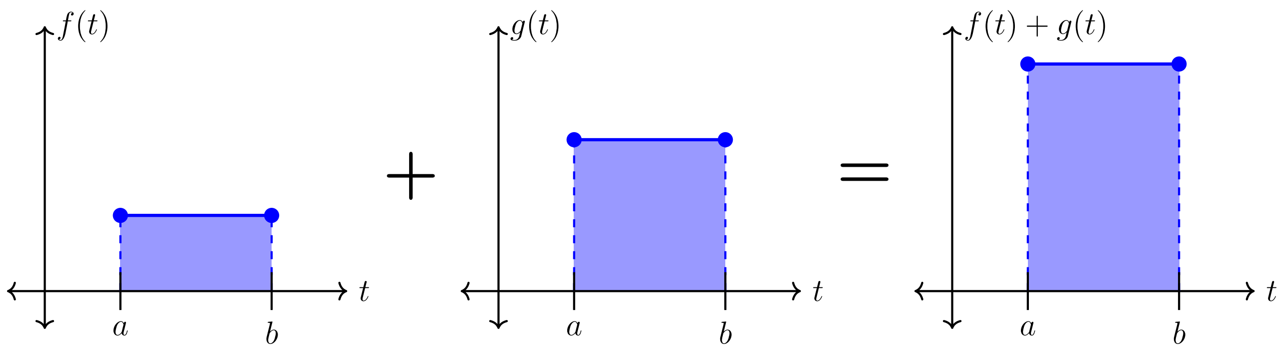


($\int$3) If f is nonnegative, the net area of the region between the graph of f and the interval $[a, b]$ should be the genuine area of the region between the graph of f and the interval $[a, b]$, and this should be a positive quantity. So, we expect that if $0 \le f(t)$ on $[a, b]$, then

$$0 \le \int_a^b f(t)\ dt.$$

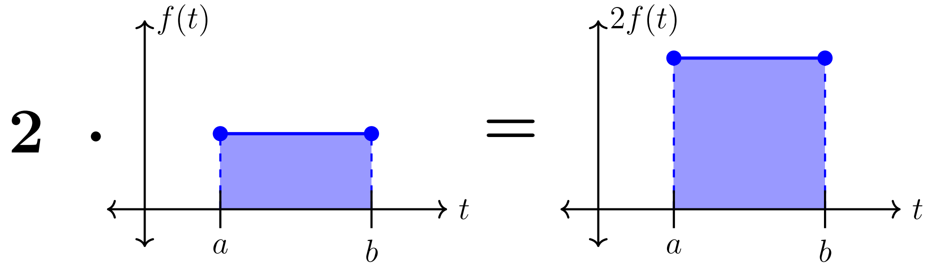
($\int$4) Adding two functions $f, g \colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ should "stack" the graphs of f and g on top of each other. Then the region between the graph of f and the interval $[a, b]$ gets "stacked" on top of region between the graph of g and the interval $[a, b]$. Consequently, the net area of the region between the graph of $f + g$ and the interval $[a, b]$ should just be the sum of these two areas:

$$\int_a^b f(t)\ dt + \int_a^b g(t)\ dt = \int_a^b \big[f(t) + g(t)\big]\ dt.$$



Next, multiplying a function $f \colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ by a constant $\alpha \in \mathbb{R}$ should somehow "scale" the net area of the region between the graph of f and the interval $[a, b]$ by that factor α. For example, the area under the graph of $2f$ over $[a, b]$ should be double the area under the graph. Consequently, the net area of the region between the graph of αf and the interval $[a, b]$ should be the product

$$\int_a^b \alpha f(t)\ dt = \alpha \int_a^b f(t)\ dt.$$



It turns out that these four properties are all that we need to prove the fundamental theorem of calculus, which we will.

Day 2: Wednesday, January 8.

Here is a more formal and less geometric approach to the integral. Let $I \subseteq \mathbb{R}$ be an interval (for the rest of today, I is *always* an interval). Denote by $\mathcal{C}(I)$ the set of all continuous real-valued functions on I. We should be able to integrate every $f \in \mathcal{C}(I)$, and we can.

2.1 Theorem. *Let $I \subseteq \mathbb{R}$ be an interval and denote by $\mathcal{C}(I)$ the set of all continuous functions from I to $\mathbb{R}$. There exists a map*

$$\int : \{(f,a,b) \mid f \in \mathcal{C}(I),\ a,b \in I\} \to \mathbb{R} \colon (f,a,b) \mapsto \int_a^b f$$

with the following properties.

($\int$1) [Constants] *If a, $b \in I$, then*

$$\int_a^b 1 = b - a.$$

($\int$2) [Additivity of the domain] *If $f \in \mathcal{C}(I)$ and a, b, $c \in I$, then*

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

($\int$3) [Monotonicity] *If $f \in \mathcal{C}(I)$ and a, $b \in I$ with $a \le b$ and $0 \le f(t)$ for all $t \in [a,b]$, then*

$$0 \le \int_a^b f.$$

If in particular $0 < f(t)$ for all $t \in [a,b]$ and if $a < b$, then

$$0 < \int_a^b f.$$

($\int$4) [Linearity in the integrand] *If f, $g \in \mathcal{C}(I)$, a, $b \in I$, and $\alpha \in \mathbb{R}$, then*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b \alpha f = \alpha \int_a^b f.$$

The number $\int_a^b f$ is the **DEFINITE INTEGRAL OF** f **FROM** a **TO** b.

Properties ($\int$4) encodes the linearity of the integral as an operator on the integrand with the limits of integration fixed, while property ($\int$2) is its **ADDITIVITY** over subintervals with the integrand fixed. Property ($\int$3) encodes the idea that a nonnegative function should have a nonnegative integral, while property ($\int$1) defines the one value of the integral that it most certainly should have from the point of view of area.

Specifically, we can express the definite integral as a limit of Riemann sums—among them, the right-endpoint sums:

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right). \tag{2.1}$$

That this limit exists is a fundamental result about continuous functions, which we will not prove. From (2.1) we can prove properties ($\int$1), ($\int$3), and ($\int$4) quite easily. Property ($\int$2) is not so obvious from (2.1), and in fact this property hinges on expressing $\int_a^b f$ as a "limit" of several kinds of Riemann sums, not just the right-endpoint sum. And then there is still the challenge of ensuring that limits of all sorts of "well-behaved" Riemann sums for f (including, but not limited to, left and right endpoint and midpoint sums) all converge to the same number. Moreover, it is plausible that one might want to integrate functions that are not continuous. (We will eventually have to handle this.)

