Set Theory, Quantifiers, and Functions

This document outlines essential concepts, vocabulary, and notation that we will use frequently and without much comment.

Set theory.

1 Undefinition. *A* **SET** *is a collection of objects, called* **ELEMENTS**. *If x is an element of the set U, then we write $x \in U$, and if y is not an element of the set A, then we write $y \notin U$. We assume that there is a notion of equality among elements of U such that the expression $x = y$ makes sense for x, $y \in U$.*

This is an undefinition, not a definition, because we have not defined what "collections" or "objects" or "equals" really means. And we will not. If a set U consists of only finitely many elements, then we may denote U by listing those elements between curly braces. For example, the set consisting precisely of the numbers 1, 2, and 3 is $\{1, 2, 3\}$; the set consisting precisely of the number 1 is $\{1\}$, and $1 \in \{1\}$. If we have enumerated these elements, say as $x_1, \ldots, x_n$, then we may also write $U = \{x_k\}_{k=1}^{n}$.

2 Example. Let $U = \{1, 2, 3\}$. Then $1 \in U$ but $4 \notin U$. And if $x_k = k$ for $k = 1$, 2, 3, then $U = \{x_k\}_{k=1}^{3}$.

If U is a set, and if $P(x)$ is a statement that is either true or false for each $x \in U$, then we denote the set of all elements x of U for which $P(x)$ is true by

$$\{x \in U \mid P(x)\}.$$

We read the expression $P(x)$ as "it is the case that $P(x)$" or "it is the case that $P(x)$ is true."

3 Example. If $U = \{1, 2, 3, 4\}$, then

$$\{x \in U \mid x \text{ is even}\} = \{2, 4\}.$$

Here $P(x)$ is the statement "x is even."

One of the most important actions that we can perform on a set is to compare it to another set. Frequently we want to show that every element of a given set is an element of another set; this encapsulates the logic of showing that if a given property is true, then another property is true (and this logic is almost all of math).

4 Definition. *A set A is a* **SUBSET** *of a set B if for each $x \in A$, it is the case that $x \in B$. That is, every element of A is an element of B. If A is a subset of B, we write $A \subseteq B$; if A is not a subset of B, then we write $A \nsubseteq B$.*

In symbols,

$$A \subseteq B \iff (x \in A \Longrightarrow x \in B).$$

5 Example. $\{1,2\} \subseteq \{1,2,3\}$ and $\{1,2,3\} \subseteq \{1,2,3\}$, but $\{1,2\} \not\subseteq \{1,3\}$.

A special instance of comparing sets is determining when they are the same, or equal.

6 Definition. *Two sets A and B are* **EQUAL**, *written $A = B$, if every element of A is an element of B, and if every element of B is an element of A. In symbols,*

$$A = B \iff A \subseteq B \text{ and } B \subseteq A \iff (x \in A \iff x \in B).$$

7 Hypothesis. **(i)** *There exists a set $\varnothing$ that contains no element. That is, if x is an element of any set U, then $x \notin \varnothing$. We call this set the* **EMPTY SET**.

(ii) *Let U be a set. An element $x \in U$ cannot be equal to the set $\{x\} \subseteq U$ whose only element is x. That is, $x \neq \{x\}$.*

(iii) *If we define a set by listing its elements within curly braces, repetition or reordering of the elements does not change the set. For example, $\{1,2,3\} = \{1,2,3,1\} = \{1,3,2\}$.*

There are several fundamental, almost "algebraic" ways in which two or more sets interact.

8 Definition. *Let U be a set and A, $B \subseteq U$. The* **UNION** *of A and B is the set*

$$A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\},$$

the **INTERSECTION** *of A and B is the set*

$$A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\},$$

and the **COMPLEMENT** *of A in B is the set*

$$B \setminus A := \{x \in B \mid x \notin A\}.$$

That is, $A \cup B$ is the set of all elements in either A or B (or both), $A \cap B$ is the set of all elements in both A and B, and $B \setminus A$ is the set of all elements in B but not in A.

