MATH 3260: LINEAR ALGEBRA I

Daily Log for Lectures and Readings

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Contents

How to Use This Daily Log

This document is our primary reference for the course. It contains all of the material that we discuss in class along with some supplementary remarks that may not be mentioned in a class meeting. Each individual day has references, when applicable, to relevant material from the text. These references are spread throughout a day's notes, and you should be consulting both the daily log and Strang's text more or less simultaneously.

The document contains several classes of problems, which interact intimately with the material and which supplement (but certainly do not replace) the problems in the textbook.

(!) Problems marked (!) are meant to be attempted *immediately*. They will directly address or reinforce something that we covered (or perhaps omitted) in class. It will be to your great benefit to pause and work (!)-problems as you encounter them.

(⋆) Problems marked (⋆) are intentionally more challenging and deeper than (!)-problems. The (⋆)-problems will summarize and generalize ideas that we have discussed in class and give you broader, possibly more abstract perspectives. You should attempt the (⋆)-problems on a second rereading of the lecture notes, after you have completed the (!)-problems. Completing all of the (⋆)-problems constitutes the *minimal* preparation for exams.

(+) Problems marked (+) are meant to be more challenging than the (!)- and (⋆)-problems and will take you deeper into calculations and proofs and make connections to concepts across and beyond the course. It will not be necessary to do any (+)-problems to master the essential material of the course, but your experience may be richer (and more meaningful, and more fun) by considering them. If you have done all of the (!)- and (⋆)-problems, and the required and recommended problems from the textbook, and if you're still feeling bored or wondering if something is "missing," check out the (+)-problems. Sometimes a (!)- or (⋆)-problem will reference a (+)-problem; you should read the statement of that (+)-problem, but feel no obligation to do it.

Day 1: Monday, August 18.

One of the most important questions to ask in mathematics is not *What does this mean?* or *How do I do it?* or *Why is this true?* but *Who cares?* You should care about linear algebra because it is used *everywhere*. Linear algebra will show up in virtually every math problem that you could pose—even calculus, and especially multivariable calculus. And unlike much of the calculus you learned prior to this course, there is a very good chance you'll use linear algebra in your career.

Nearly every part of linear algebra has an application. As we tend to do in mathematics, we'll start this course from the very beginning, so applications might not be obvious right away. That being said, here are some meaningful applications that are particularly relevant to contemporary applied mathematics.

• How do we visualize, encode, or arrange a large set of data, say, "n-dimensional" data with $n \geq 4$? How do we optimize the storage of that data, possibly by extracting the most important parts or principal components, or otherwise approximating it?

• How do we fit a line, or polynomial, or some kind of well-behaved "curve" to data? Do we want the curve merely to pass through all of the data points or do we want the curve to be a good approximation to the "behavior" of the data (even if it doesn't pass through any of the data points at all)?

• How do we find the minimum distance between objects, possibly where those objects are sets of other objects? Or how do we find the best approximation to an object within a specialized class of other objects?

• How do machine learning models work? How does data science work? (All the cool kids sure do want to be data scientists these days.)

We will discuss some of these applications later, but we need to build up quite a body of knowledge first. Specifically, to get to the answers, we need to introduce and study two fundamental objects: vectors and matrices. I caution you that this course *might* start off simple. While the majority of the calculations that we do will remain "simple" for the sake of being able to do them by hand—I will make the numbers much nicer than they would be in any "real" application, for which you'd be using a computer to do the calculations—the course *will not* remain that way. Concepts quickly build on each other, and it is easy to get overwhelmed. *Vocabulary* will be a major part of this course, possibly much more so than you're used to from prior mathematics classes. *Do not slack off.*

Most of the punchy applications boil down to solving linear systems of equations; these are incredibly versatile problems, much more so than they might look at first glance. More accurately, our task is to *understand* linear systems of equations. Having a solution formula for something is not the same as understanding that thing. The challenge is finding the right system to model your application and then understanding it.

We can tease out a tremendous amount of structure and theory from very simple motivating examples, and here will be our favorite for some time. Let's try to solve the **LINEAR**

SYSTEM

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11. \end{cases} \tag{1.1}$$

It's a system of equations because there is more than one equation, and it's linear because the unknowns only appear as the "linear powers" x and y, not x^2 or xy or $\cos(x+y)$.

By the way, you didn't need to get out of bed today and come to class to figure out how to solve it, but imagine if the system had 50 variables and 50 equations. You'd probably want a precise and systematic way of approaching it.

1.1 Problem (!). Try to solve (1.1). What does your gut instinct say to do? (If you're reading these notes for the first time and haven't been in class, *don't* read below this problem for our approach just yet—try it by yourself.)

Before we do anything to (1.1), here are some questions that we should ask.

1. Does it have a solution? That is, do there exist numbers x and y that make the two equalities in (1.1) true?

2. If not, why not? Can we quantify or qualify *failure* to solve a linear system?

3. And if there is no solution, can we somehow *approximate* the problem by one that does have a solution? Will that approximation be meaningful or helpful?

4. Is there only one solution? Is there only one way to choose the values of x and y to make the two equalities in (1.1) true? That is, is the solution **UNIQUE**?

5. If not, why not? Can we quantify or qualify why a linear system might have more than one solution?

We will solve (1.1) by transforming it into an "equivalent" system of equations that is much easier to solve—actually, several "equivalent" systems. We'll say that two systems are **EQUIVALENT** if they have precisely the same solutions. And we'll do this via algebra—more precisely, two tricks with algebra. (A trick is just a technique that doesn't feel natural yet.)

Recall that if a and b are real numbers, then

$$a = b \iff ca = cb \text{ for all } c \neq 0.$$

That is, if you know $a = b$, then you also know $ac = bc$ for all nonzero c (when $c = 0$, this is still true, as $a0 = b0 = 0$). And if you know $ac = ab$ for all nonzero c (actually, for just *one* nonzero c), then you can divide by c to get $a = b$ (you want $c \neq 0$ so you can divide by c). In the context of linear systems, scaling *both* sides of the *same* equation by the *same* **nonzero** number doesn't change things. Let's multiply the first equation by the very convenient number $c = -3$:

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \iff \begin{cases} -3x + 6y = -3 \\ 3x + 2y = 11. \end{cases} \tag{1.2}$$

That was the first trick.

Now we'll use another property of algebra. Recall that if a and b are real numbers, then

$$a = b \iff a + c = b + c \text{ for all } c.$$

That is, if you know $a = b$, then you can add c to both sides to get $a+c = b+c$. And if you know $a+c = b+c$ for all c, then just subtract c from both sides (or add $-c$ to both sides) to get $a = b$. (Actually, we just need $a+c = b+c$ for one particular c, not for all c, to allow us to do this subtraction. But if you know $a+c = b+c$ for all c, just take $c = 0$ to get $a = b$.)

So, we can add any c that we like to both sides of the second equation in the system on the right in (1.2) to find the equivalence

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \iff \begin{cases} -3x + 6y = -3 \\ 3x + 2y + c = 11 + c \end{cases}$$

Is there a particularly helpful choice of c? Let's work backwards (a great direction to work in math!).

If we know that x and y solve

$$\begin{cases} -3x + 6y = -3 \\ 3x + 2y + c = 11 + c, \end{cases}$$

then we must have $-3x + 6y = -3$. After all, that's just the first equation here. So, we could take $c = -3$, which is the same as $c = -3x + 6y$. Since the variables show up on the left, let's use $c = -3x + 6y$ on the left and $c = -3$ on the right.

We get

$$3x + 2y + (-3x + 6y) = 8y \quad \text{and} \quad 11 + (-3) = 8.$$

Then we have the arrow going one way:

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \Longrightarrow \begin{cases} -3x + 6y = -3 \\ 8y = 8 \end{cases} \tag{1.3}$$

The second equation on the right in (1.3) is pretty nice: $8y = 8$, so $y = 1$. What's really nice is that $8y = 8$ has only the unknown y, not both x and y. The first equation then becomes $-3x + 6 = -3$, so $-3x = -9$, and therefore $x = 3$. It looks like we solved our original problem (1.1). Did we? We can always plug $x = 3$ and $y = 1$ into (1.1) and make sure everything's equal.

1.2 Problem (!). Do that.

What if we didn't feel like doing that? Without actually knowing the values of x and y, can we figure out why any solution to the second system in (1.3) is also a solution to the original problem (1.1)? First, it's helpful to notice that

$$\begin{cases} -3x + 6y = -3 \\ 8y = 8 \end{cases} \iff \begin{cases} x - 2y = 1 \\ 8y = 8. \end{cases}$$

That is, we didn't need to keep the factor of -3 on each of the terms in the first equation. We only put that -3 in to help simplify the second equation; that -3 actually makes the first equation worse.

We also know

$$\begin{cases} x - 2y = 1 \\ 8y = 8. \end{cases} \iff \begin{cases} x - 2y = 1 \\ 8y + c = 8 + c. \end{cases}$$

for any choice of c. Nothing new here, right?

If we know that x and y solve

$$\begin{cases} x - 2y = 1 \\ 8y + c = 8 + c. \end{cases}$$

then the first equation says $x - 2y = 1$, so multiplying both sides of that by 3 gives $3x - 6y = 3$ (which looks familiar). Then we could take $c = 3x - 6y$ on the left in the second equation and $c = 3$ on the right in the second equation to get back to where we were:

$$\begin{cases} x - 2y = 1 \\ 8y = 8. \end{cases} \Longrightarrow \begin{cases} x - 2y = 1 \\ 3x + 2y = 11. \end{cases} \tag{1.4}$$

This is the reverse of the arrow in (1.3).

When we combine (1.2), (1.3), and (1.4), we get

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \iff \begin{cases} x - 2y = 1 \\ 8y = 8 \end{cases} \iff \begin{cases} x = 3 \\ y = 1. \end{cases} \tag{1.5}$$

This is an existence and uniqueness result for (1.1): there exists a solution ($x = 3$ and $y = 1$), and it is the only solution. Specifically, existence is the logic of $\Longleftarrow$: plug these values in and check that true equalities result. Uniqueness is the logic of $\Longrightarrow$: if x and y solve the problem, then we must have $x = 3$ and $y = 1$.

The preceding work illustrates two incredibly important operations in solving linear systems: multiply both sides of one equation by the same number, and subtract (or add) a multiple of one equation to another equation. There's a third operation—interchanging two equations, which sounds silly but actually is worthwhile—that we'll meet later. Eventually we will encode and view these operations at pretty high and abstract levels.

Day 2: Wednesday, August 20.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Equality of column vectors, zero vector in $\mathbb{R}^n$, linear combination, scalar multiple

The preceding work also illustrates something that is incredibly *unimportant* about linear systems: what we call the variables. As long as you are consistent, it doesn't matter if you write x and y, or x_1 and x_2, or α and β, and so on. What matters are the *coefficients* on the variables and the *numbers on the right*.

We are going to stack these numbers together as **COLUMN VECTORS**, which we'll just call "lists of numbers" right now. Here are the three important vectors in (1.1), and we'll also write them as ordered pairs to make typesetting easier:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = (1, 3), \qquad \begin{bmatrix} -2 \\ 2 \end{bmatrix} = (-2, 2), \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 11 \end{bmatrix} = (1, 11).$$

But

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

The object on the right is a **ROW VECTOR**, and we'll talk about them eventually.

We'll do a lot of arithmetic with (column) vectors, and much of it will happen "componentwise." We add vectors by adding their corresponding components, so

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + (-2) \\ 3 + 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}. \tag{2.1}$$

2.1 Remark. *Strictly speaking, we are overworking the role of the symbol + in* (2.1)*. The + in*

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

is the addition of column vectors, while the + in

$$\begin{bmatrix} 1 + (-2) \\ 3 + 2 \end{bmatrix}$$

is the addition of real numbers. Nobody ever thinks like that in practice, but as you open yourself to new ideas in this course, you should be aware that the same symbol can mean different things, depending on context.

2.2 Problem (!). Compute

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then we can rewrite the original problem (1.1) as

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \iff \begin{bmatrix} x \\ 3x \end{bmatrix} + \begin{bmatrix} -2y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Big deal, right? All we have done is introduced some new notation; this tells us absolutely nothing about solving (1.1) that we did not already know. Let's do one more bit of arithmetic. There are "common factors" of x and y in some of those vectors, and our gut instinct should be to factor them out.

So, we define multiplication of a vector by a number (we do *not* multiply two vectors) componentwise:

$$2\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}2(1)\\2(3)\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}.$$

We often refer to this kind of multiplication as "scalar" multiplication to emphasize that one of the factors is a "scalar"—which is to say, a real number. When multiplying a vector by a number, we always write the number first:

$$2\begin{bmatrix}1\\3\end{bmatrix}, \text{ not } \begin{bmatrix}1\\3\end{bmatrix}2 \quad \text{and} \quad c\mathbf{v}, \text{ not } \mathbf{v}c.$$

2.3 Problem (!). Compute

$$-1\begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{and} \quad 0\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Content from Strang's *ILA* 6E. See the pictures on pp. v–vi for how to interpret vector addition and scalar multiplication in two dimensions. I'll talk about this later. Page 3 has another good picture that contrasts $\mathbf{v}$ and $-\mathbf{v}$ as a kind of "reflection." Also look at Figure 1.2 (a) on p. 8 to see the effect of "averaging" the sum of two vectors. There is more componentwise arithmetic on pp. 1–2.

We rewrite (1.1) once again as

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \iff x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}.$$

Again, this offers absolutely no insights into actually solving (1.1)—yet.

The expression

$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix}$$

is something that we'll see often: it's a LINEAR COMBINATION of the vectors $(1, 3)$ and $(-2, 2)$. By the way, the way I wrote the previous sentence is an example of typesetting a column vector as an ordered pair to save space. Many important ideas can be phrased in the language of linear combinations.

Content from Strang's *ILA* 6E. Page 3 has some pictures of linear combinations. See also a linear system on p. 3 that is written in vector form and then solved with elimination, as we did (1.1).

Here are some more precise (well, mostly precise) definitions of concepts from our first pass at linear systems and vectors. Throughout, we use the following set-theoretic terminology as a convenient abbreviation: if S is a set and x is an element of S, then we write $x \in S$. If a set has only finitely many elements, we may write those elements out between curly braces; the order or repetition of the elements doesn't matter. For example, $\{1,2,3\} = \{2,1,3\} = \{1,1,2,3\}$ and $1 \in \{1,2,3\}$. In particular, we denote by $\mathbb{R}$ the set of all real numbers, so $1 \in \mathbb{R}$. For a set S, we write $x \notin S$ to mean that x is not an element of S; for example, $0 \notin \{1,2,3\}$.

2.4 Undefinition. *Let $n \geq 1$ be an integer.*

(i) *A* COLUMN VECTOR *of length n is an "ordered list" of n real numbers, which we call the* ENTRIES *or the* COMPONENTS *of $\mathbf{v}$. If $\mathbf{v}$ is a column vector of length n with entries $v_1, \ldots, v_n$ in that order, then we write*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad \text{or} \qquad \mathbf{v} = (v_1, \ldots, v_n).$$

(ii) *The set of all column vectors of length n is $\mathbb{R}^n$, and we write*

$$\mathbb{R}^n = \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \;\middle|\; v_1, \ldots, v_n \in \mathbb{R} \right\}.$$

We typically work with $n \geq 2$, and we do not typically distinguish $\mathbb{R}^1$ and $\mathbb{R}$, so $\mathbb{R}^1 = \mathbb{R}$.

(iii) *Two vectors $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$ are* EQUAL *if and only if their corresponding entries are equal:*

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \iff v_j = w_j, \ j = 1, \ldots, n.$$

(iv) *We define vector addition and multiplication by real numbers componentwise, regardless of the length of the vectors. In particular, if $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then $\mathbf{v} + \mathbf{w} \in \mathbb{R}^n$ is given by adding the corresponding components of $\mathbf{v}$ and $\mathbf{w}$, and $c\mathbf{v} \in \mathbb{R}^n$ is given by multiplying each component of $\mathbf{v}$ by c. However, if $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$ with $n \neq m$, then $\mathbf{v} + \mathbf{w}$ is not defined.*

Why is this an "undefinition," not a definition? Because we didn't give a rigorous definition of "ordered list."

2.5 Remark. *I like to think of column vectors of length n as functions from the set $\{1, \ldots, n\}$ to $\mathbb{R}$. That is, if $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, then $\mathbf{v}$ is the same as the function $f \colon \{1, \ldots, n\} \to \mathbb{R}$ such that $f(j) = v_j$ for $j = 1, \ldots, n$. And since functions are really sets of ordered pairs, $f = \{(j, v_j)\}_{j=1}^n$. This is probably a useless way to think about column vectors for day-to-day purposes, but it comforts me to know that there is deeper math behind* *that undefinition. If it doesn't comfort you, it's okay to move on and never think about this again.*

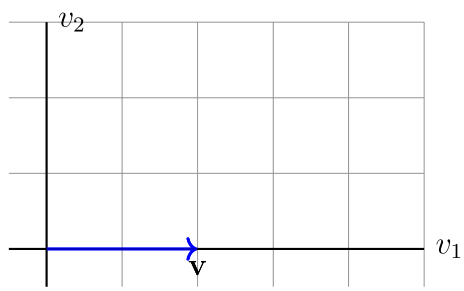
2.6 Problem (!). Does the expression

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

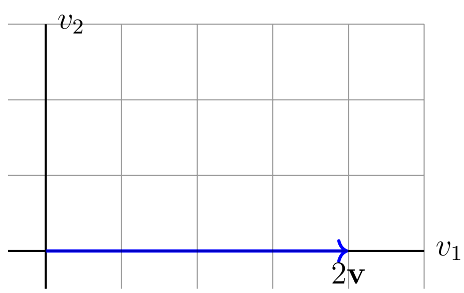
make sense?

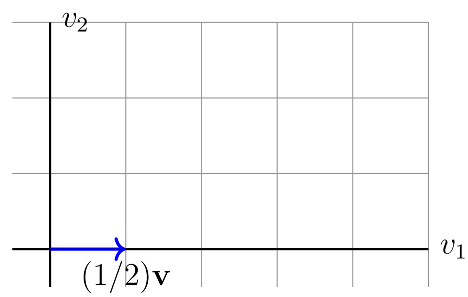
When $n \geq 4$, we have no way (as far as I know) of drawing vectors. Even $n = 3$ is hard for me, but sometimes drawing pictures for $n = 2$ can lead to insight. When I do so, I'll follow the convention that the vector $\mathbf{v} = (v_1, v_2)$ is represented by the "directed line segment" (= arrow, but fancier) from the origin $(0, 0)$ to the point (v_1, v_2). This leads to some nice geometric interpretations of vector addition and scalar multiplication.

2.7 Example. **(i)** Here's a drawing of the vector $\mathbf{v} = (2, 0)$.



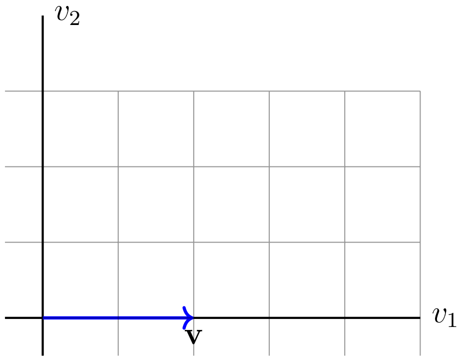
And here are drawings of $2\mathbf{v} = (4, 0)$ and $(1/2)\mathbf{v} = (1, 0)$.

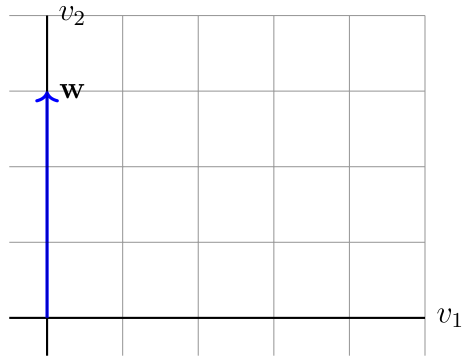




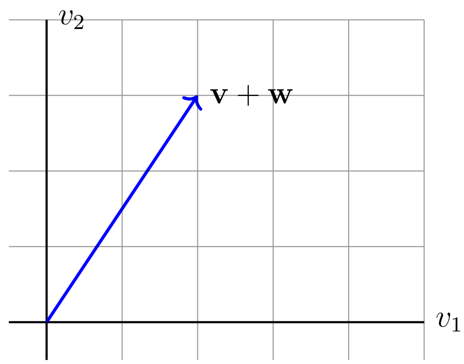
I hope you feel that $2\mathbf{v}$ is a "stretching" of $\mathbf{v}$ and $(1/2)\mathbf{v}$ is a "shrinking" (which, I guess, is also a kind of stretching).

(ii) Here are side-by-side drawings of the vectors $\mathbf{v} = (2, 0)$ from before and $\mathbf{w} = (0, 3)$, which is new.

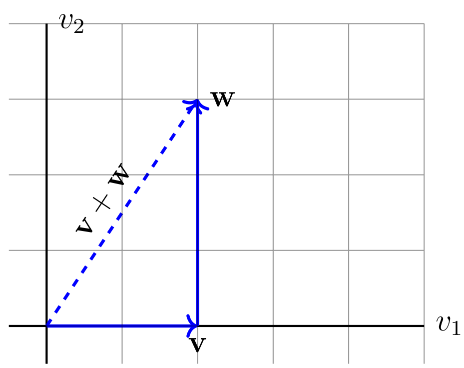




And here's the sum $\mathbf{v} + \mathbf{w} = (2, 3)$.



A cartoonish, but helpful, way to visualize the action of vector addition is that we placed the "tip" of $\mathbf{w}$ at the "tail" (the arrow end) of $\mathbf{v}$ to get the sum $\mathbf{v} + \mathbf{w}$.



2.8 Example. We compute

$$0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0(1) \\ 0(2) \\ 0(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

I hope it's obvious why we want to call the vector on the right the "zero vector in $\mathbb{R}^3$."

2.9 Definition. *The* ZERO VECTOR *in* $\mathbb{R}^n$ *is the vector* $\mathbf{0}$ *whose entries are all* 0. *Often we will write* $\mathbf{0}_n$ *to emphasize that this is the zero vector with* n *entries.*

For example,

$$\mathbf{0}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{but} \qquad \mathbf{0}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

2.10 Problem (!). **(i)** Let $\mathbf{v} \in \mathbb{R}^n$. What is $\mathbf{v} + \mathbf{0}_n$?

(ii) Does $\mathbf{0}_2 + \mathbf{0}_3$ make sense?

(iii) Generalize Example 2.8 by computing $0\mathbf{v}$ for an arbitrary $\mathbf{v} \in \mathbb{R}^n$.

(iv) Suppose you know that $c\mathbf{v} = \mathbf{0}_n$ for some $\mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}_n$. Why must it be the case that $c = 0$? [Hint: *what goes wrong if $c \neq 0$? Think componentwise.*]

(v) Let $\mathbf{v} \in \mathbb{R}^n$. What is $1\mathbf{v}$?

The most important (and only) arithmetical operations that we've defined for vectors are "vector addition" of two vectors in $\mathbb{R}^n$ to get a third vector in $\mathbb{R}^n$ and "scalar multiplication" of a number (a "scalar") in $\mathbb{R}$ and a vector in $\mathbb{R}^n$ to get a second vector in $\mathbb{R}^n$. When we do (possibly) both simultaneously, we get a new structure.

2.11 Definition. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ *and* $c_1, \ldots, c_n \in \mathbb{R}$. *The* LINEAR COMBINATION *of* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ WEIGHTED *by* $c_1, \ldots, c_n$ *is the vector* $\mathbf{v} \in \mathbb{R}^m$ *defined by*

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n. \tag{2.2}$$

We may also express this in sigma notation:

$$\mathbf{v} = \sum_{j=1}^{n} c_j\mathbf{v}_j.$$

If $\mathbf{v} \in \mathbb{R}^m$ *has the form* (2.2), *then we often say that* $\mathbf{v}$ *is a linear combination of* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *without mentioning the weights. If* $n = 1$, *then the linear combination of* $\mathbf{v}_1$ *weighted by* c_1 *is just the scalar multiplication* $c_1\mathbf{v}_1$; *if* $\mathbf{w} = c\mathbf{v}$ *for some* $c \in \mathbb{R}$ *and* $\mathbf{v} \in \mathbb{R}^m$, *then we say that* $\mathbf{w}$ *is a* SCALAR MULTIPLE *of* $\mathbf{v}$.

2.12 Problem (!). Convince yourself that, in the notation of the previous definition, we do indeed have $\mathbf{v} \in \mathbb{R}^m$. Also, what are the integers m and n encoding in that definition?

2.13 Example. We have

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and so $(2,3,0)$ is a linear combination of $(1,0,0)$ and $(0,1,0)$. So is $(1,0,0)$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

2.14 Problem (!). Here is a generalization of this example. Let

$$\mathbf{e}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{e}_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Explain why any $\mathbf{v} \in \mathbb{R}^3$ is a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

Content from Strang's *ILA* 6E. There are examples of linear combinations with $n = 2$ on p. vi and p. 2.

2.15 Problem ($\star$). Let $\mathbf{v} = (2,0)$ and $\mathbf{w} = (0,3)$. You might want to look at the pictures in Example 2.7.

(i) Convince yourself that we can write any vector $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. You can do algebra or draw pictures.

(ii) Let $\mathbf{u} = (2,3)$. Convince yourself that we can also write any vector in $\mathbb{R}^2$ as a linear combination of $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$, but try to make the coefficient on $\mathbf{u}$ as simple as possible.

(iii) Why can't we write any vector in $\mathbb{R}^2$ as just a scalar multiple of $\mathbf{v}$, $\mathbf{w}$, or $\mathbf{u}$?

(iv) More generally, what goes wrong if we try to find a single vector $\mathbf{x} \in \mathbb{R}^2$ such that every vector in $\mathbb{R}^2$ is a scalar multiple of $\mathbf{x}$? Try to explain this both "algebraically" and by drawing pictures.

(v) Given any two distinct vectors in $\mathbb{R}^2$, can we always write a third vector in $\mathbb{R}^2$ as a linear combination of those two?

So far, none of this (mostly) more precise terminology tells us anything new about solving linear systems, and, honestly, none of the following is going to help, either. The goal is to build more terminology so that we can ask questions about linear systems *in the right language*.

Here is a major step toward that right language. Recall that our original problem (1.1) can be written as a system of linear equations or as a vector equation involving a linear combination:

$$\begin{cases} x_1 & - & 2x_2 & = & 1 \\ 3x_1 & + & 2x_2 & = & 11 \end{cases} \iff x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Let's put the coefficient vectors together into a MATRIX:

$$A := \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}.$$

I hope you'll agree that this is a SQUARE matrix: it has the same number of columns and rows (two each). We most often think of matrices in terms of columns (though rows are sometimes useful). If we put

$$\mathbf{a}_1 := \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 := \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

then we will also write A as

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}.$$

This is sort of a "row vector" of column vectors.

Here is where we are going with all of this. Abbreviate $\mathbf{x} = (x_1, x_2)$ and $\mathbf{b} = (1, 11)$. Our goal is to define a notion of "matrix-vector multiplication" so that if $A\mathbf{x}$ is the "product" of A and $\mathbf{x}$, then our original problem compresses to

$$A\mathbf{x} = \mathbf{b}.$$

Day 3: Friday, August 22.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Matrix-vector product, dot product

First, of course, we need some more terminology. We control the "sizes" or "dimensions" of matrices by counting the numbers of rows and the numbers of columns—and we always list *rows before columns*. We'll say $A \in \mathbb{R}^{2\times2}$ for the matrix A above, and I hope you believe that

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \in \mathbb{R}^{2\times3} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \in \mathbb{R}^{3\times2}.$$

More generally, we say the following.

3.1 Undefinition. *Let m, $n \geq 1$ be integers.*

(i) *An $m \times n$* **MATRIX** *is a rectangular array of numbers with m rows and n columns.*

(ii) *We denote the set of all $m \times n$ matrices by $\mathbb{R}^{m\times n}$.*

(iii) *Since a matrix with m rows and 1 column is really just an ordered list of m numbers, we will typically not distinguish $\mathbb{R}^{m\times 1}$ and $\mathbb{R}^m$, and so usually $\mathbb{R}^{m\times 1} = \mathbb{R}^m$. Also, $\mathbb{R}^{1\times 1} = \mathbb{R}$. But we do not equate $\mathbb{R}^{1\times n}$ and $\mathbb{R}^n$.*

(iv) *The (i,j)-***ENTRY** *of a matrix is the entry in row i, column j of that matrix. Sometimes we will write A_{ij} for the (i,j)-entry of A, although with large matrices it might be clearer to write $A_{i,j}$, or maybe even $A(i,j)$. Two matrices are* **EQUAL** *if and only if they have the same number of rows and columns and if all of their corresponding entries are equal.*

3.2 Example. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Then $A \in \mathbb{R}^{3\times 2}$. The $(1,2)$-entry of A is 2, and the $(2,1)$-entry of A is 3.

Content from Strang's *ILA* 6E. A 3×2 matrix appears on p. vi, and a larger one (what size?) on p. vii.

3.3 Problem ($\star$). Undefinition 3.1 defines matrix equality "entrywise." Explain why we can also define matrix equality "columnwise." That is, explain why matrices A, $B \in \mathbb{R}^{m\times n}$ are equal precisely when the jth column of A equals the jth column of B for $j = 1, \ldots, n$. (Here by equality of columns I mean equality of vectors as in Undefinition 2.4.)

Regarding our choice not to identify $\mathbb{R}^{1\times n}$ and $\mathbb{R}^n$, we have things like

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (1,2,3).$$

3.4 Problem (!). Reread that sentence until it makes sense.

3.5 Remark. *As with column vectors, our attempt at defining a matrix is really an undefinition because we did not rigorously define "rectangular array" of numbers. If you really want to, you can think of $A \in \mathbb{R}^{m\times n}$ as the function $f\colon I \to \mathbb{R}$ such that $f(i,j) = A_{ij}$, where $I = \{(i,j) \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$. Or as the function $g\colon \{1, \ldots, n\} \to \mathbb{R}^m$ such that $g(j) = \mathbf{a}_j$, where $\mathbf{a}_j$ is the jth column of A, i.e., $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$. (Strictly* *speaking, f and g are not the same function, as they have different domains and ranges!) Neither way of thinking will make any of the following any easier.*

And as with column vectors, we add matrices and multiply them by real numbers componentwise.

3.6 Problem (!). Compute

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad 2\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We are finally ready to think about linear systems. With

$$A := \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

we are going to define the symbol $A\mathbf{x}$ so that

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11 \end{cases} \iff x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \iff A\mathbf{x} = \mathbf{b}.$$

The answer is pretty much staring us in the face: we should put

$$A\mathbf{x} := x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

This is something new. This is not a componentwise definition of multiplication. *Instead, the idea behind matrix-vector multiplication is that we take a linear combination of the columns of the matrix weighted by the entries of the vector.* If we write

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}, \qquad \mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

then we are saying

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := x_1\mathbf{a}_1 + x_2\mathbf{a}_2.$$

By the way, I'm writing := above to indicate that I'm making a definition; until now, we had never specified what the symbol $A\mathbf{x}$ should mean, and now I'm telling you what it is. I'll do that a lot in this course and write $X := Y$ to indicate that I'm defining the new thing X in terms of the hopefully familiar thing Y.

Let's do some computations with this definition of matrix-vector multiplication in words first: take the linear combination of the columns of the matrix with the weights as the entries from the vector, all appearing in order.

3.7 Problem (!). Convince yourself that for this to work, the number of columns of the matrix has to equal the number of entries of the vector.

3.8 Example. **(i)** Let

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3.$$

Then

$$A\mathbf{v} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+0+5 \\ 2+0+6 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}.$$

(ii) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Then

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+8 \\ 0+0 \\ (-2)+(-4) \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -6 \end{bmatrix}.$$

And now for the definition in symbols.

3.9 Definition. *Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{v} \in \mathbb{R}^n$ with*

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

The MATRIX-VECTOR PRODUCT *of A and $\mathbf{v}$ is*

$$A\mathbf{v} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} := v_1\mathbf{a}_1 + \cdots + v_n\mathbf{a}_n = \sum_{j=1}^{n} v_j\mathbf{a}_j.$$

Again, in words, the matrix-vector product $A\mathbf{v}$ is the linear combination of the columns of the matrix A weighted by the entries of the vector $\mathbf{v}$.

Content from Strang's *ILA* 6E. Page 1 has examples of matrix-vector multiplication.

3.10 Problem (!). Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{v} \in \mathbb{R}^n$. Use the definition of $A\mathbf{v}$ from Definition 3.9 to count the number of entries in $A\mathbf{v}$.

3.11 Problem ($\star$). Let $A \in \mathbb{R}^{m \times n}$. Use the definition of $A\mathbf{0}_n$ from Definition 3.9 to show that $A\mathbf{0}_n = \mathbf{0}_m$.

3.12 Problem ($\star$). **(i)** Prove that matrix-vector multiplication is LINEAR in the following sense: if $A \in \mathbb{R}^{m \times n}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$, and $c \in \mathbb{R}$, then

$$A(\mathbf{v}+\mathbf{w}) = A\mathbf{v} + A\mathbf{w} \quad \text{and} \quad A(c\mathbf{v}) = c(A\mathbf{v}).$$

This could involve a lot of $\cdots$ that might obscure the actual arithmetic going on; if it makes things more transparent, do it for $n = 2$ or $n = 3$ first. *However you do it, use the definition of matrix-vector multiplication from Definition 3.9.*

(ii) This is one of the very rare times when we'll use calculus, and it's mostly for the sake of an example. Think about the following rules for differentiation: if f and g are differentiable functions and if $c \in \mathbb{R}$, then

$$(f+g)' = f' + g' \quad \text{and} \quad (cf)' = cf'.$$

And think about the following rules for integration: if f and g are continuous on the interval $[a, b]$ and if $c \in \mathbb{R}$, then

$$\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx \quad \text{and} \quad \int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx.$$

You use(d) these rules all the time in calculus, maybe so often that you barely noticed you were using them. Why do these rules show that differentiation and integration are "linear" operations?

Every linear system compresses as a matrix-vector equation. Suppose there are m equations in n unknowns. Let $\mathbf{x}$ be the column vector of length n that contains all of these unknowns. Let A be the $m \times n$ matrix containing all of the coefficients, so the (i, j)-entry of A is the coefficient on the jth unknown in the ith equation. Let $\mathbf{b}$ be the column vector of length m that contains the right sides of these equations. Then the problem is

$$A\mathbf{x} = \mathbf{b}.$$

3.13 Example. Here is a review of how all of this works for our toy problem (1.1). We have

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \overset{(1)}{\iff} x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$\overset{(2)}{\iff} \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$\overset{(3)}{\iff} \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$\overset{(4)}{\iff} \left\{ \begin{array}{rcrcr} x_1 & - & 2x_2 & = & 1 \\ 3x_1 & + & 2x_2 & = & 11 \end{array} \right.$$

- Equality (1) is the definition of matrix-vector multiplication.
- Equality (2) is the componentwise definition of scalar multiplication.
- Equality (3) is the componentwise definition of vector addition.
- Equality (4) is the componentwise definition of vector equality.