2.2 Problem ($\star$). Let $I \subseteq \mathbb{R}$ be an interval and f, $g \colon I \to \mathbb{R}$ be continuous. Let a, b, $c \in I$ and $\alpha \in \mathbb{R}$. Using only Theorem 2.1, prove the following. *You should not use the Riemann sum formula* (2.1) *at all. The goal is to see how other properties of the integral follow directly from the essential features of Theorem 2.1.*

(i) [Generalization of ($\int$1)] $\displaystyle\int_a^b \alpha = \alpha(b-a)$

(ii) $\displaystyle\int_a^a f = 0$

(iii) $\displaystyle\int_a^b f = -\int_b^a f$

2.3 Problem (+). Use induction to generalize additivity as follows. Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{C}$ be continuous. If $t_0, \ldots, t_n \in I$, then

$$\int_{t_0}^{t_n} f = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f.$$

2.4 Problem ($\star$). Let $I \subseteq \mathbb{R}$ be an interval.

(i) Suppose that f, $g: I \to \mathbb{R}$ are continuous and a, $b \in \mathbb{R}$ with $a \le b$. If $f(t) \le g(t)$ for all $t \in [a,b]$, show that

$$\int_a^b f \le \int_a^b g. \tag{2.2}$$

(ii) Continue to assume a, $b \in I$ with $a \le b$. Prove the **TRIANGLE INEQUALITY**

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

[Hint: *recall that if x, $r \in \mathbb{R}$ with $r \ge 0$, then $-|x| \le x \le |x|$ and $|x| \le r$ if and only if $-r \le x \le r$. Use this to estimate $f(t)$ in terms of $\pm|f(t)|$ and then apply part (i).*]

(iii) Continue to assume a, $b \in I$ with $a \le b$. Suppose that $f: I \to \mathbb{R}$ is continuous and there are m, $M \in \mathbb{R}$ such that $m \le f(t) \le M$ for all $t \in [a,b]$. Show that

$$m(b-a) \le \int_a^b f \le M(b-a). \tag{2.3}$$

(iv) Show that if we remove the hypothesis $a \le b$, then the triangle inequality becomes

$$\left|\int_a^b f\right| \le \left|\int_a^b |f|\right|.$$

Why is the extra absolute value on the right necessary here?

We now have only a handful of results about the definite integral, and yet they are enough to prove the fundamental theorem of calculus. (Conversely, by themselves, they do not help us evaluate integrals more complicated than $\int_a^b \alpha$ for $\alpha \in \mathbb{C}$!) This is our first rigorous verification that an integral gives a meaningful representation of a function. Specifically, integrals represent antiderivatives.

2.5 Theorem (FTC1). *Let $f: I \to \mathbb{C}$ be continuous and fix $a \in I$. Define*

$$F: I \to \mathbb{C}: t \mapsto \int_a^t f$$

Then F is an antiderivative of f on I.

Proof. Fix $t \in I$. We need to show that F is differentiable at t with $F'(t) = f(t)$. That is, we want

$$\lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = f(t),$$

equivalently,

$$\lim_{h \to 0} \frac{F(t+h) - F(t) - hf(t)}{h} = 0.$$

We first compute

$$\begin{aligned} F(t+h) - F(t) &= \int_a^{t+h} f - \int_a^t f \\ &= \int_a^{t+h} f + \int_t^a f \\ &= \int_t^{t+h} f. \end{aligned}$$

The first two terms of the numerator of the difference quotient have now collapsed into a single integral, so it would be nice if that third term, $-hf(t)$, were also an integral. First we cleverly rewrite h:

$$h = (t+h) - t = \int_t^{t+h} 1.$$

Then we use linearity of the integral to compute

$$hf(t) = f(t) \int_t^{t+h} 1 = \int_t^{t+h} f(t).$$

It may help at this point to introduce a variable of integration. Recall that t has been fixed throughout this proof, so we should not overwork it. Instead, we use τ, and so we have We then have

$$F(t+h) - F(t) - hf(t) = \int_t^{t+h} f(\tau)\, d\tau - \int_t^{t+h} f(t)\, d\tau = \int_t^{t+h} \big[f(\tau) - f(t)\big]\, d\tau.$$

It therefore suffices to show that

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \big[f(\tau) - f(t)\big]\, d\tau = 0, \qquad (2.4)$$

and we do that in the following lemma. ■

2.6 Problem (!). Reread, and maybe rewrite, the preceding proof. Identify explicitly each property of or result about integrals that was used without reference.

This is a specific instance of a more general phenomenon in manipulating difference quotients and doing "derivatives by definition." The difference quotient has h in the denominator, and we are sending $h \to 0$, so the denominator is small. A quotient of the form $1/h$ with $h \approx 0$ is large, and large numbers are problematic in analysis. The limit as $h \to 0$ of the difference quotient exists because the numerator is sufficiently small compared to the denominator for the numerator to "cancel out" the effects of that h. In particular, to show

$$\lim_{h \to 0} \frac{F(t+h) - F(t) - hf(t)}{h} = 0,$$

we want the numerator $F(t+h) - F(t) - hf(t)$ to be even smaller than the denominator. *The answer to small denominators is smaller numerators.*

2.7 Lemma. *Let $I \subseteq \mathbb{R}$ be an interval and let $f \colon I \to \mathbb{C}$ be continuous. Then*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \big[f(\tau) - f(t)\big] \, d\tau = 0$$

for any $t \in I$.

Proof. We use the squeeze theorem. The triangle inequality implies

$$\left|\frac{1}{h} \int_t^{t+h} \big[f(\tau) - f(t)\big] \, d\tau\right| \le \frac{1}{|h|}|t+h-h| \max_{0 \le s \le 1} |f((1-s)t+s(t+h))-f(t)| = \max_{0 \le s \le 1} |f(t+sh)-f(t)|.$$

We now need to show that

$$\lim_{h \to 0} \max_{0 \le s \le 1} |f(t+sh) - f(t)| = 0.$$

This will involve the definition of continuity.

Let $\epsilon > 0$, so our goal is to find $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\max_{0 \le s \le 1} |f(t+sh) - f(t)| < \epsilon. \tag{2.5}$$

Since f is continuous at t, there is $\delta > 0$ such that if $|t - \tau| < \delta$, then $|f(\tau) - f(t)| < \epsilon$. Suppose $0 < |h| < \delta$. Then if $0 \le s \le 1$, we have

$$|(t+sh) - t| = |sh| \le |h| < \delta,$$

thus (2.5) holds. ■

2.8 Problem (⋆). Prove that the left limit in (2.5) holds. What specific changes are needed when $h < 0$?

2.9 Problem (⋆). Prove the following "averaging" identity. Let $I \subseteq \mathbb{R}$ be an interval,

$x \in \mathcal{I}$, and $f \in \mathcal{C}(I)$. Then

$$f(x) = \lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} f.$$

This says that the value of f at at point x is the limit of the average values of f on intervals centered at x as the width of those intervals shrinks to 0. [Hint: *with x fixed, put*

$$F_1(r) := \int_0^{x+r} f \quad \text{and} \quad F_2(r) := \int_0^{x-r} f$$

and show that

$$\frac{1}{2r} \int_{x-r}^{x+r} f = \frac{1}{2} \left(\frac{F_1(r) - F_1(0)}{r} - \frac{F_2(r) - F_2(0)}{r} \right).$$

Now think about difference quotients. How does this help?]

With FTC1, we can prove a second version that facilitates the computation of integrals via antiderivatives, but first we need to review the mean value theorem, which we state but do not prove.

2.10 Theorem (Mean value). *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f: [a, b] \to \mathbb{R}$ be continuous with f differentiable on (a, b). Then there is $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

2.11 Problem ($\star$). **(i)** Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $f: I \to \mathbb{R}$ is differentiable with $f'(t) = 0$ for all $t \in I$. Show that f is constant on I. [Hint: *fix $t_0 \in I$ and let $t \in I \setminus \{t_0\}$. Assuming that $t > t_0$, use the mean value theorem to express the difference quotient $(f(t) - f(t_0))/(t - t_0)$ as a derivative, which must be 0. What happens if $t < t_0$?*]

(ii) Give an example of a function f defined on the set $[-1, 1] \setminus \{0\}$ that is differentiable with $f'(t) = 0$ for all t but f is not constant. [Hint: *go piecewise.*]

2.12 Problem ($\star$). Suppose that y solves the ODE $y' = ry$ for some $r \neq 0$ on some interval $I \subseteq \mathbb{R}$. That is, $y'(t) = ry(t)$ for all $t \in I$. Prove that $u(t) := y(t)e^{-rt}$ is constant. Explain why this justifies (what is hopefully!) our expectation that all solutions to this ODE are multiples of an exponential.