9 Example. Let

$$A = \{1,2,3\} \quad \text{and} \quad B = \{2,4,6\}.$$

Then

$$A \cup B = \{1,2,3,4,6\},$$

$$A \cap B = \{2\},$$

and

$$B \setminus A = \{4, 6\}.$$

10 Problem (!). Let A and B be as in Example 9. Determine the elements of the following sets.

(i) $A \setminus B$

(ii) $(A \setminus B) \cup B$

(iii) $(A \cap B) \setminus A$

(iv) $A \setminus \varnothing$

(v) $\varnothing \setminus B$

We have agreed that when listing the elements of a (finite) set in curly braces, order does not matter: $\{1, 2\} = \{2, 1\}$. However, there are situations in which a notion of order is essential. One way to accomplish this is via the concept of the ordered pair, which is fundamental to a rigorous definition of function and to constructing many interesting and useful sets out of existing sets (often these interesting and useful constructs are sets of functions!).

11 Undefinition. *Let A and B be sets, and let $x \in A$ and $y \in B$. The* **ORDERED PAIR** *(x, y) should satisfy the following property: if $s \in A$ and $t \in B$, then $(x, y) = (s, t)$ if and only if $x = s$ and $y = t$.*

This property is not a definition, because it specifies what an ordered pair *does* rather than what an ordered pair *is*. But very often what things *do* defines what things *are*.

12 Problem (+). Let A and B be sets and let $x \in A$ and $y \in B$. Show that defining

$$(x, y) := \big\{x, \{x, y\}\big\}$$

satisfies the essential property of an ordered pair in the sense that if $s \in A$ and $t \in B$, then

$$\big\{x, \{x, y\}\big\} = \big\{s, \{s, t\}\big\}$$

if and only if $x = s$ and $y = t$.

13 Example. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Then

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}.$$

14 Problem (!). Let A and B be as in Example 13. Determine the elements of the following sets.

(i) $B \times A$

(ii) $\varnothing \times A$

While we could give a definition of ordered n-tuple similar to that of ordered pair, with the essential property being that $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ if and only if $x_k = y_k$ for $k = 1, \ldots, n$, we will not, and we will instead construct tuples using functions later.

Quantifiers.

Quantifiers tell us how elements of different sets may interact with each other and with other overarching properties of these sets. We use the symbols $\forall$, $\exists$, and $\exists!$ to abbreviate three very common phrases. Let A be a set and, for $x \in A$, let $P(x)$ be a property that is either true or false.

- The string of symbols $\forall x \in A : P(x)$ is read as "for all $x \in A$ it is the case that $P(x)$ is true." For $\forall x \in A : P(x)$ to be true, we need to show that picking any $x \in A$ results in $P(x)$ being true.

- The string of symbols $\exists x \in A : P(x)$ is read as "there exists $x \in A$ such that $P(x)$ is true." For $\exists x \in A : P(x)$ to be true, we just need to show that $P(x)$ is true for one $x \in A$. Perhaps $P(x)$ is true for all $x \in A$ ($\forall x \in A : P(x) \Longrightarrow \exists x \in A : P(x)$), but determining that is unnecessary.

- The string of symbols $\exists! x \in A : P(x)$ is read as "there exists a unique $x \in A$ such that $P(x)$ is true." For $\exists! x \in A : P(x)$ to be true, we need to show that $P(x)$ is true for one $x \in A$ and no other. Often we do this by assuming that $P(x_1)$ and $P(x_2)$ are true for some x_1, $x_2 \in A$, and then we show that $x_1 = x_2$, whatever "=" means in the context at hand.

15 Example. **(i)** The statement "For all real numbers x it is the case that x^2 is nonnegative" compresses to

$$\forall x \in \mathbb{R} : x^2 \geq 0.$$

Here $\mathbb{R}$ denotes the set of all real numbers. This is a true statement.

(ii) The statement "There exists a real number x such that $x^2 = 4$" compresses to

$$\exists x \in \mathbb{R} : x^2 = 4.$$

This too is true.

(iii) The statement "There exists a unique positive real number x such that $x^2 = 4$" compresses to

$$\exists! x \in (0, \infty) : x^2 = 4.$$

And this is also true, although the statement $\exists! x \in \mathbb{R} : x^2 = 4$ is false. In the first part of the compressed symbolic form, we could also have written

$$\exists! x > 0 : x^2 = 4$$

and used the equivalence of $x \in (0, \infty)$ and $x > 0$ to phrase things differently.

We can chain quantifiers together as much as necessary, and we will not be too picky about saying "such that" every single time we write in English words.