3.14 Problem (!). Rewrite each linear system below as a matrix-vector equation $A\mathbf{x} = \mathbf{b}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Specify the values of m and n in each case.

(i) $$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & 3x_4 & = & 1 \\ & & & & x_3 & + & 4x_4 & = & 2 \end{array} \right.$$

(ii) $$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & 7x_4 & = & 1 \\ 2x_1 & + & 4x_2 & + & 2x_3 & + & 14x_4 & = & 2 \\ & & & & 2x_3 & + & 8x_4 & = & 3 \end{array} \right.$$

(iii) $$\left\{ \begin{array}{rrrcr} x_1 & & & = & 1 \\ 2x_1 & & & = & 2 \\ & x_2 & & = & 3 \\ & & x_3 & = & 4 \end{array} \right.$$

(iv) $$\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & 1 \\ 2x_1 & + & 4x_2 & & & = & 2 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 3 \\ 7x_1 & + & 14x_2 & + & 8x_3 & = & 4 \end{array} \right.$$

3.15 Problem ($\star$). Compute each matrix-vector product and then describe in words the effect of this multiplication. For your description in words, pretend that you are talking out loud to a classmate about this multiplication, and you do not have any paper or board to write on; try to use as few symbols as possible in your description.

(i) $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for any $c, x_1, x_2, x_3 \in \mathbb{R}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for any $x_1, x_2, x_3 \in \mathbb{R}$

(iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for any $x_1, x_2 \in \mathbb{R}$

Our original questions remain the same—how to solve it, how to understand failure to solve it. The new question is probably *Why is writing it as $A\mathbf{x} = \mathbf{b}$ any better than the original way?*

Content from Strang's *ILA* 6E. Read all of p. 2 right now.

The goal of the course is the same as always: solve $A\mathbf{x} = \mathbf{b}$, and when we can't solve it, understand why. Eventually this will take us into understanding just A, apart from any linear systems. For now, we should try to understand $A\mathbf{x}$ as best as we can. There is another way of computing matrix-vector products in addition to Definition 3.9. We'll tease it out in an example.

3.16 Example. We compute

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

This is just checking that $x_1 = 3$ and $x_2 = 1$ solves our original problem

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 2x_2 = 11 \end{cases}$$

from (1.1), right?

Here is another way of looking at this arithmetic:

$$3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(1) + 1(-2) \\ 3(3) + 1(2) \end{bmatrix}.$$

Do you see how the vectors $(3, 1)$ and $(1, -2)$ appear in the first component on the right? And how $(3, 1)$ and $(3, 2)$ appear in the second component? It's almost as though the vector by which we're multiplying the matrix, and the rows of the matrix *viewed as column vectors*, are doing all of the arithmetic.

Let's introduce a new structure: the **DOT PRODUCT** of vectors in $\mathbb{R}^2$. (Just $\mathbb{R}^2$ now for starters.) Put

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := v_1 w_1 + v_2 w_2.$$

So we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1(3) + (-2)(1) = 3 - 2 = 1$$

and

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3(3) + 2(1) = 9 + 2 = 11.$$

Here is the takeaway in words: we can compute a matrix-vector product by taking the dot product of the rows of the matrix—*viewed as column vectors*—with the vector in the product.

Content from Strang's *ILA* 6E. Equation (1) on p. 9 defines the dot product of vectors in $\mathbb{R}^2$. See the box above on p. 9 for more dot products.

Let's generalize this example.

3.17 Definition. *The dot product of* $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ *is*

$$\mathbf{v} \cdot \mathbf{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} := v_1 w_1 + \cdots + v_n w_n = \sum_{j=1}^{n} v_j w_j.$$

Content from Strang's *ILA* 6E. This is equation (2) on p. 9. We won't talk about anything else from Section 1.2 for quite a while. The dot product turns out to be the key to a deeper *geometric* understanding of $\mathbb{R}^n$, in particular an understanding of *angles*, but we won't need that for some time.

3.18 Example. $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3(1) + 4(0) + 5(0) = 3$

I will do my best to reserve the symbol $\cdot$ for the dot product and use "juxtaposition" to denote multiplication of real numbers, e.g., $3(1)$, not $3 \cdot 1$. But I guess the dot product in $\mathbb{R}^1 = \mathbb{R}$ is just ordinary multiplication, so no big deal.

3.19 Problem ($\star$). Prove that the dot product is **COMMUTATIVE** in the sense that $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. This is how we expect multiplication to behave, that $xy = yx$ for all numbers x and y, right?

We can use the dot product to "extract" components of a vector. This will be a hugely useful operation.

3.20 Example. Here is how this works in $\mathbb{R}^3$. (I like $\mathbb{R}^3$: it's big enough to be interesting but not so big that it's intimidating.) Put

$$\mathbf{e}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{e}_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

These are the STANDARD BASIS VECTORS for $\mathbb{R}^3$, and we will use them a lot. I claim that if $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, then

$$\mathbf{v} \cdot \mathbf{e}_1 = v_1, \qquad \mathbf{v} \cdot \mathbf{e}_2 = v_2, \qquad \text{and} \qquad \mathbf{v} \cdot \mathbf{e}_3 = v_3.$$

We basically did the first equality in Example 3.18, so here is the second:

$$\mathbf{v} \cdot \mathbf{e}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = v_1(0) + v_2(1) + v_3(0) = v_2. \tag{3.1}$$

I'll let you check the third.

Now here is another nice identity: start with $\mathbf{v}$ and "expand it":

$$\begin{aligned} \mathbf{v} &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} \\ &= v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \\ &= (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}_3)\mathbf{e}_3. \end{aligned} \tag{3.2}$$

This is a really clean representation of a vector in terms of its components and some other, simpler vectors (and it also solves Problem 2.14, right?). Among other things, it tells us that the ith component of $\mathbf{v}$ is $\mathbf{v} \cdot \mathbf{e}_i$. We'll return to such representations many times in the future.

Day 4: Monday, August 25.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Standard basis vectors in $\mathbb{R}^n$, column space of a matrix

The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ from the previous example are phenomenally useful.

4.1 Definition. *The* STANDARD BASIS VECTORS IN $\mathbb{R}^n$ *are the vectors* $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ *defined as follows: the components of* $\mathbf{e}_j$ *are all* 0*, except for the component in row* j*, which is* 1.

We will only ever use the symbol $\mathbf{e}_j$ in this course for a standard basis vector. Unfortunately, the notation doesn't make clear what the value of n is, so $\mathbf{e}_1$ could be $(1,0)$ or $(1,0,0,0)$, and you will have to pay attention to context to understand that.

4.2 Problem ($\star$). **(i)** Write out the standard basis vectors in $\mathbb{R}^5$. You should make clear what all of their entries are.

(ii) Prove that

$$\mathbf{e}_j \cdot \mathbf{e}_k = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

Your proof should work regardless of the underlying choice of n. This generalizes the calculation in (3.1).

(iii) Let $\mathbf{v} \in \mathbb{R}^n$. Prove that

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{v} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

This generalizes the calculation in (3.2).

Now that we have an understanding of the mechanics of dot product calculations, we can examine how the dot product arises in matrix-vector multiplication. All of the ideas are in Example 3.16. We'll work with a matrix with three columns to see this a little more abstractly. Let $A \in \mathbb{R}^{m \times 3}$ and write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ * & * & * \end{bmatrix}.$$

I just want to focus on the first row of A, so I've listed that out explicitly. The symbols $*$ below denote the remaining $m-1$ rows of A. The exact values of the entries in those rows are wholly unimportant right now. (If it makes you feel better, take $m = 2$ and replace each $*$ with 0.)

Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. Then

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ * & * & * \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} a_{11} \\ * \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ * \end{bmatrix} + v_3 \begin{bmatrix} a_{13} \\ * \end{bmatrix} = \begin{bmatrix} v_1a_{11} + v_2a_{12} + v_3a_{13} \\ * \end{bmatrix}.$$

At the risk of being annoying, I am using the same symbol $*$ to denote rows 2 through m of the columns of A and then the vector $A\mathbf{v}$; I sincerely don't care what's going on there right now. Here is what we have shown: the first component of $A\mathbf{v}$ is

$$v_1a_{11} + v_2a_{12} + v_3a_{13} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} \cdot \mathbf{v},$$

which is the dot product of the first row of A *viewed as a column vector* with $\mathbf{v}$.

This generalizes substantially; the proof is just good bookkeeping and good notation.

4.3 Theorem. *Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{v} \in \mathbb{R}^n$. The ith component of $A\mathbf{v}$ is the dot product of row i of A viewed as a column vector and $\mathbf{v}$.*

4.4 Example. We compute

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 4(0) + 7(1) \\ 2(1) + 5(0) + 8(1) \\ 3(1) + 6(0) + 9(1) \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix}.$$

What do you get if you use Definition 3.9?

Content from Strang's *ILA* 6E. Read about the "row picture" and the "column picture" on p. 19; the matrix is A given on p. 18. Strang says it best: to *compute* $A\mathbf{v}$ by hand for "small" A and $\mathbf{v}$, use dot products, but to *understand* $A\mathbf{v}$, use the "linear combination of columns" definition. (There will be at least one exception to this, when we study orthogonality. I'll bring it up when it arises.) This is morally similar to the derivative: to compute it by hand, use the product rule or chain rule or something like that, but to understand it, use the limit definition.

4.5 Problem ($\star$). Go back and redo each of the matrix-vector products in Example 3.8 and Problem 3.15 with dot products. What do you find easier for work by hand: Definition 3.9 or Theorem 4.3?

We started thinking about matrices *statically*: they encode data, specifically the coefficients of a linear system of equations. Now that we can multiply matrices and vectors, we can think *dynamically*: matrices act on vectors to produce new vectors. We might even associate a matrix $A \in \mathbb{R}^{m \times n}$ with a "map" (dare I say "function"?) that associates each vector $\mathbf{v} \in \mathbb{R}^n$ with a new vector $A\mathbf{v} \in \mathbb{R}^m$.

Matrix-vector multiplication tells us useful things about matrices, not just vectors. I first claim that matrix-vector multiplication can "extract" the columns of a matrix. Let's start small. As before, we'll write

$$\mathbf{e}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{e}_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \in \mathbb{R}^{m \times 3}$. It's important that A has only three columns, but here the number of rows doesn't matter. We compute

$$A\mathbf{e}_1 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{0}_m + \mathbf{0}_m = \mathbf{a}_1.$$

In words, *multiplying by* $\mathbf{e}_1$ *extracted the first column of* A*.*

4.6 Problem (!). With $A \in \mathbb{R}^{m \times 3}$ as above, show that $A\mathbf{e}_2 = \mathbf{a}_2$ and $A\mathbf{e}_3 = \mathbf{a}_3$.

This generalizes nicely.

4.7 Theorem. *Let* $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ *be the standard basis vectors for* $\mathbb{R}^n$*: the components of* $\mathbf{e}_j$ *are all* 0*, except for the component in row* j*, which is* 1*. Let* $A \in \mathbb{R}^{m \times n}$*. Then* $A\mathbf{e}_j$ *is the* j*th column of* A*.*

4.8 Problem ($\star$). Prove it!

4.9 Problem ($\star$). Let $I_n \in \mathbb{R}^{n \times n}$ be the matrix whose jth column is $\mathbf{e}_j$. We might write $I_n = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$. Prove that $I_n\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^n$. We therefore call I_n the IDENTITY MATRIX: multiplying $\mathbf{v}$ by I_n just tells you what $\mathbf{v}$ is. [Hint: *prove it for* $n = 3$ *first to see the pattern of the arithmetic before doing it for* n *arbitrary.*]

Content from Strang's *ILA* 6E. Look at the four matrices on p. 18: identity, diagonal, triangular, symmetric. Why are the last three called what they are?

We can go further than data extraction via matrix-vector multiplication. I like to say that *what things do defines what things are.* And what matrices do is multiply vectors! Recall that two matrices A, $B \in \mathbb{R}^{m \times n}$ are equal if their corresponding entries are all equal: $A_{ij} = B_{ij}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. That is a "static" way of viewing matrix equality (and not a bad way at all). Here is the "dynamic" way: A and B are equal if they always do the same thing to the same vector.

4.10 Theorem. *Let* A*,* $B \in \mathbb{R}^{m \times n}$*. Then* $A = B$ *if and only if* $A\mathbf{v} = B\mathbf{v}$ *for all* $\mathbf{v} \in \mathbb{R}^n$*.*

Proof. This is an "if and only if" statement, so we need to prove two things. First we want to assume that $A = B$ and then show that $A\mathbf{v} = B\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. This feels pretty silly, right? We should just be able to "substitute" A in for B. If we want to be pickier and more precise about what $=$ means here, $A = B$ means that A and B have equal entries, so also equal columns. That is, $A = \begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}$ and $B = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}$ with $\mathbf{a}_j = \mathbf{b}_j$ for all j. (And what does $\mathbf{a}_j = \mathbf{b}_j$ mean? Componentwise equality.) So, if $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, then

$$A\mathbf{v} = \begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}\begin{bmatrix}v_1 \\ \vdots \\ v_n\end{bmatrix} = v_1\mathbf{a}_1 + \cdots + v_n\mathbf{a}_n = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}\begin{bmatrix}v_1 \\ \vdots \\ v_n\end{bmatrix} = B\mathbf{v}.$$

Now we want to show that if $A\mathbf{v} = B\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, then $A = B$. The key words here are "for all." We can pick any $\mathbf{v} \in \mathbb{R}^n$ that we like, and we will have the equality $A\mathbf{v} = B\mathbf{v}$. If we want to extract data about A and B, there are good, specific choices for $\mathbf{v}$: take $\mathbf{v} = \mathbf{e}_j$. Then $A\mathbf{e}_j = B\mathbf{e}_j$ for each j, and so the jth column of A equals the jth column of B. That means $A = B$. ■

4.11 Problem ($\star$). A matrix $A \in \mathbb{R}^{n \times n}$ is **DIAGONAL** if $A_{ij} = 0$ for $i \neq j$.

(i) What does an arbitrary diagonal matrix $A \in \mathbb{R}^{4 \times 4}$ look like? Write it down. [Hint: *you need at most* 5 *numbers here.*]

(ii) Describe in words the effect of multiplying a vector by a diagonal matrix. What happens to the components of that vector?

We now have as good an understanding of matrix-vector multiplication as we're going to get without doing anything new. Remember that our goal in this course is to understand the problem $A\mathbf{x} = \mathbf{b}$ as best as we can. Our work so far has focused on understanding $A\mathbf{x}$. Now it is time to relate $\mathbf{b}$ to A.

By definition, $A\mathbf{x}$ is a linear combination of the columns of A weighted by the entries of $\mathbf{x}$. To have $A\mathbf{x} = \mathbf{b}$, we therefore want to be able to express $\mathbf{b}$ as a linear combination of the columns of A. The set of all $\mathbf{b}$ that can be expressed in this way has a special name.

4.12 Definition. *The* **COLUMN SPACE** *of $A \in \mathbb{R}^{m \times n}$ is the set of all linear combinations of the columns of A. We denote it by $\mathbf{C}(A)$, and every vector in $\mathbf{C}(A)$ is a vector in $\mathbb{R}^m$. Equivalently,*

$$\mathbf{C}(A) := \{A\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n\}.$$

For a vector $\mathbf{b} \in \mathbb{R}^m$, we write $\mathbf{b} \in \mathbf{C}(A)$ to mean that $\mathbf{b}$ is a vector in $\mathbf{C}(A)$; equivalently, $\mathbf{b}$ has the form $\mathbf{b} = A\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^n$.

We say column *space*, not column *set*. The *set* of columns of $A = \begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix} \in \mathbb{R}^{m \times n}$ is just the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of at most n vectors (maybe fewer than n, if some of the columns of A are repeated). We'll see, and appreciate, how $\mathbf{C}(A)$ is a "dynamic" object that has some more structure than a plain old set of vectors, which is we call it a *space*.

Content from Strang's *ILA* 6E. The column space is defined at the bottom of p. 20.

4.13 Example. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then

$$\mathbf{C}(A) = \{A\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^2\} = \left\{ v_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} \;\middle|\; v_1,\ v_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2v_1 \\ 3v_2 \end{bmatrix} \;\middle|\; v_1,\ v_2 \in \mathbb{R} \right\}.$$

To be able to solve $A\mathbf{x} = \mathbf{b}$ for as many $\mathbf{b}$ as possible, we want $\mathbf{C}(A)$ to be as "large" as possible. Ideally (perhaps) we would have $\mathbf{C}(A) = \mathbb{R}^m$. What does "=" mean here? (We've only seen "=" for equality of numbers, vectors, and matrices, but now we are talking about the equality of two sets, $\mathbf{C}(A)$ and $\mathbb{R}^m$.) This means that every vector in $\mathbf{C}(A)$ is a vector in $\mathbb{R}^m$ (that's true by definition of $\mathbf{C}(A)$: boring!), and, more excitingly, that every vector in $\mathbb{R}^m$ is a vector in $\mathbf{C}(A)$. So then every $\mathbf{b} \in \mathbb{R}^m$ would be a linear combination of the columns of A, and so we could solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^m$.

4.14 Example. With A as in Example 4.13, we claim that $\mathbf{C}(A) = \mathbb{R}^2$. We need to take an arbitrary $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ and show $\mathbf{b} \in \mathbf{C}(A)$. That is, we need to find $\mathbf{v} \in \mathbb{R}^2$ such that $A\mathbf{v} = \mathbf{b}$. From Example 4.13, it suffices to find v_1, $v_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 2v_1 \\ 3v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Looking at componentwise equalities, this is equivalent to

$$2v_1 = b_1 \quad \text{and} \quad 3v_2 = b_2,$$

and that is the same as

$$v_1 = \frac{b_1}{2} \quad \text{and} \quad v_2 = \frac{b_2}{3}.$$

This tells us what $\mathbf{v}$ should be for us to have $\mathbf{b} = A\mathbf{v}$, and we get something more: there is only one way to define $\mathbf{v}$ in terms of $\mathbf{b}$, because there is only one way to define v_1 and v_2 in terms of b_1 and b_2.

This approach to understanding $\mathbf{C}(A)$ didn't tell us anything about the mechanics of solving $A\mathbf{x} = \mathbf{b}$ that we didn't already know. In fact, to show that every $\mathbf{b} \in \mathbb{R}^2$ is in $\mathbf{C}(A)$, we just solved $A\mathbf{x} = \mathbf{b}$. With more work, and patience, and maybe some trust, the payoff will be that we can control $\mathbf{C}(A)$ *without explicitly* solving $A\mathbf{x} = \mathbf{b}$.

Day 5: Wednesday, August 27.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Matrix with dependent columns (N)

As discussed before this example, we're going to see the symbol $=$ used for more than just equality of numbers or vectors or matrices. I want to emphasize this a little more abstractly.

5.1 Definition. *Let $\mathcal{V}$ and $\mathcal{W}$ be sets of vectors in $\mathbb{R}^m$. We say that $\mathcal{V} = \mathcal{W}$ if $\mathcal{V}$ and $\mathcal{W}$ contain precise the same vectors. That is, if $\mathbf{v} \in \mathcal{V}$, then $\mathbf{v} \in \mathcal{W}$, and if $\mathbf{w} \in \mathcal{W}$, then $\mathbf{w} \in \mathcal{V}$.*

5.2 Example. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

For $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, we have

$$A\mathbf{v} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 0 \end{bmatrix}.$$

So, if $\mathbf{b} = (b_1, b_2) \in \mathbf{C}(A)$, then $b_2 = 0$. Thus $\mathbf{C}(A) \neq \mathbb{R}^2$, as $\mathbf{e}_2 = (0, 1) \in \mathbb{R}^2$ but $\mathbf{e}_2 \notin \mathbf{C}(A)$.

We can be a little more precise about what $\mathbf{C}(A)$ is, rather than what it isn't. We showed above that

$$\mathbf{C}(A) = \left\{ \begin{bmatrix} 2v_1 \\ 0 \end{bmatrix} \,\middle|\, v_1 \in \mathbb{R} \right\},$$

and I claim that we can be even simpler:

$$\mathbf{C}(A) = \{c\mathbf{e}_1 \mid c \in \mathbb{R}\} =: \mathcal{V}.$$

That is, $\mathbf{C}(A)$ is the set of all scalar multiples of $\mathbf{e}_1$. Here's why, using Definition 5.1.

First let $\mathbf{b} \in \mathbf{C}(A)$. Then $\mathbf{b} = (2v_1, 0)$ for some $v_1 \in \mathbb{R}$. This says $\mathbf{b} = 2v_1\mathbf{e}_1 \in \mathcal{V}$, if we take $c = 2v_1$.

Now let $\mathbf{b} \in \mathcal{V}$. Then $\mathbf{b} = c\mathbf{e}_1 = (c, 0)$ for some $c \in \mathbb{R}$. If we take $v_1 = c/2$, then $\mathbf{b} = (2v_1, 0) \in \mathbf{C}(A)$.

5.3 Problem ($\star$). **(i)** Prove that

$$\mathbf{C}\left(\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}\right) = \mathbb{R}^2.$$

[Hint: *repeat the work that gave us the equivalent systems in* (1.5), *but instead of having the right side of that system be* $(1, 11)$, *use an arbitrary* $\mathbf{b} = (b_1, b_2)$.]

(ii) What is

$$\mathbf{C}\left(\begin{bmatrix} 1 & -2 & 4 \\ 3 & 2 & 5 \end{bmatrix}\right)?$$

[Hint: *don't reinvent the wheel. You know the column space from the previous part, and you know that this column space is the set of all linear combinations of the form*

$$v_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + v_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + v_3 \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Is there an "easy" value that you can pick for v_3 *to relate this linear combination to what would appear in the previous part?*]

5.4 Problem ($\star$). Let $A \in \mathbb{R}^{m \times n}$.

(i) Explain why $\mathbf{0}_m \in \mathbf{C}(A)$. [Hint: *you want to find* $\mathbf{x} \in \mathbb{R}^n$ *such that* $A\mathbf{x} = \mathbf{0}_m$. *Is there an* $\mathbf{x}$ *that immediately comes to mind?*]

(ii) Suppose that $\mathbf{C}(A) = \{\mathbf{0}_m\}$. What does this tell you about A? [Hint: *for any standard basis vector* $\mathbf{e}_j$ *of* $\mathbb{R}^n$, *what do you know about* $A\mathbf{e}_j$*?*]

Failure in math and life teaches us a lot, and there is a lot to be learned from what happens when $\mathbf{C}(A) \neq \mathbb{R}^m$ for $A \in \mathbb{R}^{m \times n}$. Here are some problematic A.

5.5 Example. **(i)** Earlier we saw that if

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

then $\mathbf{C}(A)$ is the set of all scalar multiples of $\mathbf{e}_1 = (1, 0)$, and that is not $\mathbb{R}^2$.

(ii) Let

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}.$$

For v_1, $v_2 \in \mathbb{R}$, we have

$$v_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + v_2 \begin{bmatrix} -2 \\ -6 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2v_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (v_1 - 2v_2) \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

The last equality is true by distribution:

$$c_1\mathbf{v} + c_2\mathbf{v} = (c_1 + c_2)\mathbf{v} \quad \text{for any} \quad c_1,\ c_2 \in \mathbb{R},\ \mathbf{v} \in \mathbb{R}^n.$$

This calculation says that every $\mathbf{b} \in \mathbf{C}(A)$ is a multiple of $(1,3)$. Is every vector in $\mathbb{R}^2$ a multiple of $(1,3)$? Surely not: something like $(0,1)$ cannot be written as

$$\begin{bmatrix}0\\1\end{bmatrix} = c\begin{bmatrix}1\\3\end{bmatrix}.$$

What goes wrong in an equality like that?

(iii) Let

$$A = \begin{bmatrix}1&0&3\\0&2&4\\0&0&0\end{bmatrix}.$$

I think you'll agree that any $\mathbf{b} = (b_1, b_2, b_3) \in \mathbf{C}(A)$ has $b_3 = 0$; the deadly thing is that row of all 0. If not, let's use dot products for a change:

$$\begin{bmatrix}1&0&3\\0&2&4\\0&0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}1(v_1)+0(v_2)+3v_3\\0(v_1)+2v_2+4v_3\\0(v_1)+0(v_2)+0(v_3)\end{bmatrix} = \begin{bmatrix}v_1+3v_3\\2v_2+4v_3\\0\end{bmatrix}.$$

What really was going on in the previous example? The rows of zeros in the first and third matrices were problematic, but the column space is about *columns*.

5.6 Example. Let's take another look at those matrices from Example 5.5.

(i) I think it's easier to start with

$$\begin{bmatrix}1&-2\\3&-6\end{bmatrix}$$

and recall our arithmetic to see that the second column is -2 times the first column.

(ii) Then maybe we'll recognize that the second column of

$$\begin{bmatrix}2&0\\0&0\end{bmatrix}$$

is 0 times the first column. Even though these matrices have two columns, only one matters—somehow there is "redundant" data in the matrix!

(iii) Is there redundancy in

$$\begin{bmatrix}1&0&3\\0&2&4\\0&0&0\end{bmatrix}?$$

I claim that no column is a multiple of another—this is annoying to check, but it builds character, so you should do it. (Here's how to get started: if the first column is c times the second column, won't some of the zero and nonzero entries interact badly?) But maybe, if we're lucky, we'll notice patterns relating the third column to the first and second. Because life is short, I'll tell you those patterns:

$$\begin{bmatrix}3\\4\\0\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix} + \begin{bmatrix}0\\4\\0\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\2\\0\end{bmatrix}.$$

The third column is a linear combination of the first two. This is redundancy again: at a "linear" level, the third column can be recovered from the first two.

In fact, the third column just disappears when looking at linear combinations

$$v_1\begin{bmatrix}1\\0\\0\end{bmatrix} + v_2\begin{bmatrix}0\\2\\0\end{bmatrix} + v_3\begin{bmatrix}3\\4\\0\end{bmatrix} = v_1\begin{bmatrix}1\\0\\0\end{bmatrix} + v_2\begin{bmatrix}0\\2\\0\end{bmatrix} + v_3\left(3\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\2\\0\end{bmatrix}\right)$$
$$= (v_1+3v_3)\begin{bmatrix}1\\0\\0\end{bmatrix} + (v_2+2v_3)\begin{bmatrix}0\\2\\0\end{bmatrix}.$$

5.7 Problem (!). Draw pictures in $\mathbb{R}^2$ of the column spaces of the matrices in parts (i) and (ii) of Example 5.5. Explain how these pictures illustrate why you can't "reach" any vector in $\mathbb{R}^2$ via a linear combinations of the columns of these matrices. How does the specific "redundancy" in each matrix show up in your pictures?

Here is a first glimpse of how redundancy impacts the column space. Say that $A \in \mathbb{R}^{m\times 2}$ and the second column of A is a scalar multiple of the first: $A = \begin{bmatrix}\mathbf{a}_1 & c\mathbf{a}_1\end{bmatrix}$. If $\mathbf{b} \in \mathbf{C}(A)$, then $\mathbf{b} = A\mathbf{v}$ for some $\mathbf{v} = (v_1, v_2)$. But

$$A\mathbf{v} = \begin{bmatrix}\mathbf{a}_1 & c\mathbf{a}_1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = v_1\mathbf{a}_1 + v_2(c\mathbf{a}_1) = v_1\mathbf{a}_1 + (v_2c)\mathbf{a}_1 = (v_1+v_2c)\mathbf{a}_1.$$

Although A has two columns, we only need one of them to describe the column space: $\mathbf{C}(A)$ is the set of all scalar multiples of $\mathbf{a}_1$. (Strictly speaking, in the sense of Definition 5.1, this needs proof. Why is every scalar multiple of $\mathbf{a}_1$ in the column space? Because $c\mathbf{a}_1 = c\mathbf{a}_1 + \mathbf{0}_m = c\mathbf{a}_1 + 0\mathbf{a}_2 \in \mathbf{C}(A)$. Adding zero is a great trick.)

The same thing happens if the redundancy is more complicated. Say that $A \in \mathbb{R}^{m\times 3}$ and the third column of A is a linear combination of the first two:

$$A = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & (c_1\mathbf{a}_1 + c_2\mathbf{a}_2)\end{bmatrix}.$$

If $\mathbf{v} = (v_1, v_2, v_3)$ and you're patient with the algebra, I claim that

$$A\mathbf{v} = (v_1 + c_1v_3)\mathbf{a}_1 + (v_2 + c_2v_3)\mathbf{a}_2,$$

and so the column space of A is just the set of linear combinations of columns 1 and 2.

Here was the problem with the matrices in Examples 5.5 and 5.6: one of their columns was a linear combination of the other columns. Informally, from the point of view of the column structure of the matrices, there was redundant data. This is bad by itself: we're storing more information than we need to. But it's worse from the point of view of solving linear systems: somehow redundant data prevented the column space from being as large as possible. Our job is to understand why.

The problem with the matrices in Examples 5.5 and 5.6 was twofold: these were matrices in $\mathbb{R}^{m\times m}$ but their column spaces were not all of $\mathbb{R}^m$ (so we could not always solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$), and one of their columns was in the span of the others. Somehow these problems are related. We first give a name to the latter situation and then make a conjecture.

5.8 Definition. *The columns of a matrix $A \in \mathbb{R}^{m\times n}$ are* **DEPENDENT** *if (at least) one column is a linear combination of the other columns. If $n = 1$ and the matrix only has one column, we say its column is dependent if it is the zero vector.*

We do not say that "the matrix is dependent" or that the column that's a linear combination of the others is "dependent" or "dependent on the other columns." We only talk about the dependence of the columns in totality, relative to the whole matrix.

The inclusion of the special case of the zero vector when there is only one column (and when it does not make sense to talk about "span of the *others*," because there are no "other" columns when $n = 1$) is a bit of a technicality that will be helpful later. For $n \geq 2$, here is the importance of quantifiers: all that it takes for a matrix to have dependent columns is for *one* column to be "bad." Also, dependence is a relative thing: we're talking about vectors being dependent in the context of the rest of the columns of a matrix.

5.9 Example. **(i)** The columns of

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are dependent because the fourth column is the linear combination $1\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$.

(ii) The columns of the identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are not dependent (so they're "independent"?) because none is a linear combination of the other columns. This is annoying to check column by column, but if, say, $\mathbf{e}_1 = c_1\mathbf{e}_2 + c_3\mathbf{e}_3$, we get

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix},$$

so $c_2 = c_3 = 0$, and then $\mathbf{e}_1 = \mathbf{0}$. That's wrong.

5.10 Problem ($\star$). Show that the columns of

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

are dependent. [Hint: $\mathbf{v} = 1\mathbf{v}$. *What does this say about a repeated column being a linear combination of the other columns?*]

5.11 Problem ($\star$). Explain why if $A \in \mathbb{R}^{m \times n}$ satisfies one of the following conditions, its columns are dependent. I'm not saying these things *always* happen when a matrix has dependent columns, but they do guarantee dependence.

(i) The same column appears at least twice in A.

(ii) The zero vector (in $\mathbb{R}^m$) is a column of A.

(iii) One column of A is a multiple of another column.

5.12 Problem ($\star$). If the columns of a matrix are dependent, is *every* column in the matrix a linear combination of the other columns (not including itself in that linear combination)? [Hint: *go back through the various examples of matrices with dependent columns.*]

Dependence is exactly the right condition to encode redundancy from the point of view of the column space. If a matrix has dependent columns, then you don't need all of those columns to describe its column space.

5.13 Theorem (Removal). *Let $A \in \mathbb{R}^{m \times n}$ with $n \geq 2$ and suppose that one of the columns of A is a linear combination of the others. Let $\widetilde{A} \in \mathbb{R}^{m \times (n-1)}$ contain the $n-1$ columns of A except the one that is the linear combination of the others. Then $\mathbf{C}(A) = \mathbf{C}(\widetilde{A})$.*

Proof. I'll do all of this this for $n = 3$ to keep the notation simple. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \in \mathbb{R}^{m \times 3}$ and, to make things even simpler, assume that $\mathbf{a}_3$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, there are x_1, $x_2 \in \mathbb{R}$ such that $\mathbf{a}_3 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$. We'll put $\widetilde{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$, and we want to show $\mathbf{C}(A) = \mathbf{C}(\widetilde{A})$.