2.13 Corollary (FTC2). *Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{C}$ be continuous. If F is any antiderivative of f on I, then*

$$\int_a^b f = F(b) - F(a)$$

for all $a, b \in I$.

Proof. Let $G(t) := \int_a^t f$, so G is an antiderivative of f by FTC1. Put $H = G - F$, so $h' = 0$ on I. Since I is an interval, the mean value theorem mplies that H is constant. The most important inputs here are a and b, so we note that $H(a) = H(b)$, and so

$$G(a) - F(a) = G(b) - F(b).$$

But $G(a) = \int_a^a f = 0$, so this rearranges to

$$G(b) = F(b) - F(a),$$

and $G(b) = \int_a^b f$. ∎

The fundamental theorems of calculus are, of course, the keys to both substitution and integration by parts, two of the most general techniques for evaluating integrals in terms of simpler functions. Recall that substitution involves turning the more complicated integral $\int_a^b f(\varphi(t))\varphi'(t)\, dt$ into the simpler integral $\int_{\varphi(a)}^{\varphi(b)} f(u)\, du$. For this to make sense, the function φ should be defined and continuous on an interval containing a and b, and f should be defined and continuous on an interval containing $\varphi(a)$ and $\varphi(b)$, and also φ should map the interval containing a and b to the domain of f, so $f \circ \varphi$ is defined and continuous. Also, the product $(f \circ \varphi)\varphi'$ should be continuous, and that requires φ' to be continuous on I.

2.14 Definition. *Let $I \subseteq \mathbb{R}$ be an interval. A function $\varphi\colon I \to \mathbb{R}$ is* **CONTINUOUSLY DIFFERENTIABLE** *if φ is differentiable on I (and thus continuous itself on I) and if also φ' is continuous on I. We denote the set of all continuously differentiable functions on I by $\mathcal{C}^1(I)$.*

2.15 Theorem (Substitution). *Let I, $J \subseteq \mathbb{R}$ be intervals with a, $b \in I$. Let $\varphi \in \mathcal{C}^1(I)$ and $f \in \mathcal{C}(J)$ with $\varphi(t) \in J$ for all $t \in I$. Then*

$$\int_a^b (f \circ \varphi)\varphi' = \int_{\varphi(a)}^{\varphi(b)} f.$$

Proof. Let $F(\tau) := \int_{\varphi(a)}^{\tau} f$. The chain rule implies that $F \circ \varphi$ is an antiderivative of $(f \circ \varphi)\varphi'$; indeed, by FTC1,

$$(F \circ \varphi)' = (F' \circ \varphi)\varphi' = (f \circ \varphi)\varphi'.$$

Then FTC2 implies

$$\int_a^b (f \circ \varphi)\varphi' = F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f - \int_{\varphi(a)}^{\varphi(a)} f = \int_{\varphi(a)}^{\varphi(b)} f.$$ ∎

A recurring theme of our subsequent applications of integrals will be that we are trying to estimate or control some kind of difference (this is roughly 90% of analysis), and it turns out to be possible to rewrite that difference in a tractable way by introducing an integral. It may be possible to manipulate further terms under consideration by rewriting them as integrals, too. The fundamental identity that we will use in the future is (2.6) below.

2.16 Example. FTC2 allows us to rewrite a functional difference as an integral. When we incorporate substitution, we can get a very simple formula for that difference. Suppose that $I \subseteq \mathbb{R}$ is an interval, $f \in \mathcal{C}^1(I)$, and a, $b \in I$. Then

$$f(b) - f(a) = \int_a^b f'.$$

We will reverse engineer substitution and make the limits of integration simpler and the integrand more complicated. (This turns out to be a good idea.)

Define

$$\varphi\colon [0,1] \to \mathbb{R}\colon t \mapsto (1-t)a + tb = a + (b-a)t.$$

Then $\varphi(0) = a$, $\varphi(1) = b$, and $a \le \varphi(t) \le b$ for all t if $a \le b$, and otherwise $b \le \varphi(t) \le a$ for all t if $b \le a$. (Here is a proof of the first case, assuming $a \le b$. Then $b - a \ge 0$ and $t \ge 0$, so $(b-a)t \ge 0$, thus $a \le a + (b-a)t$. But also $(1-t)a \le (1-t)b$ since $1 - t \ge 0$ and $a \le b$, thus $(1-t)a + tb \le (1-t)b + tb = b$.) In other words, we think of φ as "parametrizing" the line segment between the points a and b on the real line.

Substitution implies

$$\int_a^b f' = \int_0^1 f'(\varphi(t))\varphi'(t)\ dt,$$

and we calculate $\varphi'(t) = b - a$. Thus

$$\int_a^b f' = (b-a)\int_0^1 f'(a + (b-a)t)\ dt.$$

In conclusion, if $f \in \mathcal{C}^1(I)$ and a, $b \in I$, then

$$f(b) - f(a) = (b-a)\int_0^1 f'(a + (b-a)t)\ dt. \tag{2.6}$$

This represents explicitly how $f(b) - f(a)$ depends on the quantity $b - a$; if we know how to control f' (maybe f' is bounded on an interval containing a and b), then we have an estimate for the size of $f(b) - f(a)$ in terms of $b - a$. While the mean value theorem would allow us to rewrite $(f(b) - f(a))/(b-a)$ in terms of f', that result is existential and not nearly as explicit as (2.6).

2.17 Problem (!). Prove the following variant of Example 2.16: if $I \subseteq \mathbb{R}$ is an interval, $f \in \mathcal{C}^1(I)$, and t, $t + h \in I$, then

$$f(t+h) - f(t) = h\int_0^1 f'(t + \tau h)\ d\tau.$$

2.18 Problem ($\star$). Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and p-PERIODIC for some $p \in \mathbb{R}$, in the sense that $f(t+p) = f(t)$ for all $t \in \mathbb{R}$. Then the integral of f over any interval of length p is the same:

$$\int_a^{a+p} f = \int_0^p f$$

for all $a \in \mathbb{R}$. Give two proofs of this identity as follows.

(i) Define

$$F\colon \mathbb{R} \to \mathbb{R}\colon a \mapsto \int_a^{a+p} f$$

and use FTC1 and the p-periodicity of f to show that $F'(a) = 0$ for all a. Since F is also defined on an interval (the interval here is $\mathbb{R}$), F must be constant.

(ii) First explain why

$$\int_a^{a+p} f = \int_0^p f + \left(\int_p^{a+p} f - \int_0^a f\right).$$

Then substitute $u = t - p$ to show

$$\int_p^{a+p} f = \int_0^a f(t-p)\, dt$$

and use the p-periodicity of f.

2.19 Problem (!). Let $I \subseteq \mathbb{R}$ be an interval and f, $g \in \mathcal{C}^1(I)$ and a, $b \in I$. Prove the INTEGRATION BY PARTS identity

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g. \tag{2.7}$$

[Hint: *this is equivalent to an identity for* $\int_a^b (f'g + fg')$ *and that integrand is a perfect derivative by the product rule.*]

2.20 Problem ($\star$). Let $f \in \mathcal{C}^2(\mathbb{R})$ with $f(0) = 0$. Prove that

$$f(x+y) - f(x) - f(y) = xy \int_0^1 \int_0^1 f''(sx + ty)\, ds\, dt$$

for all x, $y \in \mathbb{R}$. What happens in the case $f(\tau) = \tau^2$?