16 Example. **(i)** The statement "For all real numbers x, there is a real number y such that there is a unique positive real number z with $z^2 = xy$" compresses to

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \exists! z > 0 : z^2 = xy.$$

What is y?

(ii) The statement "For all real numbers x, if x is nonnegative, then there exists a unique nonnegative y such that $y^2 = x$" compresses to

$$\forall x \in \mathbb{R} : x \geq 0 \Longrightarrow \exists! y \geq 0 : y^2 = x.$$

However, a shorter, and (importantly!) still logically equivalent version of the above is

$$\forall x \geq 0\ \exists! y \geq 0 : y^2 = x.$$

When writing a statement with symbolic quantifiers, we do not always need to replicate verbatim every part of that statement, as long as we have a logically equivalent form.

17 Problem ($\star$). Let $I \subseteq \mathbb{R}$ be an interval and let f be a real-valued function defined on I. Translate each of the following statements into (a logically equivalent) symbolic form using $\forall$, $\exists$, and/or $\exists!$ whenever possible.

(i) For all $x \in I$ and all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in I$ with $|x - y| < \delta$, then

$$|f(x) - f(y)| < \epsilon.$$

(ii) For all $\epsilon > 0$, there exists $\delta > 0$ such that if x, $y \in I$ with $|x - y| < \delta$, then

$$|f(x) - f(y)| < \epsilon.$$

(iii) There exist $x \in I$ and $\delta > 0$ such that if $y \in I$ with $|x - y| < \delta$, then

$$|f(y)| > \frac{f(x)}{2}.$$

(iv) For all a, $b \in I$ and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$, there exists $x \in I$ such that $f(x) = c$.

(v) For all $y \in \mathbb{R}$ there exists a unique $x \in I$ such that $f(x) = y$.

When *negating* quantified statements, it can be helpful to write them out symbolically and then formally flip each $\forall$ to $\exists$ and each $\exists$ to $\forall$. More precisely, the negation of the statement $\forall x : P(x)$ is $\exists x :\sim P(x)$, where $\sim P(x)$ is an abbreviation for the statement "it is not the case that $P(x)$" or "it is not the case that $P(x)$ is true" (or "it is the case that $P(x)$ is false"). Likewise, the negation of $\exists x : P(x)$ is $\forall x :\sim P(x)$. There is no standard way to flip $\exists!$ in a negation, as the negation of unique existence is either nonunique existence or nonexistence.

18 Example. We abbreviate the statement "For all real numbers x, there is a real number y such that the product xy is nonnegative" by "$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : xy \geq 0$." Symbolically, its negation is $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : xy < 0$." In words, this negation reads "There is $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ the product xy is negative." (Which is true, the original statement or its negation?)

Frequently we need to negate quantified statements that are couched in "if-then" language. Recall that the statement "If P, then Q" is true if the statement "It is not the case that P is true and Q is false." (This reflects the intuition behind the compression $P \implies Q$: we want the truth of P to force the truth of Q.) So, the negation of "If P, then Q" is "P and not Q." Negating that Q often involves manipulating quantifiers.

19 Example. Consider the statement "If x is a real number, then there exists a real number y such that $y^3 = x$." (This number y is actually unique, but we will not include the uniqueness quantifier here to make negation easier.) One way to compress this symbolically is

$$x \in \mathbb{R} \implies \exists y \in \mathbb{R} : y^3 = x.$$

The negation of this statement is then

$$x \in \mathbb{R} \text{ and } \forall y \in \mathbb{R} : y^3 \neq x.$$

20 Problem ($\star$). Negate all but the last quantified symbolic statements from Problem 17 and then write out those negations as complete sentences using English words and none of the symbols $\forall$, $\exists$, and/or $\exists!$. [Hint: *recall that the negation of $P \implies Q$ is "P and not Q."*]

Functions.

Set-theoretically, functions relate elements of one set to elements of another set; practically, functions govern almost all behaviors in mathematics.

21 Undefinition. *A* FUNCTION *from a set A to a set B is a rule or operation that pairs (or associates, or maps) every element of A with one and only one element of B.*

The problem with this definition (which is why it is an undefinition) is the use of weasel words: "rule," "operation," "pairs," "associates," "maps." What do these words mean? We will make this annoyingly precise, but first we consider some examples to see how broad functions can be.

22 Example. The following should all be functions.

(i) The pairing of real numbers x with their doubles $2x$ is a function: every real number is paired with another number, and only one number at that.

