Instructions. The following problems are designed to test your mastery of essential material from MATH 1190: Calculus I, MATH 2202: Calculus II, MATH 2203: Calculus III, and MATH 2306: Ordinary Differential Equations that we will need in our PDE course. I won't collect or grade your work, and you should not be too alarmed if you struggle with some of these problems, especially if it has been some time since you took these classes. In particular, we will review various aspects of definite integrals, improper integrals, and series throughout the term. However, any struggles here will indicate things that you need to review. If you don't know what a word or concept means, you should look it up in a relevant textbook or online. In some cases, I specifically direct you to look something up first, and you should definitely do that!

Calculus I/II (MATH 1190/2202).

1. Suppose that f is a function defined on the interval $[0, 2]$. Explain in words what the equalities

$$\lim_{x\to 0^+} f(x) = 1, \qquad \lim_{x\to 1} f(x) = 2, \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = 3$$

mean.

2. Draw the graph of a function f defined on the interval $[-1, 1]$ that is continuous at all points in $[-1, 1]$ except -1, 0, and 1 and such that

$$\lim_{x\to a^+} f(x) \quad \text{and} \quad \lim_{x\to a^-} f(x)$$

exist for every number a in $[-1, 1]$.

3. Let f and g be continuous on the interval I and let a, b, and c be points in I. Let α be any real number. Match each of the integrals (I) through (VI) in the left column with the quantities (A) through (F) in the right column.

(I) $\int_a^b \big(f(t) + g(t)\big)\, dt$	(A) $\int_a^b f(t)\, dt$
(II) $\int_a^b \alpha f(t)\, dt$	(B) $b - a$
(III) $\int_a^c f(t) + \int_c^b f(t)\, dt$	(C) $\int_a^b f(t)\, dt + \int_a^b g(t)\, dt$
(IV) $\int_a^a f(t)\, dt$	(D) $-\int_a^b f(t)\, dt$
(V) $\int_b^a f(t)\, dt$	(E) 0
(VI) $\int_a^b 1\, dt$	(F) $\alpha \int_a^b f(t)\, dt$

4. It is a fact that if f is a continuous function on an interval $[a, b]$ with $f(x) \geq 0$ for all

$a \le x \le b$, and if $f(x_0) > 0$ for some point x_0 in $[a, b]$, then

$$\int_a^b f(x)\ dx > 0.$$

Draw a picture that explains why this is true.

5. Suppose that f is a continuous function on the interval $[0, 1]$ and

$$\int_0^1 f(x)\ dx = 2.$$

Show that

$$\int_0^1 x f(x^2)\ dx = 1.$$

[Hint: *substitute.*]

6. Suppose that f is a differentiable function on the interval $[-\pi, \pi]$ and f' is continuous on $[-\pi, \pi]$. Suppose also that

$$\int_{-\pi}^{\pi} \sin(x) f'(x)\ dx = -1.$$

Show that

$$\int_{-\pi}^{\pi} \cos(x) f(x)\ dx = 1.$$

[Hint: *integrate by parts.*]

7. Suppose that f is continuous on $(-\infty, \infty)$. Look up the definition of the improper integral

$$\int_{-\infty}^{\infty} f(x)\ dx.$$

Use this definition to calculate

$$\int_{-\infty}^{\infty} e^{-|x|}\ dx.$$

Then explain why $f(x) = x$ is not improperly integrable on $(-\infty, \infty)$.

8. Look up the comparison test for improper integrals and use it to show that the function

$$\operatorname{sech}(x) := \frac{2}{e^x + e^{-x}}$$

is improperly integrable on $(-\infty, \infty)$. This function sech is the **HYPERBOLIC SECANT**.

9. Suppose that f is an improperly integrable function on $(-\infty, \infty)$ and abbreviate

$$\mathcal{I} := \int_{-\infty}^{\infty} f(x)\ dx.$$

Let a be a real number. Use the definition of the improper integral to show that

$$\int_{-\infty}^{\infty} f(x+a)\ dx = \mathcal{I}.$$

If $a \neq 0$, show that

$$\int_{-\infty}^{\infty} f(ax)\ dx = \frac{\mathcal{I}}{a}.$$

10. Let (a_k) be a sequence of real numbers. Here we assume $k \geq 0$ is an integer. Explain in words what it means for the series

$$\sum_{k=0}^{\infty} a_k$$

to converge to some real number S.

11. Look up the comparison test for series and what a p-series is (specifically, for which real numbers p does the series $\sum_{k=0}^{\infty} k^p$ converge?). Use these results to show that the series

$$\sum_{k=0}^{\infty} \frac{1}{1+k^2}$$

converges.