First, let $\mathbf{b} \in \mathbf{C}(A)$, so $\mathbf{b} = A\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^3$. Do algebra:

$$\begin{aligned} \mathbf{b} &= A\mathbf{v} \\ &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3 \\ &= v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3(x_1\mathbf{a}_1 + x_2\mathbf{a}_2) \\ &= v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3x_1\mathbf{a}_1 + c_3x_2\mathbf{a}_2 \end{aligned}$$

$$
\begin{aligned}
&= (v_1 + v_3x_1)\mathbf{a}_1 + (v_2 + v_3x_2)\mathbf{a}_2 \\
&= \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\end{bmatrix}\begin{bmatrix}v_1 + v_3x_1 \\ v_2 + v_3x_2\end{bmatrix} \\
&= \widetilde{A}\begin{bmatrix}v_1 + v_3x_1 \\ v_2 + v_3x_2\end{bmatrix} \\
&\in \mathbf{C}(\widetilde{A}).
\end{aligned}
$$

That is, we have shown

$$\mathbf{b} \in \mathbf{C}(A) \Longrightarrow \mathbf{b} \in \mathbf{C}(\widetilde{A}).$$

Now we use the trick of adding zero. Let $\widetilde{\mathbf{b}} \in \mathbf{C}(\widetilde{A})$. Then $\widetilde{\mathbf{b}} = \widetilde{A}\widetilde{\mathbf{v}}$ for some $\widetilde{\mathbf{v}} \in \mathbb{R}^2$, so

$$
\begin{aligned}
\widetilde{\mathbf{b}} &= \widetilde{A}\widetilde{\mathbf{v}} \\
&= \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}_1 \\ \widetilde{v}_2\end{bmatrix} \\
&= \widetilde{v}_1\mathbf{a}_1 + \widetilde{v}_2\mathbf{a}_2 \\
&= \widetilde{v}_1\mathbf{a}_1 + \widetilde{v}_2\mathbf{a}_2 + \mathbf{0}_m \\
&= \widetilde{v}_1\mathbf{a}_1 + \widetilde{v}_2\mathbf{a}_2 + 0\mathbf{a}_3 \\
&= \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}\widetilde{v}_1 \\ \widetilde{v}_2 \\ 0\end{bmatrix} \\
&= A\begin{bmatrix}\widetilde{v}_1 \\ \widetilde{v}_2 \\ 0\end{bmatrix} \\
&\in \mathbf{C}(A).
\end{aligned}
$$

We have shown

$$\widetilde{\mathbf{b}} \in \mathbf{C}(\widetilde{A}) \Longrightarrow \widetilde{\mathbf{b}} \in \mathbf{C}(A).$$

By Definition 5.1, we conclude $\mathbf{C}(A) = \mathbf{C}(\widetilde{A})$. ■

Day 6: Friday, August 29.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Null space of a matrix

6.1 Example. We can "iterate" this theorem several times. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

The zero vector is always a linear combination of any other vectors under consideration:

$$\mathbf{0}_4 = 0\mathbf{a}_2 + 0\mathbf{a}_3 + 0\mathbf{a}_4 + 0\mathbf{a}_5 + 0\mathbf{a}_6.$$

So

$$\mathbf{C}(A) = \mathbf{C}(\begin{bmatrix}\mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6\end{bmatrix}).$$

Next, $\mathbf{a}_3 = 2\mathbf{a}_2 = 2\mathbf{a}_2 + 0\mathbf{a}_4 + 0\mathbf{a}_5 + 0\mathbf{a}_6$, so

$$\mathbf{C}(\begin{bmatrix}\mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6\end{bmatrix}) = \mathbf{C}(\begin{bmatrix}\mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6\end{bmatrix}).$$

What jumps out to me next is that $\mathbf{a}_4 = 2\mathbf{a}_2 + 3\mathbf{a}_5 = 2\mathbf{a}_2 + 3\mathbf{a}_5 + 0\mathbf{a}_6$, so

$$\mathbf{C}(\begin{bmatrix}\mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6\end{bmatrix}) = \mathbf{C}(\begin{bmatrix}\mathbf{a}_2 & \mathbf{a}_5 & \mathbf{a}_6\end{bmatrix}).$$

And if you think about it long enough, none of these columns are combinations of the other two. We only needed three of the original six columns to control the column space:

$$\mathbf{C}(A) = \mathbf{C}(\begin{bmatrix}\mathbf{a}_2 & \mathbf{a}_5 & \mathbf{a}_6\end{bmatrix}).$$

6.2 Problem ($\star$). **(i)** In the previous example, explain why none of the columns $\mathbf{a}_2$, $\mathbf{a}_5$, or $\mathbf{a}_6$ is a linear combination of the other two. (You need to check three things here.)

(ii) Repeat the method of this example to show that $\mathbf{C}(A) = \mathbf{C}(\begin{bmatrix}\mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_6\end{bmatrix})$. The lesson is that we can express the column space in multiple efficient ways—but there is a certain "threshold" of efficiency that we can't beat. Here, it's *three* columns.

Now that we have some experience with dependent columns, we can make the conjecture that Examples 5.5 and 5.6 motivated.

6.3 Conjecture. *If the columns of $A \in \mathbb{R}^{m \times m}$ are dependent, then $\mathbf{C}(A) \neq \mathbb{R}^m$, and so we can't always solve $A\mathbf{x} = \mathbf{b}$.*

Unfortunately, it's only a conjecture right now, and we don't yet have the tools to prove it. And even when we know it's true, we probably want a way of verifying that the columns of a matrix are dependent—hopefully a more systematic way than just "getting lucky" and noticing that one column is a linear combination of the others.

6.4 Problem (!). We can talk about a nonsquare matrix with dependent columns, but the conjecture was only for a square matrix. Here's why. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that $\mathbf{C}(A) = \mathbb{R}^2$ and that the columns of A are dependent.

Our current definition of dependence is good, I think, because it encapsulates the inefficient redundancies in Examples 5.5 and 5.6. It's less good, however, because it singles out a particular column for blame. One column has to be "guilty" and expressible as a linear combination of the other columns. Finding that guilty column could be hard if the matrix is large. There's another test for dependence (actually, there are many, and you will come to know them and love them).

6.5 Example. Let

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}.$$

As you know, we studied A in Examples 5.5 and 5.6. We know that the second column of A is -2 times the first column. With $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$, we have $\mathbf{a}_2 = -2\mathbf{a}_1$. Or, if you prefer (and I do), $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}_2$. Working backwards,

$$\mathbf{0}_2 = 2\mathbf{a}_1 + \mathbf{a}_2 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = A\mathbf{x}, \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Look at that: we solved the problem $A\mathbf{x} = \mathbf{0}_2$ in an "interesting" way—by finding a nonzero solution $\mathbf{x}$. Of course $\mathbf{x} = \mathbf{0}_2$ always meets $A\mathbf{x} = \mathbf{0}_2$. Now we have an extra, "nontrivial" solution.

This "nontrivial" solution to the "homogeneous" problem $A\mathbf{x} = \mathbf{0}$ turns out to "characterize" dependence. By "characterize" I mean that it gives another way of checking dependence—we could have started with this matrix-vector product situation. Here's the technical result.

6.6 Theorem. *The columns of a matrix $A \in \mathbb{R}^{m\times n}$ are dependent if and only if there exists a vector $\mathbf{x} \in \mathbb{R}^n$ such that both $\mathbf{x} \neq \mathbf{0}_n$ and $A\mathbf{x} = \mathbf{0}_m$.*

Proof. This is an "if and only if" statement, so we need to give proofs going in both directions.

($\Longrightarrow$) Start by assuming that the columns of A are dependent. We want to find a solution $\mathbf{x} \neq \mathbf{0}_n$ to $A\mathbf{x} = \mathbf{0}_m$, and we know that one column is a linear combination of the others.

1. To keep the notation simple and to help us see the general structure of the argument, let's start with a very special situation:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \in \mathbb{R}^{m \times 4} \quad \text{with} \quad \mathbf{a}_3 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_4\mathbf{a}_4$$

for some x_1, x_2, $x_4 \in \mathbb{R}$. By the way, the number of rows m here is irrelevant.

Now rearrange:

$$\mathbf{0}_m = (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_4) - \mathbf{a}_3 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + (-1)\mathbf{a}_3 + x_4\mathbf{a}_4 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ -1 \\ x_4 \end{bmatrix}.$$

So with $\mathbf{x} = (x_1, x_2, -1, x_4)$, we have $A\mathbf{x} = \mathbf{0}_m$. And we definitely have $\mathbf{x} \neq \mathbf{0}_4$ because of that entry of -1. Maybe $x_1 = x_2 = x_4 = 0$, but still $\mathbf{x} \neq \mathbf{0}_4$.

2. Here's the full proof: say that the jth column of A is a linear combination of the other columns. So

$$\mathbf{a}_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} x_k\mathbf{a}_k$$

for some $x_k \in \mathbb{R}$. Then

$$\mathbf{0}_m = \sum_{\substack{k=1 \\ k \neq j}}^{n} x_k\mathbf{a}_k + (-1)\mathbf{a}_j = A\mathbf{x},$$

where $\mathbf{x}$ is the vector whose kth entry is x_k for $k \neq j$, and whose jth entry is -1. That jth entry is nonzero, so $\mathbf{x} \neq \mathbf{0}_n$.

($\Longleftarrow$) Now we assume that $A\mathbf{x} = \mathbf{0}_m$ for some $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}_n$. We want to show that (at least) one column of A is a linear combination of the others.

1. Again, we'll start with a special case: $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \in \mathbb{R}^{m \times 4}$, $\mathbf{x} = (x_1, x_2, x_3, x_4)$, and $x_2 \neq 0$. So we have

$$\mathbf{0}_m = A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4,$$

and so

$$-x_2\mathbf{a}_2 = x_1\mathbf{a}_1 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4.$$

Since $x_2 \neq 0$, we may divide to find

$$\mathbf{a}_2 = \left(-\frac{x_1}{x_2}\right)\mathbf{a}_1 + \left(-\frac{x_3}{x_2}\right)\mathbf{a}_3 + \left(-\frac{x_4}{x_2}\right)\mathbf{a}_4.$$

And so $\mathbf{a}_2$ is a linear combination of the other columns.

2. Here's the full proof: let $A\mathbf{x} = \mathbf{0}_m$ and suppose that the jth entry of $\mathbf{x}$ is nonzero. Then

$$\mathbf{0}_m = A\mathbf{x} = \sum_{k=1}^{n} x_k\mathbf{a}_k = x_j\mathbf{a}_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} x_k\mathbf{a}_k,$$

so

$$\mathbf{a}_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(-\frac{x_k}{x_j}\right) \mathbf{a}_k$$

is a linear combination of the other columns. ■

What is special here is not that we have a solution to the homogeneous problem $A\mathbf{x} = \mathbf{0}_m$. We always do: $A\mathbf{0}_n = \mathbf{0}_m$. What is special is that this solution is *not* the zero vector in $\mathbb{R}^n$.

Content from Strang's *ILA* 6E. The blurb on "Independent columns" at the bottom of p. 30 effectively proves this theorem. Forget about the matrix A in that discussion and just think that it's telling you about the (in)dependence of the columns of a matrix C.

This theorem transfers the burden of guilt from a particular column of the matrix to a solution to the matrix-vector equation $A\mathbf{x} = \mathbf{0}_m$. Since this course is all about understanding (and maybe sometimes even solving) $A\mathbf{x} = \mathbf{0}_m$, that should make us happy.

The collection of all solutions to $A\mathbf{x} = \mathbf{0}_m$ has a special name, and in many ways it's the moral complement of the column space.

6.7 Definition. *The* NULL SPACE *of $A \in \mathbb{R}^{m \times n}$ is the collection of all solutions to $A\mathbf{x} = \mathbf{0}_m$. We denote it by $\mathbf{N}(A)$, and every vector in $\mathbf{N}(A)$ is a vector in $\mathbb{R}^n$. Equivalently,*

$$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}_m\}.$$

Eventually we will develop a detailed procedure for figuring out exactly what vectors are in $\mathbf{N}(A)$. This will be part of our broader program of developing a detailed procedure for solving $A\mathbf{x} = \mathbf{b}$, which should be unsurprising, since studying the null space amounts to studying $A\mathbf{x} = \mathbf{b}$ in the special case of $\mathbf{b} = \mathbf{0}_m$.

6.8 Example. Here's an easy null space to study: $\mathbf{N}(I_2) = \{\mathbf{0}_2\}$. For if $I_2\mathbf{x} = \mathbf{0}_2$, then $\mathbf{x} = \mathbf{0}_2$, and certainly $I_2\mathbf{0}_2 = \mathbf{0}_2$.

6.9 Problem (!). At the very least we always know one vector in the null space. Let $A \in \mathbb{R}^{m \times n}$. Explain why $\mathbf{0}_n \in \mathbf{N}(A)$.

6.10 Problem ($\star$). Describe as precisely as possible the null spaces of

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}.$$

[Hint: *for the first matrix, some of your work in Problem 5.3 might help.*]

6.11 Problem (!). Which of the column and null spaces do you feel is more "explicitly" described and which more "implicitly"? (All feelings are valid, but for me there are definitely incorrect feelings here.)

Now we can recast Theorem 6.6 and facts about dependence in the language of the null space.

6.12 Corollary. *The columns of a matrix* $A \in \mathbb{R}^{m \times n}$ *are dependent if and only if* $\mathbf{N}(A) \neq \{\mathbf{0}_n\}$. *(Since* $\mathbf{0}_n \in \mathbf{N}(A)$ *always, saying* $\mathbf{N}(A) \neq \{\mathbf{0}_n\}$ *means there is some* $\mathbf{v} \in \mathbf{N}(A)$ *with* $\mathbf{v} \neq \mathbf{0}_n$*.)*

6.13 Problem (!). Convince yourself that this corollary is true.

6.14 Problem (!). Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find a nonzero vector $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{0}_3$. [Hint: *part (iii) of Example 5.6.*]

Day 7: Wednesday, September 3.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Dependent list of vectors (N), independent list of vectors (N)

We know that the column space of A controls existence of solutions to $A\mathbf{x} = \mathbf{b}$: we can solve this problem precisely when $\mathbf{b} \in \mathbf{C}(A)$. We know that $\mathbf{N}(A)$ controls solutions to $A\mathbf{x} = \mathbf{0}_m$, but does $\mathbf{N}(A)$ tell us anything useful about the more general problem $A\mathbf{x} = \mathbf{b}$? It sure does!

Remember that when we are solving equations, we don't just want to know if solutions exist. We want to know if they're unique, too. That is, if given $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, we have a solution $\mathbf{x} \in \mathbb{R}^n$ to $A\mathbf{x} = \mathbf{b}$, is this the only $\mathbf{x}$? Could my $\mathbf{x}$ be different from yours?

Here's the trick. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ solve $A\mathbf{x} = \mathbf{b}$. That is,

$$A\mathbf{x}_1 = \mathbf{b} \quad \text{and} \quad A\mathbf{x}_2 = \mathbf{b}.$$

For solutions to be unique, we want $\mathbf{x}_1 = \mathbf{x}_2$. Is this true? If $\mathbf{x}_1 = \mathbf{x}_2$, then $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}_n$. What if solutions aren't unique? Then $\mathbf{x}_1 \neq \mathbf{x}_2$, and so $\mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}_n$. And what is $\mathbf{x}_1 - \mathbf{x}_2$ doing?

All we know about about $\mathbf{x}_1$ and $\mathbf{x}_2$ is how they behave under multiplication by A, so let's try

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}_m. \tag{7.1}$$

Thus $\mathbf{x}_1 - \mathbf{x}_2 \in \mathsf{N}(A)$. And if $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\mathsf{N}(A)$ is bigger than just $\{\mathbf{0}_n\}$. By the way, the first equality here, that of $A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2$, was linearity of matrix-vector multiplication (Problem 3.12). This is linear algebra, after all!

This argument started by looking at nonunique solutions to $A\mathbf{x} = \mathbf{b}$ and deduced information about $\mathsf{N}(A)$. Let's start with $\mathsf{N}(A)$ and suppose it's bigger than just $\{\mathbf{0}_n\}$. Let $\mathbf{z} \in \mathsf{N}(A)$ with $\mathbf{z} \neq \mathbf{0}_n$, so $A\mathbf{z} = \mathbf{0}_m$. Suppose that we also have a solution $\mathbf{x}_\star$ to $A\mathbf{x}_\star = \mathbf{b}$. Finally, let $c \in \mathbb{R}$. Then

$$A(\mathbf{x}_\star + c\mathbf{z}) = A\mathbf{x}_\star + cA\mathbf{z} = \mathbf{b} + c\mathbf{0}_m = \mathbf{b} + \mathbf{0}_m = \mathbf{b}. \tag{7.2}$$

Again, as in (7.1), it's the linearity of matrix-vector multiplication that makes the first equality possible.

So what's the point of (7.2) and that c? For each c, we get a new vector $\mathbf{x}_\star + c\mathbf{z}$. And so from (7.2) we get a new solution to $A\mathbf{x} = \mathbf{b}$.

7.1 Problem (!). Prove it: if $c_1 \neq c_2$, explain why $\mathbf{x}_\star + c_1\mathbf{z} \neq \mathbf{x}_\star + c_2\mathbf{z}$. [Hint: *what happens if, instead,* $\mathbf{x}_\star + c_1\mathbf{z} = \mathbf{x}_\star + c_2\mathbf{z}$*? Here it's important that* $\mathbf{z} \neq \mathbf{0}_n$.]

Here's the point: if $\mathsf{N}(A) \neq \{\mathbf{0}_n\}$, then solutions to $A\mathbf{x} = \mathbf{b}$ are never unique. (Okay, maybe $A\mathbf{x} = \mathbf{b}$ doesn't have *any* solutions. But, in the situation of $\mathsf{N}(A)$ being bigger than $\{\mathbf{0}_n\}$, when solutions do exist, there are infinitely many of them.)

These arguments lead to an important result.

7.2 Theorem. *Let* $A \in \mathbb{R}^{m \times n}$.

(i) *Suppose that for some* $\mathbf{b} \in \mathbb{R}^m$*, the problem* $A\mathbf{x} = \mathbf{b}$ *has at least two different solutions. Then* $\mathsf{N}(A) \neq \{\mathbf{0}_n\}$.

(ii) *Suppose that* $\mathsf{N}(A) \neq \{\mathbf{0}_n\}$*. If the problem* $A\mathbf{x} = \mathbf{b}$ *has a solution (maybe it doesn't!), then it has infinitely many solutions.*

(iii) *If the columns of* A *are dependent, then solutions to* $A\mathbf{x} = \mathbf{b}$ *(which may or may not exist!) can't be unique.*

(iv) *If for some* $\mathbf{b} \in \mathbb{R}^m$*, the problem* $A\mathbf{x} = \mathbf{b}$ *has at least two different solutions, then the columns of* A *are dependent.*

7.3 Problem (⋆). The proof of this theorem is contained in the paragraphs between Problem 6.13 and the statement of the theorem. I acknowledge that those paragraphs are a little chatty. Reread them and write a proof of this theorem in your own (possibly more efficient) words. (Also, and this isn't part of the problem, just something to think about: I love how we can say so much about the solution behavior to $A\mathbf{x} = \mathbf{b}$ even though we still don't have any algorithmic or formulaic procedures for finding those solutions. Having a formula for something isn't the same as understanding that thing!)

Our discussion of the null space has focused on (non)uniqueness of solutions to $A\mathbf{x} = \mathbf{b}$, and we've concluded that dependent columns prevent uniqueness of solutions. This still hasn't resolved Conjecture 6.3 on how dependent columns affect *existence*. That's going to take more work, and the work starts with thinking more positively. The opposite of dependence, which seems to be bad, is probably independence.

7.4 Definition. *The columns of a matrix $A \in \mathbb{R}^{m \times n}$ are* **INDEPENDENT** *if they are not dependent.*

If "dependent" means "at least one column is a linear combination of the others," then "not dependent" has to mean "no column is a linear combination of the others." So that's what "independent" means. If dependence means "one column is guilty" (of being a linear combination of the others, of being redundant), then independence means "all columns are innocent." Either way, that's a lot to check—find one guilty column, or prove that every column is innocent. Fortunately, Corollary 6.12, that great equalizer, offers a different take.

7.5 Corollary. *The columns of a matrix $A \in \mathbb{R}^{m \times n}$ are* **INDEPENDENT** *if and only if $\mathbf{N}(A) = \{\mathbf{0}_n\}$, that is, precisely when the only solution to $A\mathbf{x} = \mathbf{0}_m$ is $\mathbf{x} = \mathbf{0}_n$.*

7.6 Example. The columns of the $n \times n$ identity matrix $I_n = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$ are independent, for if $I_n\mathbf{x} = \mathbf{0}_n$, then since $I_n\mathbf{x} = \mathbf{x}$, we immediately have $\mathbf{x} = \mathbf{0}_n$.

7.7 Problem ($\star$). Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$. Prove that the columns of A are independent if and only if whenever $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}_m$, we must have $c_j = 0$ for $j = 1, \ldots, n$. [Hint: *what is the definition of matrix-vector multiplication?*]

Remember that we started the study of dependent columns because we observed that square matrices with dependent columns seemed to have too small of a column space. That led us to make Conjecture 6.3. Here is the more upbeat analogue for a square matrix with independent columns.

7.8 Conjecture. *If the columns of $A \in \mathbb{R}^{m \times m}$ are independent, then $\mathbf{C}(A) = \mathbb{R}^m$.*

7.9 Problem (!). Again, the conjecture is only for square matrices. Explain why the columns of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

are independent but $\mathbf{C}(A) \neq \mathbb{R}^3$.

As with Conjecture 6.3, we do not yet have the tools to prove Conjecture 7.8, nor do we have a particularly efficient way of verifying that a matrix's columns are independent beyond the definition. Both conjectures beg the question of which columns in a matrix really matter—which ones are redundant and which ones are essential for describing the column space. This will lead to a more general definition of dependence and independence that can be given beyond the context of matrices. So, what columns really matter?

Content from Strang's *ILA* 6E. Answer: the "independent ones," as alluded to on pp. 20–22. This will require us to broaden the definition of independence to allow only *some* of the columns of the matrix to be independent—that is, some of the columns of a matrix with dependent columns can still be independent, if we define "independent" correctly. Now is also a good time to (re)read pp. v–vii up to, but not including, the $A = CR$ section.

To describe what columns really matter, it will help to have a variation on our notions of dependence and independence that frees us from thinking solely about vectors as columns of matrices. Naturally, we'll start by thinking about vectors as columns of matrices.

7.10 Example. We have some experience telling us that the columns of the matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

are dependent. Among other ways to see this, we have $\mathbf{a}_3 = 3\mathbf{a}_1 + 2\mathbf{a}_2$, and from this the removal theorem gives $\mathbf{C}(A) = \mathbf{C}(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix})$. But we really *need* those columns: $\mathbf{C}(A) \neq \mathbf{C}(\begin{bmatrix} \mathbf{a}_1 \end{bmatrix})$ and $\mathbf{C}(A) \neq \mathbf{C}(\begin{bmatrix} \mathbf{a}_2 \end{bmatrix})$. (Why?)

I'm sure you'll agree that the columns of

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are independent. It's no accident that we needed *just* these columns to control $\mathbf{C}(A)$, but it turns out that we had other options: $\mathbf{C}(A) = \mathbf{C}(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix})$. You probably won't be surprised that the columns of $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$ are independent and $\mathbf{C}(A) \neq \mathbf{C}(\begin{bmatrix} \mathbf{a}_3 \end{bmatrix})$.

7.11 Problem (!). Fill in the gaps from Example 7.10.

(i) Prove that $\mathbf{C}(A) = \mathbf{C}(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}) = \mathbf{C}(\begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix})$. [Hint: *don't work directly with the definition of the column space; use the removal theorem instead.*]

(ii) Prove that the columns of $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$, $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$, and $\begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ are independent.

(iii) Prove that $\mathbf{C}(A) \neq \mathbf{C}(\begin{bmatrix} \mathbf{a}_1 \end{bmatrix})$, $\mathbf{C}(A) \neq \mathbf{C}(\begin{bmatrix} \mathbf{a}_2 \end{bmatrix})$, and $\mathbf{C}(A) \neq \mathbf{C}(\begin{bmatrix} \mathbf{a}_3 \end{bmatrix})$.

The important columns in A from Example 7.10 were $\mathbf{a}_1$ and $\mathbf{a}_2$. (Or maybe $\mathbf{a}_1$ and $\mathbf{a}_3$. Or maybe $\mathbf{a}_2$ and $\mathbf{a}_3$.) We could describe $\mathbf{C}(A)$ using no more, and no less, than these two, and when we stuck them in a matrix, that matrix had independent columns. There's something clunky in our language here, and it's that last sentence. We don't need to "stick them in a matrix" to appreciate their independence. Going forward, it will save us some time if we can think about (in)dependence of vectors without introducing a matrix.

7.12 Definition. *A list of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ *is* DEPENDENT *if (at least) one vector in this list is a linear combination of the others. The list is* INDEPENDENT *if it is not dependent.*

7.13 Example. In this example we'll think about the standard basis vectors in $\mathbb{R}^3$.

(i) The list $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_1 + 2\mathbf{e}_2$ is dependent because the third vector is a linear combination of the first two.

(ii) The list $\mathbf{e}_1$, $\mathbf{e}_2$, $2\mathbf{e}_1$ is dependent because the third vector is a linear combination of the first two: $2\mathbf{e}_1 = 2\mathbf{e}_1 + 0\mathbf{e}_2$.

(iii) The list $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_1$ is dependent because the third vector is, again, a linear combination of the first two: $\mathbf{e}_1 = 1\mathbf{e}_1 + 0\mathbf{e}_2$. *Any list with a repeated vector is dependent.*

(iv) The list $\mathbf{e}_1$, $\mathbf{0}_3$, $\mathbf{e}_2$ is dependent because $\mathbf{0}_3 = 0\mathbf{e}_1 + 0\mathbf{e}_2$. *Any list containing the zero vector is dependent.*

Day 8: Friday, September 5.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Span of a list of vectors, rank of a matrix

There are lots of equivalent ways for a list to be (in)dependent, and it's all basically the same logic as with columns of a matrix.

8.1 Theorem (Equivalent conditions for a dependent list). *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$. *The following are equivalent (so if one of the things below is true, both are true; if one is false, both are false).*

(i) *One vector in the list* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is a linear combination of the others.*

(ii) *There exist scalars* $c_1, \ldots, c_n \in \mathbb{R}$ *such that at least one of the scalars is nonzero and* $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}_m$.

8.2 Problem (!). Reread the proof of Theorem 6.6 and use that proof to explain why Theorem 8.1 is true.

Here's the happier version, for independence.

8.3 Corollary (Equivalent conditions for an independent list). *Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$. The following are equivalent (so if one of the things below is true, both are true; if one is false, both are false).*

(i) *No vector in the list $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a linear combination of the others.*

(ii) *Let $c_1, \ldots, c_n \in \mathbb{R}$ satisfy $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}_m$. Then $c_j = 0$ for $j = 1, \ldots, n$. That is, the only linear combination of the list that adds up to the zero vector is the "trivial" linear combination with weights all zero.*

8.4 Example. In this example we'll think again about the standard basis vectors in $\mathbb{R}^3$.

(i) The list $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is independent because if $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = \mathbf{0}_3$, then

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and so $c_1 = c_2 = c_3$.

(ii) We check the (in)dependence of the list $\mathbf{e}_1$, $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$. Assume $c_1\mathbf{e}_1 + c_2(\mathbf{e}_1 + \mathbf{e}_2) + c_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \mathbf{0}_3$. This reads

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and that's a system of linear equations (for which we still don't have a systematic solution procedure!). But this is not too hard to solve: we want

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{8.1}$$

and looking at the third entries, that says $c_3 = 0$, thus $0 = c_2 + c_3 = c_2$, and last $0 = c_1 + c_2 + c_3 = c_1$.

By the way, the *vector* equation (8.1) is equivalent to the *matrix-vector* equation

$$\begin{bmatrix} 1 & 1 & 1 & \\ 0 & 1 & 1 & \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}_3.$$

This is the dream: an *upper-triangular* matrix (because below the diagonal it's all 0) with nonzero entries on the diagonal. Such a system is easy because we can "back-solve" or "back-substitute" starting from the bottom, which we just did ($c_3 = 0$ to $c_2 = 0$ to $c_1 = 0$). Very soon we will develop versatile procedures for converting *any* problem $A\mathbf{x} = \mathbf{b}$ to "upper-triangular" form.

Content from Strang's *ILA* 6E. Pages 116–117 (just read Example 1 on p. 117) define independence and give examples. I know this is a big jump ahead in the book, but you can read it now. Ignore for now the remark at the bottom of p. 116 about the "free variable" and "special solution."

8.5 Problem (!). Try to give answers to the following questions that are a little more direct than just repeating the definition of (in)dependence.

(i) When is a list of length 1 dependent? Independent?

(ii) When is a list of length 2 dependent? Independent?

8.6 Problem ($\star$). If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a list in $\mathbb{R}^m$, we'll say that $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is a **REORDERING** of that list if for each j between 1 and n, there is a unique k between 1 and n such that $\mathbf{w}_j = \mathbf{v}_k$. For example, the list $\mathbf{e}_2$, $\mathbf{e}_1$, $\mathbf{e}_3$ is a reordering of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, but the list $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_2$ is not.

(i) Suppose that a list is dependent. Prove that any reordering of that list is dependent, too. [Hint: *to get a sense of how the argument should go, assume that* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ *is dependent, and then explain why* $\mathbf{v}_4$, $\mathbf{v}_3$, $\mathbf{v}_2$, $\mathbf{v}_1$ *is also dependent.*]

(ii) Do the same for independence.

(iii) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a list in $\mathbb{R}^m$ with $n \le m$, and suppose that the vectors in this list are (some of) the standard basis vectors for $\mathbb{R}^m$. No standard basis vector appears two or more times in this list (and not every standard basis vector has to be in the list: maybe $n < m$). Prove that this list is independent.

We introduced lists so that we can talk about (in)dependence without having to bring up a matrix all the time. In turn, this involves talking a lot about linear combinations of lists.

8.7 Definition. *The* **SPAN** *of the vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ *is the set of all linear combinations of these vectors, and we denote it by* $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. *If a set* $\mathcal{V}$ *of vectors in* $\mathbb{R}^m$ *equals* $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, *then we say that the list* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ **SPANS** $\mathcal{V}$.

Content from Strang's *ILA* 6E. The span of a list of vectors is defined in the box on p. 21.

8.8 Example. **(i)** Let $\mathbf{v} \in \mathbb{R}^m$. Then $\text{span}(\mathbf{v}) = \{c\mathbf{v} \mid c \in \mathbb{R}\}$. So the span of a single vector is the set of all scalar multiples of that vector.

(ii) With $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ as the standard basis vectors for $\mathbb{R}^3$ (context matters!), Example 3.20 gives $\text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

(iii) Let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$. Then $\mathbf{C}\left(\begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}\right) = \text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$. So every column space is a span, and every span is a column space—but talking about spans cuts down on some chatter and avoids unnecessarily introducing a matrix just to have a column space.

8.9 Problem (!). Prove that $\text{span}(\mathbf{0}) = \{\mathbf{0}\}$. That is, the only vector in the span of $\mathbf{0}$ is $\mathbf{0}$ itself.

8.10 Problem ($\star$). Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$. Prove that $\mathbf{0}_m \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

8.11 Problem (!). Explain why a list and any reordering of that list have the same span. [Hint: *does the order in which you add vectors ever matter? Nope.*]

Now here is how the language of (in)dependent lists and spans gives us insight into efficient representations of column spaces.

8.12 Example. In Example 7.10 and earlier, we worked with a matrix of the form $A = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$. Then $\mathbf{C}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. That's true but not very helpful or insightful. But what was special about this A was that $\mathbf{a}_3 = 3\mathbf{a}_1 + 2\mathbf{a}_2$. That is, $\mathbf{a}_3 \in \text{span}(\mathbf{a}_1, \mathbf{a}_2)$. Then the removal theorem says $\mathbf{C}(A) = \mathbf{C}(\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\end{bmatrix}) = \text{span}(\mathbf{a}_1, \mathbf{a}_2)$. Notationally, I think it's easier to say $\mathbf{C}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2)$ than to introduce that "truncated" matrix from the removal theorem.

We also saw the list $\mathbf{a}_1$, $\mathbf{a}_2$ was independent, which might seem incidental, but Problem 7.11 suggests that something special is happening. That problem showed that we could write $\mathbf{C}(A)$ as the span of any two linearly independent columns of A but not as the span of any one column of A. Finally, no list of more than two columns of A is independent: such a list either contains a repeated column or is just (a rearrangement of) the dependent list $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.

This example suggests that there is a "threshold" for independence, spans, and column spaces: there is a number of columns of a matrix that is "just right" to span the column space efficiently without any redundancy.

8.13 Definition. *If $A \in \mathbb{R}^{m \times n}$ has all zero columns (and therefore is the "zero matrix"), then the* RANK *of A is* 0. *Otherwise, if A has at least one nonzero column, the* RANK *of A is the length of the longest list of linearly independent columns of A. Sometimes we write this number as* $\text{rank}(A)$.

8.14 Example. Each of the following matrices has rank 1. What a waste of storage space!

(i) $\mathbf{C}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \{\mathbf{0}_2\}$

(ii) $\mathbf{C}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

(iii) $\mathbf{C}\left(\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ as each column is a multiple of the first.

8.15 Example. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the standard basis vectors for $\mathbb{R}^3$. The rank of each of the following matrices is 2, each column space is the set

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; v_1,\ v_2 \in \mathbb{R} \right\}, \tag{8.2}$$

and we can write the column spaces as the spans of lists of two independent vectors but not lists of just one vector.

(i) $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \end{bmatrix}$: $(\mathbf{e}_1, \mathbf{e}_2)$ spans the column space.

(ii) $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & 2\mathbf{e}_2 \end{bmatrix}$: $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_1, 2\mathbf{e}_2)$ span the column space.

(iii) $\begin{bmatrix} \mathbf{e}_1 & (\mathbf{e}_1+\mathbf{e}_2) & \mathbf{e}_2 \end{bmatrix}$: $(\mathbf{e}_1, \mathbf{e}_2)$, $(\mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2)$, and $(\mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2)$ all span the column space.

8.16 Problem ($\star$). Convince yourself that each of the six lists in the previous example is independent and that the span of each list is the set (8.2).

Content from Strang's *ILA* 6E. Work through the example on p. 23 with the matrix A_6. We won't talk about this for some time in class, but the "row rank = column rank" calculations for 2×2 and 3×3 rank-1 matrices are good practice, so check the details yourself.

8.17 Problem (!). Let

$$A = \begin{bmatrix} 2 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 6 \\ 0 & 4 & 8 & 8 \end{bmatrix}.$$

What is $\operatorname{rank}(A)$? Find three different lists of independent columns of A where each list has length equal to $\operatorname{rank}(A)$. To make life interesting, if you find one such list, a reordering of that list (Problem 8.6) doesn't count as one of your three lists.

Here, more precisely, is how we might guess that rank is a "threshold" for efficient representations of the column space.

8.18 Conjecture. *Let* $A \in \mathbb{R}^{m \times n}$.

(i) *If* $\mathbf{C}(A)$ *equals the span of a list* r *independent columns, then any other list of* r *independent columns of* A *also spans* $\mathbf{C}(A)$.

(ii) *But no list of fewer than* r *columns can span* $\mathbf{C}(A)$.

(iii) *Any list of more than* r *columns is dependent (and so redundant for spanning purposes: you don't need all of that list to span the column space).*

Day 9: Monday, September 8.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Matrix-matrix product

There is another way to check for independence that works well by hand for "small" matrices and that more generally reinforces the underlying idea of linear combinations. It's a bit technical, so I'll tell you how it goes, then we'll do some examples and think about it, and last we'll prove it. I like calling it the "linear independence lemma" and so that's how it's labeled below.

9.1 Lemma (Linear independence). *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *be a list in* $\mathbb{R}^m$. *The following are equivalent.*

(i) *The list* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is independent.*

(ii) *The following two conditions hold. First,* $\mathbf{v}_1 \neq \mathbf{0}_m$. *Second, if* $n \geq 2$, *then for* $j = 2, \ldots, n$, *the* j*th vector in the list is not a linear combination of the first* $j-1$ *vectors. In more symbols and fewer words, this means* $\mathbf{v}_1 \neq \mathbf{0}_m$ *and, if* $n \geq 2$, *then* $\mathbf{v}_j \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$ *for* $j = 2, \ldots, n$.