(ii) The pairing of people in a room with the date (1 through 31) on which they were born. Everyone has only one birthday.

(iii) The pairing of people in a room with the color of the chair in which they are seated (assuming everyone is sitting in a chair and every chair has a discernible color). This last function does not involve numbers at all!

The better definition of function involves ordered pairs.

23 Definition. *Let A and B be sets. A* **FUNCTION** f **FROM** A **TO** B *is a set of ordered pairs with the following properties.*

(i) *If $(x, y) \in f$, then $x \in A$ and $y \in B$. That is, $f \subseteq A \times B$.*

(ii) *For each $x \in A$, there is a unique $y \in B$ such that $(x, y) \in f$.*

We often use the notation $f\colon A \to B$ to mean that f is a function from A to B. If $(x, y) \in f$, then we write $y = f(x)$. The set A is the **DOMAIN** *of f, and the set B is the* **CODOMAIN** *of f. The* **IMAGE** *or* **RANGE** *of f is the set*

$$f(A) := \{f(x) \mid x \in A\}.$$

More generally, if $E \subseteq A$, then the **IMAGE OF** E **UNDER** f *is*

$$f(E) := \{f(x) \mid x \in E\}.$$

The first condition in this definition encodes the act of pairing: elements of A are paired with elements of B as ordered pairs. The second condition encodes the idea that *every* element of A is pair with *one and only one* element of B.

24 Example. Let

$$f = \{(1, -1), (2, 1), (3, -1), (4, 1)\}.$$

Then f is clearly a set of ordered pairs. We study possible domains and codomains of f.

(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, -1\}$. Then for each $x \in A$, there is one and only one $y \in B$ such that $(x, y) \in f$, and so f is a function from A to B. Moreover, $f(A) = B$. It happens that $f(1) = f(3)$, and also $f(2) = f(4)$, but that does not violate any part of the definition of function.

(ii) Let $A = \{1, 2, 3\}$ and $B = \{1, -1\}$. Since $(4, 1) \in f$ but $4 \notin A$, f cannot be a function from A to B; the first condition in the definition of function is violated.

(iii) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, -1\}$. Since $5 \in A$ but $(5, y) \notin f$ for all $y \in B$, f cannot be a function from A to B; part of the second condition in the definition of function is violated.

(iv) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, -1, 0\}$. Again, for each $x \in A$, there is one and only one $y \in B$ such that $(x, y) \in f$, and so f is a function from A to B. It happens that $f(A) \neq B$, since $0 \notin f(A)$, but that does not violate any part of the definition of function.

25 Problem (!). Suppose that A and B are sets, $x \in A$, and $f : A \to B$ is a function. Which, if any, of the objects x, $\{x\}$, $f(x)$, $f(\{x\})$, and $\{f(x)\}$ are equal?

26 Problem ($\star$). **(i)** Why is $\{(1, -1), (1, 1), (2, 1), (3, -1), (4, 1)\}$ not a function from $\{1, 2, 3, 4\}$ to $\{1, -1\}$?

(ii) With f defined in Example 24, determine the elements of $f(\{1, 2\})$ and $f(\{1\})$.

(iii) Let $f = \big\{(x, x^2) \bigm| x \in \mathbb{R}\big\}$. Let $I = [0, \infty)$. Show that $f(I) = I$.

(iv) Why is $\big\{(x, y) \bigm| x,\ y \in \mathbb{R} \text{ and } y^2 = x\big\}$ not a function from $\mathbb{R}$ to $\mathbb{R}$?

27 Problem ($\star$). Let A be a set and let $n \geq 1$ be an integer. Suppose that $x_1, \ldots, x_n \in A$. Define

$$(x_1, \ldots, x_n) := \{(k, x_k)\}_{k=1}^{n}.$$

That is, $(x_1, \ldots, x_n)$ is the function $f : \{1, \ldots, n\} \to A$ such that $f(k) = x_k$ for each k.

(i) With this definition of "ordered n-tuple," prove that

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$$

if and only if $x_k = y_k$ for $k = 1, \ldots, n$.

(ii) In the special case $n = 2$, how does this definition of the ordered pair (x, y) compare with the original one?