2.21 Problem (+). Suppose that $f \in \mathcal{C}^2(\mathbb{R})$. Suppose also that $f'(0) = 0$ and there is $M > 0$ such that

$$|f''(t)| \leq M \text{ for all } t \in \mathbb{R}.$$

Show that

$$|f(x) - f(y)| \le M\big(|x| + |y|\big)|x - y|.$$

By considering the special case $f(x) = x^2$, explain why we might call this a "difference of squares" estimate. [Hint: *use Example 2.16 to rewrite the difference* $f(x) - f(y)$ *as an integral involving* f' *and expose the factor* $x - y$. *That is,* $f(x) - f(y) = (x - y)\mathcal{I}(x, y)$, *where* $\mathcal{I}(x, y)$ *represents this integral. Since* $f'(0) = 0$, *we have* $\mathcal{I}(x, y) = \mathcal{I}(x, y) - (x - y)\int_0^1 f'(0)\ dt$ *Rewrite this difference as an integral from* 0 *to* 1 *of some integrand (which involves* f'*) and apply Example 2.16 again to that integrand so that, in the end,* $\mathcal{I}(x, y)$ *is a double integral involving* f''.]

Day 3: Friday, January 10.

No class due to weather. You should read the material below on your own and work through it line by line.

We have now built enough machinery to study elementary ODE, all of which will reappear in our study of genuine PDE. It will We proceed through three kinds of first-order problems—specifically, all are initial value problems (IVP).

The first is the **DIRECT INTEGRATION** problem

$$\begin{cases} y' = f(t), \ t \in I \\ y(0) = y_0. \end{cases} \tag{3.1}$$

Here $I \subseteq \mathbb{R}$ is an interval with $0 \in I$, $f \in \mathcal{C}(I)$ is a given function, and $y_0 \in \mathbb{R}$ is also specified. The goal is to find a differentiable function y on I such that $y'(t) = f(t)$ for all $t \in I$. (In general, when solving an ODE, one wants a differentiable function y defined on an interval that "makes the ODE true" when values from that interval are substituted in. Also, the domain of a solution should be an interval to reflect the physical ideal that time should be "unbroken"—and because it makes things nice mathematically. In particular, the interval should contain 0 so that we can evaluate $y(0)$ and find $y(0) = y_0$. Last, the derivative should be continuous to reflect the physical ideal that the rates of change do not vary too much—and because it makes things nice mathematically.)

We work backwards. Assume that the problem has a solution y, so $y'(t) = f(t)$ for all $t \in I$. For $t \in I$ fixed, integrate both sides of this equality from 0 to t to find

$$\int_0^t y'(\tau)\ d\tau = \int_0^t f(\tau)\ d\tau.$$

Be very careful to change the variable of integration from t to τ (or anything other than t), since t is now in the limit of integration. We cannot do anything more for the integral on the right, but on the left FTC2 gives

$$\int_0^t y'(\tau)\ d\tau = y(t) - y(0) = y(t) - y_0.$$

That is,

$$y(t) - y_0 = \int_0^t f(\tau)\ d\tau,$$

and so

$$y(t) = y_0 + \int_0^t f(\tau)\, d\tau. \tag{3.2}$$

Thus if y solves the IVP (3.1), then y has the form above. This is a *uniqueness* result: the only possible solution is this one. But is this really a solution?

3.1 Problem (!). Use FTC1 and properties of integrals to check that defining y by (3.2) solves (3.1).

We write this up formally.

3.2 Theorem. *Let $I \subseteq \mathbb{R}$ be an interval with $0 \in I$, let $f \in \mathcal{C}(I)$, and let $y_0 \in \mathbb{R}$. The only solution to*

$$\begin{cases} y' = f(t) \\ y(0) = y_0 \end{cases}$$

is

$$y(t) = y_0 + \int_0^t f(\tau)\, d\tau.$$

3.3 Example. To solve

$$\begin{cases} y' = e^{-t^2} \\ y(0) = 0, \end{cases}$$

we integrate:

$$y(t) = 0 + \int_0^t e^{-\tau^2}\, d\tau = \int_0^t e^{-\tau^2}\, d\tau.$$

We stop here, because we cannot evaluate this integral in terms of "elementary functions." (Long ago with times tables, working with t^2 was hard; then that got easier, but we got older and wiser and sadder and took trig, and working with $\sin(t)$ was hard. Now we are even older, and by the end of the course, working with $\int_0^t f(\tau)\, d\tau$ should feel just as natural as working with any function defined in more "elementary" terms.)

Day 4: Monday, January 13.

Material from *Basic Partial Differential Equations* by Bleecker & Csordas

Pages 4–6 review first-order linear ODE via integrating factors. This is not the method that we used in class, and I don't think it will be very helpful when we want to apply these ODE techniques to PDE. You might try redoing the textbook's examples with variation of parameters.

Now we make the ODE more complicated and introduce y-dependence on the left side:

we study the **LINEAR FIRST-ORDER** problem

$$\begin{cases} y' = ay + f(t) \\ y(0) = y_0. \end{cases}$$

Again, $f \in \mathcal{C}(I)$ for some interval $I \subseteq \mathbb{R}$ with $0 \in I$, and a, $y_0 \in \mathbb{R}$. The function f is sometimes called the **FORCING** or **DRIVING** term. And, again, the expression $y' = ay + f(t)$ means that we want y to satisfy $y'(t) = ay(t) + f(t)$ for all t in the domain of y (which hopefully will turn out to be I). If $a = 0$, this reduces to a direct integration problem, and it would be nice if our final solution formula will respect that.

To motivate our solution approach, we first suppose $f = 0$ and consider the exponential growth problem

$$y' = ay.$$

Calculus intuition suggests that all solutions have the form $y(t) = Ce^{at}$, where necessarily $C = y(0) = y_0$. Problem 2.12 proves this using a (nonobvious) algebraic trick, but we will also see this as a consequence of the more general result below that includes the driving term.

The valuable, if surprising, idea that has come down to us through the generations is to replace the constant C with an unknown function u and guess that

$$y(t) = u(t)e^{at}$$

solves the more general problem $y' = ay + f(t)$. This is the first appearance of an **ANSATZ** in this course—that is, we have made a *guess* that a solution has a particular form.

Now the goal is to solve for u. Under the ansatz $y(t) = u(t)e^{at}$, we compute, with the product rule,

$$y'(t) = u'(t)e^{at} + u(t)ae^{at},$$

and we substitute that into our ODE $y' = ay + f(t)$. Then we need

$$u'(t)e^{at} + u(t)ae^{at} = au(t)e^{at} + f(t).$$

The same term $u(t)ae^{at}$ appears on both sides (this is a hint that we made the right ansatz), and we subtract it, leaving

$$u'(t)e^{at} = f(t).$$

We solve for things by getting them by themselves, so divide to find

$$u'(t) = e^{-at}f(t).$$

This is an ODE for u, but it would be nice if it had an initial condition. We know $y(t) = u(t)e^{at}$ and $y(0) = y_0$, so

$$y_0 = y(0) = u(0)e^{a0} = u(0).$$

That is, u must solve the direct integration problem

$$\begin{cases} u' = e^{-at}f(t) \\ u(0) = y_0, \end{cases}$$

and so, from our previous work,

$$u(t) = y_0 + \int_0^t e^{-a\tau} f(\tau)\ d\tau.$$

Returning to the ansatz $y(t) = u(t)e^{at}$, we have

$$y(t) = e^{at}\left(y_0 + \int_0^t e^{-a\tau} f(\tau)\ d\tau\right),$$

and so we have proved another theorem. By the way, we call it "variation of parameters" because we have "varied" the parameter y_0 in the solution to the linear homogeneous IVP (i.e., the solution $y(t) = y_0 e^{at}$ when $f = 0$) via the ansatz $y(t) = u(t)e^{at}$, with u replacing y_0.