I think we better do a concrete example right away.

9.2 Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Certainly

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \mathbf{0}_3.$$

Next, we want to check that $\mathbf{v}_2 \notin \operatorname{span}(\mathbf{v}_1)$. Otherwise, we would have

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for some $c \in \mathbb{R}$. Equating the second components, this would mean $3 = 0$. Last, we want to check that $\mathbf{v}_3 \notin \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. Otherwise, we would have

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

Equating the third components, we would have $6 = 0$. And so the list $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is independent.

9.3 Problem (!). Use Lemma 9.1 to explain why the columns of the matrix

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are dependent.

9.4 Remark. *In the linear independence lemma, what's up with the focus on the first vector being nonzero? Well, we know that if the first vector is the zero vector, then the list is dependent. If the list has only one vector (boring!), then dependence happens if that vector is zero.*

And why don't we say that subsequent vectors in the list beyond the first can't be the zero vector? Because the condition $\mathbf{v}_j \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$ *precludes that: we know* $\mathbf{0}_m \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$ *by Problem 8.10, so if* $\mathbf{v}_j \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$, *then it's definitely the case that* $\mathbf{v}_j \neq \mathbf{0}_m$.

Content from Strang's *ILA* 6E. Reread p. 20, this time paying attention to dependence and independence. Then work through the (in)dependence tests on p. 21 for the matrices A_4 and A_5. There is one thing here that we have not yet discussed: what does it mean for only "some" of the columns of A to be independent?

Proof time!

Proof (Of Lemma 9.1). Very often when thinking about independence, it's worth asking "What goes wrong if the thing that I want to be true is false?" We'll do that a lot in this argument.

($\Longrightarrow$) First suppose that the list $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is independent. We want to prove that the second, technical part of the lemma is true.

Since the list is independent, no vector can be the zero vector, or the list would be dependent. In particular, $\mathbf{v}_1 \neq \mathbf{0}_m$. Then if $n = 1$, we're done. Otherwise, suppose $n \geq 2$. If the jth vector for some $j \geq 2$ is a linear combination of the previous $j - 1$ vector, then that vector is a linear combination of *all of the other* vectors in the list. Just put the weights on the remaining vectors to be zero. And then the list is dependent, which is wrong.

1. Here's how this works for a small n with more symbols and fewer words. Say $n = 4$ and $\mathbf{v}_3 \in \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. Then $\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ for some c_1, $c_2 \in \mathbb{R}$. But then by the great trick of adding zero,

$$\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \mathbf{0}_m = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_4,$$

and so $\mathbf{v}_3$ is in the span of the other vectors in the list. That's dependence for you.

We also want to rule out the cases $\mathbf{v}_2 \in \text{span}(\mathbf{v}_1)$ and $\mathbf{v}_4 \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Can you adapt the argument in the preceding paragraph to show why either of these cases imply dependence?

2. Here's how this works in general. Say that for some $j \geq 2$ we have $\mathbf{v}_j \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$. So

$$\mathbf{v}_j = c_1\mathbf{v}_1 + \cdots + c_{j-1}\mathbf{v}_{j-1}$$

for some $c_1, \ldots, c_{j-1} \in \mathbb{R}$. Rearrange this to read

$$c_1\mathbf{v}_1 + \cdots + c_{j-1}\mathbf{v}_{j-1} + (-1)\mathbf{v}_j = \mathbf{0}_m.$$

If $j = n$, then the list $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is dependent. If $j < n$, add zero to find

$$c_1\mathbf{v}_1 + \cdots + c_{j-1}\mathbf{v}_{j-1} + (-1)\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}_m,$$

and so the list is again dependent.

($\Longleftarrow$) Now suppose that the technical condition in the second part of the lemma holds. We need to prove that the list is independent. If the list has only one vector, then that vector is nonzero, so we're done. Say that the list has at least two vectors. One way to prove independence is to assume $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}_m$ and show $c_j = 0$ for all j. What goes wrong if at least one coefficient is nonzero?

Well, if $c_2 = \cdots = c_n = 0$, then we get $c_1\mathbf{v}_1 = \mathbf{0}_m$. Since at least one coefficient has to be nonzero, that has to be c_1 here: $c_1 \neq 0$. Then $\mathbf{v}_1 = \mathbf{0}_m$, a contradiction.

Otherwise, suppose that a coefficient with index 2 or higher is nonzero (maybe there's more than one such coefficient). Here's the trick: look at the *highest-indexed* coefficient that's nonzero. I'll show you what I mean in the case $n = 4$ first.

1. Say that $n = 4$. What if $c_4 \neq 0$? Then we have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}_m$, and this rearranges to give

$$\mathbf{v}_4 = \left(-\frac{c_1}{c_4}\right)\mathbf{v}_1 + \left(-\frac{c_2}{c_4}\right)\mathbf{v}_2 + \left(-\frac{c_3}{c_4}\right)\mathbf{b}_3 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3).$$

That contradicts the technical condition that we're assuming.

Now what if $c_4 = 0$ but $c_3 \neq 0$? Then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}_3$, and, as above,

$$\mathbf{v}_3 = \left(-\frac{c_1}{c_3}\right)\mathbf{v}_1 + \left(-\frac{c_2}{c_3}\right)\mathbf{v}_2 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2).$$

That's a contradiction.

Do you see what to do if $c_3 = c_4 = 0$ but $c_2 \neq 0$?

2. More generally, let j be the largest index such that $c_j \neq 0$. In the case above, we had $j = 4 = n$, but maybe $j < n$. If $j = 1$, this means that $c_2 = 0, \ldots, x_n = 0$, and then $\mathbf{0}_m = c_1\mathbf{v}_1$. Since $c_1 \neq 0$, we get $\mathbf{v}_1 = \mathbf{0}_m$, a contradiction. Otherwise, if $2 \leq j \leq n$, then we rearrange

$$\mathbf{0}_m = c_1\mathbf{v}_1 + \cdots + c_{j-1}\mathbf{v}_{j-1} + c_j\mathbf{v}_j$$

into

$$\mathbf{v}_j = \left(-\frac{c_1}{c_j}\right)\mathbf{v}_1 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right)\mathbf{v}_{j-1} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}),$$

yet another contradiction. ■

The linear independence lemma allows us to winnow down a list to its essential components from the point of view of spans. The following concept and lemma yield the list analogue of (repeated iterations of) the removal theorem for matrices (Theorem 5.13).

9.5 Problem ($\star$). If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a list in $\mathbb{R}^m$, and if $p \leq n$, then we'll say that a SUBLIST of the original list is a list of the form $\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_p}$, where $1 \leq j_1 < j_2 < \cdots < j_p \leq n$. This is a painful definition, so here's an example. Start with the list $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. Then $\mathbf{v}_2$, $\mathbf{v}_4$ is a sublist, as is the list $\mathbf{v}_1$ with one entry, but the list $\mathbf{v}_1$, $\mathbf{v}_1$, $\mathbf{v}_2$ is not a sublist, nor is $\mathbf{v}_2$, $\mathbf{v}_1$.

(i) Suppose that a list is independent. Prove that any sublist is independent, too. [Hint: *if the list* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *is independent and your sublist is* $\mathbf{v}_1$, $\mathbf{v}_2$, *you want to show that if* $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, *then* $c_1 = c_2 = 0$. *But you only know stuff about linear combinations of the whole list. How can you get* $\mathbf{v}_3$ *to show up in* $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$? *What's the right coefficient to slap on* $\mathbf{v}_3$?]

(ii) Suppose that a list contains a dependent sublist. Prove that the whole list is dependent, too.

(iii) Is every sublist of a dependent list always dependent, too?

9.6 Lemma. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a list in $\mathbb{R}^m$, and suppose there is at least one nonzero vector in the list. Then this list has an independent sublist with the same span: there is a sublist $\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r}$ of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that* $\operatorname{span}(\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r}) = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Proof (of Lemma 9.6). We reduce the list as follows. Let $\mathbf{v}_{j_1}$ be the first nonzero vector in the list. (At least one exists.) So $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j_1}) = \operatorname{span}(\mathbf{v}_{j_1})$. Also, the list $\mathbf{v}_{j_1}$ is independent because $\mathbf{v}_{j_1} \neq \mathbf{0}$.

Let $\mathbf{v}_{j_2}$ be the first vector in the list that is a multiple of $\mathbf{v}_{j_1}$, i.e., $\mathbf{v}_{j_2} \notin \operatorname{span}(\mathbf{v}_{j_1})$. So $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j_2}) = \operatorname{span}(\mathbf{v}_{j_1}, \mathbf{v}_{j_2})$. Also, the list $\mathbf{v}_{j_1}$, $\mathbf{v}_{j_2}$ is independent because $\mathbf{v}_{j_1} \neq \mathbf{0}$ and $\mathbf{v}_{j_2} \notin \operatorname{span}(\mathbf{v}_{j_1})$.

Let $\mathbf{v}_{j_3}$ be the first vector in the list that is not in $\operatorname{span}(\mathbf{v}_{j_1}, \mathbf{v}_{j_2})$. So $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j_3}) = \operatorname{span}(\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3})$. And the list $\mathbf{v}_{j_1}$, $\mathbf{v}_{j_2}$, $\mathbf{v}_{j_3}$ is independent since $\mathbf{v}_{j_3} \notin \operatorname{span}(\mathbf{v}_{j_1}, \mathbf{v}_{j_2})$.

Now turn the crank and keep going: eventually we run out of vectors in the list. More precisely, either $\mathbf{v}_n$ is not in the span of the previous vectors in the list, and we take $j_r = n$, or for some $j_r < n$, all of the vectors $\mathbf{v}_{j_r+1}, \ldots, \mathbf{v}_n$ are in the span of the sublist stopping with $\mathbf{v}_{j_r}$. ■

9.7 Problem (!). Go back to Example 6.1 and think of the columns of the matrix there as the original list (so $n = 6$). Convince yourself that in the notation of Lemma 9.6, the results of Example 6.1 gave $r = 3$, $j_1 = 2$, $j_2 = 5$, and $j_3 = 6$.

Content from Strang's *ILA* 6E. Think once more about the matrices A_1 through A_5 on pp. 20–21. Apply the algorithm in Lemma 9.6 to extract the linearly independent columns that span the column spaces.

9.8 Remark. *Please don't read this.*

(i) *I keep using the word "list," but I never defined it. A list is not quite a set: the list $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_1$ in $\mathbb{R}^3$ has three entries, although the third repeats the first (so the list is what, independent or dependent?). But the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\}$ is the same as $\{\mathbf{e}_1, \mathbf{e}_2\}$, because repeating an element when describing a set doesn't yield a different set. Also, $\{\mathbf{e}_1, \mathbf{e}_2\} = \{\mathbf{e}_2, \mathbf{e}_1\}$, because changing the order of elements when describing a set doesn't change that set. Thus*

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\} = \{\mathbf{e}_1, \mathbf{e}_2\} = \{\mathbf{e}_2, \mathbf{e}_1\}.$$

So the concept of a list should both allow repetition and encode order: when we talk about the columns of the matrix $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \end{bmatrix}$, we want to respect their order:

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \end{bmatrix} \neq \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \end{bmatrix} \neq \begin{bmatrix} \mathbf{e}_2 & \mathbf{e}_1 \end{bmatrix}.$$

(ii) *I think that the most precise (which is not the same as most useful) way to think of a list is as a set of ordered pairs: the list* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is the set* $\{(k, \mathbf{v}_k)\}_{k=1}^n$. *This encodes order* ($\mathbf{v}_1$ *comes before* $\mathbf{v}_2$) *and allows repetition: the list* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_1$ *is the set* $\{(1, \mathbf{e}_1), (2, \mathbf{e}_2), (3, \mathbf{e}_1)\}$, *which is not the same as the list* $\{(1, \mathbf{e}_1), (2, \mathbf{e}_2)\}$ *or the list* $\{(1, \mathbf{e}_2), (2, \mathbf{e}_1)\}$.

(iii) *I would feel more comfortable if I could write a list enclosed in parentheses: instead of talking about the list* $\mathbf{v}_1, \ldots, \mathbf{v}_n$, *I'd talk about the list* $(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{(k, \mathbf{v}_k)\}_{k=1}^n$. *This could lead to a weird overworking of ordered pairs in the case of a list of length* 2: $(\mathbf{v}_1, \mathbf{v}_2) = \{(1, \mathbf{v}_1), (2, \mathbf{v}_2)\}$. *And a list of length* 1 *would be* $(\mathbf{v}_1) = \{(1, \mathbf{v}_1)\}$. *The upshot, though, is that drilling into what it means for two sets to be equal, we have* $(\mathbf{v}_1, \ldots, \mathbf{v}_n) = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ *if and only if* $n = m$ *and* $\mathbf{v}_k = \mathbf{w}_k$ *for* $k = 1, \ldots, n$; *that is, two lists are equal precisely when they have the same length and when the corresponding entries are equal.*

(iv) *This is all very similar to how I talked about vectors as sets of ordered pairs in Remark 2.5 and even more similar to one of the definitions of a matrix that I attempted in Remark 3.5. In fact, the former remark suggested defining a vector* $\mathbf{v} = (v_1, \ldots, v_n)$ *as a list in* $\mathbb{R}$ *(which means that a list of vectors would be a list of lists!), and the latter remark suggested defining the matrix* $A = \begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix} \in \mathbb{R}^{m \times n}$ *as* $A = \{(k, \mathbf{a}_k)\}_{k=1}^n$, *and by the definition of list above, that would have* $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$. *That is, a matrix would just be the list of its columns...and if you've read this remark this far, you're probably wondering why, then, we bothered to talk about lists instead of just sticking with matrices.*

But please do read and attempt this. This problem is a summary of everything we've built so far.

9.9 Problem (!). Let $A \in \mathbb{R}^{m \times n}$. The goal of life is to solve $A\mathbf{x} = \mathbf{b}$ given $\mathbf{b} \in \mathbb{R}^m$. Fill in the blanks below.

(i) We can always solve $A\mathbf{x} = \mathbf{b}$ precisely when $\mathbf{C}(A) =$ [what: $\mathbb{R}^n$, $\mathbb{R}^m$, $\{\mathbf{0}_n\}$, $\{\mathbf{0}_m\}$, or something else entirely?], where

$$\mathbf{C}(A) = \text{[what's the definition?]}$$

is the column space. So for existence of solutions we want $\mathbf{C}(A)$ to be as [large or small?] as possible.

(ii) Matrix-vector multiplication is a linear combination of [what kind of stuff?]. So $\mathbf{C}(A)$ is the set of all linear combinations of [what?]. To describe $\mathbf{C}(A)$ as efficiently as possible, we only want to use [what kind of columns?] If the columns of A are [independent or dependent?] and A is square, then we expect that solutions [will or will not?] always exist.

(iii) If a solution to $A\mathbf{x} = \mathbf{b}$ exists, then it's unique precisely when $\mathbf{N}(A) =$ [what: $\mathbb{R}^n$, $\mathbb{R}^m$, $\{\mathbf{0}_n\}$, $\{\mathbf{0}_m\}$, or something else entirely?], where

$$\mathbf{N}(A) = \text{[what's the definition?]}$$

is the null space. So for uniqueness of solutions we want $\mathbf{N}(A)$ to be as [large or small?] as possible. If the columns of A are [independent or dependent], then solutions, if they exist, [will or won't be?] unique [and is this just a conjecture or did we prove it already?].

The time has come to find actual solutions to $A\mathbf{x} = \mathbf{b}$ at last, not just talk about their properties. (It does make me happy how much we've been able to say and conjecture *without* having formulas for solutions or computational solution procedures.) We already know how to do it, and everything we do will pretty much be a bigger and badder version of the work on our very first problem (1.1). The right way to *encode* that work involves, unsurprisingly, matrices—specifically, a notion of *multiplying matrices*.

We've already seen that matrices are both *static* and *dynamic*. (I like to say this a lot.) They *statically* encode data about linear systems (= life), and they *dynamically* act on vectors to produce new vectors. We will teach matrices how to act on other matrices to produce new—and simpler, and more informative—matrices. This action will be *matrix-matrix multiplication*. And once we know how to multiply matrices, we can reverse-engineer the process and *factor* matrices to reveal even more useful data about matrices. (And everyone sure does want to be a *data* scientist these days.)

Think about the factoring that you've already done in life before linear algebra. You've factored integers into products of powers of primes:

$$12 = 2^2(3).$$

And you've factored polynomials into simpler polynomials:

$$x^2 - 4x + 4 = (x-2)^2.$$

Both kinds of factorizations reveal (potentially) useful information: what the essential components of an integer are, how to find zeros and maybe graph polynomials. If we know how to multiply matrices, perhaps we can factor them so that only the most important information comes out in the factorization.

For example, maybe we could have something like

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},$$

where the first matrix (okay, column vector) is the only column that matters, and the second matrix (okay, row vector) contains the data needed for constructing all of the columns out of this first column. This hinges on defining matrix products in such a way that the definition yields the equality above. How do we do it?

Content from Strang's *ILA* 6E. The real goal is to answer the questions posed at the end of p. 22. We'll get there.

So, what is a "good" definition of matrix multiplication? Starting small might help: let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Then we know how to compute $A\mathbf{b} \in \mathbb{R}^m$. Let's do something a little weird and think about the matrix $\begin{bmatrix}\mathbf{b}\end{bmatrix} \in \mathbb{R}^{n \times 1}$. If the product $A\begin{bmatrix}\mathbf{b}\end{bmatrix}$ has any meaning *as a matrix*, it probably should be $\begin{bmatrix}A\mathbf{b}\end{bmatrix} \in \mathbb{R}^{m \times 1}$. So the matrix-matrix product $A\begin{bmatrix}\mathbf{b}\end{bmatrix}$ is just the matrix whose only column is the vector $A\mathbf{b}$.

What if the second factor has more columns? Let $B = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_p\end{bmatrix}$. (I'm intentionally not telling you how many rows B has just yet.) If we want to compute AB and continue the pattern above, then we might want to multiply each column of B by A. But if $A \in \mathbb{R}^{m \times n}$, then each column of B needs to be in $\mathbb{R}^n$ so that we can do that multiplication. And the matrix-vector product A times column of B yields a vector in $\mathbb{R}^m$. We don't care how many columns of B there are, so p can be arbitrary. Thus $AB \in \mathbb{R}^{m \times p}$.

9.10 Definition. *Let $A \in \mathbb{R}^{m \times n}$ and $B = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_p\end{bmatrix} \in \mathbb{R}^{n \times p}$. The* **MATRIX PRODUCT** *AB is*

$$AB := \begin{bmatrix}A\mathbf{b}_1 & \cdots & A\mathbf{b}_p\end{bmatrix} \in \mathbb{R}^{m \times p}.$$

Content from Strang's *ILA* 6E. Matrix multiplication is defined in equation (1) on p. 27. Work through the examples on that page and p. 28, noting the appearance of the dot product.

Here is the first reason for this definition and the restriction on the sizes of A and B: we want this definition to return the usual definition of matrix-vector multiplication when B is a column vector. Honestly, this probably isn't the most compelling motivation for that definition. Trust me for a bit that this is a good thing to do and let's just practice some computations—I don't want to overwhelm you just yet with too much motivation when the definition is still so new. The reveal will be worth the wait.

Day 10: Wednesday, September 10.

10.1 Example. **(i)** Let

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Both A, $B \in \mathbb{R}^{2 \times 2}$, so the product AB is defined and $AB \in \mathbb{R}^{2 \times 2}$ as well.

We compute

$$A\mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$A\mathbf{b}_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Feel free to do this with the original definition of matrix-vector multiplication or dot products. Thus

$$AB = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2\end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}.$$

(ii) Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Since $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 3}$, the product AB is defined and $AB \in \mathbb{R}^{2\times 3}$.

We compute

$$A\mathbf{b}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

$$A\mathbf{b}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

and

$$A\mathbf{b}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

Thus

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}.$$

(iii) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Since $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 2}$, the product AB is defined and $AB \in \mathbb{R}^{3\times 2}$.

We compute

$$A\mathbf{b}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

and

$$A\mathbf{b}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 18 \end{bmatrix}.$$

Thus

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 10 \\ 9 & 18 \end{bmatrix}.$$

10.2 Problem (!). Describe in words the effects of computing the three products in the previous example. [Hint: *for part (i), think about subtraction.*] Compare your response to patterns that you observed in Problem 3.15.

Coming out of these examples is a nice fact that helps when computing "small" products AB by hand.

10.3 Theorem. *Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$. Then the (i, j)-entry of AB is the dot product of row i of A (considered as a column vector in $\mathbb{R}^n$) with column j of B.*

Proof. We know what AB is at the level of columns: column j of AB is the matrix-vector product of A with column j of B. So the entry in row i of column j of AB is the dot product of row i of A (considered as a column vector in $\mathbb{R}^n$) with column j of B. ■

10.4 Problem (!). Redo the matrix products in Example 10.1 using dot products.

10.5 Problem ($\star$). Suppose that A and B are matrices such that the product AB is defined.

(i) If a whole row of A has all zero entries, what do you know about AB?

(ii) If a whole column of B has all zero entries, what do you know about AB?

10.6 Problem ($\star$). Suppose that A and B are matrices such that the product AB is defined.

(i) Prove that if $\mathbf{v} \in \mathbf{C}(AB)$, then $\mathbf{v} \in \mathbf{C}(A)$.

(ii) Give an example of A and B for which $\mathbf{C}(AB) \neq \mathbf{C}(A)$.

Here is something less nice. We expect that the order in which we multiply real numbers doesn't matter: if $x, y \in \mathbb{R}$, then $xy = yx$. Not so for matrices: you should expect $AB \neq BA$ in general.

10.7 Problem ($\star$). **(i)** Explain why even if the matrix product AB is defined, the product BA may not be defined. What do you need to know about A and B for both products AB and BA to be defined?

(ii) Use the matrices A and B from part (i) of Example 10.1 to show that we may have $AB \neq BA$ even when these products are both defined.

Is this that big a deal? Is our definition of matrix multiplication wrong because it doesn't commute ($AB \neq BA$, typically, even when both products are defined)? I say no. When we see that matrix multiplication encodes the right "dynamic" action, we won't care about the loss of commutativity. Heck, most dynamic actions in life don't "commute"—order matters when putting on shoes vs. socks.

Content from Strang's *ILA* 6E. Check the multiplication in equation (6) on p. 28 for further reinforcement that $AB \neq BA$ in general. Answer the question at the bottom of the page.

10.8 Problem ($\star$). The noncommutativity of matrix multiplication ($AB \neq BA$ in general, even when both products are defined) was probably something of a surprise. After all, for a, $b \in \mathbb{R}$, we always have $ab = ba$. Here's another surprise: we can have AB equal the "zero matrix" (= the matrix whose entries are all 0) even when both A and B are nonzero matrices (i.e., A and B each have at least one entry that's not 0).

(i) Cook up an example of this yourself by working with 2×2 matrices. [Hint: *you can do this with diagonal matrices if you play the entries off each other carefully.*]

(ii) Now we're going to be very general. Suppose that $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ with AB equal to the zero matrix. Explain why if $\mathbf{v} \in \mathbf{C}(B)$, then $\mathbf{v} \in \mathbf{N}(A)$ as well.

10.9 Example. Let

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 \end{bmatrix}.$$

The columns of B are vectors in $\mathbb{R}^1$, and of course we usually think of these as real numbers: $\mathbb{R} = \mathbb{R}^1 (= \mathbb{R}^{1\times 1})$. But here it is helpful, if silly, to keep the vector point of view. That is, we think that

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}, \quad \text{where} \quad \mathbf{b}_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 3 \end{bmatrix}.$$

Now we compute AB by multiplying A against the columns of B. Of course we are going to have

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 & A\mathbf{b}_4 \end{bmatrix},$$

so what are these matrix-vector products? When in doubt, suck it up and go back to the definition: $A\mathbf{b}_j$ is the linear combination of the columns of A weighted by the entries of $\mathbf{b}_j$. But we are being too generous with the plurals. Since A has only one column and $\mathbf{b}_j$ has only one entry, $A\mathbf{b}_j$ is the scalar multiplication of that single column of A with that one entry of $\mathbf{b}_j$.

We start with

$$A\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The second equality is the definition of matrix-vector multiplication $A\mathbf{v}$ for $A \in \mathbb{R}^{n\times 1}$ and $\mathbf{v} \in \mathbb{R}^1$: it's a linear combination of one vector. Pretty silly, I know.

Let's do it again in gory detail:

$$A\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

I think you'll agree that

$$A\mathbf{b}_3 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \text{and} \quad A\mathbf{b}_4 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

All together,

$$AB = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 0 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & 3\\2 & 0 & 4 & 6\\3 & 0 & 6 & 9\end{bmatrix}.$$

Just look at that matrix: it's so redundant! Every column is a multiple of the first, so $\text{rank}(AB) = 1$, and the entries of B tell us how to do that multiplication. Far better to keep the matrix factored as AB so that we can see the important data: the one column in A and the multipliers in B.

This pattern generalizes nicely: for any $\mathbf{a} \in \mathbb{R}^m$ and $c_1, \ldots, c_{n-1} \in \mathbb{R}$

$$\begin{bmatrix}\mathbf{a} & c_1\mathbf{a} & \cdots & c_{n-1}\mathbf{a}\end{bmatrix} = \begin{bmatrix}\mathbf{a}\end{bmatrix}\begin{bmatrix}1 & c_1 & \cdots & c_{n-1}\end{bmatrix}. \tag{10.1}$$

10.10 Problem (!). Stare at this equality until you believe it. Maybe write something, too.

Content from Strang's *ILA* 6E. Read and work through all of the calculations on pp. 29–30 under "Rank One Matrices and $A = CR$."

The factorization (10.1) is the sort of factorization that we desire. It takes a matrix with a lot of redundant data and breaks it up into the important chunks. The first factor in (10.1) contains the only independent column of the product, and the second factor tells you how to build the other columns out of that one and only one necessary column. Can we get something like this for a matrix with more than one independent column and for which the "building" might be more complicated?

This is one of those times where it's helpful to know where you're going before you get there.

10.11 Example. We compute

$$\begin{bmatrix}1 & 0\\0 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\end{bmatrix}$$

and we do so by thinking very intentionally about columns this time (not dot products, please, even though that's faster):

$$\begin{bmatrix}1 & 0\\0 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\2\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix},$$

$$\begin{bmatrix}1 & 0\\0 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\2\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

We've seen this matrix before. We know that, going from left to right, its first two columns are linearly independent, and its third column is a linear combination of the two, thus $\text{rank}(AB) = 2$. The product on the left makes that explicit: the first factor in the product contains the linearly independent columns, and the second factor tells you how to put those columns together.

The structure of the second factor is interesting. The first two columns of the identity matrix $I_2 \in \mathbb{R}^{2\times 2}$ (Problem 4.9) are there, and I will block them off as follows:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right].$$

The matrix on the right is an example of a **PARTITIONED MATRIX**. This is totally euphemistic and just a nice way of breaking matrices into "submatrices" so that you see more meaningful patterns in the data.

Let me put

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} I_2 & F \end{bmatrix}.$$

This is a **BLOCK MATRIX**: a matrix whose entries are other matrices. Again, totally euphemistic, just a convenient way of seeing the most important parts from a bird's-eye perspective.

The factorization above is the dream. We want to take a matrix $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$ and write it as a product $A = CR$. The columns of the matrix C should be independent, and they should be columns of A, as long an independent list of columns of A as you can get. In particular, we want $C \in \mathbb{R}^{m\times r}$. We want the matrix $R \in \mathbb{R}^{r\times n}$ to tell us how to build the columns of A out of linear combinations of the columns of C. So C times the jth column of R should give us the jth column of A. Think of C as the "ingredients" factor and R as the "recipe" factor: the recipe tells you how to get a meal out of the ingredients.

Content from Strang's *ILA* 6E. Read "C Contains the First r Independent Columns of A" on p. 30 and "Matrix Multiplication C times R on pp. 31–32. Check the calculations in Example 2, equation (10), equation (11), and the box on p. 32. Also jump ahead to Example 5 on pp. 34–35 (you don't have to read about that "columns × rows" way of multiplying matrices). For yet another example, go back to "Matrix Multiplication $A = CR$ on p. vii. You do not have to feel that you could see these CR-factorizations immediately; you should agree that the given matrix multiplication works out.

Unfortunately, we still do not have the tools to do prove that such a factorization exists or to develop an algorithm for computing it "reasonably" and effectively for all but the most trivial and obvious matrices. This is just like we don't have efficient tools for checking dependence or independence of vectors. Let me at least state this factorization as a dream.

10.12 Conjecture. *Let $A \in \mathbb{R}^{m\times n}$ have rank r. Then there exist matrices $C \in \mathbb{R}^{m\times r}$ and $R \in \mathbb{R}^{r\times n}$ such that $A = CR$. In particular, the columns of C are the first r independent columns of A ("first" meaning the first that appear in order from left to right) and* $\operatorname{rank}(C) = r$. *From Conjecture 8.18, we also expect that A and C have the same column spaces.*

Content from Strang's *ILA* 6E. If you're curious, read pp. 32–33 to learn more about computing R. Feel free to skip that for now. We will revisit this in extensive detail in the future.

Ideally, we could write

$$R = \begin{bmatrix} I_r & F \end{bmatrix} \tag{10.2}$$

with r as the $r \times r$ identity matrix. If $r = n$ and all of the columns of A are independent, then we just have $R = I_n$, and there is no F block present, since no column is a linear combination of the others in this case. You should think of that F block being possibly fictitious. Then because we multiply CR by doing the matrix-vector product of C with each column of R, we get

$$CR = C\begin{bmatrix} I_r & F \end{bmatrix} = \begin{bmatrix} CI_r & CF \end{bmatrix} = \begin{bmatrix} C & CF. \end{bmatrix}$$

10.13 Problem (!). If $A \in \mathbb{R}^{m\times n}$ can be written as $A = CR$ with $C \in \mathbb{R}^{m\times r}$ and R as in (10.2), and if $r < n$, what are the dimensions of that F? In particular, is the product CF defined?

However, R doesn't always have the structure (10.2). The snag can be that the first r columns of A may not be an independent list, and maybe the list of the first r independent columns is "interspersed" throughout A.

10.14 Problem (!). Let

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Compute $A := CR$ and comment on the structure of A. Specifically, are the columns of C the first two columns of A?

Content from Strang's *ILA* 6E. This is what Strang means by the parenthetical remark "in correct order" on p. in the displayed equations after "$A = CR$ becomes."

To get around this, the reality is that we'll have to write

$$R = \begin{bmatrix} I_r & F \end{bmatrix} P$$

for a "permutation" matrix P that will reshuffle the columns appropriately. We'll get around to talking about permutation matrices later, but this will mean that A would have the factorization

$$A = C \begin{bmatrix} I_r & F \end{bmatrix} P.$$

Is that allowed? Can we multiply three matrices at once? Will it matter which two matrices we multiply first?

Nope!

Day 11: Friday, September 12.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Upper-triangular matrix (N)

11.1 Theorem. *Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{p \times q}$. Then $(AB)C = A(BC)$.*

This theorem says that the order in which you *group* matrices during multiplication doesn't matter: matrix multiplication is **ASSOCIATIVE**. Thus we just write ABC and eliminate the parentheses. The *order* still totally matters, and we should not expect $ABC = ACB$ or some nonsense like that.

Content from Strang's *ILA* 6E. Read "AB times $C = A$ times BC" on p. 29.

11.2 Problem (!). Let

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Compute

$$C \begin{bmatrix} I_2 & F \end{bmatrix} P$$

and compare your result to Problem 10.14.

The proof of Theorem 11.1 is largely a thankless exercise in juggling parentheses, so I will leave that for you to suss out over the course of two problems. You're not going to believe this now, but the associative property of matrix multiplication is a major indication of why this definition of matrix multiplication is the *right* definition. You can thank me later; I'll wait.

11.3 Problem ($\star$). Let $A \in \mathbb{R}^{m \times n}$, $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix} \in \mathbb{R}^{n \times p}$, and $\mathbf{v} = (v_1, \ldots, v_p) \in \mathbb{R}^p$. Explain why each of the following four equalities is true:

$$\begin{aligned} A(B\mathbf{v}) &= A(v_1\mathbf{b}_1 + \cdots + v_p\mathbf{b}_p) \\ &= v_1 A\mathbf{b}_1 + \cdots + v_p A\mathbf{b}_p \text{ [Hint: } \textit{Problem 3.12}\text{]} \\ &= \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{bmatrix} \mathbf{v} \\ &= (AB)\mathbf{v}. \end{aligned}$$

11.4 Problem (+). **(i)** Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{p \times q}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_q$ be the standard basis vectors for $\mathbb{R}^q$ (Problem 4.2). Explain why to prove that $(AB)C = A(BC)$, it suffices to show that $((AB)C)\mathbf{e}_j = (A(BC))\mathbf{e}_j$.

(ii) Use only the fact that $(DE)\mathbf{v} = D(E\mathbf{v})$ for matrices D, E and vectors $\mathbf{v}$ for which both sides of that equality are defined (as proved in Problem 11.3), justify each equality below:

$$((AB)C)\mathbf{e}_j = (AB)(C\mathbf{e}_j) = A(B(C\mathbf{e}_j)) = A((BC)\mathbf{e}_j) = (A(BC))\mathbf{e}_j.$$

All this being said, we still have no idea of how to compute that "CR-factorization" of a matrix unless we are really lucky and see the dependence relations among the columns from the get-go. There is quite a systematic way of doing that, and it is related to proving Conjectures 6.3 and 7.8, and to developing an explicit algorithm for solving $A\mathbf{x} = \mathbf{b}$ when we can actually solve it. That is, all of our dreams will come true through very related techniques.

Content from Strang's *ILA* 6E. At this point we have learned all the matrix-vector mechanics that we need to actually solve linear systems (and to understand our failure when we can't solve them). Just to be safe, read "Review of AB on p. 29 and make sure you have no doubts there. Then read "Thoughts on Chapter 1" on p. 38 for a summary of everything that we've done and a hint of what's to come.

We are almost ready to start solving linear systems. It will be helpful to know where we're going before we get there, so we'll pause (briefly) on matrix manipulations and look at three linear systems, each of which is in a very nice form, and which together illustrate the scope of possibilities for solution behavior to $A\mathbf{x} = \mathbf{b}$.

11.5 Example. **(i)** We consider

$$\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}.$$

As a linear system, this reads

$$\begin{cases} x_1 & - & 2x_2 & = & 1 \\ & & 8x_2 & = & 8 \end{cases}$$

Look familiar? This was our very first problem!