28 Problem (+). Let A, B, C, and D be sets and let $f : A \to B$ and $g : C \to D$ be functions. Prove that $f = g$ if and only if $A = C$ and $f(x) = g(x)$ for all $x \in A$ (equivalently, for all $x \in C$). [Hint: *remember that f and g are sets of ordered pairs. To prove the forward implication, if $f = g$, we want to show $x \in A \iff x \in C$ and $f(x) = g(x)$ for all $x \in A$. So, take some $x \in A$ and obtain $(x, f(x)) \in g$. Why does this force $x \in C$ and $g(x) = f(x)$? To prove the reverse implication and show $f = g$, we want to establish $(x, y) \in f \iff (x, y) \in g$. If $(x, y) \in f$, why do we have $x \in A$ and thus*

$x \in C$? Since $f(x) = g(x)$, why does this lead to $(x, y) \in g$?]

Life starts with sets and then we connect them with functions (which are themselves sets). Naturally, we may also want to consider sets of functions. If A and B are sets, we denote by

$$B^A$$

the set of all functions from A to B.

29 Example. The set $\{1,2\}^{\{1\}}$ is the set of all functions from $\{1\}$ to $\{1,2\}$. Any function from $\{1\}$ to $\{1,2\}$ must be a set consisting of a single ordered pair whose first coordinate is 1 and whose second coordinate is either 1 or 2. So,

$$\{1,2\}^{\{1\}} = \big\{\{(1,1)\}, \{(1,2)\}\big\}.$$

30 Problem ($\star$). What are all the elements of $\{1,-1\}^{\{1,2,3,4\}}$? [Hint: *there are eight.*]

31 Example. Several annoying ambiguities often appear when discussing functions.

(i) It is not always clear what the codomain of a function is. Let A, B, and C be sets with $B \subseteq C$. Then $B^A \subseteq C^A$, for if $f \subseteq A \times B$, then also $f \subseteq A \times C$. Thus any function $f\colon A \to B$ is also a function $f\colon A \to C$. For example, any real-valued function is also a complex-valued function. Sometimes this sort of distinction matters, and sometimes it does not.

(ii) The unfortunate phrase "well-defined function" is frequently used when one tries to define a function by some kind of choice. For example, suppose that we try to defined $f\colon [0,\infty) \to \mathbb{R}$ by specifying that $y = f(x)$ if $y^2 = x$. That is,

$$f = \big\{(x,y) \in [0,\infty) \times \mathbb{R} \;\big|\; y^2 = x\big\} = \big\{(y^2,y) \in \mathbb{R} \times \mathbb{R} \;\big|\; y \in \mathbb{R}\big\}.$$

Then f is not a function, since both $(1,1) \in f$ and $(1,-1) \in f$. We would say that defining f in this way does not yield a well-defined function. More precisely, f is simply not a function!

What usually happens is that one starts with a **RELATION** from a set A to a set B, i.e., a set $f \subseteq A \times B$, and then one shows that f is a function. It is usually obvious that for every $x \in A$, there is at least one $y \in B$ such that $(x,y) \in f$, so the typical task is to show that if (x,y_1), $(x,y_2) \in f$, then $y_1 = y_2$.

If $f\colon A \to B$ is a function and if there is an "explicit" formula for $f(x)$, then we often write

$$f\colon A \to B\colon x \mapsto f(x).$$

For example, we might study the function

$$f\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x^2,$$

and this is the function $\{(x, x^2) \mid x \in \mathbb{R}\}$.

32 Problem (!). Why are the functions

$$f\colon [0,1] \to \mathbb{R}\colon x \mapsto x^2 \quad \text{and} \quad g\colon [-1,1] \to \mathbb{R}\colon x \mapsto x^2$$

not the same?

We have previously used the concept of image to examine the behavior of a function on a subset of its domain. We could also consider a function "restricted" to that subset as another kind of function.

33 Definition. *Let A and B be sets and let $f\colon A \to B$ be a function. Let $E \subseteq A$. The* **RESTRICTION** *of f to E is the function*

$$f\big|_E\colon E \to B\colon x \mapsto f(x).$$

That is,

$$f\big|_E = \{(x, f(x)) \mid x \in E\}\,.$$

34 Problem (⋆). **(i)** Check that the restriction of a function to a subset of its domain is indeed a function (in the strict sense of Definition 23.

(ii) Let A and B be sets, $f \in B^A$, and $E \subseteq A$. Prove that $f\big|_E = f$ if and only if $E = A$. How does this shed new light on Problem 32?