4.1 Theorem (Variation of parameters). *Let $f \in \mathcal{C}(I)$ for some interval $I \subseteq \mathbb{R}$ with $0 \in I$ and a, $y_0 \in \mathbb{R}$. Then the only solution to*

$$\begin{cases} y' = ay + f(t) \\ y(0) = y_0 \end{cases} \tag{4.1}$$

is

$$y(t) = e^{at}\left(y_0 + \int_0^t e^{-a\tau} f(\tau)\ d\tau\right). \tag{4.2}$$

Is it?

4.2 Problem (!). **(i)** Check that the function y in (4.2) actually solves (4.1). (Does y satisfy $y'(t) = ay(t) + f(t)$ for all t in some interval containing 0? Do we have $y(0) = y_0$? Is y' continuous?)

(ii) Check that we recover the direct integration result of Theorem 3.2 from Theorem 4.1 when $a = 0$.

By the way, the ODE $y' = ay + f(t)$ is sometimes more precisely called a **FIRST-ORDER CONSTANT COEFFICIENT LINEAR ODE**. It is constant-coefficient because the coefficient a on y is a constant real number. This ODE is **HOMOGENEOUS** if $f(t) = 0$ for all t and otherwise **NONHOMOGENEOUS**. The uniqueness part of Theorem 4.1 proves that all solutions to $y' = ay$ have the form $y(t) = y(0)e^{at}$. Sometimes this is established with separation of variables, which we will consider shortly.

4.3 Example. We study

$$\begin{cases} y' = 2y + 3e^{-4t} \\ y(0) = 1, \end{cases}$$

and rather than just use the formula from (4.2), we repeat the "variation of parameters" argument with the concrete data at hand. The corresponding homogeneous problem is $y' = 2y$, which has the solutions $y(t) = Ce^{2t}$, and so we guess that our nonhomogeneous problem has the solution $y(t) = u(t)e^{2t}$. Substituting this into both sides of the ODE, we want

$$u'(t)e^{2t} + u(t)(2e^{2t}) = 2u(t)e^{2t} + 3e^{-4t},$$

thus

$$u'(t)e^{2t} = 3e^{-4t},$$

and so

$$u'(t) = 3e^{-6t}.$$

With the initial condition $u(0) = y(0) = 1$, this is the direct integration problem

$$\begin{cases} u' = 3e^{-6t} \\ u(0) = 1, \end{cases}$$

and the solution to that is

$$u(t) = 1 + \int_0^t 3e^{-6\tau}\, d\tau = 1 + \frac{3e^{-6\tau}}{-6}\bigg|_{\tau=0}^{\tau=t} = 1 + \frac{3e^{-6t} - 3}{-6} = \frac{3}{2} + \frac{e^{-6t}}{2}.$$

Thus the solution to the original IVP is

$$y(t) = e^{2t}\left(\frac{3}{2} + \frac{e^{-6t}}{2}\right).$$

4.4 Problem ($\star$). We probably expect physically that two objects in motion that start "close" together should remain "close" together, at least for "some" time. We might call this "continuous dependence on initial conditions." Suppose that u and v solve

$$\begin{cases} u' = au + f(t) \\ u(0) = u_0 \end{cases} \quad \text{and} \quad \begin{cases} v' = av + f(t) \\ v(0) = v_0. \end{cases}$$

That is, u and v solve the same ODE but with possibly different initial conditions.

(i) Prove that if $T \ge 0$ and $0 \le t \le T$, then

$$|u(t) - v(t)| \le e^{aT}|u_0 - v_0|. \tag{4.3}$$

This estimate controls how close u and v are over the time interval $[0, T]$ in terms of how close u_0 and v_0 are.

(ii) Suppose $a < 0$. What does (4.3) say about the behavior of different solutions to $y' = ay + f(t)$ as $t \to \infty$?

Our experience with ODE in general, and our concrete work with the linear problem, tell us that initial conditions should determine solutions uniquely. But sometimes in both ODE and PDE, one is less concerned with the initial state of the solution and more with its behavior at a "boundary." For example, what is the long-time asymptotic behavior of a solution? Does it have a limit at infinity, or does it settle down into some coherent shape? Here is a toy problem of how boundary behavior determines the solution.

4.5 Example. Let $f \in \mathcal{C}(\mathbb{R})$. All solutions to $y' = f(t)$ are

$$y(t) = y(0) + \int_0^t f(\tau)\ d\tau.$$

What, if any, choices for the initial condition $y(0)$ guarantee

$$\lim_{t\to\infty} y(t) = 0?$$

We want

$$\lim_{t\to\infty}\left(y(0) + \int_0^t f(\tau)\ d\tau\right) = 0.$$

By the basic algebra of limits, this happens if and only if (1) the limit

$$\lim_{t\to\infty}\int_0^t f(\tau)\ d\tau$$

exists and (2) the identity

$$y(0) + \lim_{t\to\infty}\int_0^t f(\tau)\ d\tau = 0$$

holds.

We have discovered something new: the condition (1) must be met. That is to say, the improper integral $\int_0^\infty f(\tau)\ d\tau$ must converge. Nowhere in the statement of the "end behavior problem"

$$\begin{cases} y' = f(t) \\ \lim_{t\to\infty} y(t) = 0 \end{cases} \tag{4.4}$$

was it made explicit that f must be improperly integrable on $[0,\infty)$. But if this is true, we have shown that any solution to (4.4) must satisfy $y(0) = -\int_0^\infty f(\tau)\ d\tau$, and so any solution to (4.4) satisfies the IVP

$$\begin{cases} y' = f(t) \\ y(0) = -\int_0^\infty f(\tau)\ d\tau. \end{cases}$$

Is the reverse true?

4.6 Problem (!). Assume the following.

(i) $f \in \mathcal{C}(\mathbb{R})$ is improperly integrable on $[0, \infty)$, i.e., $\lim_{b \to \infty} \int_0^b f$ exists.

$f \in \mathcal{C}(\mathbb{R})$ is improperly integrable on $[0, \infty)$, and that improper integrals respect the "algebraic" properties of definite integrals from Theorem 2.1, and that the derivative of $t \mapsto \int_t^\infty f$ is $-f(t)$, prove that the only solution to

$$\begin{cases} y' = f(t) \\ \lim_{t \to \infty} y(t) = 0 \end{cases}$$

is

$$y(t) = -\int_t^\infty f(\tau)\, d\tau.$$

Day 5: Wednesday, January 15.

Material from *Basic Partial Differential Equations* by Bleecker & Csordas

Pages 2–3 review separation of variables for ODE.

5.1 Example. Now we consider the more general "end behavior" problem

$$\begin{cases} y' = ay + f(t) \\ \lim_{t \to \infty} y(t) = 0, \end{cases}$$

where $a > 0$. Any solution to the ODE must meet

$$y(t) = e^{at}\left(y(0) + \int_0^t e^{-a\tau} f(\tau)\, d\tau\right),$$

and so our solution is the product of two functions, one of which blows up as $t \to \infty$ (since $\lim_{t \to \infty} e^{at} = \infty$ for $a > 0$). We probably want the other factor in the product to tend to 0 as $t \to \infty$; if that factor limited, say, to a nonzero constant, then the whole limit would be ∞ times that constant, which would definitely not be 0.