Of course, we "back-solve" or "back-substitute" to get first $x_2 = 1$ and then $x_1 - 2 = 1$, so $x_1 = 3$. The problem has only one solution:

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(ii) We consider

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}.$$

Write it out, and don't laugh:

$$\begin{cases} x_1 & - & 2x_2 & = & 1 \\ & & 0 & = & 8. \end{cases}$$

Of course this system has no solution, because $0 \neq 8$.

(iii) We consider

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Write it out, keep laughing:

$$\begin{cases} x_1 & - & 2x_2 & = & 1 \\ & & 0 & = & 0. \end{cases}$$

There is really not much to do, since the second equation is both true and doesn't involve unknowns. There's not much more we can do with the first equation, since we don't know the value for x_2.

Here is the right, if not obvious, thing to do: rewrite $x_1 = 1 + 2x_2$. This says that every choice of $x_2 \in \mathbb{R}$ gives x_1 via this formula. You can pick any x_2 that you want, so there are infinitely many solutions. At the level of vectors, we could write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Every value of x_2 gives a different solution, and so this problem has infinitely many solutions.

Content from Strang's *ILA* 6E. Work through the three systems on p. 40, which have the same properties as the three above.

The three examples above are paradigmatic in the sense that a linear system has only one of three general solution "behaviors": only one solution, no solution, or infinitely many solutions. This is actually very easy to prove using matrix notation—which is why we use that notation, to make our lives easier. But the other thing to take from this example is that the *structure* of the linear systems was very nice: all of the matrices were "upper-triangular" in the sense that their entries were 0 below the diagonal. This made back-solving/substituting very, very easy.

Content from Strang's *ILA* 6E. For a very broad overview of where we're going, read p. 39. It's okay if you don't understand everything on a first pass. Then read the first three paragraphs on p. 83.

We formalize the situations of Example 11.5. The method of proof here has a lot in common with Theorem 7.2, so you should reread that theorem and the discussion preceding it.

11.6 Theorem. *Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then one, and only one, of the following is true.*

(i) *There exists a unique solution $\mathbf{x} \in \mathbb{R}^n$ to the problem $A\mathbf{x} = \mathbf{b}$. That is, we can solve the problem, and if $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$ for some $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^n$, then $\mathbf{x}_1 = \mathbf{x}_2$.*

(ii) *There is no solution to the problem $A\mathbf{x} = \mathbf{b}$. That is, $A\mathbf{x} \neq \mathbf{b}$ for every $\mathbf{x} \in \mathbb{R}^n$.*

(iii) *There are infinitely many solutions to $A\mathbf{x} = \mathbf{b}$.*

Proof. We want one, and only, one, of three possibilities to hold. One way for this to work out is to assume that the first two are false and then show why the third must be true. So, assume that $A\mathbf{x} = \mathbf{b}$ has a solution (so the second part is false) but this solution is not unique (so the first part is false). That is, there are $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^n$ such that $A\mathbf{x}_1 = \mathbf{b}$, $A\mathbf{x}_2 = \mathbf{b}$, and $\mathbf{x}_1 \neq \mathbf{x}_2$.

Our goal is to find infinitely many different $\mathbf{x} \in \mathbb{R}^n$ that satisfy $A\mathbf{x} = \mathbf{b}$. Here is the trick. Like most tricks in math, it may not be obvious at first glance, so you should reread this proof until it becomes obvious.

Put $\mathbf{z} := \mathbf{x}_1 - \mathbf{x}_2$. Then $\mathbf{z} \neq \mathbf{0}_n$, since $\mathbf{x}_1 \neq \mathbf{x}_2$. And

$$A\mathbf{z} = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}_m.$$

The second equality is the linearity of matrix-vector multiplication (Problem 3.12).

Now let $c \in \mathbb{R}$ be arbitrary and $\mathbf{x} = \mathbf{x}_1 + c\mathbf{z}$. Then

$$A\mathbf{x} = A(\mathbf{x}_1 + c\mathbf{z}) = A\mathbf{x}_1 + A(c\mathbf{z}) = A\mathbf{x}_1 + c(A\mathbf{z}) = \mathbf{b} + c\mathbf{0}_m = \mathbf{b} + \mathbf{0}_m = \mathbf{b}.$$

The second and third equalities are, again, the linearity of matrix-vector multiplication. Make sure you understand why all of the other equalities are true.

So why does this give infinitely many solutions? Maybe we should have put $\mathbf{x}_c := \mathbf{x}_1 + c\mathbf{z}$ instead to emphasize the dependence of $\mathbf{x}_c$ on the parameter c. (I guess there might be conflicts of notation with $c = 1$ and $c = 2$?) The point is that each different $c \in \mathbb{R}$ generates a different $\mathbf{x}_1 + c\mathbf{z} \in \mathbb{R}^n$: you can, and should, check that if $c_1 \neq c_2$, then $\mathbf{x}_1 + c_1\mathbf{z} \neq \mathbf{x}_1 + c_2\mathbf{z}$. ■

11.7 Problem (!). By considering the vector $\mathbf{z} = (2, -1)$, explain how the proof of this theorem generalizes the situation in part (iii) of Example 11.5.

Content from Strang's *ILA* 6E. After you do the problem above, reread Example 3 on p. 40. The vector that Strang calls $\boldsymbol{X}$ is what I call $\mathbf{z}$.

The time has come to systematically solve linear systems! We go all the way back to our very first example, in which we showed that

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \iff \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}. \tag{11.1}$$

The latter system was easy to solve with "back-substitution":

$$\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \iff \begin{cases} x_1 - 2x_2 = 1 \\ \phantom{x_1 - {}} 8x_2 = 8 \end{cases} \iff \begin{cases} x_1 - 2x_2 = 1 \\ \phantom{x_1 - 2{}} x_2 = 1 \end{cases}$$

$$\iff \begin{cases} x_1 - 2 = 1 \\ \phantom{x_1 - {}} x_2 = 1 \end{cases} \iff \begin{cases} x_1 = 3 \\ x_2 = 1. \end{cases}$$

What's up with the $\iff$ in (11.1) Why are these two problems *equivalent* in the sense that $\mathbf{x}$ solves one of them precisely when it solves the other? That's what we need to tease out more precisely and comprehensively here.

We start by identifying the nice form of the second problem.

11.8 Definition. *A matrix $U \in \mathbb{R}^{m \times m}$ is* **UPPER-TRIANGULAR** *if all of the entries of U below the diagonal are 0. That is, the (i, j)-entry of U is 0 when $i > j$.*

11.9 Example. Each matrix below is upper-triangular:

$$\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Content from Strang's *ILA* 6E. For a longer example of why upper-triangular matrices are nice for back-substitution, read p. 41 through the "Special note" in the box. I expect that you are comfortable with this back-substitution method for solving linear systems, and I will not do examples with it here.

How do you do this? How do you "convert" $A \in \mathbb{R}^{m\times m}$ into an upper-triangular matrix U so that we have the equivalence of the problems

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad U\mathbf{x} = \mathbf{c}$$

for some appropriate $\mathbf{c}$? The point is that the arrows go *both* ways:

$$A\mathbf{x} = \mathbf{b} \Longrightarrow U\mathbf{x} = \mathbf{c} \quad \text{and} \quad U\mathbf{x} = \mathbf{c} \Longrightarrow A\mathbf{x} = \mathbf{b}.$$

Having an arrow go one way in math doesn't always mean it goes the other way.

The good news is that we already know how to do this. It's all contained in the manipulations that we did on our very first problem at the level of equations and variables. The big idea was subtracting a multiple of one equation from another. We can do all of this at the level of matrices (and cut out the variables) by subtracting a multiple of one row of a matrix from another. And the bigger idea is that we'll encode this via matrix multiplication.

Specifically, we'll compress the operations of *Gaussian elimination* into a matrix E, so that EA is upper-triangular. Let $U := EA$ and $\mathbf{c} := E\mathbf{b}$. Then if $A\mathbf{x} = \mathbf{b}$, we can apply E to both sides to get $EA\mathbf{x} = E\mathbf{b}$, thus $U\mathbf{x} = \mathbf{c}$. (So any solution to our original problem solves this new problem.) Next, we'll show that E is *invertible*: there is a matrix E^{-1} such that $E^{-1}E = I_n$. So if you know $U\mathbf{x} = \mathbf{c}$, then $E^{-1}U\mathbf{x} = E^{-1}\mathbf{c}$. From what U and $\mathbf{c}$ are, this says $E^{-1}EA\mathbf{x} = E^{-1}E\mathbf{b}$, and so $A\mathbf{x} = \mathbf{b}$. (So any solution to the new problem solves the original problem—the thing we actually care about.) And last, why is this new problem $U\mathbf{x} = \mathbf{c}$ so nice? Because U is upper-triangular, which (when the diagonal entries of U are nonzero) permits us to solve $U\mathbf{x} = \mathbf{c}$ by *back-substitution*.

Day 12: Monday, September 15.

Specifically, to turn

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

into

$$U = \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix},$$

we want to subtract 3 times the first row of A from the second row of A. The revolution of linear algebra is that we can encode this via matrix multiplication. Whenever we want to "do something" in this class, you should ask yourself how we can accomplish this by multiplying by a suitable matrix.

So, what matrix E satisfies

$$EA = U?$$

At the very least we need $E \in \mathbb{R}^{m\times 2}$ since $A \in \mathbb{R}^{2\times 2}$. And we really want $m = 2$ since $EA = U \in \mathbb{R}^{2\times 2}$. So, $E \in \mathbb{R}^{2\times 2}$.

Here is where it is wise to think about matrix multiplication as E times the columns of A. What is E doing to each column? We want

$$E\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix}. \tag{12.1}$$

How can we view the vector on the right as a linear combination with weights given by v_1 and v_2? The vectors in that linear combination will be the columns of E.

So, work backwards:

$$\begin{bmatrix} v_1 \\ v_2 - 3v_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ -3v_1 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If we put

$$E := \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix},$$

then we have the desired equality (12.1).

12.1 Problem (!). Check that. Then compute $EA = U$ with A and U as above.

Here is how we're thinking. Assume $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (1, 11)$. Then $EA\mathbf{x} = E\mathbf{b}$. Compute $EA = U$ with U as above and $E\mathbf{b} = (1, 8) =: \mathbf{c}$. Then solve $U\mathbf{x} = \mathbf{c}$. That *should* give a solution to the original problem $A\mathbf{x} = \mathbf{b}$, and we can always plug it in and check that it does.

Going in reverse requires a little more thought. Why does solving $EA\mathbf{x} = E\mathbf{b}$ give a solution to $A\mathbf{x} = \mathbf{b}$? It would be nice if we could "cancel" the factor of E from both sides. We can, and that's called inverting a matrix, and we'll do that nice and abstractly soon.

12.2 Problem ($\star$). In fact, you can do that right now. Put

$$F := \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

First explain in words the effect of multiplying $F\mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^2$. Then check that $FE\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$. Finally, suppose that $EA\mathbf{x} = E\mathbf{b}$, multiply both sides by F, and explain why $A\mathbf{x} = \mathbf{b}$.

It feels like we're doing "elimination" twice: we multiplied EA and then $E\mathbf{b}$ separately. We can combine all of the data of our problem into one "augmented" matrix: put

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right].$$

I like to draw a line separating the $\mathbf{b}$ column when I'm working with actual numbers. Then do one matrix multiplication:

$$E \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} EA & E\mathbf{b} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right] = \begin{bmatrix} U & \mathbf{c} \end{bmatrix}.$$

From here, solve $U\mathbf{x} = \mathbf{c}$ by back-substitution.

I'm going to tell you the path forward, even if it isn't obvious right now. Here is the cartoon for $A \in \mathbb{R}^{3\times3}$. We want to turn A into an upper-triangular matrix U by multiplying

A by the "right" matrices. In the "ideal" case, at the level of rows, we are going to subtract multiples of row 1 to create 0 entries in rows 2 and below of column 1. Specifically, the multiples will be based on the $(1,1)$-entry, which for now we hope is nonzero.

So we have the conversion

$$\begin{bmatrix} \circledast & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \to \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

I've written the changed entries in blue. Now subtract a multiple of the second row from the third row to create zeros in the second column below the second row. Again, in the "ideal" case, the multiple will be based on the $(2,2)$-entry, which we should hope is nonzero:

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \to \begin{bmatrix} * & * & * \\ 0 & \circledast & * \\ 0 & 0 & * \end{bmatrix}.$$

Again, the blue entries are new or changed. Because both the second and third rows had 0 in their first column, subtracting a multiple of the second row from the third row did not destroy that 0 in the first column of the third row. This is the nice upper-triangular structure that is ideal for back-solving.

How do we accomplish this multiplication? I am going to tell you the answer, which generalizes all our work with E above. Let $A \in \mathbb{R}^{m \times n}$ and $\ell \in \mathbb{R}$. To subtract ℓ times row j of A from row i of A (with $i \neq j$), multiply A by the **ELIMINATION MATRIX** $E_{ij} \in \mathbb{R}^{m \times m}$ whose entries are 1 on the diagonal, $-\ell$ in the (i,j)-position, and 0 elsewhere. So, E_{ij} is "almost" the identity matrix, except for the (i,j)-entry.

12.3 Problem (!). Prove that this formula for E_{ij} works by computing the following very special case and explaining the effect in words:

$$E_{21}\mathbf{v}, \qquad \text{where} \qquad E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -\ell & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{v} := \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Then spend at least five minutes thinking about how using dot products could help you prove the more general result stated in the paragraph above this problem.

We do an example in glacially slow detail.

12.4 Example. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}.$$

We want to multiply A by "elimination" matrices like the 2×2 situation above so that 0 appears in the second and third rows of the first column. To get 0 in the $(2,1)$-entry, we should subtract 2 times the first row from the second. The matrix

$$E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

accomplishes this, and here is what we get:

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 8 & 7 & 9 \end{bmatrix}.$$

I'll use the idiosyncratic notation

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix} \xrightarrow[E_{21}]{\mathsf{R2} \mapsto \mathsf{R2}-2\times\mathsf{R1}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 8 & 7 & 9 \end{bmatrix}, \qquad E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

to represent this. Saying $\mathsf{R2} \mapsto \mathsf{R2} - 2 \times \mathsf{R1}$ means that row 2 is replaced by row 2 minus 2 times row 1.

Now we want to clear out the $(3,1)$-entry, and we can do this by subtracting 4 times row 1 from row 3. So, we multiply

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 8 & 7 & 9 \end{bmatrix} \xrightarrow[E_{31}]{\mathsf{R3} \mapsto \mathsf{R3}-4\times\mathsf{R1}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix}, \qquad E_{31} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

Finally, we want to clear out the 3 in the $(3,2)$-entry:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix} \xrightarrow[E_{32}]{\mathsf{R3} \mapsto \mathsf{R3}-3\times\mathsf{R2}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad E_{32} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

We're done! Let's abbreviate $E = E_{32}E_{31}E_{21}$. The product

$$EA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} =: U$$

is upper-triangular. If we wanted to solve $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^3$, it would suffice to solve $U\mathbf{x} = E\mathbf{b}$ instead.

Content from Strang's *ILA* 6E. Read and work through everything on p. 42 right now. This is hugely important. Then read p. 45 up to and including equation (7). This is another example of elimination. Last, read all of p. 49 (but don't worry about inverses for now).

The process in Example 12.4 is called **Gaussian elimination**, and we're going to use it (and sometimes do it) a lot. We start with a matrix A, multiply A by a bunch of matrices that we collect in one product E, and find that $U = EA$ has zeros in some very nice places. This is a leitmotif of our subject. Per the magisterial *Numerical Linear Algebra* by Trefethen & Bau, "The algorithms of numerical linear algebra are mainly built upon one technique used over and over again: putting zeros into matrices" (p. 191).

We are going to focus on "reducing" A to an upper-triangular form, and I am going to leave practicing with back-substitution to you. It's mostly just a longer version of part (i) of Example 11.5.

12.5 Problem (!). Use the method of Example 12.4 to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (0, 1, 5)$. I suggest that you do the three elimination steps on the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$. Then you'll get $E\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} U & \mathbf{c} \end{bmatrix}$, with U as we found above and $\mathbf{c} = E\mathbf{b}$. (I don't suggest multiplying out E and then computing $E\mathbf{b}$ from that.) Now solve $U\mathbf{x} = \mathbf{c}$.

12.6 Problem (⋆). We prefer upper-triangular matrices, in part for consistency, but "lower-triangular" matrices can be equally nice. Solve

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

12.7 Problem (+). We usually expect that matrix multiplication is not commutative. However, sometimes it is.

(i) Let ℓ_1, $\ell_2 \in \mathbb{R}$ and put

$$E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -\ell_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{31} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell_2 & 0 & 1 \end{bmatrix}.$$

Explain in words what E_{21} and E_{31} "do" (i.e., what is the effect of multiplying $E_{21}\mathbf{v}$ and $E_{31}\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^3$?). Then explain why you think this means that $E_{21}E_{31} = E_{31}E_{21}$. Do the actual matrix multiplication to convince yourself that this is true.

(ii) Let $\ell_3 \in \mathbb{R}$ and

$$E_{32} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\ell_3 & 1 \end{bmatrix}.$$

Without doing any calculations, explain why you should expect E_{31} and E_{32} *not* to commute. Then do the multiplication to check $E_{31}E_{32} \neq E_{32}E_{31}$.

12.8 Remark. *Associativity of matrix multiplication (Theorem 11.1) is key to how matrices act.*

We defined the matrix product AB in such a way that $(AB)\mathbf{v} = A(B\mathbf{v})$, at least when A, B, and $\mathbf{v}$ are sized appropriately so that all of the products involved are defined (this was Problem 11.3). If we think about matrices as dynamic objects, we could have the matrix B act on the vector $\mathbf{v}$ first to get the vector $B\mathbf{v}$, and then we could have the matrix A act on the vector $B\mathbf{v}$ to get the vector $A(B\mathbf{v})$. Or we could have the matrix A act on the matrix B all at once, and we get the new matrix AB. Then the matrix AB acts on the vector $\mathbf{v}$ to get the vector $(AB)\mathbf{v}$. Our choice of the definition for the symbol AB ensures that the two vectors $(AB)\mathbf{v}$ and $A(B\mathbf{v})$ are the same.

At the level of Gaussian elimination, this allows us to collect all of the elimination matrices (and their forthcoming relatives) into one big matrix that acts on A all at once. Associativity of matrix multiplication ensures that the order in which we group those factors doesn't matter.

Elimination can break down in two ways. The first is not so bad and just requires a new kind of matrix to correct things. The second is worse and will prevent us from solving the linear system.

12.9 Example. What if at the jth step of elimination, the (j, j)-entry is 0, but an entry further down in column j is not 0? All hope is not lost. Consider

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 3 \\ 8 & 9 & 9 \end{bmatrix} \xrightarrow[E_{21}]{\text{R2} \mapsto \text{R2}-2\times\text{R1}} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \\ 8 & 9 & 9 \end{bmatrix} \xrightarrow[E_{31}]{\text{R3} \mapsto \text{R3}-4\times\text{R1}} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix}.$$

The matrices E_{21} and E_{31} are the same as before in Example 12.4, so I didn't write them out again.

The problem is that the $(2, 2)$-entry is now 0. We can't use that to eliminate the 3 in the $(3, 2)$-entry. But if we could "flip" rows 2 and 3, we'd be done. (This is totally legitimate: you can interchange the order of equations in a system of equations and not change the solution structure at all.) If only there were a matrix $P \in \mathbb{R}^{3\times3}$ such that

$$P \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

What we really want is that

$$P \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_3 \\ v_2 \end{bmatrix}.$$

We can get P by working backwards and thinking of matrix-vector multiplication as a linear combination:

$$\begin{bmatrix} v_1 \\ v_3 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Are you okay with how I got the last equality? Maybe it would help to rearrange the sum so that v_1, v_2, and v_3 come in order:

$$v_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + v_3 \begin{bmatrix}0\\1\\0\end{bmatrix} + v_2 \begin{bmatrix}0\\0\\1\end{bmatrix} = v_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + v_2 \begin{bmatrix}0\\0\\1\end{bmatrix} + v_3 \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Here is the result:

$$\begin{bmatrix}2 & 2 & 1\\0 & {\color{blue}0} & 1\\0 & 1 & 5\end{bmatrix} \xrightarrow[P_{23}]{\text{R3} \mapsto \text{R2, R2} \mapsto \text{R3}} \begin{bmatrix}2 & 2 & 1\\0 & 1 & 5\\0 & 0 & 1\end{bmatrix}, \qquad P_{23} := \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}.$$

I am calling this P_{23} to emphasize that we get it by interchanging columns 2 and 3 of the identity matrix. We'll call such a matrix formed by swapping columns of the identity a **PERMUTATION MATRIX**.

What we get is that

$$EA = U, \qquad E := P_{23}E_{31}E_{21}$$

with U upper-triangular. The matrix E is now a little more complicated than in Example 12.4, as we have to include a factor of a permutation matrix, not just an elimination mtrix.

In general, to interchange rows i and j of $A \in \mathbb{R}^{m \times m}$, multiply $P_{ij}A$, where $P_{ij} \in \mathbb{R}^{m \times m}$ is the matrix whose columns are those of the $m \times m$ identity matrix with columns i and j interchanged. Such a matrix P_{ij} is, again, a **PERMUTATION MATRIX**. So, if at some stage of elimination, the diagonal entry that you want to use to eliminate entries below is 0, but other entries in that column are nonzero, just "permute" the rows to bring that nonzero entry up to the row that you want. Then eliminate as usual in the remaining rows.

Content from Strang's *ILA* 6E. Read "Possible breakdown of elimination" on p. 43 up to but not including the "Caution!" paragraph. Then read p. 45 after equation (1) and look at the calculation in "Exchange rows 2 and 3." These P_{ij} permutation matrices are special cases of a more general permutation matrix structure, which is the identity matrix with its columns (equivalently, rows) rearranged in various ways. See pp. 64–65. We won't need those more general permutation matrices for a while.

12.10 Problem (!). Explain in words (no need for any calculations) why $P_{ij}A = P_{ji}A$.

12.11 Problem ($\star$). Let $P_{13} \in \mathbb{R}^{3\times 3}$ be the permutation matrix that interchanges columns 1 and 3 of the 3×3 identity matrix. Compute $P_{13}A$ and AP_{13} for an arbitrary $A \in \mathbb{R}^{3\times 3}$. Then conjecture about what the different effects of multiplying $P_{ij}A$ and AP_{ij} are for an arbitrary $A \in \mathbb{R}^{m\times m}$ and an arbitrary permutation matrix $P_{ij} \in \mathbb{R}^{m\times m}$ that interchanges columns i and j of the $m \times m$ identity matrix. (You do not have to prove your conjecture.)

12.12 Problem (+). Let $A \in \mathbb{R}^{m\times n}$ and let $S \in \mathbb{R}^{n\times d}$ be a matrix whose columns are some of the columns of the $n \times n$ identity matrix. Here $d \geq 1$ is any integer, and the columns of the identity may be repeated, and some columns of the identity may not appear at all. Describe in words the structure of the matrix AS. [Hint: *the letter S might stand for "selection" matrix—what is being "selected" here?*]

Day 13: Wednesday, September 17.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Invertible matrix (N)

Here is the nastier breakdown of elimination: what if at some step, the diagonal entry that you want to use to eliminate entries below is 0 and *all* other entries in that column are 0, too? Good news is that you don't have to do any more elimination on entries in that column, as they're already 0. Bad news is that you won't be able to solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$. Here's a particular example of why.

13.1 Example. Here is a problematic matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

We eliminate:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[E_{21}]{\text{R2} \mapsto \text{R2}-2\times\text{R1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Maybe it doesn't look so problematic right now. We would want to use the $(2,2)$-entry in $E_{21}A$ to eliminate the $(3,2)$-entry, but the $(3,2)$-entry is already 0. So, $E_{21}A$ is already upper-triangular! Why is this not enough for us to be happy?

Let's actually try to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ arbitrary. If $A\mathbf{x} = \mathbf{b}$, then $E_{21}A\mathbf{x} = E_{21}\mathbf{b} = (b_1, b_2 - 2b_1, b_3)$. Thus we want

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}.$$

At the level of actual equations, this is

$$\begin{cases} x_1 + 2x_2 + 3x_3 = b_1 \\ 0 = b_2 - 2b_1 \\ 5x_3 = b_3. \end{cases}$$

Look at that second equation: it says $b_2 - 2b_1 = 0$, equivalently, $b_2 = 2b_1$. Think about the logic here. We assumed that $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (b_1, b_2, b_3)$, and we deduced that $b_2 = 2b_1$. This means that $\mathbf{b}$ cannot be just any vector in $\mathbb{R}^3$; it has to satisfy this "solvability condition" of $b_2 = 2b_1$. Surely not every vector in $\mathbb{R}^3$ does this—for example, take $\mathbf{b} = (1, 0, 0)$. So we can't always solve $A\mathbf{x} = \mathbf{b}$.

It's worth interpreting this in the context of the column space. Look at the structure of A: the second row is twice the first row. More precisely,

$$A\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 4x_2 + 6x_3 \\ 5x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2(x_1 + 2x_2 + 3x_3) \\ 5x_3 \end{bmatrix}.$$

So, if $\mathbf{b} = (b_1, b_2, b_3) \in \mathbf{C}(A)$, then $b_2 = 2b_1$. This is exactly the solvability condition that we deduced from elimination.

13.2 Problem ($\star$). Does the "arrow go the other way"? The previous example showed

$$\mathbf{b} \in \mathbf{C}(A) \Longrightarrow b_2 = 2b_1.$$

Do we have

$$b_2 = 2b_1 \Longrightarrow \mathbf{b} \in \mathbf{C}(A)?$$

Yes! If $b_2 = 2b_1$, then $A\mathbf{x} = \mathbf{b}$ is the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = b_1 \\ 2x_1 + 4x_2 + 6x_3 = 2b_1 \\ 5x_3 = b_3. \end{cases}$$

Use the third equation to solve for x_3, take x_2 to be any number that you like, and then use the first equation to write x_1 in terms of the values forced on x_3 and chosen for x_2. Why does this also satisfy the second equation automatically?

Content from Strang's *ILA* 6E. Read the rest of "Possible Breakdown of Elimination" on p. 43 starting with "Caution!"

These results will follow and support us for the rest of the course and beyond. Here is an abstraction of our elimination procedure.

13.3 Theorem (Gaussian elimination). *Let $A \in \mathbb{R}^{m \times m}$. Then there exist matrices E, $U \in \mathbb{R}^{m \times m}$ with the following properties.*

(i) $EA = U$.

(ii) *U is upper-triangular.*

(iii) *E is the product of elimination matrices E_{ij} and/or permutation matrices P_{ij}.*

Proof. If the $(1,1)$-entry of A is nonzero, multiply A by elimination matrices $E_{21}, \ldots, E_{m1}$ to subtract multiples of row 1 of A from rows 2 through m of A. Call the product of these elimination matrices E_1. If $m = 2$, then $E_1 A$ is upper-triangular. If $m \geq 3$ and the $(2,2)$-entry of $E_1 A$ is nonzero, multiply $E_1 A$ by elimination matrices $E_{32}, \ldots, E_{m2}$ to subtract multiples of row 2 of $E_1 A$ from rows 3 through m of $E_1 A$. Call the product of these elimination matrices E_2. If $m = 3$, then $E_2 E_1 A$ is upper-triangular. Otherwise, turn the crank and keep going.

If at any stage the (j,j)-entry is zero and the entries in column j in rows $j+1$ through m are zero, just proceed to the next step and consider the $(j+1, j+1)$-entry. If the (j,j)-entry is zero and some entry in rows $j+1$ through m of column j is nonzero, multiply by a permutation matrix so that this nonzero entry is now the (j,j)-entry. Then eliminate as before. Call the product of the elimination matrices and the permutation matrices E_j. ■

What this result says is that if $A\mathbf{x} = \mathbf{b}$, then $EA\mathbf{x} = E\mathbf{b}$, and so $U\mathbf{x} = E\mathbf{b}$. The upper-triangular system $U\mathbf{x} = E\mathbf{b}$ is much easier to solve, and so we like it. At least, we like it *when the diagonal entries of U are nonzero.*

13.4 Theorem. *Let $U \in \mathbb{R}^{m \times m}$ be an upper-triangular matrix whose diagonal entries are nonzero. Then for any $\mathbf{c} \in \mathbb{R}^m$, there exists a unique $\mathbf{x} \in \mathbb{R}^m$ such that $U\mathbf{x} = \mathbf{c}$.*

Proof. This is really back-substitution in the abstract. Here's the proof for $m = 3$. Take

$$U = \begin{bmatrix} u_{11} & * & * \\ 0 & u_{22} & * \\ 0 & 0 & u_{33} \end{bmatrix},$$

where u_{11}, u_{22}, and u_{33} are nonzero. So if you want to solve $U\mathbf{x} = \mathbf{c}$ with $\mathbf{c} = (c_1, c_2, c_3)$, first you'd look at

$$u_{33} x_3 = c_3.$$

Since $u_{33} \neq 0$, we can divide to find that x_3 must be

$$x_3 = \frac{c_3}{u_{33}}.$$

Go back up a step and look at

$$u_{22} x_2 + \text{stuff depending on } x_3 = c_2.$$

The point is that we know what this "stuff" is because we know x_3 exactly. Solve this as

$$x_2 = \frac{c_2 - \text{stuff}}{u_{22}}.$$

This is the only choice for x_2. Do the same for x_1. ■

But are we really sure that if $EA = U$, then a solution to $U\mathbf{x} = E\mathbf{b}$ is also a solution to $A\mathbf{x} = \mathbf{b}$? For small problems, we can check it by plug-and-chug, but why is this true in general?

The time has come to be sure that we can "invert" E, and this is a good reason to study matrix inverses in general. We will overall be much more concerned with properties of inverses than formulas for inverses. There's an algorithm that will let you do that, and we'll see it briefly, but we'll mostly abide by the slogan "What things do defines what things are."

Content from Strang's *ILA* 6E. Read the first two paragraphs on p. 50.

Here is what we want: why does $EA\mathbf{x} = E\mathbf{b}$ imply $A\mathbf{x} = \mathbf{b}$? More abstractly, if $E \in \mathbb{R}^{m\times m}$ and $E\mathbf{v} = E\mathbf{b}$ for some $\mathbf{v}$, $\mathbf{b} \in \mathbb{R}^m$, do we necessarily have $\mathbf{v} = \mathbf{b}$? It would be nice if we could "undo" the "action" of E by multiplying by another matrix. Is there $F \in \mathbb{R}^{m\times m}$ such that $F(E\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^m$? If so, then assuming $E\mathbf{v} = E\mathbf{b}$ gives $F(E\mathbf{v}) = F(E\mathbf{b})$, and thus $\mathbf{v} = \mathbf{b}$ as desired.

Look more closely at the equation $F(E\mathbf{w}) = \mathbf{w}$. This just says $(FE)\mathbf{w} = \mathbf{w}$. What does that tell us about the matrix product FE? If $(FE)\mathbf{w} = \mathbf{w}$ for all $\mathbf{v} \in \mathbb{R}^m$, then we could take $\mathbf{w} = \mathbf{e}_j$ as the standard basis vectors. We find $(FE)\mathbf{e}_j = \mathbf{e}_j$, and so the jth column of FE must be $\mathbf{e}_j$: the jth column of the $m \times m$ identity matrix. That is, we want $FE = I_m$.

We are actually going to ask for a little bit more in the following definition: that $EF = I_m$ as well. For me, this is an artifact of our intuition from multiplication of real numbers (if $ab = 1$ for a, $b \in \mathbb{R}$, then of course $ba = 1$), but it's necessary to require here since matrix multiplication isn't commutative. (That is, just because we have $FE = I_m$ *shouldn't* automatically imply that $EF = I_m$. Surprisingly, and gloriously, it does, but that takes some work.)

13.5 Definition. *A matrix $E \in \mathbb{R}^{m\times m}$ is* **INVERTIBLE** *if there exists a matrix $F \in \mathbb{R}^{m\times m}$ such that*

$$FE = I_m \quad \text{and} \quad EF = I_m. \tag{13.1}$$

13.6 Example. **(i)** Let

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

be the elimination matrix that subtracts 2 times the first row from the second row. Can we invert E? We're done if we find $F \in \mathbb{R}^{2\times 2}$ such that $EF = FE = I_2$. What *should* F be?

This is where it might help to think about E *dynamically*: what does E do? We just said it: E subtracts 2 times the first row from the second row. So undoing E should add two times the first row to the second row. That is,

$$E\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 - 2v_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 \\ (v_2 - 2v_1) + 2v_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

So maybe

$$F = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

works. Check it yourself.

(ii) Let

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

be the permutation matrix that interchanges rows 1 and 2. Undoing P should interchange those rows again: we want $F \in \mathbb{R}^{2\times 2}$ such that if

$$P\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}, \quad \text{then} \quad F\begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This looks like we should just take $F = P$. I suggest that you check that $P^2 = I_2$. By the way, this is the first time we're using "power" notation for matrix multiplication: $P^2 = PP$.

13.7 Problem ($\star$). Talking about invertible matrices is existential language: there just has to be one matrix F such that $EF = FE = I_m$. (Shortly we'll prove that there can be at most one such F.) Talking about noninvertible matrices is universal language: you have to show $EF \neq I_m$ or $FE \neq I_m$ for all F. Explain why the matrix in $\mathbb{R}^{m\times m}$ whose entries are all zero is not invertible.

Content from Strang's *ILA* 6E. Read Examples 4 and 5 on p. 52 about inverting elimination matrices. Skip the remarks about the inverse of FE in Example 5 for now.

Day 14: Friday, September 19.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Inverse of an invertible matrix

Example 13.6 should be comforting in that it suggests that elimination and permutation matrices are invertible. We'd probably like to say that their "inverses" are what we expect: invert subtracting by adding, invert permuting by permuting again. What gives us the right to say that a matrix has only one inverse? A (nonzero) real number has only one reciprocal to undo multiplication, but why is this true for matrices?

Here's why. Suppose that E has "two" inverses F_1 and F_2, so

$$F_1E = EF_1 = F_2E = EF_2 = I_m. \tag{14.1}$$

We need to show that $F_1 = F_2$. Here is a great trick: multiply by 1. You know that $1x = x$ for any $x \in \mathbb{R}$, and the same is true for matrices.

14.1 Problem (!). Check that $AI_m = I_mA = A$ for any $A \in \mathbb{R}^{m\times m}$.

So,

$$F_1 = F_1 I_m = F_1(EF_2) = (F_1E)F_2 = I_m F_2 = F_2. \tag{14.2}$$

Here is the formal result.

14.2 Theorem. *Let $E \in \mathbb{R}^{m \times m}$. There exists at most one $F \in \mathbb{R}^{m \times m}$ satisfying* (13.1).

Content from Strang's *ILA* 6E. This is Note 2 on p. 50.