12. Suppose that (a_k) is a bounded sequence in the sense that there is $M > 0$ such that $|a_k| \leq M$ for all k. Let $r > 0$. Explain why the series

$$\sum_{k=0}^{\infty} a_k e^{-krt} \cos(kx)$$

converges for all real numbers x and all $t > 0$. [Hint: *think about the series* $\sum_{k=0}^{\infty}(e^{-rt})^k$ *as a geometric series, and look up results on geometric series as needed. If* r, $t > 0$, *why do we have* $|e^{-rt}| < 1$*? Put this together with the comparison test.*]

Calculus III (MATH 2203).

13. Suppose that u is a function defined on $\mathbb{R}^2$. We write $u = u(x,t)$ to indicate that we will write the independent variables of u as x and t in that order. Explain in words what the equality

$$\lim_{(x,t)\to(0,0)} u(x,t) = 1$$

means.

14. Explain why the function

$$u(x,t) := \begin{cases} \dfrac{x^2-t^2}{x^2+t^2}, & (x,t) \neq (0,0) \\ \\ 0, & (x,t) = (0,0) \end{cases}$$

is not continuous at $(0,0)$. [Hint: *consider the behavior of* u *along the coordinate axes.*]

15. Suppose that $u = u(x,t)$ is a function. Give the limit definitions of the partial derivatives $u_x(x,t)$ and $u_t(x,t)$ at points (x,t) in $\mathbb{R}^2$. Note that these partial derivatives may be written as

$$\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial t}$$

although we will not do that in our class. We will, however, sometimes write $\partial_x[u](x,t)$ and $\partial_t[u](x,t)$.

16. Let $u(x,t) = \cos(2x - 3t)$. Compute all of the partial derivatives

$$u_x, \qquad u_t, \qquad u_{xt}, \qquad u_{tx}, \qquad u_{xx}, \qquad \text{and} \qquad u_{tt}.$$

Find numbers a and b such that $au_t + bu_x = 0$.

17. For a function $u = u(x,t)$, how would you interpret the partial derivatives u_x and u_t as rates of change? In what direction is the change being measured?

18. With $u(x,t) = \cos(2x - 3t)$ as before, compute the GRADIENT ∇u.

19. If $\mathbf{v} = (v_1, v_2)$ is a vector in $\mathbb{R}^2$, what does the DIRECTIONAL DERIVATIVE of a function $u = u(x,t)$ in the direction of $\mathbf{v}$ measure? What is the limit definition of this directional derivative? How can you use the gradient to compute the directional derivative quickly? Do so for $u(x,t) = \cos(2x - 3t)$ and $\mathbf{v} = (1,2)$.

20. Look up what the multivariable chain rule says and use it to compute $H'(0)$, where $H(s) = h(f(s), g(s))$ and $h = h(x,t)$, f, and g are functions satisfying

$$h_x(1,2) = 3, \qquad h_t(1,2) = 4, \qquad f(0) = 1, \qquad g(0) = 2, \qquad f'(0) = 5, \qquad \text{and} \qquad g'(0) = 6.$$

Ordinary Differential Equations (MATH 2306).

21. How would you solve a SEPARABLE ODE of the form $y'(t) = g(t)h(y(t))$? Here g and h are given functions and y is the unknown.

22. Let y_0 be a real number. What is the largest open interval containing 0 on which the solution to

$$\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$$

is defined? Your answer will involve y_0.

23. How would you solve a FIRST-ORDER LINEAR ODE of the form $y'(t) = a(t)y(t) + b(t)$. Here a and b are given functions, and y is the unknown.

24. Solve

$$\begin{cases} y' = 2y + 3e^{4t} \\ y(0) = y_0 \end{cases}$$

for $y_0 = 0$ and $y_0 = 1$.

25. Let a and y_0 be real numbers and let f be a function. Solve

$$\begin{cases} y' = ay + f(t) \\ y(0) = y_0. \end{cases}$$

Your answer will involve a, y_0, and a definite integral from 0 to t with f appearing in the integrand.

26. How would you solve a **SECOND-ORDER LINEAR, CONSTANT-COEFFICIENT, HOMOGENEOUS** ODE of the form $ay'' + by' + cy = 0$, where a, b, and c are real numbers and $a \neq 0$?

27. Find all solutions to

$$y'' - y = 0, \qquad y'' + y = 0, \qquad \text{and} \qquad y'' + y' = 0.$$

28. For which real numbers p are all solutions y to the ODE

$$y'' + py' + y = 0$$

bounded? (A solution y is **BOUNDED** if there is a number $M > 0$ such that $|y(t)| \leq M$ for all t.)