Indeed, we can see this using the definition of limit: if we assume $\lim_{t \to \infty} y(t) = 0$, then there is $M > 0$ such that if $t \geq M$, then $|y(t)| \leq 1$. From the formula for y, we find

$$\left| y(0) + \int_0^t e^{-a\tau} f(\tau)\, d\tau \right| \leq e^{-at}.$$

Since $a > 0$, this inequality and the squeeze theorem imply

$$\lim_{t \to \infty} \left(y(0) + \int_0^t e^{-a\tau} f(\tau)\, d\tau\right) = 0,$$

and thus

$$y(0) = -\int_0^\infty e^{-a\tau} f(\tau)\, d\tau \quad \text{and} \quad y(t) = -e^{at} \int_t^\infty e^{-a\tau} f(\tau)\, d\tau.$$

This directly generalizes the case of $a = 0$. In fact, we get a little more freedom here, in that for $a > 0$, it is easier for $\int_0^\infty e^{-a\tau} f(\tau)\, d\tau$ to exist (see below). We leave the case $a < 0$ as a (possibly surprising) exercise.

5.2 Problem ($\star$). Suppose that y solves $y' = ay + f(t)$ with $a < 0$ and $\lim_{t\to\infty} y(t) = 0$. As in the previous example, there is $M > 0$ such that for all $t \geq M$, we have

$$\left| y(0) + \int_0^t e^{-a\tau} f(\tau)\, d\tau \right| \leq e^{-at}.$$

However, since $-a > 0$, this does not imply any convergence of the integral term to $y(0)$ as $t \to \infty$.

(i) What if we consider $t \to -\infty$? Adapt the work in Example 5.1 to relate $\lim_{t\to-\infty} y(t)$, if this limit exists, and $y(0)$, in the case that $y' = ay + f(t)$ with $a < 0$.

(ii) Consider the concrete problem

$$y' = -2y + 3e^{-t}.$$

Show that every solution to this problem satisfies $\lim_{t\to\infty} y(t) = 0$, and thus the boundary condition as $t \to \infty$ is of no help in specifying the initial condition.

5.3 Problem (+). We clarify a remark from the previous example about improper integrals. In the following, let $a > 0$.

(i) Suppose that $f \in \mathcal{C}(\mathbb{R})$ is absolutely integrable on $[0, \infty)$; that is,

$$\int_0^\infty |f| := \lim_{b\to\infty} \int_0^b |f|$$

converges. Show that $\int_0^\infty e^{-a\tau} f(\tau)\, d\tau$ converges as well.

(ii) Suppose that $f \in \mathcal{C}(\mathbb{R})$ is bounded on $[0, \infty)$; that is, there is $M > 0$ such that

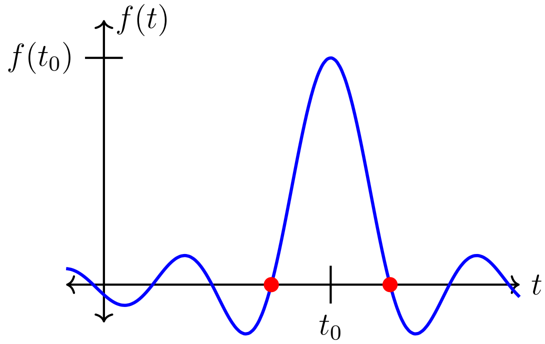
$$|f(t)| \leq M$$

for all $t \geq 0$. Show that $\int_0^\infty e^{-a\tau} f(\tau)\, d\tau$ still converges. Give an example to show that f need not be absolutely integrable on $[0, \infty)$.

Now we move to **SEPARABLE** ODE. Before defining and solving this kind of ODE in general, we do a pedestrian, but illustrative, example. And before doing that, we need to review a fact about continuity that will resurface many times in this course.

5.4 Lemma. *Let $I \subseteq \mathbb{R}$ be an interval and let $f \in \mathcal{C}(I)$. Suppose that $f(t_0) \neq 0$ for some $t_0 \in I$. Then there exists $\delta > 0$ such that $f(t) \neq 0$ for $t \in (t_0 - \delta, t_0 + \delta) \cap I$.*

Proof. We start with "proof by picture," which is always a good way to get an idea for the "real" proof. By continuity, the graph of f "near" t_0 should be "close" to $f(t_0)$, and so the graph should be above the t-axis.



Here is the more rigorous proof. By continuity and the assumption $f(t_0) > 0$, there is $\delta > 0$ such that if $t \in (t_0 - \delta, t_0 + \delta) \cap I$, then $|f(t) - f(t_0)| < f(t_0)/2$. (In the language of classical δ-ϵ proofs, we are taking $\epsilon = f(t_0)/2$ here.) This inequality is equivalent to

$$\frac{f(t_0)}{2} < f(t) < \frac{3f(t_0)}{2},$$

and so $f(t) > f(t_0)/2 > 0$ for $t \in (t_0 - \delta, t_0 + \delta) \cap I$.

If $f(t_0) < 0$, put $g(t) := |f(t)|$ and use the previous argument to conclude that $g(t) > 0$ for $t \in t_0 - \delta, t_0 + \delta) \cap I$, thus $f(t) < 0$ for those t. ■

5.5 Example. We study

$$\begin{cases} y' = y^2 \\ y(0) = y_0. \end{cases}$$

If $y_0 = 0$, then we can take $y(t) = 0$ for all t to get a solution; indeed, $y'(t) = 0$ and $(y(t))^2 = 0^2 = 0$ for all t.

Otherwise, suppose $y_0 \neq 0$. That is, $y(0) \neq 0$. By continuity—since a solution y to this IVP is defined and differentiable at 0, thus continuous at 0—for $t \approx 0$ we have $y(t) \neq 0$. Thus we can divide to find

$$\frac{y'(t)}{(y(t))^2} = 1. \tag{5.1}$$

This is the first big idea of separation of variables: "separate the variables" so that all appearances of the unknown function y and its derivative are together on one side. The second big idea is to integrate.

Specifically, since (5.1) holds for all $t \approx 0$, we can integrate

$$\int_0^t \frac{y'(\tau)}{(y(\tau))^2}\, d\tau = \int_0^t 1\, d\tau. \tag{5.2}$$

Note our good grammar: we are integrating from 0 to t, so we have changed the independent variable from t in (5.1) to τ above. We substitute $u = y(\tau)$ on the left and use the initial condition $y(0) = y_0$ to find

$$\int_0^t \frac{y'(\tau)}{(y(\tau))^2}\, d\tau = \int_{y(0)}^{y(t)} \frac{du}{u^2} = \int_{y_0}^{y(t)} u^{-2}\, du = -u^{-1}\big|_{u=y_0}^{u=y(t)} = \frac{1}{y_0} - \frac{1}{y(t)}.$$

Returning to (5.2), we find

$$\frac{1}{y_0} - \frac{1}{y(t)} = t,$$

and so we solve for $y(t)$ as

$$y(t) = \left(\frac{1}{y_0} - t\right)^{-1}.$$

Recalling that a formula alone is not sufficient to describe a function, we also establish the domain of this solution. As a formula alone, y above is defined on $\mathbb{R} \setminus \{y_0^{-1}\}$, but that is not an interval. Remember that we want the domain of the solution to this IVP to be an interval containing 0. The largest intervals in $\mathbb{R} \setminus \{y_0^{-1}\}$ (go big or go home) are $(-\infty, y_0^{-1})$ and (y_0^{-1}, ∞). Which interval we use depends on whether $y_0 < 0$ or $y_0 > 0$; if $y_0 < 0$, then $y_0^{-1} < 0$, too, so $0 \notin (-\infty, y_0^{-1})$ but $0 \in (y_0^{-1}, \infty)$. The reverse holds when $y_0 > 0$, and so there we take the domain to be $(-\infty, y_0^{-1})$.

Both situations illustrate a "blow-up in finite time." If we send t to the boundary of the domain, then the solution explodes to $\pm\infty$. For example, when $y_0 > 0$, the solution is defined on $(-\infty, y_0^{-1})$, and we have

$$\lim_{t \to (y_0^{-1})^-} y(t) = \lim_{t \to (y_0^{-1})^-} \frac{1}{y_0^{-1} - t} = \infty.$$

Note that here we are only using the limit from the left.