We can now talk about "the" inverse of a matrix.

14.3 Definition. *Let $E \in \mathbb{R}^{m \times m}$ be invertible. The* **INVERSE** *of E is the unique matrix F satisfying*

$$FE = EF = I_m,$$

and we write $F = E^{-1}$.

Let's generalize Example 13.6.

14.4 Theorem. **(i)** *Let $E_{ij} \in \mathbb{R}^{m \times m}$ be the elimination matrix that subtracts ℓ times row j from row i (so 1's along the diagonal, $-\ell$ in the (i, j)-entry, and 0 everywhere else). Then E_{ij} is invertible, and E_{ij}^{-1} is the elimination matrix that adds ℓ times row j to row i (so 1's along the diagonal, ℓ in the (i, j)-entry, and 0 everywhere else).*

(ii) *Let $P_{ij} \in \mathbb{R}^{m \times m}$ be the permutation matrix that interchanges rows i and j (so P_{ij} is the $m \times m$ identity matrix with columns i and j interchanged). Then P_{ij} is invertible and $P_{ij}^{-1} = P_{ij}$.*

Now go back and look very carefully at the calculation in (14.2). We did not use all of the equalities in (14.1). Instead, we only needed that $F_1E = I_m$ and $EF_2 = I_m$. We might call F_1 a **LEFT INVERSE** and F_2 a **RIGHT INVERSE**. Here is what we have proved.

14.5 Corollary. *Let $E \in \mathbb{R}^{m \times m}$ have left and right inverses in the sense that there are F_1, $F_2 \in \mathbb{R}^{m \times m}$ such that*

$$F_1E = I_m \quad \text{and} \quad EF_2 = I_m.$$

Then E is invertible and $F_1 = F_2 = E^{-1}$.

Proof. Okay, maybe this needs a teensy bit of proof. First, the calculation in (14.2) shows $F_1 = F_2$. Put $F = F_1$. Then the hypotheses give $FE = F_1E = I_m$ and $EF = EF_2 = I_m$, and so F satisfies Definition 14.3. ■

14.6 Problem (!). We probably expect that undoing the undoing of an action does that action. Totally makes sense, right? More precisely, if $E \in \mathbb{R}^{m \times m}$ is invertible, we should expect that E^{-1} is also invertible and $(E^{-1})^{-1} = E$. (That's how exponents work, right?)

Prove this by showing that E satisfies the definition of inverse for E^{-1}. *What things do defines what things are.*

14.7 Problem ($\star$). Let E, $A \in \mathbb{R}^{m \times m}$. Suppose that $EA = I_m$ and E is invertible. Prove that A is invertible, too.

We are particularly interested in inverting a matrix that is a product of elimination matrices and permutation matrices. We know that any elimination or permutation matrix is invertible. More generally, is the product of invertible matrices invertible?

Yes. Suppose that A, $B \in \mathbb{R}^{m \times m}$ are invertible. We will show that AB is invertible. Think about action: first you do B to a vector $\mathbf{v}$ by multiplying $B\mathbf{v}$, and then you do A by multiplying $A(B\mathbf{v}) = (AB)\mathbf{v}$. To undo AB, you probably want to undo A first and then B. (Getting dressed, socks go on first, then shoes; getting undressed, shoes come off first, then socks.) So we might guess that $(AB)^{-1} = B^{-1}A^{-1}$. The good news is that we can check this using the definition:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_mB = B^{-1}B = I_m.$$

14.8 Problem (!). Check that $(AB)(B^{-1}A^{-1}) = I_m$ as well.

Here is the formal result.

14.9 Theorem. *Let A, $B \in \mathbb{R}^{m \times m}$ be invertible. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Content from Strang's *ILA* 6E. Read "The Inverse of a Product AB" on pp. 51–52. Then go back to Example 5 on p. 52. The point for our larger story is that multiplying elimination matrices together when getting $EA = U$ is not the best of ideas, whereas computing E^{-1} is more meaningful.

This seems to be everything that we want. Theorem 13.3 tells us that for any $A \in \mathbb{R}^{m \times m}$, we can always find a product of elimination and/or permutation matrices, which we call E, such that $EA = U$ is upper-triangular. Now we know that E is invertible. Given $\mathbf{b} \in \mathbb{R}^m$, it is usually easier to solve $U\mathbf{x} = E\mathbf{b}$, and then we have $E^{-1}U\mathbf{x} = E^{-1}(E\mathbf{b})$, where

$$E^{-1}U = E^{-1}(EA) = (E^{-1}E)A = I_mA = A \quad \text{and} \quad E^{-1}(E\mathbf{b}) = (E^{-1}E)\mathbf{b} = \mathbf{b}.$$

Thus $A\mathbf{x} = \mathbf{b}$, which is what we always wanted to be sure of.

Invertibility is another way of asking about solvability of linear systems. Suppose that $A \in \mathbb{R}^{m \times m}$ is invertible. I claim that $A\mathbf{x} = \mathbf{b}$ always has a solution, and that solution is unique. For uniqueness, work backwards and assume $A\mathbf{x} = \mathbf{b}$; then $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$, and so $\mathbf{x} = A^{-1}\mathbf{b}$. To check that this is actually a solution, plug in: $A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$.

14.10 Theorem. *Let $A \in \mathbb{R}^{m \times m}$ be invertible and $\mathbf{b} \in \mathbb{R}^m$. Then the problem $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*

Content from Strang's *ILA* 6E. This is Note 3 on p. 50.

14.11 Problem (!). Hugely important: convince yourself of the following for an invertible $A \in \mathbb{R}^{m \times m}$.

(i) $\mathbf{N}(A) = \{\mathbf{0}_m\}$. (So if $\mathbf{N}(A)$ is bigger than $\{\mathbf{0}_m\}$, then A is not invertible.)

(ii) $\mathbf{C}(A) = \mathbb{R}^m$. (So if $\mathbf{C}(A)$ is smaller than $\mathbb{R}^m$, then A is not invertible.)

(iii) The columns of A are independent. (So if the columns of A are dependent, then A is not invertible.)

Does the logic go the other way? If $A\mathbf{x} = \mathbf{b}$ always has a unique solution, is A invertible? If $\mathbf{N}(A) = \{\mathbf{0}_m\}$ (uniqueness guaranteed, maybe not existence), is A invertible? If $\mathbf{C}(A) = \mathbb{R}^m$ (existence guaranteed, maybe not uniqueness), is A invertible? If the columns of A are independent, is A invertible? Yes, yes, yes, and yes. But establishing all of that needs some preparation.

14.12 Problem ($\star$). Often knowing that a matrix is invertible is more useful than having a formula for that inverse. Here's a situation in which the presence of an invertible matrix "keeps things the same." Let $A \in \mathbb{R}^{m \times n}$ be any matrix and let $B \in \mathbb{R}^{n \times n}$ be invertible. Show that $\mathbf{C}(AB) = \mathbf{C}(A)$ as follows. First, explain why $AB\mathbf{v} \in \mathbf{C}(A)$ for any $\mathbf{v} \in \mathbb{R}^n$. Next, justify the equality $A\mathbf{w} = (AB)(B^{-1}\mathbf{w})$ and explain how that shows that anything in $\mathbf{C}(A)$ is in $\mathbf{C}(AB)$.

Content from Strang's *ILA* 6E. I am not going to talk about determinants now, or much later (I hope!), but you should read Note 6 on p. 50 and Example 2 on p. 51 and also think about the four 2×2 matrices in Example 3 on p. 51. Determinants are a quick and easy way of understanding 2×2 matrices, which arise in a lot of applications (e.g., ordinary differential equations). Try using Note 6 to solve our original problem

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Using the solution formula $\mathbf{x} = A^{-1}\mathbf{b}$ from Theorem 14.10 in practice requires us to compute A^{-1}. This turns out to be "expensive" computationally, rather more so than elimination and back-substitution.

Content from Strang's *ILA* 6E. Read "The Cost of Elimination" on pp. 57–58. The following link to a section from the fifth edition elaborates on this:

https : //math.mit.edu/ gs/linearalgebra/ila5/linearalgebra5_11 − 1.pdf.

The point is that using A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ for $A \in \mathbb{R}^{m\times m}$ might take around m^3 arithmetical operations, but using elimination would take only around $m^3/3$ operations. If this excites you, take a numerical linear algebra class. Read the beautiful book by Trefethen & Bau, too.

Day 15: Monday, September 22.

Let's go back to elimination in the context of inverses. How does being able to solve a linear system $A\mathbf{x} = \mathbf{b}$ via elimination say anything about the invertibility of A?

We'll start with the nicest case: upper-triangular. I claim that we can eliminate "upwards" on an upper-triangular matrix with nonzero diagonal entries to find an invertible matrix E such that $EU = I_m$. Then $U = E^{-1}$, and so U is invertible. Here is how this works.

15.1 Example. Let

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

We met this matrix in Example 12.4. I want to turn U into I_3 starting from the bottom.

The first thing to do is to make that entry of 2 in the $(3,3)$-slot into a 1. This requires division by 2 in the third row. Of course we want to encode this, like everything else, via matrix multiplication. What matrix $D \in \mathbb{R}^{3\times 3}$ does that? We want

$$D \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3/2 \end{bmatrix}.$$

I think you know what to do by now: expand the vector on the right as a linear combination weighted by v_1, v_2, and v_3, and you'll see that D should be

$$D_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

I'm calling it D_{33} now because the action is happening in the $(3,3)$-entry.

So, we have the transformation

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[D_{33}]{\text{R3} \mapsto (1/2)\times\text{R3}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

This **scaling matrix** D_{33}, along with the elimination and permutation matrices, is the last of the so-called **elementary matrices** that we need to encode "row operations" on matrices.

Now we eliminate "upwards." We want the other entries in column 3 to be 0, so we subtract multiples of row 3 from rows 1 and 2. (Well, multiples of 1.) We get

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{23}]{\text{R2} \mapsto \text{R2}-\text{R3}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{23} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[E_{13}]{\text{R1} \mapsto \text{R1}-\text{R3}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{13} := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

And then we'll subtract a multiple of row 2 from row 1 to make that $(1, 2)$-entry 0:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{12}]{\text{R1} \mapsto \text{R1}-\text{R2}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{12} := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Last, we rescale the first row:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[D_{11}]{\text{R1} \mapsto (1/2)\times\text{R1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3, \qquad D_{11} := \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We conclude

$$D_{11}E_{12}E_{13}E_{23}D_{33}U = I_3,$$

so putting

$$E := D_{11}E_{12}E_{13}E_{23}D_{33}$$

gives $EU = I_3$. Certainly E is invertible, as all elimination matrices are invertible, and scaling matrices are invertible when their diagonal entries are nonzero. Then $U = E^{-1}I_3 = E^{-1}$, and so U is invertible with $U^{-1} = E$.

15.2 Problem ($\star$). Let $D \in \mathbb{R}^{m \times m}$ be **DIAGONAL**: the (i, j)-entry of D is 0 for $i \neq j$. Prove that if all of the diagonal entries of D are nonzero, then D is invertible; give an explicit formula for D^{-1}.

The arithmetic in Example 15.1 is called **GAUSS–JORDAN ELIMINATION**. I'll state how this works in the abstract.

15.3 Theorem (Gauss–Jordan elimination). *Let $U \in \mathbb{R}^{m \times m}$ be upper-triangular with nonzero diagonal entries. Then there exists an invertible matrix $E \in \mathbb{R}^{m \times m}$, which is the product of elimination and/or scaling matrices (but not permutation matrices), such that $EU = I_m$. Consequently, $U = E^{-1}$ is invertible.*

Proof. This should feel basically the same as the proof of Theorem 13.3. Multiply U by a scaling matrix D_{mm} to divide row m by $u_{mm} \neq 0$ so that the (m, m)-entry of $D_{mm}U$ is 1.

Then subtract multiples of row m from rows $m-1$ through 1 to create zeros in rows $m-1$ through 1 of column m. Go to the $(m-1, m-1)$-entry: rescale so that it's 1, create zeros in rows $m-2$ through 1 of column $m-1$ through elimination. Repeat. Let E be the product of all of the scaling and/or elimination matrices used, in the order that you use them from the bottom up at each stage. No need for permutation matrices because all of the diagonal entries are nonzero. ■

15.4 Remark. *Previously we used "Gaussian elimination" on an arbitrary $A \in \mathbb{R}^{m\times m}$ to find an invertible matrix $E \in \mathbb{R}^{m\times m}$ such that $EA = U$ with U upper-triangular. Now, in the special case that the diagonal entries of U were nonzero, we used "Gauss–Jordan elimination" to find another invertible matrix $\widetilde{E}$ such that $\widetilde{E}U = I_m$, thus $\widetilde{E}EA = I_m$, so A is invertible with $A^{-1} = (\widetilde{E}E)^{-1}$.*

Content from Strang's *ILA* 6E. Page 57 offers an algorithm for computing A^{-1} by hand if you really need to do it for a small A. I will never ask you to do that, and Strang gives only one problem asking for an explicit calculation (Problem 29 in Section 2.2 if you're curious)—that's how deprecated the method is. Far better to *understand* A^{-1} than have a general formula for it.

15.5 Problem (!). Explain why the matrix A from Example 12.4 is invertible. What is A^{-1}? (Don't actually compute it—no one really cares—but express A^{-1} as the product of the inverses of a bunch of elimination, scaling, and/or permutation matrices.)

Day 16: Wednesday, September 24.

We've learned a lot about invertible matrices—in particular that we can always solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} = A^{-1}\mathbf{b}$ when A is invertible, but that we probably shouldn't because computing A^{-1} is computationally expensive. The alternative is that we do elimination on A so that $U := EA$ is upper-triangular with nonzero diagonal entries, and then we solve $U\mathbf{x} = E\mathbf{b}$ via back-substitution. That requires us to compute $E\mathbf{b}$, too. There is a variation on this approach that is still computationally less expensive than computing A^{-1} and that gives us some new insights into matrix multiplication, so it's worth learning. We start with a very concrete example.

16.1 Example. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}.$$

We saw in Example 12.4 that

$$EA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U,$$

where

$$E := E_{32}E_{31}E_{21}$$

and

$$E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{31} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad E_{32} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

We went further in Example 15.1 and found $\widetilde{E}$ such that $\widetilde{E}U = I_3$, but that's less important here. Rather, the new thing to focus on is the *factorization*

$$A = E^{-1}U.$$

Recall that we originally talked about multiplying matrices with the goal of *factoring* matrices: breaking matrices into products of simpler matrices to reveal meaningful properties. What is simpler about the matrices E^{-1} and U, and what is meaningful about the factorization $A = E^{-1}U$?

Certainly U is simpler than A because U is upper-triangular: U has a nice structure with a lot of simple data—many zero entries. What about E^{-1}? A *bad* idea is to compute E as the product $E = E_{32}E_{31}E_{21}$ and then try to compute E^{-1} from that. Go ahead and try it and see how opaque the work is. (I mean, we haven't really computed a matrix inverse other than Remark 15.4.) But we do know that

$$E^{-1} = (E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}.$$

And we know what each of these inverses are because they are inverses of elimination matrices:

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \qquad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now think about what they are doing. Multiplying by E_{31}^{-1} says "Add 4 times row 1 to row 3":

$$E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}.$$

says add 4 times row 1 to row 3. Multiplying by E_{21}^{-1} says "Add 2 times row 1 to row 2":

$$E^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = E_{21}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} =: L.$$

Just look at that matrix L. It's **LOWER-TRIANGULAR**, because every entry above the diagonal is 0. And the entries below the diagonal are the negatives of the multipliers from the original elimination step. *This is no accident.*

How does this factorization $A = LU$ help? Let's solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (0, 1, 5)$. Ideally you did this in Problem 12.5. This problem is the same as $LU\mathbf{x} = \mathbf{b}$. Now here is the trick: abbreviate $\mathbf{c} := U\mathbf{x}$. Then we want $L\mathbf{c} = \mathbf{b}$. The clever idea is to view $\mathbf{c}$ as an unknown; then we can solve $L\mathbf{c} = \mathbf{b}$ using back-substitution, and then we solve $U\mathbf{x} = \mathbf{c}$ with another round of back-substitution. Nowhere does elimination hit $\mathbf{b}$.

Let's go: $L\mathbf{c} = \mathbf{b}$ is the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix},$$

equivalently

$$\begin{cases} c_1 & & & & & = & 0 \\ 2c_1 & + & c_2 & & & = & 1 \\ 4c_1 & + & 3c_2 & + & c_3 & = & 5 \end{cases}$$

The first equation immediately gives $c_1 = 0$, so the second reduces to $c_2 = 1$, and then the third is $3 + c_3 = 5$, thus $c_3 = 2$. Hence

$$\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Next, $U\mathbf{x} = \mathbf{c}$ is the system

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

equivalently

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 1 \\ & & & & 2x_3 & = & 2 \end{cases}$$

The third equation is $2x_3 = 2$, thus $x_3 = 1$. Then the second equation is $x_2 + 1 = 1$, so $x_2 = 0$. And the first equation is then $2x_1 + 0 + 1 = 0$, so $2x_1 = -1$, and therefore $x_1 = -1/2$. That is, $\mathbf{x} = (-1/2, 0, 1)$.

This example has a number of lessons for us. First, if we can factor $A = LU$, with L lower-triangular and U upper-triangular, and where both L and U have all nonzero entries on their diagonals, then we can solve $A\mathbf{x} = \mathbf{b}$ easily by back-substitution *and without doing any elimination calculations on* $\mathbf{b}$. Second, we *might* be able to achieve this "LU-factorization" if we can reduce A to upper-triangular form using only elimination, not permutation, matrices. In particular, finding that factor of L involved inverting the product of elimination matrices that governed that reduction—but we did not multiply all those elimination matrices together and then calculate the inverse; instead, we used properties of inverses of products and *what elimination matrices do*. (It pains me to say this, but brute force isn't always the best force.)

All of this turns out to be more generally true.

16.2 Theorem (LU-factorization). *Suppose that $A \in \mathbb{R}^{m \times m}$ can be reduced to upper-triangular form using only elimination, not permutation, matrices. That is, there is $E \in \mathbb{R}^{m \times m}$ such that $EA = U$, where U is upper-triangular and E is a product of only elimination matrices. Then $L := E^{-1}$ is lower-triangular, the diagonal entries of L are all 1, and $A = LU$. Moreover, for any $\mathbf{b} \in \mathbb{R}^m$, there is $\mathbf{x} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ if and only if there is $\mathbf{c} \in \mathbb{R}^m$ such that*

$$\begin{cases} L\mathbf{c} = \mathbf{b} \\ U\mathbf{x} = \mathbf{c}. \end{cases} \tag{16.1}$$

Proof. We are only going to prove the last sentence. The proof that L is lower-triangular when E is a product of only elimination matrices is essentially an abstraction of the calculation in Example 16.1. (Try replacing the multipliers 2, 4, and 3 with arbitrary ℓ_{21}, ℓ_{31}, $\ell_{32} \in \mathbb{R}$ and watch the same lower-triangular structure appear. Or check out the readings in Strang mentioned below.)

Here is the proof of that last sentence, assuming that we have the factorization $A = LU$. First, if there is $\mathbf{x} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$, then $LU\mathbf{x} = \mathbf{b}$. Put $\mathbf{c} = U\mathbf{x}$ to find $L\mathbf{c} = \mathbf{b}$. So, both equations in (16.1) are true.

Now suppose that both equations in (16.1) are true. Work backwards:

$$\mathbf{b} = L\mathbf{c} = L(U\mathbf{x}) = (LU)\mathbf{x} = A\mathbf{x}.$$

By the way, the proof of that last sentence did not use *at all* the fact that L and U are triangular or that L has diagonal entries equal to 1. However, if we wanted to start by solving (16.1) and end up with a solution to $A\mathbf{x} = \mathbf{b}$, it would be necessary for L and U to have all nonzero diagonal entries. ■

16.3 Problem ($\star$). This is yet another perspective on our original toy problem (1.1). Let

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}.$$

Find matrices L, $U \in \mathbb{R}^{2 \times 2}$ such that L is lower-triangular, U is upper-triangular, and $A = LU$. Let $\mathbf{b} = (1, 11)$. Solve $A\mathbf{x} = \mathbf{b}$ by first solving $L\mathbf{c} = \mathbf{b}$ for some $\mathbf{c} \in \mathbb{R}^2$ and then solving $U\mathbf{x} = \mathbf{c}$ for $\mathbf{x} \in \mathbb{R}^2$.

Content from Strang's *ILA* 6E. Here are sketches of the existence of the LU-factorization. First, reread Example 5 on p. 52 to see again how inverting products of elimination matrices works. Think carefully about the two bold sentences on "feels an effect" and "feels no effect." Do you understand exactly what this means? Then read p. 53 and contrast the calculations in equations (10) and (11). Which do you like better? Read all of p. 59—and think about the last paragraph on p. 58: "A proof means that we have not just seen that pattern and believed it and liked it, but understood it." *This is why we prove things.* Another proof of

LU appears on p. 60, using the matrix multiplication technique discussed on p. 34.

So who cares? The work in Example 16.1 probably felt no more efficient than a routine back-substitution approach (which you did in Problem 12.5, right?) Maybe it felt more inefficient! That's a valid feeling. All of our examples in this class are effectively toy problems designed so that the on-the-fly arithmetic is easy.

But what if you need to solve $A\mathbf{x} = \mathbf{b}_j$ for many $\mathbf{b}_j$? If you have only a finite number of $\mathbf{b}_j$, maybe you could work with a large augmented matrix $\begin{bmatrix} A & \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix}$, do elimination on A via the matrix E, so $EA = U$, and then study $\begin{bmatrix} U & E\mathbf{b}_1 & \cdots & E\mathbf{b}_p \end{bmatrix}$. Then you would have to solve $U\mathbf{x} = E\mathbf{b}_j$ by back-substitution. However, it is arguably less computationally expensive to solve $LU = \mathbf{b}_j$ by the two-step process above. In particular, it may be the case* that solving $A\mathbf{x} = \mathbf{b}_j$ is part of a larger *iterative* process: at the jth step, you get a new $\mathbf{b}_j$, but A stays the same. If you want to keep doing this *indefinitely*, the elimination calculations $E\mathbf{b}_j$ may become expensive. Doing the elimination just once to get $A = LU$, and then solving $LU\mathbf{x} = \mathbf{b}_j$ via the two-step process, may be less expensive.

The LU-factorization works when no row interchanges are needed, i.e., when we can write $EA = U$ with U upper-triangular and E as a product only of elimination matrices, not permutation matrices. Basically, it's possible to "almost" commute permutation and elimination matrices so that we have $PA = LU$ with P a product of permutation matrices, L lower-triangular, and U upper-triangular. Figuring out how to get that P factor out front is a little tricky, and I think this is better covered in a numerical linear algebra course. But once you know $PA = LU$, to solve $A\mathbf{x} = \mathbf{b}$, first permute $PA\mathbf{x} = P\mathbf{b}$, and then solve $LU\mathbf{x} = P\mathbf{b}$ as we did above.

Content from Strang's *ILA* 6E. See p. 65. This is wholly optional reading and requires a little more knowledge of permutation matrices than I expect or desire right now.

16.4 Problem ($\star$). One challenge in teaching this course is writing good problems! Even the most innocent-looking matrix can involve nasty arithmetic when it comes to Gaussian elimination because of *all those fractions*. Here's a way to get good exam problems and generate your own practice.

I like matrices with integer entries for which Gaussian elimination doesn't require any division when doing the elimination steps to get to the upper-triangular form. Write down a square (3×3 or 4×4) matrix L whose diagonal entries are all 1 and whose entries below the diagonal are nonnegative integers between 0 and 9. (Don't make all of these entries 0.) Let U be a square upper-triangular matrix of the same size as L whose entries on and above the diagonal are also nonnegative integers between 0 and 9. (Again, don't make all of these entries 0.) Multiply LU and call that product A. Do Gaussian elimination on A to reduce it to upper-triangular form. How do L and U show up?

* I found this StackExchange post really helpful:

`https://math.stackexchange.com/questions/266355/necessity-advantage-of-lu-decomposition over-gaussian-elimination`.

Day 17: Friday, September 26.

You took Exam 1.

Day 18: Monday, September 29.

Gauss–Jordan elimination says that if the diagonal entries of an upper-triangular matrix are nonzero, then that matrix is invertible. The arrow of our logic goes the other way, too.

18.1 Lemma. *Let $U \in \mathbb{R}^{m \times m}$ be upper-triangular.*

(i) *If U has a zero diagonal entry, then $\mathbf{N}(U) \neq \{\mathbf{0}_m\}$, and so U is not invertible.*

(ii) *If $\mathbf{N}(U) = \{\mathbf{0}_m\}$, then all of the diagonal entries of U are nonzero, and so U is invertible.*

Proof. I want to consider two possible structures of U: one where the first diagonal entry is zero and one where it isn't, but a zero diagonal entry occurs further down along the diagonal.

1. *The first diagonal entry is zero.* I'll do a specific case and a more notationally burdensome general case.

(i) *The case $m = 4$.* Here U has the form

$$U = \begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

I think it's pretty obvious that $U\mathbf{e}_1 = \mathbf{0}_4$, and since $\mathbf{e}_1 \neq \mathbf{0}_4$, so $\mathbf{e}_1 \in \mathbf{N}(U)$, and therefore $\mathbf{N}(U) \neq \{\mathbf{0}_4\}$, thus U is not invertible.

(ii) *The general case.* Here U has the form

$$U = \begin{bmatrix} \mathbf{0}_m & \widetilde{U} \end{bmatrix},$$

where $\widetilde{U}$ is "the rest" of U (columns 2 through m). Again, since $U\mathbf{e}_1$ is the first column of U, we have $U\mathbf{e}_1 = \mathbf{0}_m$ and $\mathbf{e}_1 \neq \mathbf{0}_m$, so $\mathbf{e}_1 \in \mathbf{N}(U)$, and therefore $\mathbf{N}(U) \neq \{\mathbf{0}_m\}$, thus U is not invertible.

2. *An entry on the diagonal in rows 2 or below is zero.* We'll look at the *first* zero entry on the diagonal (first starting from the top row), and so now this zero entry has to occur in row 2 or below. That is, $u_{jj} = 0$ for some $j \geq 2$ but $u_{ii} \neq 0$ for $1 \leq i \leq j - 1$.

(i) *An example in the case $m = 4$.* Here is one such possibility when $m = 4$:

$$U = \begin{bmatrix} ☺ & * & ☹ & * \\ 0 & ☺ & ☹ & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

By ☺ I mean nonzero entries, and in the notation above $j = 3$. Now look at the upper-triangular matrix

$$\widehat{U} := \begin{bmatrix} ☺ & * \\ 0 & ☺ \end{bmatrix}.$$

This has nonzero diagonal entries and so we can find x_1, $x_2 \in \mathbb{R}$ such that

$$\widehat{U} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ☹ \\ ☹ \end{bmatrix}.$$

But then

$$x_1 \begin{bmatrix} ☺ \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} * \\ ☺ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ☹ \\ ☹ \\ 0 \\ 0 \end{bmatrix},$$

and so the third column of U is in the span of the first two. Thus the columns of U are dependent, so $\mathbf{N}(U) \neq \{\mathbf{0}_m\}$.

(ii) *The general case.* As before, assume that there is $j \geq 2$ such that $u_{jj} = 0$ but $u_{ii} \neq 0$ for $1 \leq i \leq j-1$. Write

$$U = \left[\begin{array}{cc|c} \widehat{U} & \widehat{\mathbf{u}}_j & \widetilde{U} \\ 0 & \mathbf{0}_{m-(j-1)} & \end{array}\right].$$

This block expression for U contains a lot of condensed notation: $\widehat{U}$ is a $(j-1)\times(j-1)$ upper-triangular matrix with nonzero diagonal entries, $\widehat{\mathbf{u}}_{j-1} \in \mathbb{R}^{j-1}$, $\widetilde{U}$ contains the remaining columns of U, and, irritatingly, 0 means a matrix whose entries are all 0. If $j = m$, then $\widetilde{U}$ isn't there. Since the diagonal entries of $\widehat{U}$ are nonzero, we can find $\widehat{\mathbf{x}}_{j-1} \in \mathbb{R}^{j-1}$ such that $\widehat{U}\widehat{\mathbf{x}}_{j-1} = \widehat{\mathbf{u}}_{j-1}$. From this, you can show that the jth column of U is in the span of the first $j-1$ columns, so the columns of U are dependent, and therefore $\mathbf{N}(U) \neq \{\mathbf{0}_m\}$. ■

We can combine Gauss–Jordan elimination (Theorem 15.3) and this lemma to conclude the following.

18.2 Theorem. *An upper-triangular matrix is invertible if and only if all of its diagonal entries are nonzero.*

There's another nasty consequence for an upper-triangular matrix U with a zero diagonal entry: you can't always solve $U\mathbf{x} = \mathbf{c}$ for any $\mathbf{c} \in \mathbb{R}^m$. This happened in Example 13.1, which you should reread right now.

18.3 Lemma. *Let $U \in \mathbb{R}^{m \times m}$ be upper-triangular.*

(i) *If U has a zero diagonal entry, then $\mathbf{C}(U) \neq \mathbb{R}^m$, and so U is not invertible.*

(ii) *If $\mathbf{C}(U) = \mathbb{R}^m$, then all of the diagonal entries of U are nonzero, and so U is invertible.*

Proof. First, why do we need to prove this? We know from Lemma 18.1 that since U has a zero diagonal entry, U is not invertible. *But did we ever show that if a square matrix is not invertible, then its column space isn't all of $\mathbb{R}^m$?* **No**. This lemma turns out to be the key step in figuring that out.

As in the proof of Lemma 18.1, we'll consider two different locations for the zero diagonal entry—just sort of "flipped" from that proof.

1. *The (m, m)-entry of U is zero.* That is, the "last" entry on the diagonal is zero.

(i) *The case $m = 4$.* Here U has the form

$$U = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Do you see the problem? If $\mathbf{b} = (b_1, b_2, b_3, b_4) \in \mathbf{C}(U)$, then $b_4 = 0$. For example, $\mathbf{e}_4 \notin \mathbf{C}(U)$, and so $\mathbf{C}(U) \neq \mathbb{R}^4$.

(ii) *The general case.* If the (m, m)-entry of U is zero, then the mth (last) row of U has all zero entries. This is because U is upper-triangular and the other entries in that row are all below the diagonal. Suppose that $\mathbf{b} \in \mathbf{C}(U)$, so $\mathbf{b} = U\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^m$. Then the mth entry of $\mathbf{b}$ is $b_m = 0$ because this entry is the dot product of the mth row of U with $\mathbf{x}$. That dot product is zero since the mth row of U is the zero vector. And so not every $\mathbf{b} \in \mathbb{R}^m$ can be in $\mathbf{C}(U)$; for example, $\mathbf{e}_m \notin \mathbf{C}(U)$.

2. *The (j, j)-entry of U is zero for some $j < m$.* That is, an entry "further up" the diagonal is zero. Additionally, we'll assume that the (j, j)-entry is the first zero entry on the diagonal from the bottom, so $u_{ii} \neq 0$ for $j + 1 \leq i \leq m$.

(i) *The case $m = 4$.* One possibility here is that U has the form

$$U = \begin{bmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & ☺ & * \\ 0 & 0 & 0 & ☺ \end{bmatrix}.$$

Here $☺ \neq 0$, and so the $(2, 2)$-entry is the first nonzero entry on the diagonal when you go up from the bottom.

Use those nonzero diagonal entries to do Gauss–Jordan elimination in rows 1 and 2 of columns 3 and 4. Then there is an invertible matrix E such that

$$EU = \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, if $\mathbf{b} \in \mathbf{C}(U)$, then there is $\mathbf{v} \in \mathbb{R}^4$ such that $U\mathbf{v} = \mathbf{b}$, and then $EU\mathbf{v} = E\mathbf{b}$. Consequently, the second entry of $E\mathbf{b}$ is 0. But that doesn't happen for all $\mathbf{b}$: take $\mathbf{b} = E^{-1}\mathbf{e}_2$. That is, $E^{-1}\mathbf{e}_2 \notin \mathbf{C}(U)$. Something like this happened in Example 13.1, right?

(ii) *The general case.* Here U has the form

$$U = \begin{bmatrix} * & * \\ \overrightarrow{\mathbf{0}} & \overrightarrow{\mathbf{u}} \\ 0 & \widehat{U} \end{bmatrix}.$$

Here $*$ is just some matrix, the symbol $\overrightarrow{\mathbf{0}}$ is a *row vector* whose entries are all 0, the symbol $\overrightarrow{\mathbf{u}}$ is a row vector, the symbol 0 is (irritatingly!) a matrix whose entries are all 0, and, critically, $\widehat{U}$ is upper-triangular with nonzero entries on the diagonal. (If you're still reading, a good exercise for you is to figure out the dimensions of all of that junk. I'll tell you that $\widehat{U}$ has $m-(j+1)$ rows and columns.) Use the nonzero entries of $\widehat{U}$ to do elimination in the rows above $\widehat{U}$. Then there's an invertible matrix E such that

$$EU = \begin{bmatrix} * & 0 \\ \overrightarrow{\mathbf{0}} & \overrightarrow{\mathbf{0}} \\ 0 & I \end{bmatrix}.$$

The important thing is that row j of EU is all zero. Then if $\mathbf{b} \in \mathbf{C}(U)$, the jth entry of $E\mathbf{b}$ is 0, and, as before, this can't happen for every $\mathbf{b} \in \mathbb{R}^m$. ■

Here is the payoff for all of our work on linear systems and inverses. It's only valid for square systems, because it talks about inverses and hinges on those technical, demanding results for upper-triangular matrices, but what a payoff for this case!

18.4 Theorem (Invertible matrix theorem). *Let $A \in \mathbb{R}^{m \times m}$. The following statements are equivalent in the sense that if any one of them is true, then all of the others are true, and if any one of them is falls, then all of the others are false.*

(i) *A is invertible.*

(ii) *For each $\mathbf{b} \in \mathbb{R}^m$, there is exactly one $\mathbf{x} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$. (The problem $A\mathbf{x} = \mathbf{b}$ always has a unique solution.)*

(iii) $\mathbf{C}(A) = \mathbb{R}^m$. *(A solution to the problem $A\mathbf{x} = \mathbf{b}$ always exists.)*

(iv) $\mathbf{N}(A) = \{\mathbf{0}_m\}$. *(If the problem $A\mathbf{x} = \mathbf{b}$ has a solution, it's unique.)*

(v) *The columns of A are independent. (The problem $A\mathbf{x} = \mathbf{b}$ has no redundant data.)*

Proof. That part (i) implies (ii) is Theorem 14.10. That part (i) implies parts (iii), (iv), and (v) is Problem 14.11. If we can prove that either of parts (iii) or (iv) implies part (i), then we'll know that part (ii) implies part (i), since parts (iii) and (iv) together imply part (ii).

So, we focus on showing that the last three parts imply the very first. Throughout, we'll use Gauss–Jordan elimination to write $EA = U$ for some invertible E and upper-triangular U. Equivalently, $A = E^{-1}U$, and so for A to be invertible, we just need U to be invertible. And since U is upper-triangular, Theorem 18.2 says that we just need the diagonal entries of U to be nonzero. So that's what we'll think about.

(iii) $\Longrightarrow$ (i) I claim that $\mathbf{C}(U) = \mathbb{R}^m$, too. Here's why: if $\mathbf{b} \in \mathbb{R}^m$, then since $\mathbf{C}(A) = \mathbb{R}^m$, there is $\mathbf{v} \in \mathbb{R}^m$ such that $A\mathbf{x} = E^{-1}\mathbf{b}$. Then $U\mathbf{x} = EA\mathbf{x} = \mathbf{b}$. (This might feel like Problem 14.12. What's different here?) So, $\mathbf{C}(U) = \mathbb{R}^m$. Lemma 18.3 then says that all of the diagonal entries of U are nonzero.

(iv) $\Longrightarrow$ (i) I claim that $\mathbf{N}(U) = \{\mathbf{0}_m\}$, too. Here's why: if $U\mathbf{x} = \mathbf{0}_m$, then $A\mathbf{x} = E^{-1}U\mathbf{x} = \mathbf{0}_m$, too. So $\mathbf{x} = \mathbf{0}_m$, as desired, and $\mathbf{N}(U) = \{\mathbf{0}_m\}$. Lemma 18.1 then says that all of the diagonal entries of U are nonzero.

(v) $\Longrightarrow$ (i) We've known for a long time that independent columns imply that the null space is as small as possible. Specifically, from Corollary 7.5, if the columns of A are independent, then $\mathbf{N}(A) = \{\mathbf{0}_m\}$. (This, by the way, is true even when A is not square.) So part (iv) is true, which implies the invertibility of A. (But when A is not square, it doesn't make sense to talk about an inverse.) ■

I hope you're surprised that existence and uniqueness by themselves, separately, are enough to imply existence and uniqueness together! This is a special property of square systems that nonsquare problems need not share, as we will see. We will give examples of nonsquare problems for which existence is always true but uniqueness fails, and for which existence sometimes fails but uniqueness is always true.

18.5 Problem (!). Uniqueness can never fail "only sometimes." Let $A \in \mathbb{R}^{m \times n}$ and let $\mathbf{b}_1$, $\mathbf{b}_2 \in \mathbb{R}^m$. Suppose that there are two different $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^n$ such that $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_1$. Explain why if the problem $A\mathbf{x} = \mathbf{b}_2$ has a solution, it really has infinitely many solutions. [Hint: $\mathbf{x}_1 - \mathbf{x}_2 \in \mathbf{N}(A)$. *Adapt the proof of Theorem 11.6.*]

18.6 Problem (!). Use the invertible matrix theorem to prove our longstanding Conjectures 6.3 and 7.8. (We still don't have the tools to prove Conjecture 8.18.)

18.7 Problem ($\star$). Let $A \in \mathbb{R}^{m \times m}$. Recall that $E \in \mathbb{R}^{m \times m}$ is a **LEFT INVERSE** of A if $EA = I_m$ and $F \in \mathbb{R}^{m \times m}$ is a **RIGHT INVERSE** of A if $AF = I_m$. (Important: we're not assuming that E or F is invertible here. Also, reread Corollary 14.5 right now.)

(i) Prove that if A has a left inverse, then A is invertible. [Hint: *if* $A\mathbf{x} = \mathbf{0}_m$, *what is* $EA\mathbf{x}$*?*]

(ii) Prove that if A has a right inverse, then A is invertible. [Hint: *if* $\mathbf{b} \in \mathbb{R}^m$, *what is* $AF\mathbf{b}$*?*]

18.8 Remark. *The nonzero diagonal entries of an upper-triangular matrix are sometimes called its* **PIVOTS**. *The pivots of a general* $A \in \mathbb{R}^{m \times m}$ *are the nonzero diagonal entries of the upper-triangular matrix to which* A *can always be transformed by elimination and row interchanges, i.e., by Theorem 13.3. This language is a little perilous, as we never proved that the matrix* U *from Theorem 13.3 was unique—could we write* $E_1A = U_1$ *and* $E_2A = U_2$ *with* U_1 *and* U_2 *both upper-triangular,* $U_1 \neq U_2$*, and* E_1 *and* E_2 *as the product of elimination and/or permutation matrices? What's important from the point of view of invertibility is not the exact value of these "pivots" but rather whether they are all nonzero or not.*

18.9 Problem (+). Let $A \in \mathbb{R}^{m \times m}$ and suppose that $E_1, E_2 \in \mathbb{R}^{m \times m}$ are invertible with E_1A and E_2A both upper-triangular. Prove that if E_1A has no nonzero diagonal entries, then E_2A also has no nonzero diagonal entries. [Hint: *what goes wrong if* E_2A *has some nonzero diagonal entries?*] We will eventually prove that E_1A and E_2A must have the same number of nonzero diagonal entries, although we need more technology for that.

Content from Strang's *ILA* 6E. Page 41 introduces the terminology "pivot." I personally feel that the phrase "nonzero pivot" is redundant. Informally, you should think of the pivots as "the nonzero things that you multiply by when doing elimination." Because we can permute rows even when we don't need to avoid zero diagonal entries, we can select an "ideal" pivot at any state of elimination—see "'Partial Pivoting' to Reduce Roundoff Errors" on p. 66 and think once more about taking a numerical linear algebra class after this one.

Day 19: Wednesday, October 1.

Our best successes in this course arguably come from square systems: $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{m \times m}$ and $\mathbf{b} \in \mathbb{R}^m$, same number of equations as unknowns. We will see that it is with square systems alone that we have a chance (not a guarantee) for both existence and uniqueness of solutions—it is possible both to be able to solve the problem and have only one solution for it. With nonsquare systems—$A\mathbf{x} = \mathbf{b}$, $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $m \neq n$—we will show that either existence or uniqueness always fails (maybe both). Understanding how to quantify and qualify our failures, and how to move on from them, will be the central part of our forthcoming story. We can see this happen with relatively small systems using relatively few numbers.

19.1 Example. We consider our favorite problem $A\mathbf{x} = \mathbf{b}$ for the variety of A below.

(i) It's hard to get nicer than

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

because then the unique solution to $A\mathbf{x} = \mathbf{b}$ is always just $\mathbf{x} = \mathbf{b}$ (for $\mathbf{b} \in \mathbb{R}^2$).

(ii) It's easy to get less nice, though. Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then existence fails for $\mathbf{b}$ with $b_2 \neq 0$, while uniqueness also fails. Inspired by Problem 11.7 and the fact that $A\mathbf{e}_2 = \mathbf{0}_2$, where $\mathbf{e}_2 = (0, 1)$, we can check that

$$A\left(\begin{bmatrix} b_1 \\ 0 \end{bmatrix} + c\mathbf{e}_2\right) = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

for any b_1, $c \in \mathbb{R}$. Thus solutions, when they exist, are never unique. The dependence of the columns affects both existence and uniqueness here, per the invertible matrix theorem.

(iii) With

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

existence fails for those $\mathbf{b} \in \mathbb{R}^3$ with $b_3 \neq 0$. But solutions, when they exist are unique, because the only solution to $A\mathbf{x} = \mathbf{0}_3$ is $\mathbf{x} = \mathbf{0}_2$. We saw this in Problem 11.7, and it's closely related to the independence of the columns. We'll revisit this soon.

(iv) With

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

both existence and uniqueness fail, since we can't solve $A\mathbf{x} = (0, 1, 0)$, while when we can solve $A\mathbf{x} = (b_1, 0, 0)$ with $\mathbf{x} = (b_1, 0)$, we can also solve it with $\mathbf{x} = (b_1, c)$ for any $c \in \mathbb{R}$. And the columns are dependent.

(v) With

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

we have existence but not uniqueness: take $\mathbf{x} = (b_1, b_2, c)$ to solve $A\mathbf{x} = (b_1, b_2)$. Again, dependent columns.

(vi) Last, existence and uniqueness fail for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Once again, we can only solve $A\mathbf{x} = \mathbf{b}$ for $b_2 = 0$, and when we can, $\mathbf{x} = (b_1, c_2, c_3)$ is a solution for any c_2, $c_3 \in \mathbb{R}$. Dependent columns are the worst.

Here is what the previous example suggests as we move beyond square systems.

19.2 Conjecture. *Let $A \in \mathbb{R}^{m \times n}$.*

(i) *If $m > n$ (more equations than unknowns, more rows than columns, A is taller than it is wide), then we will always fail to solve $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^m$. That is, $\mathbf{C}(A) \neq \mathbb{R}^m$. It may or may not be possible to get unique solutions.*

(ii) *If $m < n$ (more unknowns than equations, more columns than rows, A is wider than it is tall), then we will never be able to solve $A\mathbf{x} = \mathbf{b}$ uniquely. Solutions may or may not exist in the first place.*

Additionally, unlike with the invertible matrix theorem for square matrices, neither existence nor uniqueness by itself is enough to guarantee the other condition.

Content from Strang's *ILA* 6E. Now is a good time to reread p. 38.

Our goal is to prove Conjecture 19.2 and do a bit more: to qualify and quantify our inherent failure to obtain existence and uniqueness for nonsquare problems. Previously, we focused on existence first (the column space) and then uniqueness (the null space). This is my preference: I think it's more important to talk about having *some* solutions to work with first and then ask if they are the *only* solutions. Now that we are a little more experienced, we are going to do the reverse. First we'll study the specific problem $A\mathbf{x} = \mathbf{0}_m$ for $A \in \mathbb{R}^{m \times n}$ (usually with $m \neq n$), and from that we'll learn techniques for approaching the more general problem $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \in \mathbb{R}^m$ arbitrary.

Day 20: Friday, October 3.

We are going to calculate a lot of null spaces for increasingly more complicated matrices whose structures nonetheless have a very special underlying form.

20.1 Example. We first reconsider two matrices from Example 19.1.

(i) The null space of I_2 is just $\{\mathbf{0}_2\}$, for if $I_2\mathbf{v} = \mathbf{0}_2$, then since $I_2\mathbf{v} = \mathbf{v}$, we just have $\mathbf{v} = \mathbf{0}_2$.

(ii) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The system $A\mathbf{x} = \mathbf{0}_3$ (for $\mathbf{x} \in \mathbb{R}^2$) is just

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ 0 = 0, \end{cases}$$

so $\mathbf{N}(A) = \{\mathbf{0}_2\}$ once again.

20.2 Problem (+). Here's the generalization of Example 20.1. Prove that

$$\mathbf{N}(I_n) = \{\mathbf{0}_n\} \quad \text{and} \quad \mathbf{N}\left(\begin{bmatrix} I_n \\ 0 \end{bmatrix}\right) = \{\mathbf{0}_n\}.$$

In the second, block matrix, you should interpret the symbol 0 as representing one or more rows of zeros. [Hint: *convince yourself that*

$$\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ B\mathbf{x} \end{bmatrix}$$

whenever $A \in \mathbb{R}^{m_1 \times n}$, $B \in \mathbb{R}^{m_2 \times n}$, *and* $\mathbf{x} \in \mathbb{R}^n$.]

When we first met the null space in the context of independence, we didn't have many tools for actually solving systems. Now we do, so we can say a lot more about the structure and behavior of null spaces. We'll start with a bunch of examples where the matrices have an eerily nice form. Then we'll think about that form and how to achieve it for arbitrary matrices.

20.3 Example. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}.$$

For $\mathbf{x} \in \mathbb{R}^4$, we have $A\mathbf{x} = \mathbf{0}_2$ if and only if

$$\begin{cases} x_1 \phantom{{}+x_2} + 2x_3 + 3x_4 = 0 \\ \phantom{x_1+{}} x_2 \phantom{{}+2x_3} + 4x_4 = 0 \end{cases}$$

This is not as nice as the square upper-triangular systems that we have previously studied. There's no equation with just one variable in it!

The right, if not immediately obvious, strategy is to solve for what we can easily solve for. The unknowns x_1 and x_2 have coefficients of 1 on them, so solving for those two variables in terms of x_3 and x_4 is easier, comparatively speaking, than solving for x_3 or x_4.

Gotta solve for something, anyway. We get

$$\begin{cases} x_1 = -2x_3 - 3x_4 \\ x_2 = -4x_4, \end{cases}$$

and if we put $\mathbf{x} = (x_1, x_2, x_3, x_4)$, then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3x_4 \\ -4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_4 \\ -4x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix}.$$

Think about that for a moment. We have shown that every $\mathbf{x} \in \mathsf{N}(A)$ is a linear combination of those two vectors on the right. More compactly,

$$\mathsf{N}\left(\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}\right) = \mathsf{C}\left(\begin{bmatrix} -2 & -3 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

(Strictly speaking, we have shown that *if* $\mathbf{x} \in \mathsf{N}(A)$, *then* $\mathbf{x}$ is in the column space of that 4×2 matrix. You should check your work and show that each column in that 4×2 matrix is in $\mathsf{N}(A)$.)

I hope it looks like there are two degrees of freedom in describing the null space, which come from those two variables x_3 and x_4 whose values we didn't (couldn't) specify. You might call those "free" variables, because the values of x_1 and x_2 were specified. And our matrix has rank 2, right? This can't be an accident.

Do you see the pattern here? Our original matrix A had the block structure

$$A = \begin{bmatrix} I_2 & F \end{bmatrix}, \qquad F := \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix},$$

and its null space is

$$\mathsf{N}(A) = \mathsf{C}\left(\begin{bmatrix} -F \\ I_2 \end{bmatrix}\right).$$

This can't be an accident.

Content from Strang's *ILA* 6E. This example is basically the same as Example 1 on p. 93. Strang calls the columns of

$$\begin{bmatrix} -2 & -3 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the "special solutions" for $A\mathbf{x} = \mathbf{0}_2$. What is "special" about these solutions is that they are linearly independent, and every solution to $A\mathbf{x} = \mathbf{0}_2$ is in the span of these solutions.

20.4 Problem (+). Here's the generalization of Example 20.3. Let $n > m$ and $F \in \mathbb{R}^{m \times (n-m)}$. Prove that

$$\mathbf{N}\left(\begin{bmatrix} I_m & F \end{bmatrix}\right) = \mathbf{C}\left(\begin{bmatrix} -F \\ I_m \end{bmatrix}\right) \quad \text{and so} \quad \mathbf{N}\left(\begin{bmatrix} I_m & F \end{bmatrix}\right) \neq \{\mathbf{0}_n\}.$$

[Hint: *write any* $\mathbf{x} \in \mathbb{R}^n$ *as* $\mathbf{x} = (\mathbf{x}_m, \mathbf{x}_{n-m})$ *with* $\mathbf{x}_m \in \mathbb{R}^m$ *and* $\mathbf{x}_{n-m} \in \mathbb{R}^{n-m}$, *and argue that*

$$\begin{bmatrix} A & B \end{bmatrix} \mathbf{x} = \begin{bmatrix} A\mathbf{x}_m & B\mathbf{x}_{n-m} \end{bmatrix}$$

when $A \in \mathbb{R}^{m \times m}$ *and* $B \in \mathbb{R}^{m \times (n-m)}$.]

20.5 Problem (!). Putting more zero rows into the matrix doesn't change the null space. We saw this already in Example 20.1. Adapt the work of Example 20.3 to express the null space of

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

as a column space.

20.6 Problem (+). Here's the generalization of Problem 20.5. Let $1 \le r < n$ and $F \in \mathbb{R}^{r \times (n-r)}$. Prove that

$$\mathbf{N}\left(\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix}\right) = \mathbf{C}\left(\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}\right) \quad \text{and so} \quad \mathbf{N}\left(\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix}\right) \neq \{\mathbf{0}_n\}.$$

Here the two occurrences of the symbol 0 are meant to represent matrices whose entries are all the number 0. (What are the dimensions of those matrices?)

20.7 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

We proceed as in Example 20.3: assume $A\mathbf{x} = \mathbf{0}_2$ and write this as the linear system

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 0 \\ x_3 + 4x_4 = 0. \end{cases}$$

We solve for the variables with the simples coefficients of 1; these are now x_1 and x_3:

$$\begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = -4x_4. \end{cases}$$

Vectorizing, we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_4 \\ 0 \\ -4x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Thus

$$\mathsf{N}\left(\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}\right) = \mathsf{C}\left(\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix}\right). \tag{20.1}$$

Prior examples and problems about finding null spaces had the identity matrix show up in a pretty obvious way. It looks like the 2×2 identity is jumbled here, as compared to the matrix in Example 20.3. How can we sort it out?

We've handled "jumbled" matrices before. Recall that a permutation matrix $P \in \mathbb{R}^{m \times m}$ is a matrix formed by reordering the columns of the $m \times m$ identity matrix. If $B \in \mathbb{R}^{m \times n}$, then PB reorders the rows of B per the ordering of the columns in P. With our A in this example, however, it's a matter of reordering the *columns* of the nicer-looking matrix from Example 20.3. We'd be happier if the columns of the 2×2 identity matrix appeared first in A. Equivalently, how do we turn the matrix

$$B := \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

from Example 20.3 into our current matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}?$$

If multiplying on the left by a permutation matrix reorders rows, we might guess that multiplying on the right reorders columns. This turns out to be true.

Here A, $B \in \mathbb{R}^{2 \times 4}$, so if we multiply on the right by a permutation matrix P, we better have $P \in \mathbb{R}^{4 \times 4}$. What we want is $BP = A$, and we know that if $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{bmatrix}$, then

$$BP = \begin{bmatrix} B\mathbf{p}_1 & B\mathbf{p}_2 & B\mathbf{p}_3 & B\mathbf{p}_4 \end{bmatrix}.$$

So, we want

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 & \mathbf{a}_2 & \mathbf{a}_4 \end{bmatrix} = A = BP = \begin{bmatrix} B\mathbf{p}_1 & B\mathbf{p}_2 & B\mathbf{p}_3 & B\mathbf{p}_4 \end{bmatrix}.$$

We know that the columns of P are going to be columns of I_4, and we know that $B\mathbf{e}_j = \mathbf{b}_j$, where $\mathbf{e}_j$ is the jth column of I_4, i.e., the jth standard basis vectors. All together, this says that we should take

$$P = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{e}_2 & \mathbf{e}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to find

$$BP = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{e}_2 & \mathbf{e}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} = A.$$

This nice relationship between A and B shows up at the level of the null spaces. Example 20.3 taught us that

$$\mathbf{N}(B) = \mathbf{N}\left(\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}\right) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right),$$

and now we have shown that

$$\mathbf{N}(BP) = \mathbf{N}(A) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix}\right).$$

Recall that multiplying on the left by P interchanges rows 2 and 3, so

$$\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix} = P \begin{bmatrix} -2 & -3 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So here is the conclusion:

$$\mathbf{N}\left(\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix} P\right) = \mathbf{N}\left(\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}\right) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix}\right) = \mathbf{C}\left(P \begin{bmatrix} -2 & -3 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Last thing: two degrees of freedom in this null space, right? And, again, $\text{rank}(A) = 2$.

Content from Strang's *ILA* 6E. This was basically Example 2 on p. 94.

20.8 Problem (+). Here's the generalization of Example 20.7. Let $m < n$ and $F \in \mathbb{R}^{m \times (n-m)}$. Let $P \in \mathbb{R}^{n \times n}$ be invertible (note that the P in that example was its own inverse). Prove that

$$\mathbf{N}\left(\begin{bmatrix} I_m & F \end{bmatrix} P\right) = \mathbf{C}\left(P^{-1}\begin{bmatrix} -F \\ I_m \end{bmatrix}\right) \quad \text{and so} \quad \mathbf{N}\left(\begin{bmatrix} I_m & F \end{bmatrix} P\right) \neq \{\mathbf{0}_n\}. \tag{20.2}$$

[Hint: *you want to solve* $\begin{bmatrix} I_m & F \end{bmatrix} P\mathbf{x} = \mathbf{0}_m$. *Put* $\mathbf{y} = P\mathbf{x}$. *Now you want to solve* $\begin{bmatrix} I_m & F \end{bmatrix} \mathbf{y} = \mathbf{0}_m$. *You know how to do this from Problem 20.4. Along the way, check that the matrix product in the column space above is actually defined.*]

20.9 Problem (!). Adapt the work of Example 20.7 to express the null space of

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

as a column space.

20.10 Problem (+). Here's the generalization of Problem 20.9. Let $1 \leq r < n$ and $F \in \mathbb{R}^{r \times (n-r)}$. Let $P \in \mathbb{R}^{n \times n}$ be invertible. Prove that

$$\mathbf{N}\left(\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P\right) = \mathbf{C}\left(P^{-1}\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}\right) \quad \text{and so} \quad \mathbf{N}\left(\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P\right) \neq \{\mathbf{0}_n\}. \tag{20.3}$$

As before, the symbols 0 denote matrices whose entries are all 0.

Content from Strang's *ILA* 6E. The expressions (20.2) and (20.3) for the null space appear in the box on p. 97 and the subsequent "Review" paragraph. You should now be able to understand and appreciate *all* of p. 142 and *all* of pp. 410–411, including #3. Interpret P^{T} as P^{-1} for now, as we haven't talked about transposes.

Day 21: Monday, October 6.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

RREF (N—be able to give examples of matrices not in RREF and explain why), pivot column of a matrix, free column of a matrix in RREF

The conclusion from all of the recent problems and examples should be that the null space is "easy" to describe when the matrix under consideration has one of the following special forms:

$$I_n, \qquad \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} I_m & F \end{bmatrix}, \qquad \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} I_m & F \end{bmatrix} P, \quad \text{or} \quad \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P. \tag{21.1}$$

Above, P is in practice a permutation matrix (and actually a rather specific kind of permutation matrix), although the only thing that we really required in the null space calculations was the invertibility of P.

Certainly not every matrix has one of these six forms—common to all of these forms is the appearance of an identity matrix within the columns of overall matrix. But every nonzero matrix can be *reduced* to one of these forms by the elementary row operations that we already know and love—by Gauss–Jordan elimination. Once again, the major technique in computational linear algebra is putting zeros in matrices.

21.1 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}.$$

Previously we only performed elementary row operations on square matrices, but they certainly work on nonsquare matrices, too. We compute

$$\begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix} \xrightarrow[E_{21}]{\text{R2} \mapsto \text{R2}-2\times\text{R1}} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 8 \end{bmatrix}, \qquad E_{21} := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[D_{33}]{\text{R3} \mapsto (1/2)\times\text{R3}} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \qquad D_{33} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\xrightarrow[E_{13}]{\text{R1} \mapsto \text{R1}-\text{R3}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \qquad E_{12} := \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[P_{23}]{\text{R2} \mapsto \text{R3},\ \text{R3} \mapsto \text{R2}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad P_{23} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

That is,

$$EA = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad E := P_{23}E_{12}D_{33}E_{21}.$$

Now put

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}.$$

Then

$$EA = \begin{bmatrix} I_2 & F \\ 0 & 0 \end{bmatrix} P =: R_0$$

where the symbol 0 denotes the matrix $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

We saw in Problem 20.9 that

$$\mathbf{N}(R_0) = \mathbf{N}\left(\begin{bmatrix} I_2 & F \\ 0 & 0 \end{bmatrix} P\right) = \mathbf{C}\left(P^{-1}\begin{bmatrix} -F \\ I_2 \end{bmatrix}\right) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix}\right).$$

This is helpful here because we have $EA = R_0$ with E invertible. If $\mathbf{v} \in \mathbf{N}(A)$, then $A\mathbf{v} = \mathbf{0}_2$, so $E(A\mathbf{v}) = \mathbf{0}_m$. And then

$$\mathbf{0}_2 = (EA)\mathbf{v} = R_0\mathbf{v},$$

so $\mathbf{v} \in \mathbf{N}(R_0)$. Conversely, if $\mathbf{v} \in \mathbf{N}(R_0)$, then $R_0\mathbf{v} = \mathbf{0}_2$, so

$$A\mathbf{v} = E^{-1}R_0\mathbf{v} = E^{-1}\mathbf{0}_2 = \mathbf{0}_2.$$

Thus $\mathbf{N}(A) = \mathbf{N}(R_0)$. This is a nice auxiliary fact: multiplying on the left by an invertible matrix does not change its null space.

All together, we conclude

$$\mathbf{N}\left(\begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}\right) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix}\right).$$

The example above is prototypical: Gauss–Jordan elimination "reduces" any $A \in \mathbb{R}^{m \times n}$ to a matrix of the following structure.

21.2 Definition. *A matrix $R \in \mathbb{R}^{m \times n}$ is in* REDUCED ROW ECHELON FORM (RREF) *if it has the following four properties.*

Row Property 1. *Any row of R whose entries are all zero is below any row with some nonzero entries.*

Row Property 2. *If a row contains nonzero entries, the first nonzero entry of that row is 1, called the* LEADING 1 *or the* PIVOT *for that row.*

Column Property 1. *The other entries of any column containing a leading* 1 *are* 0*. That is, a column containing a leading* 1 *is a column of the* $m \times m$ *identity matrix* I_m*, equivalently, a standard basis vector for* $\mathbb{R}^m$*. A column containing a leading* 1 *is called a* **PIVOT COLUMN***. A column that is not a pivot column is called a* **FREE COLUMN***.*

Column Property 2. *If* $1 \le i < j \le m$ *and rows* i *and* j *both contain nonzero entries, then the leading* 1 *of row* i *appears in a column before the column containing the leading* 1 *of row* j*. That is, the leading* 1 *of a given row is "to the left" of the leading* 1*'s in the rows below.*

21.3 Example. The matrix

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF. Here's why.

There is only one row of all 0 entries, and it's at the bottom of R, below the rows with nonzero entries. The rows with nonzero entries are rows 1 and 2, and their first nonzero entries are 1. Columns 1 and 3 contain these leading 1's, and the other entries of columns 1 and 3 are all 0. (That is, columns 1 and 3 are columns of I_3.) Last, row 2 contains nonzero entries, and the leading 1 of row 2 is in column 3, which is after the leading 1 in row 1 (which is in column 1).

21.4 Problem (!). Explain *all* of the reasons why

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is not in RREF.

21.5 Problem (!). Use the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

to answer the following question. If a matrix $R \in \mathbb{R}^{m \times n}$ is in RREF, is every column of the identity matrix I_m that appears in R also a pivot column?

Day 22: Wednesday, October 8.

22.1 Example. For practice with the axioms of the RREF from Definition 21.2, we construct all matrices $R \in \mathbb{R}^{3\times4}$ that are in RREF and that have pivot columns in columns 2 and 4 only. We proceed via the following steps.

1. Start with the first column (a very good place to start). If any entry is nonzero, that entry is the leading nonzero entry in its row (can't start earlier than the first column), and so column 1 is a pivot column. This is not allowed under the rules of our current game, so the first column is $\mathbf{0}_3$, and therefore

$$R = \begin{bmatrix} 0 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}.$$

2. The second column is a pivot column, so it is either $\mathbf{e}_1$, $\mathbf{e}_2$, or $\mathbf{e}_3$. Column Property 2 basically tells us that it's $\mathbf{e}_1$. Otherwise, there would be no first appearance of $\mathbf{e}_1$ before $\mathbf{e}_2$ or $\mathbf{e}_3$.

Here is another way to see this. If column 2 is $\mathbf{e}_2$, then

$$R = \begin{bmatrix} 0 & 0 & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The other two entries in row 1 (in columns 3 and 4) can't both be 0, as that would violate Row Property 1. So, at least one of them is nonzero, thus a leading nonzero entry. But then $\mathbf{e}_1$ appears in column 3 or 4, again contradicting Column Property 2.

3. We now know

$$R = \begin{bmatrix} 0 & 1 & ? & ? \\ 0 & 0 & ? & ? \\ 0 & 0 & ? & ? \end{bmatrix}.$$

Look at the third column. If it has a nonzero entry in rows 2 or 3, that is the leading nonzero entry in that row, and so column 3 is a pivot column. This is not allowed in our game. However, it doesn't look like there are any restrictions on the $(1,3)$-entry of R, since that would not be a leading nonzero entry in row 1. Let's write

$$R = \begin{bmatrix} 0 & 1 & * & ? \\ 0 & 0 & 0 & ? \\ 0 & 0 & 0 & ? \end{bmatrix}.$$

We have upgraded the $(1,3)$-entry from ? to $*$ to emphasize that it can be any number right now, zero or not.

4. The fourth column is a pivot column, so it is $\mathbf{e}_1$, $\mathbf{e}_2$, or $\mathbf{e}_3$. If it's $\mathbf{e}_1$, then

$$\begin{bmatrix} 0 & 1 & * & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

but then the 1 in the $(1,4)$-entry is not the leading nonzero entry in row 1, so column 4 is not a pivot column after all. If column 4 is $\mathbf{e}_3$, then

$$R = \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and that contradicts Row Property 1. The only choice left is that column 4 is $\mathbf{e}_2$.

We conclude that all matrices $R \in \mathbb{R}^{3\times 4}$ that are in RREF with pivot columns in columns 2 and 4 only have the form

$$R = \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the $(1,3)$-entry is arbitrary. This is a pretty restricted family of matrices.

22.2 Problem (!). Write down all possible 2×2 matrices that are in RREF. How do you know that you've found them all? (Some of the entries of your matrices will have to be more or less arbitrary real numbers, but be sure to specify if certain values are excluded.)

22.3 Problem ($\star$). Let $A \in \mathbb{R}^{m\times n}$. Why is the jth column of A the zero vector (in $\mathbb{R}^m$) precisely when the jth column of the RREF of A is the zero vector? [Hint: $EA = \mathsf{rref}(A)$ *for some invertible* E. *What are* $E\mathbf{0}_m$ *and* $E^{-1}\mathbf{0}_m$*?*]

Here is the fruit of Gauss–Jordan elimination.

22.4 Theorem. *Let* $A \in \mathbb{R}^{m\times n}$ *be nonzero (i.e.,* A *has at least one nonzero entry). There exists an invertible matrix* $E \in \mathbb{R}^{m\times m}$ *such that* EA *is in RREF with one of the following forms:*

(i) I_n, *in which case* A *is square and invertible;*

(ii) $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$, *in which case* $n < m$ *(more rows than columns);*

(iii) $\begin{bmatrix} I_m & F \end{bmatrix}$, *in which case* $m < n$ *(more columns than rows) and* $F \in \mathbb{R}^{m\times(n-m)}$*;*

(iv) $\begin{bmatrix} I_m & F \end{bmatrix} P$, *with the same conditions as in form (iii) and* P *a permutation matrix;*

(v) $\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix}$, *in which case* $1 \le r \le \min\{m,n\}$ *and* $F \in \mathbb{R}^{r\times(n-r)}$*;*

(vi) $\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P$, *with the same conditions as in form (v) and* P *a permutation matrix.*

This form is unique in the sense that if $\widetilde{E} \in \mathbb{R}^{m \times m}$ is invertible with $\widetilde{E}A$ in RREF, then $EA = \widetilde{E}A$. We write $EA = \mathsf{rref}(A)$ and call $\mathsf{rref}(A)$ the **RREF OF** *A.*

We are not going to prove this theorem in detail. Existence, again, is just Gauss–Jordan elimination. Uniqueness is surprisingly more annoying.

Proof. Start with the first nonzero column *from the left* (at least one column is nonzero since A is nonzero). If needed, multiply by a permutation matrix so the first entry in this column is nonzero. If there are more zero entries in this column, multiply by elimination matrices to make the entries in rows 2 and below all zero. Then go to the first row below row 1 that has a nonzero entry; call that row i (so $i \geq 2$). Go to the column that has the first nonzero entry in that row; call that column j (so $j \geq 2$). Repeat the permutation (if needed) and elimination operations to make the entries in rows $i+1$ and below of column j all zero. Continue until you have reached the last row of the matrix or until all rows below have only zero entries.

Now start with the first nonzero row from the bottom; call this row i. Multiply by a scaling matrix so that the first nonzero entry in this row (starting *from the left*) is one. Say that this first nonzero entry appears in column j. If $i \geq 2$, multiply by elimination matrices to make the entries in rows $i-1$ to 1 of column j all zero. Then go to the first nonzero row above row i (if there is any such row) and the first nonzero element of that row, which will be in column $j-1$ or before. Go to the column that has the first nonzero entry in that row and multiply by a scaling matrix so that the first nonzero entry is one. Use elimination to create zeros in the entries of that column that fall in the rows above. Continue until you have reached the first row of the matrix or until all rows above have only zero entries.

If the $(1,1)$-entry of the matrix is nonzero, multiply by a scaling matrix to make it one. Finally, multiply the matrix by a permutation matrix so that all rows whose entries are only zero are below all rows with nonzero entries. ■

22.5 Problem ($\star$). Example 21.1 constructs $E \in \mathbb{R}^{3\times 3}$ such that

$$E \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) By revisiting the elementary row operations in that example, explain why E in Theorem 22.4 might not be unique. [Hint: *could P_{23} or D_{33} have appeared earlier or later?*]

(ii) With E from Example 21.1, find a permutation matrix $\widetilde{P}$ such that

$$E \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \widetilde{P}.$$

Contrast this with the result of Example 21.1 and explain how this shows that P and F from Theorem 22.4 may not be unique.

(iii) Explain why there cannot exist a matrix $A \in \mathbb{R}^{3\times4}$ such that

$$\mathsf{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Conclude that not every permutation matrix P can appear in the forms of Theorem 22.4.

(iv) Give examples of two matrices $A \neq B$ that have the same RREF. [Hint: *look no further than the first form in Theorem 22.4.*]

Content from Strang's *ILA* 6E. A reduction to RREF is given at the top of p. 95 and another is done in Example 2 at the bottom of the page. A third is Example 3 on pp. 97–98, and this also includes a null space calculation and remarks on the CR-factorization (which we will revisit shortly). Page 96 gives the algorithm for computing the RREF column by column. Read p. 142 up to but not including the "Factorization" box.

22.6 Problem ($\star$). Find a matrix $A \in \mathbb{R}^{3\times4}$ whose entries are all nonzero such that

$$\mathsf{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Provide a matrix $E \in \mathbb{R}^{3\times3}$ such that $EA = \mathsf{rref}(A)$; you may express E as a product of elementary matrices, and you do not have to multiply that product out.