Now we generalize this work substantially. Let f and g be continuous functions (quite possibly on different subintervals of $\mathbb{R}$), and consider the IVP

$$\begin{cases} y' = f(t)g(t) \\ y(0) = y_0. \end{cases}$$

If $g(y_0) = 0$, then we claim that $y(t) = y_0$ is a solution to this IVP, which we call an **EQUILIBRIUM SOLUTION**.

5.6 Problem. Prove that.

Suppose that $g(y_0) \neq 0$. Since g is continuous, for y "close to" y_0, we have $g(y) \neq 0$. In fact, $g(y)$ is either positive for all y close to y_0 or negative for all y close to y_0.

Now we work backward. Assume that y solves this IVP with $g(y_0) \neq 0$. Since y is continuous and $y(0) = y_0$, for t close to 0, we have $y(t)$ close to y_0, and thus $g(y(t)) \neq 0$. We can then divide to find that for t close to 0, y must also satisfy

$$\frac{y'(t)}{g(y(t))} = f(t).$$

This is the heart of separation of variables: division. And division is only possible when the denominator is nonzero. We integrate both sides from 0 to t, still keeping t close to 0:

$$\int_0^t \frac{y'(\tau)}{g(y(\tau))}\, d\tau = \int_0^t f(\tau)\, d\tau. \tag{5.3}$$

There is not much more that we can say about the integral on the right, but on the left we take the composition $g \circ y$ as a hint to substitute $u = y(t)$. This yields

$$\int_0^t \frac{y'(\tau)}{g(y(\tau))}\, d\tau = \int_{y(0)}^{y(t)} \frac{du}{g(u)} = \int_{y_0}^{y(t)} \frac{du}{g(u)}. \tag{5.4}$$

Combining (5.3) and (5.4), we conclude that if y solves the separable IVP with $y_0 \neq 0$, then for t sufficiently close to 0, we have

$$\int_{y_0}^{y(t)} \frac{du}{g(u)} = \int_0^t f(\tau)\, d\tau.$$

We rewrite this one more time. Put

$$H(y,t) := \int_{y_0}^{y} \frac{du}{g(u)} - \int_0^t f(\tau)\, d\tau. \tag{5.5}$$

Here the domain of H is all y such that $g(u) \neq 0$ for u between y_0 and y and all t such that f is defined between 0 and t. Thus if y solves the separable IVP with $y_0 \neq 0$, then for t sufficiently close to 0, we have

$$H(y(t),t) = 0.$$

This is an **implicit equation** for y.

5.7 Problem (!). Consider the exponential growth problem

$$\begin{cases} y' = ay \\ y(0) = y_0, \end{cases}$$

where we assume $y_0 > 0$ (but place no restrictions on a). In the context of this specific problem, what is the function H from (5.5)? Use this function H, and the assumption $y_0 > 0$, to show, as expected, that $y(t) = y_0 e^{at}$.

It would be nice if, in general, we could reverse our logic and conclude that if $H(y(t),t) = 0$, then y solves the separable IVP. More generally, why should we be able to solve $H(y,t) = 0$?

5.8 Problem (+). The IMPLICIT FUNCTION THEOREM says the following. Let a, b, $c \in \mathbb{R}$ with $a < b$ and $c > 0$. Let H be defined on $\mathcal{D} := \{(y,t) \in \mathbb{R}^2 \mid a < y < b, \ |t| < c\}$, and suppose that the partial derivatives H_y and H_t exist and are continuous on $\mathcal{D}$. Suppose that $H(y_0, 0) = 0$ for some $y_0 \in (a,b)$ with $H_y(y_0, 0) \neq 0$. Then there exist δ, $\epsilon > 0$ and a continuously differentiable function $Y : (-\delta, \delta) \to (y_0 - \epsilon, y_0 + \epsilon)$ such that $H(y,t) = 0$ for $|t| < \delta$ and $|y - y_0| < \epsilon$ if and only if $y = Y(t)$. In particular, $Y(0) = y_0$.

We use the implicit function theorem to prove the existence of solutions to separable IVP.

(i) For practice, consider $H(y,t) := y^2 + t^2 - 1$. Check that $H(1,0) = 0$ and $H_y(1,0) \neq 0$ and conclude that $H(Y(t),t) = 0$ for some function Y defined on a subinterval $(-\delta, \delta)$. Then do algebra and find an explicit formula for Y.

(ii) In this part and the following, consider the separable problem

$$\begin{cases} y' = f(t)g(y) \\ y(0) = y_0, \end{cases}$$

where g is continuous on (a,b), f is continuous on $(-c,c)$, and $y_0 \in (a,b)$ with $g(y_0) \neq 0$. Without loss of generality, we will assume $g(y) > 0$ for $y \in (a,b)$. Our goal is to solve the implicit equation

$$H(y,t) := \int_{y_0}^{y} \frac{du}{g(u)} - \int_0^t f(\tau)\, d\tau = 0$$

First check that $H(y_0, 0) = 0$ and $H_y(y_0, 0) \neq 0$, and obtain the existence of a function Y meeting the conclusions of the implicit function theorem with $Y(0) = 1$. (In particular, we get $Y(0) = y_0$.)

(iii) Now we show that Y solves the original ODE. Differentiate the identity $H(Y(t),t) = 0$ with respect to t, use the multivariable chain rule and FTC1, and conclude that $Y' = f(t)g(Y)$.

(iv) It turns out that just from $H(Y(0), 0) = 0$ we can obtain $Y(0) = y_0$, even without the implicit function theorem. To see this, use properties of integrals to show that $H(Y(0), 0) = 0$ implies

$$\int_{y_0}^{Y(0)} \frac{du}{g(u)} = 0.$$

Suppose that $Y(0) \neq y_0$ and use the monotonicity of the integral and the fact that $g(u) > 0$ for u between y_0 and $Y(0)$ to obtain a contradiction.

Day 6: Friday, January 17.

Material from *Basic Partial Differential Equations* by Bleecker & Csordas

There are many examples of second-order constant-coefficient linear ODE on pp. 6–13. Example 8, while worth reading, is probably more complicated than any problem that we will encounter at this level for some time.

The final kind of ODE that we need to review for this course is the second-order constant-coefficient linear problem, which reads

$$ay'' + by' + cy = f(t),$$

with a, b, $c \in \mathbb{R}$, $a \neq 0$ (so that the problem is genuinely second-order), and f continuous on some interval containing 0. One can prove the following theorem by recasting the second-order linear problem as a first-order linear system and developing an analogue of variation of parameters for that system, which requires some matrix manipulations but not too much fuss otherwise. We will not purse the linear system/matrix approach here.

6.1 Theorem. *Let a, b, $c \in \mathbb{R}$ with $a \neq 0$ and $f \in \mathcal{C}(I)$ with $0 \in I$. There exists a unique solution $y \in \mathcal{C}^2(I)$ to the IVP*

$$\begin{cases} ay'' + by' + cy = f(t) \\ y(0) = y_0 \\ y'(0) = y_1. \end{cases}$$

This theorem does not tell us in the slightest a *formula* for y. That will come later in the case $f \neq 0$, as we simply do not need it right now. Instead, we will focus largely on the homogeneous problem with $f = 0$. The uniqueness result is a consequence of the following.

6.2 Lemma. *Let a, b, $c \in \mathbb{R}$ with $a \neq 0$. The only solution $y \in \mathcal{C}^2(\mathbb{R})$ to*

$$\begin{cases} ay'' + by' + cy = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

is $y = 0$.

6.3 Problem (!). Use Lemma 6.2 to prove Theorem 6.1. [Hint: *suppose that the IVP in the theorem has two solutions, say, u and v. What IVP does $z := u - v$ satisfy?*]

6.4 Problem (+). This problem outlines a proof of Lemma 6.2. *Content forthcoming.*

Taking Theorem 6.1 for granted, we now focus on the homogeneous case of $f = 0$. Here one studies the **CHARACTERISTIC EQUATION**

$$a\lambda^2 + b\lambda + c = 0$$

and develops solution patterns based on the root structure. They are the following.

Root structure	**Solution pattern**
Two distinct real roots $\lambda_1 \neq \lambda_2$	$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
One repeated real root λ_0	$y(t) = c_1 e^{\lambda_0 t} + c_2 t e^{\lambda_0 t}$
Two complex conjugate roots $\alpha \pm i\beta$ $(\beta \neq 0)$	$y(t) = e^{\alpha t}\big(c_1 \cos(\beta t) + c_2 \sin(\beta t)\big)$

That any of these solution patterns actually works can be checked by directly substituting it into the ODE and using the structure of a, b, and c that results from the root pattern. For example, in the repeated real root case one has $b^2 - 4ac = 0$, thus $c = b^2/4a$, and also $\lambda_0 = -b/2$. So, one would need to show that $y(t) = c_1 e^{-(b/2)t} + c_2 t e^{-(b/2)t}$ solves $ay'' + by' + (b^2/4a)y = 0$. This is mostly a lot of thankless algebra—so thankless that we do not even spell it out as a problem.

6.5 Example. **(i)** The characteristic equation of $y'' - y = 0$ is $\lambda^2 - 1 = 0$. Factoring the difference of perfect squares, we have $\lambda = \pm 1$. These are distinct real roots, so all solutions are $y(t) = c_1 e^t + c_2 e^{-t}$.

(ii) The characteristic equation of $y'' = 0$ is $\lambda^2 = 0$, so $\lambda = 0$. This is a repeated real root, so all solutions are $y(t) = c_1 e^{0t} + c_2 t e^{0t} = c_1 + c_2 t$. (Of course, we could directly integrate twice to get the same result.)

(iii) The characteristic equation of $y'' + y = 0$ is $\lambda^2 + 1 = 0$, so $\lambda^2 = -1$ and thus $\lambda = \pm i$. These are complex conjugate roots with $\alpha = 0$ (which is certainly allowed) and $\beta = 1$. All solutions are $y(t) = e^{0t}\big(c_1 \cos(t) + c_2 \sin(t)\big) = c_1 \cos(t) + c_2 \sin(t)$.

6.6 Example. Let $\lambda \in \mathbb{R}$. The IVP

$$\begin{cases} y'' + \lambda^2 y = 0 \\ y(0) = y_0 \\ y'(0) = y_1 \end{cases}$$

governs the motion of an undamped, undriven simple harmonic oscillator (at least when $\lambda > 0$). We can extract two solution formulas:

$$y(t) = \begin{cases} y_0 + y_1 t, \ \lambda = 0 \\ y_0 \cos(\lambda t) + \dfrac{y_1}{\lambda} \sin(\lambda t), \\ \lambda \neq 0. \end{cases}$$

It would be nice if these were really "the same." We might wonder what happens in the limit as $\lambda \to 0$ with t fixed. Certainly $\lim_{\lambda \to 0} y_0 \cos(\lambda t) = y_0$, and L'Hospital's rule gives

$$\lim_{\lambda \to 0} \frac{\sin(\lambda t)}{\lambda} = t.$$

This should be comforting: the solution appears to be continuous in λ. Is there a more efficient way to write it? In particular, can we make a factor of t appear in the second term when $\lambda \neq 0$? Certainly:

$$\frac{\sin(\lambda t)}{\lambda} = t\left(\frac{\sin(\lambda t)}{\lambda t}\right)$$

when $t \neq 0$. Put

$$\operatorname{sinc}(x) := \begin{cases} \sin(x)/x, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

L'Hospital's rule ensures continuity of sinc; it is, in fact, infinitely differentiable.

For $t \neq 0$, we then have

$$y(t) = y_0 \cos(\lambda t) + y_1 t \operatorname{sinc}(\lambda t).$$

This formula is also valid at $t = 0$, since it reduces to $y_0 = y(0)$ there.

6.7 Example. We can prove uniqueness of solutions to

$$\begin{cases} y'' + \lambda^2 y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

directly, without relying on Lemma 6.2. What is really valuable here is not the uniqueness result but the trick that we use to get it: multiply through by a derivative. This will resurface from time to time in our study of actual PDE.

Specifically, if $y'' + \lambda^2 y = 0$, then $y''y' + \lambda^2 yy' = 0$. It may not be obvious at first glance, since doing calculus in reverse probably feels unusual, but

$$yy' = \partial_t \left[\frac{y^2}{2}\right].$$

And, similarly,

$$y''y' = \partial_t \left[\frac{(y')^2}{2}\right]$$

Thus

$$\frac{1}{2}\partial_t[y^2 + (y')^2] = 0,$$

and so $y^2 + (y')^2$ is constant. We only know the value of y and y' at one point: $t = 0$. And so

$$(y(t))^2 + (y'(t))^2 = (y(0))^2 + (y'(0))^2 = 0$$

for all t.

Now here is another trick: if a, $b \in \mathbb{R}$, and if $a^2 + b^2 = 0$, then $a = b = 0$. Otherwise if $a \neq 0$ or $b \neq 0$, then $a^2 > 0$ or $b^2 > 0$, and then we would have $0 < a^2 + b^2 = 0$, a contradiction. In particular, $y(t) = 0$ for any t.

6.8 Problem (+). Generalize the preceding work as follows. Let $\mathcal{V} \in \mathcal{C}^1(\mathbb{R})$ with $\mathcal{V}(r) > 0$ for all $r \neq 0$, $\mathcal{V}(0) = 0$, and $\mathcal{V}'(0) = 0$. Show that the only solution to the IVP

$$\begin{cases} y'' + \mathcal{V}'(y) = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

is $y = 0$. [Hint: *for existence, be sure to explain why $y = 0$ is actually a solution. For uniqueness, suppose that y solves the IVP, multiply by y', and obtain that $(y')^2/2 + \mathcal{V}(y)$ is constant. What is its value? What does that tell you about $\mathcal{V}(y)$?*]

6.9 Problem (+). If we change the ODE from $y'' + \lambda^2 y$ to $y'' - \lambda^2 y$, the formula in Example 6.6 and the uniqueness proof in Example 6.7 will not work. We adapt them here.

(i) The **HYPERBOLIC SINE AND COSINE**, respectively, are

$$\sinh(x) := \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh(x) := \frac{e^x + e^{-x}}{2}.$$

Show that the solution to the IVP

$$\begin{cases} y'' - \lambda^2 y = 0 \\ y(0) = y_0 \\ y'(0) = y_1 \end{cases}$$

can be written in the form

$$y(t) = y_0 \cosh(\lambda t) + \frac{y_1}{\lambda} \sinh(\lambda t).$$

(ii) Prove that the only solution to

$$\begin{cases} y'' - \lambda^2 y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

is $y = 0$ using the following steps. Put $z(t) := y(t/\lambda)$ and show that $z'' - z = 0$ with $z(0) = z'(0) = 0$. Show next that $(z' + z)' = z' + z$, so $z'(t) + z(t) = Ce^t$ for some constant C. Take $t = 0$ to conclude $C = 0$, so $z' = -z$. How does this help?