22.7 Problem ($\star$). For each of the six RREF forms in Theorem 22.4, find a matrix whose RREF has that form. Construct your matrix so that it has at least two rows and at least two columns and that all of its entries are nonzero. For the forms with a permutation matrix, ensure that a permutation matrix is actually needed in your form (don't just let P be the identity, which is a permutation matrix, but a boring one). Give the exact RREF of each matrix, not just the general form that it has.

Now we can resolve Conjecture 19.2 using the RREF. Let $A \in \mathbb{R}^{m\times n}$ and $R = \mathsf{rref}(A)$. Let $E \in \mathbb{R}^{m\times m}$ be invertible with $EA = R$, so $A = E^{-1}R$.

First suppose $m > n$, so A has more rows than columns. We want to show that existence fails: $\mathbf{C}(A) \neq \mathbb{R}^m$. Look back at the possible types of RREF for A from Theorem 22.4. If the RREF is the first type (I_n), then A is square, and that isn't the case here. If the RREF is the third or fourth type, then A has more columns than rows, and that isn't the case here. So the RREF is one of the three remaining types, and each of those types always has at least one row of zeros. This means that $R\mathbf{x}$ always has a zero row (entry) for any $\mathbf{x} \in \mathbb{R}^n$, and therefore $\mathbf{C}(R) \neq \mathbb{R}^m$. For example, the vector whose entries are all 1 isn't in $\mathbf{C}(R)$.

Let $\mathbf{c} \in \mathbb{R}^m$ with $\mathbf{c} \notin \mathbf{C}(R)$. I claim that $E^{-1}\mathbf{c} \notin \mathbf{C}(A)$. Otherwise, if $E^{-1}\mathbf{c} \in \mathbf{C}(A)$, then there is $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = E^{-1}\mathbf{c}$. But then $EA\mathbf{x} = EE^{-1}\mathbf{c}$, so $R\mathbf{x} = \mathbf{c}$, and $\mathbf{c} \in \mathsf{C}(R)$ after all. That can't happen, so $E^{-1}\mathbf{c} \notin \mathsf{C}(A)$, and therefore $\mathsf{C}(A) \neq \mathbb{R}^m$.

On to part (ii): assume $m < n$ now, so A has more columns than rows. We want to show that uniqueness fails: $\mathsf{N}(A) \neq \{\mathbf{0}_n\}$. Look back at the possible types of RREF for A from Theorem 22.4. If the RREF is the first type (I_n), then A is square, and that isn't the case here. If the RREF is the second type, then A has more rows than columns, also false here. So the RREF is one of the four remaining types, and in each of those types we've deduced that the RREF has a nontrivial null space (specifically, we did this in Problems 20.4, 20.6, 20.8, and Problem 20.10), and and so A has a nontrivial null space as well. Specifically, since there is $\mathbf{x} \in \mathsf{N}(R)$ such that $\mathbf{x} \neq \mathbf{0}_n$, we have $R\mathbf{x} = \mathbf{0}_m$, thus $E^{-1}R\mathbf{x} = E^{-1}\mathbf{0}_m$, and then $A\mathbf{x} = \mathbf{0}_m$. Hence $\mathbf{x} \in \mathsf{N}(A)$ and $\mathbf{x} \neq \mathbf{0}_n$, so $\mathsf{N}(A) \neq \{\mathbf{0}_n\}$.

22.8 Theorem. *Let $A \in \mathbb{R}^{m \times n}$.*

(i) *If $m > n$, then $\mathsf{C}(A) \neq \mathbb{R}^m$, and so we can't always solve $A\mathbf{x} = \mathbf{b}$.*

(ii) *If $m < n$, then $\mathsf{N}(A) \neq \{\mathbf{0}_n\}$, and so solutions to $A\mathbf{x} = \mathbf{b}$, if they exist, are never unique.*

22.9 Problem (!). Prove that any list of more than n vectors in $\mathbb{R}^n$ is dependent.

Content from Strang's *ILA* 6E. This is discussed in the "Important" box on p. 98 and the two paragraphs preceding that. Now read Example 4 on p. 98.

Theorem 22.8 is disappointing: it tells us to expect failure (of existence and/or uniqueness). However, most of math is learning how to manage failure. That's what we'll do. If we can't solve the problem, can we solve a "related" problem? (Yes: this is least squares.) If we have too many solutions to the problem, can we pick a "best" solution? (Yes: this is the pseudoinverse). We'll build the technology to address these questions after a little more practice with the RREF.

Day 23: Friday, October 10.

We now know that if $A \in \mathbb{R}^{m \times n}$ with $m \neq n$, either existence or uniqueness of solutions to $A\mathbf{x} = \mathbf{b}$ will fail, depending on whether $m > n$ or $m < n$. We can be much more precise about this failure: what aspects of $\mathbf{b}$ will guarantee existence or nonexistence, what will solutions $\mathbf{x}$ look like when they do exist, and how nonuniqueness is encoded in the solutions.

23.1 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}.$$

We studied the null space of A in Example 21.1. Now we study the general problem $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ arbitrary. Since $A \in \mathbb{R}^{3\times 4}$, Theorem 22.8 tells us that solutions, if they exist, will never be unique, but that theorem doesn't tell us anything about whether we can solve $A\mathbf{x} = \mathbf{b}$ for *every* $\mathbf{b}$ or not.

For a small problem like this, the most efficient thing to do is to put the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ into RREF in the form $\begin{bmatrix} R_0 & E\mathbf{b} \end{bmatrix}$, where $R_0 = \mathsf{rref}(A)$ and $EA = R_0$ with E invertible. We basically repeat the steps of Example 21.1, where we were taking $\mathbf{b} = \mathbf{0}_3$ throughout. This time, however, we don't write out the elementary matrices that do all the elimination; you can just look at that example. We could do the following calculations in several different orders and end up with the same thing; I'm going to follow the pseudocode of the proof of Theorem 22.4.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 7 & b_1 \\ 2 & 4 & 2 & 14 & b_2 \\ 0 & 0 & 2 & 8 & b_3 \end{array}\right] \xrightarrow{\mathsf{R2} \,\mapsto\, \mathsf{R2}-2\times\mathsf{R1}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 7 & b_1 \\ 0 & 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 2 & 8 & b_3 \end{array}\right]$$

$$\xrightarrow{\mathsf{R3} \,\mapsto\, (1/2)\times\mathsf{R3}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 7 & b_1 \\ 0 & 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & 4 & b_3/2 \end{array}\right]$$

$$\xrightarrow{\mathsf{R1} \,\mapsto\, \mathsf{R1}-\mathsf{R3}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 - b_3/2 \\ 0 & 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & 4 & b_3/2 \end{array}\right]$$

$$\xrightarrow{\mathsf{R2} \,\mapsto\, \mathsf{R3},\ \mathsf{R3}\mapsto\mathsf{R2}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 - b_3/2 \\ 0 & 0 & 1 & 4 & b_3/2 \\ 0 & 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right].$$

Then $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$\begin{cases} x_1 + 2x_2 \qquad + 3x_4 = b_1 - b_3/2 \\ \qquad\qquad x_3 + 4x_4 = b_3/2 \\ \qquad\qquad\qquad 0 = b_2 - 2b_1. \end{cases} \tag{23.1}$$

The third equation is a "solvability condition": if $A\mathbf{x} = \mathbf{b}$, then we must have $b_2 = 2b_1$. This is not the first time that we've seen this condition, and hopefully it's apparent from the row structure of A (the second row is twice the first row). But here I want to emphasize that we can solve $A\mathbf{x} = \mathbf{b}$ if and only if we can solve the (simpler) problem (23.1); this is just because all of the row operations are reversible (elementary matrices are invertible!). And if $b_2 = 2b_1$, then we can solve (23.1). Thus $\mathbf{C}(A) = \{\mathbf{b} \in \mathbb{R}^3 \mid b_2 = 2b_1\}$.

Formula time: we get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (b_1 - b_3/2) - 2x_2 - 3x_4 \\ x_2 \\ b_3/2 - 4x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 - b_3/2 \\ 0 \\ b_3/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

This is a wonderfully transparent solution formula. The "free" variables x_2 and x_4 quantify the nonuniqueness: each choice of x_2 and x_4 gives a different solution. And the solvability condition $b_2 = 2b_1$ makes precise the lack of existence: if $b_2 \neq b_1$, then there is no solution. (By the way, only b_1 and b_3 show up in the solution, since $b_2 = 2b_1$ if we're going to be able to solve the problem.)

Look at one other thing. Assuming $b_2 = 2b_1$ and taking $x_2 = x_4 = 0$, we conclude that one solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_\star := (b_1 - b_3/2, 0, b_3/2, 0)$, while all other solutions are $\mathbf{x} = \mathbf{x}_\star + c_1\mathbf{z}_1 + c_2\mathbf{z}_2$, where $\mathbf{z}_1 = (-2, 1, 0, 0)$, $\mathbf{z}_2 = (-3, 0, -4, 1)$, and hopefully we recognize $\mathsf{N}(A) = \mathsf{C}\big(\begin{bmatrix}\mathbf{z}_1 & \mathbf{z}_2\end{bmatrix}\big)$ from Example 21.1. By the way, taking $\mathbf{b} = 0$ (i.e., $b_1 = b_2 = b_3 = 0$), we recover the null space calculation from that example. This structural pattern in the solution is, like so many other things in this course, no accident.

It probably won't hurt to do another concrete computational example.

23.2 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 2 \\ 7 & 14 & 8 \end{bmatrix}.$$

This is, of course, the transpose of the matrix from Examples 21.1 and 24.2, and we'll talk about transposes in detail someday. For now, we study $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (b_1, b_2, b_3, b_4)$. Since $A \in \mathbb{R}^{4\times 3}$, we know that we will not be able to solve $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b}$, but Theorem 22.8 doesn't tell us anything yet about uniqueness of solutions when we can solve it. (Actually, we do know about uniqueness already: column 2 is 2 times column 1, so the columns of A are dependent, and so uniqueness always fails.)

Again, we could do the following calculations in several different orders and end up with the same thing; I'm going to follow the pseudocode of the proof of Theorem 22.4. We have

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 2 & 4 & 0 & b_2 \\ 1 & 2 & 2 & b_3 \\ 7 & 14 & 8 & b_4 \end{array}\right] \xrightarrow{\text{R2} \mapsto \text{R2}-2\times\text{R1}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 1 & 2 & 2 & b_3 \\ 7 & 14 & 8 & b_4 \end{array}\right]$$

$$\xrightarrow{\text{R3} \mapsto \text{R3}-\text{R1}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 2 & b_3 - b_1 \\ 7 & 14 & 8 & b_4 \end{array}\right]$$

$$\xrightarrow{\text{R4} \mapsto \text{R4}-7\times\text{R1}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 2 & b_3 - b_1 \\ 0 & 0 & 8 & b_4 - 7b_1 \end{array}\right]$$

$$\xrightarrow{\mathsf{R4} \mapsto \mathsf{R4}-4\times\mathsf{R3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 2 & b_3 - b_1 \\ 0 & 0 & 0 & (b_4 - 7b_1) - 4(b_3 - b_1) \end{array}\right]$$

$$\xrightarrow{\mathsf{R3} \mapsto (1/2)\times\mathsf{R3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & (b_3 - b_1)/2 \\ 0 & 0 & 0 & b_4 - 4b_3 - 3b_1 \end{array}\right]$$

$$\xrightarrow{\mathsf{R2} \mapsto \mathsf{R3},\ \mathsf{R3} \mapsto \mathsf{R2}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 0 & 1 & (b_3 - b_1)/2 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_4 - 4b_3 - 3b_1 \end{array}\right].$$

Along the way, I simplified

$$(b_4 - 7b_1) - 4(b_3 - b_1) = b_4 - 4b_3 - 3b_1.$$

I'm not writing down the elementary matrices doing everything, but I think you can by now. They'll be 4×4, since $A \in \mathbb{R}^{4\times3}$.

The problem $A\mathbf{x} = \mathbf{b}$ is then equivalent to

$$\begin{cases} x_1 + 2x_2 = b_1 \\ x_3 = (b_3 - b_1)/2 \\ 0 = b_2 - 2b_1 \\ 0 = b_4 - 4b_3 - 3b_1 \end{cases}$$

We now have *two* solvability conditions:

$$b_2 - 2b_1 = 0 \quad \text{and} \quad b_4 - 4b_3 - 3b_1 = 0.$$

If these are met, then the solution $\mathbf{x}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 - 2x_2 \\ x_2 \\ (b_3 - b_1)/2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ (b_3 - b_1)/2 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ (b_3 - b_1)/2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

We should see the same solution structure as in Example 23.1: taking $\mathbf{x}_\star = (b_1, 0, (b_3 - b_1)/2)$ gives a solution $A\mathbf{x}_\star = \mathbf{b}$ when the solvability conditions are met, and then every other solution is $\mathbf{x} = \mathbf{x}_\star + x_2\mathbf{z}_1$ for some $x_2 \in \mathbb{R}$, where $\mathbf{z}_1 = (-2, 1, 0)$. A byproduct of this calculation is that

$$\mathsf{N}(A) = \mathsf{C}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right). \tag{23.2}$$

By the way, we also figured out

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

23.3 Problem (!). Interpret this null space result (23.2) in in light of Problem 20.10. [Hint: *here $n - r = 1$ and the 1×1 identity matrix is just 1, or $\begin{bmatrix}1\end{bmatrix}$, if you must.*]

Here is the structural pattern of solutions that we are seeing from Examples 23.1 and 23.2.

23.4 Theorem. *Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Suppose that $\mathbf{x}_\star \in \mathbb{R}^n$ satisfies $A\mathbf{x}_\star = \mathbf{b}$. Then any other solution $\mathbf{x} \in \mathbb{R}^n$ to $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{x} = \mathbf{x}_\star + \mathbf{z}$ for some $\mathbf{z} \in \mathbf{N}(A)$.*

23.5 Problem (!). Prove it. [Hint: *what does $\mathbf{x} - \mathbf{x}_\star$ do?*]

It looks like we have a "decomposition" from Theorem 23.4 for solutions to $A\mathbf{x} = \mathbf{b}$. Any solution $\mathbf{x}$ is the sum of one "particular" solution and a vector in the null space. As in so many other places in the course, we need to build some more tools, but eventually we will be able to say a bit more about what that "particular" solution is doing, and maybe how to choose it best when we have many options.

Content from Strang's *ILA* 6E. Read all of p. 104 again and now read p. 105.

Day 24: Monday, October 13.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Pivot column of a matrix (not necessarily in RREF), free column of a matrix (not necessarily in RREF)

Examples 23.1 and 23.2 involved matrices A that were not so nice from the point of view of solving $A\mathbf{x} = \mathbf{b}$. They had nontrivial null spaces (so long uniqueness) and solvability conditions on their column spaces (solutions don't always exist). We can be much more precise beyond saying that solutions don't always exist and, when they do, they're not unique.

To talk sensibly, we need some new vocabulary, which relies on Definition 21.2.

24.1 Definition. *Column j of a matrix is a* PIVOT COLUMN *of that matrix if column j of the RREF is a pivot column, and otherwise column j is a* FREE COLUMN *if column j in the RREF is a free column.*

24.2 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}.$$

We've known for a while (say, Example 21.1) that

$$\mathsf{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: R_0.$$

(i) The pivot columns of R_0 are columns 1 and 3, because they contain the leading 1's; the free columns are columns 2 and 4, because they are not pivot columns. The pivot columns of A are therefore columns 1 and 3 as well; the free columns are columns 2 and 4. I think it's clear that the pivot columns of R_0 are independent, since they are $\mathbf{e}_1$ and $\mathbf{e}_2$.

(ii) It may not be too hard to see that the list of pivot columns of A are independent, thanks to that 0 in the $(3,1)$-entry of A. I claim that we can see the independence of the pivot columns of A directly from the RREF without using any particular knowledge of the entries of A.

Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$. We want to show that $\mathbf{a}_1$ and $\mathbf{a}_3$ are independent, so assume $c_1\mathbf{a}_1 + c_2\mathbf{a}_3 = \mathbf{0}_3$. The goal is $c_1 = c_2 = 0$. We know $EA = R_0$ for some invertible $E \in \mathbb{R}^{3\times3}$, so we have

$$E(c_1\mathbf{a}_1 + c_2\mathbf{a}_3) = E\mathbf{0}_3,$$

and therefore

$$c_1E\mathbf{a}_1 + c_2E\mathbf{a}_3 = \mathbf{0}_3,$$

and therefore

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 = \mathbf{0}_3,$$

since $E\mathbf{a}_1 = \mathbf{e}_1$ and $E\mathbf{a}_3 = \mathbf{e}_2$. The independence of $\mathbf{e}_1$ and $\mathbf{e}_2$ therefore forces $c_1 = c_2 = 0$.

(iii) I also claim that $\mathbf{a}_2 \in \mathrm{span}(\mathbf{a}_1)$ and $\mathbf{a}_4 \in \mathrm{span}(\mathbf{a}_1, \mathbf{a}_3)$. That is, the free columns are in the span of the *preceding* pivot columns. Okay, that $\mathbf{a}_2 \in \mathrm{span}(\mathbf{a}_1)$ is pretty obvious from looking at the entries of A, but, again, say that we didn't know the exact entries of A, just the RREF. With

$$R_0 = \begin{bmatrix} \mathbf{e}_1 & 2\mathbf{e}_1 & \mathbf{e}_2 & (3\mathbf{e}_1 + 4\mathbf{e}_2) \end{bmatrix},$$

we have

$$\mathbf{a}_2 = E^{-1}(2\mathbf{e}_1) = 2E^{-1}\mathbf{e}_1 = 2\mathbf{a}_1.$$

Also,

$$\mathbf{a}_4 = E^{-1}(3\mathbf{e}_1 + 4\mathbf{e}_2) = 3E^{-1}\mathbf{e}_1 + 4E^{-1}\mathbf{e}_2 = 3\mathbf{a}_1 + 4\mathbf{a}_3.$$

I don't think this result for $\mathbf{a}_4$ is quite so obvious obvious from the entries of A.

Since $\mathbf{a}_2$, $\mathbf{a}_4 \in \mathrm{span}(\mathbf{a}_1, \mathbf{a}_3)$, we conclude

$$\mathbf{C}(A) = \mathrm{span}(\mathbf{a}_1, \mathbf{a}_3).$$

And since $\mathbf{a}_1$ and $\mathbf{a}_3$ are independent, we shouldn't be able to pare this span down any further—this *is* the most efficient way to write $\mathbf{C}(A)$ as a span of (some) columns of A.

24.3 Problem (!). Use Example 24.2 to explain why $\mathbf{C}(A) \neq \mathbf{C}(\mathrm{rref}(A))$ in general. But explain why $\mathbf{N}(A) = \mathbf{N}(\mathrm{rref}(A))$ always.

24.4 Remark. *In the problem $A\mathbf{x} = \mathbf{b}$, call the unknown x_j a* **PIVOT VARIABLE** *of A if column j of A is a pivot column, and call x_j a* **FREE VARIABLE** *if column j of A is a free variable. We can phrase the computational procedures of Examples 20.3, 20.7, and 21.1 as follows. To determine $\mathbf{N}(A)$, put A in RREF as R_0 and use the equation $R_0\mathbf{x} = \mathbf{0}_m$ to solve for the pivot variables in terms of the free variables.*

Example 24.2 illustrates a number of important truths about pivot and free columns, which we summarize in the next theorem. We will not prove this theorem, as the strategy of proof is essentially that of the example, and the proof would just involve a lot of subscripts and $+\cdots+$. The results here mostly fall under what people mean when they say "Elementary row operations preserve linear (in)dependence relations among the columns of a matrix"—reducing A to its RREF doesn't change how columns are dependent or independent.

24.5 Theorem. *Let $A \in \mathbb{R}^{m \times n}$.*

(i) *The list of pivot columns of A are independent.*

(ii) *The free columns of A are in the span of the pivot columns. More precisely, if the jth column of A is a free column, then it is in the span of the pivot columns that appear in the first $j-1$ columns of A. (In fact, the weights in the expression of column j of A as a linear combination of preceding pivot columns are the same as the weights in the expression of column j of $\mathrm{rref}(A)$ as a linear combination of the preceding pivot columns.) If the first column of A ($j = 1$) is free, then it is zero.*

(iii) *The column space of A is the span of the pivot columns of A.*

24.6 Remark. *It's not really fair to say that the independent columns of A are the pivot columns of A. If you look at A from Example 24.2, you can check that any list of one column from A is independent, as are any list of two columns except for the list $\mathbf{a}_1$, $\mathbf{a}_2$. And if you do a little more thankless work, you'll see that any list of three or more columns of A is dependent. Rather, what we are looking for is the simplest way to find independent columns of A—and that comes from taking just the pivot columns.*

The RREF can help us tackle the outstanding Conjecture 10.12 on the CR-factorization. Recall that this conjecture says that if $A \in \mathbb{R}^{m \times n}$ has r independent columns, then $A = CR$ for some $C \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$, where the columns of C are r independent columns of A. We'll give a "proof by example" first.

24.7 Example. Let $A \in \mathbb{R}^{3 \times 4}$ be any matrix whose RREF is

$$R_0 := \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here we are writing R_0, not R, so we can save R for the factor in the CR-factorization to come. Then there is an invertible $E \in \mathbb{R}^{3 \times 3}$ such that $EA = R_0$, and so $A = E^{-1}R_0$. Write

$$E^{-1} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix},$$

and be aware that the columns of E^{-1} are independent.

Then

$$A = E^{-1}R_0 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & 2\mathbf{v}_1 & \mathbf{v}_3 & (3\mathbf{v}_1 + 4\mathbf{v}_2) \end{bmatrix}.$$

Importantly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent columns of A! And no other columns of A are independent along with $\mathbf{v}_1$ and $\mathbf{v}_2$.

If we stare at this representation of A long enough, hopefully we'll see

$$A = \begin{bmatrix} \mathbf{v}_1 & 2\mathbf{v}_1 & \mathbf{v}_3 & (3\mathbf{v}_1 + 4\mathbf{v}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

That's our CR-factorization! Put

$$C = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

to see that $A = CR$, that C contains the independent columns of A, and that the dimensions of C and R check out. And it was no accident that we got R by chopping off the zero rows of R_0. We can actually do better than saying that C contains independent columns of A. Rather, C contains the *pivot* columns of A, which here are columns 1 and 3.

24.8 Theorem (CR-factorization [Strang]). *Let $A \in \mathbb{R}^{m \times n}$ be a nonzero matrix. There exist $C \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$ with the following properties.*

(i) *$A = CR$.*

(ii) *The columns of C are the pivot columns of A (and so C has independent columns).*

(iii) *The column space of A equals the column space of C:* $\mathbf{C}(A) = \mathbf{C}(C)$.

(iv) *The rows of R are the nonzero rows of the RREF of A:* $\mathsf{rref}(A) = \begin{bmatrix} R \\ 0 \end{bmatrix}$.

Proof. Let $R_0 := \mathsf{rref}(A)$, so there is an invertible matrix $E \in \mathbb{R}^{m \times m}$ such that $EA = R_0$. Then $A = E^{-1}R_0$. Write

$$R_0 = \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where $R \in \mathbb{R}^{r \times n}$ has a nonzero entry in each row. That is, R contains all of the nonzero rows of the RREF of A. (Since A is not the zero matrix, A has at least one nonzero entry, so the RREF of A has at least one row with a leading 1.)

Next, write

$$E^{-1} = \begin{bmatrix} C & V \end{bmatrix},$$

where $C \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{m \times (n-r)}$. That is, C contains the first r columns of E^{-1}; if $r = n$, then $E^{-1} = C$, and V isn't there.

Now do the arithmetic:

$$A = E^{-1}R_0 = \begin{bmatrix} C & V \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} = CR.$$

That's the CR-factorization!

Let's check that the columns of C are the pivot columns of A. Think about the jth pivot column of A. Say that this is also the kth plain old column of A. (This is going to annoy you, so look at our favorite matrix in Example . There, the first column of A is the first pivot column, and the third column of A is the second pivot column.) Since A and its RREF R_0 have the same pivot columns, the jth pivot column of R_0 occurs in its kth column. So, the kth column of R_0 is the standard basis vector $\mathbf{e}_j \in \mathbb{R}^n$. And so the kth column of R is the standard basis vector $\widetilde{\mathbf{e}}_j \in \mathbb{R}^r$ (just chop off the last $n - r$ zeros of $\mathbf{e}_j$ to get $\widetilde{\mathbf{e}}_j$).

All together now: the kth column of A is C times the kth column of R, which is $C\widetilde{\mathbf{e}}_j$, which is the jth column of C. Thus the jth column of C is indeed the jth pivot column of A. We already know that the pivot columns span $\mathbf{C}(A)$, and now the pivot columns are precisely the columns of C, so $\mathbf{C}(C) = \mathbf{C}(A)$. ■

Content from Strang's *ILA* 6E. The second half of p. 96 revisits Example 2 (from p. 95) in the context of CR. Example 3 does the same. Now read the "Review" paragraph toward the bottom of p. 97, p. 142 up to and now including the "Factorization" box, and pp. 410–411 up to but not including #3.

Day 25: Wednesday, October 15.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Matrix with full column rank (N), matrix with full row rank (N)

The number of pivot columns in a matrix is a key piece of data for that matrix. Previously (Definition 8.13) we defined the **RANK** of a matrix as the length of the longest list of linearly independent columns in that matrix (kind of an annoying thing to calculate, right?), and we conjectured that any list of columns longer than the rank would be dependent (Conjecture 8.18). We can improve on this definition and prove the conjecture at the same time.

25.1 Theorem. *Let $A \in \mathbb{R}^{m \times n}$ with $r := \operatorname{rank}(A)$. Then r is the number of pivot columns of A. That is, A has a list of r independent columns (just take the pivot columns), and any list in $\mathbf{C}(A)$ of length greater than r is dependent.*

Proof. If $r = 0$, then A is just the zero matrix and therefore has no pivot columns. Assume now that $r \geq 1$. We consider a special illustrative case and then give the general proof.

1. Suppose that $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \in \mathbb{R}^{m \times 4}$ and that columns 1 and 3 are the only pivot columns (so $r = 2$ here, and m isn't really important). We've seen a matrix like this a few times before, right? Then the list $\mathbf{a}_1, \mathbf{a}_3$ is independent, $\mathbf{a}_2 \in \operatorname{span}(\mathbf{a}_1)$, and $\mathbf{a}_4 \in \operatorname{span}(\mathbf{a}_1, \mathbf{a}_3)$. It suffices to show that any list of length 3 in $\mathbf{C}(A)$ is dependent, as then any longer list contains a dependent sublist (Problem 9.5).

Here's the trick. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a list in $\mathbf{C}(A)$. Write $\mathbf{v}_1 = w_1\mathbf{a}_1 + w_2\mathbf{a}_3$, $\mathbf{v}_2 = x_1\mathbf{a}_1 + x_w\mathbf{a}_3$, $\mathbf{v}_3 = y_1\mathbf{a}_1 + y_w\mathbf{a}_3$. Put

$$V := \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_4 \end{bmatrix} \in \mathbb{R}^{m \times 3} \quad \text{and} \quad \widetilde{A} := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix} \in \mathbb{R}^{m \times 2}.$$

Then

$$V = \begin{bmatrix} (x_1\mathbf{a}_1 + x_3\mathbf{a}_3) & (y_1\mathbf{a}_1 + y_3\mathbf{a}_3) & (u_1\mathbf{a}_1 + u_3\mathbf{a}_3) \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & u_1 \\ x_3 & y_3 & u_3 \end{bmatrix} =: \widetilde{A}B.$$

The killer is that $B \in \mathbb{R}^{2 \times 3}$, so B has more columns than rows. By Theorem 22.8, we know $\mathbf{N}(B) \neq \{\mathbf{0}_3\}$. That is, there is $\mathbf{z} \in \mathbb{R}^3$ such that $B\mathbf{z} = \mathbf{0}_2$ and $\mathbf{z} \neq \mathbf{0}_3$. But then $V\mathbf{z} = \widetilde{A}B\mathbf{z} = \mathbf{0}_3$, so $\mathbf{z} \in \mathbf{N}(V)$ with $\mathbf{z} \neq \mathbf{0}_3$. Thus $\mathbf{N}(V) \neq \{\mathbf{0}_3\}$, and therefore the columns of V are dependent.

2. Here's what happens in general. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{r+1}$ be a list in $\mathbf{C}(A)$. We want to show that this list is dependent.

Say that $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_r}$ is the list of pivot columns of A and put $\widetilde{A} = \begin{bmatrix} \mathbf{a}_{j_1} & \cdots & \mathbf{a}_{j_r} \end{bmatrix}$. Write each $\mathbf{v}_k$ as a linear combination of the pivot columns, so $\mathbf{v}_k = \widetilde{A}\mathbf{x}_k$ for some $\mathbf{x}_k \in \mathbb{R}^r$. Then

$V = \widehat{A}X$, where

$$V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{r+1} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_{r+1} \end{bmatrix} \in \mathbb{R}^{r \times (r+1)}.$$

The matrix X has more columns than rows, so $\mathbf{N}(X) \neq \{\mathbf{0}_{r+1}\}$ by Theorem 22.8. Then there is $\mathbf{z} \in \mathbb{R}^{r+1}$ such that $X\mathbf{z} = \mathbf{0}_r$ and $\mathbf{z} \neq \mathbf{0}_{r+1}$. We compute $V\mathbf{z} = \widetilde{A}X\mathbf{z} = \mathbf{0}_r$, so $\mathbf{z} \in \mathbf{N}(V)$ with $\mathbf{z} \neq \mathbf{0}_{r+1}$. Then $\mathbf{N}(V) \neq \{\mathbf{0}_{r+1}\}$, and therefore the columns of V are dependent. ■

25.2 Problem (!). What is the rank of a diagonal matrix?

25.3 Problem (!). Let $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$. We know that a list of more than r vectors in $\mathbf{C}(A)$ is always dependent. Is a list of fewer than r vectors in $\mathbf{C}(A)$ always independent?

25.4 Problem ($\star$). Let $R \in \mathbb{R}^{m \times n}$ be one of the "canonical" RREF forms from Theorem 22.4. What is the rank of R? (Your answer will involve the numbers m, n, and/or r.)

25.5 Problem ($\star$). We know that if $A \in \mathbb{R}^{m \times n}$ has n pivot columns, then the columns of A are independent. (Why?) The reverse is true: suppose that the columns of A are independent, and explain what goes wrong if one column is a free column and not a pivot column. [Hint: *part (ii) of Theorem 24.5.*]

A consequence of Theorems 24.5 and 25.1 is that you really can't beat the rank when describing the column space. The former says that you can span $\mathbf{C}(A)$ with $r = \operatorname{rank}(A)$ columns (the pivot columns), and the latter says that you don't need more than r columns to span $\mathbf{C}(A)$, as any list of more than r columns will be dependent. You might wonder if we can span $\mathbf{C}(A)$ with *fewer* than r columns (pivot or not). No! We can't span $\mathbf{C}(A)$ with fewer than r *vectors*, whether they are columns of A or not. It's basically the same story as the proof of Theorem 25.1.

Here's how it goes. Put the r pivot columns of A into the matrix $\widetilde{A} \in \mathbb{R}^{m \times r}$. Suppose that $\mathbf{C}(A) = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_s)$ for $s < r$. Let $V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_s \end{bmatrix} \in \mathbb{R}^{m \times s}$. So each column of $\widetilde{A}$ is a linear combination of the columns of V: for $j = 1, \ldots, r$, we can write $\widetilde{\mathbf{a}}_j = V\mathbf{x}_j$ for some $\mathbf{x}_j \in \mathbb{R}^s$. Let $X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_r \end{bmatrix} \in \mathbb{R}^{r \times s}$. We have shown $\widetilde{A} = VX$.

Here $s < r$, so Theorem 22.8 implies $\mathbf{N}(X) \neq \{\mathbf{0}_s\}$. Then there is $\mathbf{z} \in \mathbb{R}^r$ such that $\mathbf{z} \neq \mathbf{0}_r$ and $X\mathbf{z} = \mathbf{0}_s$. Thus $\widetilde{A}\mathbf{z} = VX\mathbf{z} = V\mathbf{0}_s = \mathbf{0}_m$, so $\mathbf{z} \in \mathbf{N}(A)$ and $\mathbf{z} \neq \mathbf{0}_r$. So, $\mathbf{N}(\widetilde{A}) \neq \{\mathbf{0}_r\}$, which means that the columns of $\widetilde{A}$ are dependent. This is a contradiction since the columns of $\widetilde{A}$ are the pivot columns of A and therefore independent.

25.6 Theorem. *Let $A \in \mathbb{R}^{m \times n}$ have rank r. Then no list of fewer than r vectors in $\mathbf{C}(A)$ can span $\mathbf{C}(A)$.*

25.7 Problem (!). Why do you think people sometimes call the list of pivot columns of a matrix both a "minimal spanning list" for the column space and a "maximal linearly independent list" in the column space?

25.8 Problem ($\star$). If you don't want to use the pivot columns, you don't really have to. Let $A \in \mathbb{R}^{m \times n}$ have rank $r \ge 1$. Prove that any list of r independent columns of A spans $\mathbf{C}(A)$. [Hint: *what goes wrong if the list doesn't span* $\mathbf{C}(A)$*? If a vector in* $\mathbf{C}(A)$ *isn't in the span of the list, add it to the list. Then you have a new list of* $r+1$ *vectors in* $\mathbf{C}(A)$*. How does this new list contradict Theorem 25.1? You may want to use the linear independence lemma (Lemma 9.1.)*]

25.9 Problem (!). Convince yourself that we have now resolved all of Conjecture 8.18.

25.10 Problem (+). Reread Problem 18.9. Now let $U \in \mathbb{R}^{m \times m}$ with $0 \le r \le n$ nonzero entries on its diagonal. Prove that $\text{rank}(U) = r$. [Hint: *if for* $j \ge 2$*, the* j*th diagonal entry of* U *is zero, show that the* j*th column of* U *is in the span of the previous* $j-1$ *columns. Use this to construct a list of* r *independent columns of* U *that span* $\mathbf{C}(U)$.]

Here are some useful bounds on rank.

25.11 Theorem. *Let* $A \in \mathbb{R}^{m \times n}$*. Then* $0 \le \text{rank}(A) \le \min\{m, n\}$*, with* $\text{rank}(A) = 0$ *only when* A *is the zero matrix.*

Proof. If $\text{rank}(A) = 0$, then A has no pivot columns, so $\mathsf{rref}(A)$ has no pivot columns and therefore is the zero matrix. Otherwise, $\mathsf{rref}(A)$ would have a leading nonzero entry in some row and thus a pivot column. Then since $A = E^{-1}\mathsf{rref}(A)$ for some invertible $E \in \mathbb{R}^{m \times m}$, A is also the zero matrix.

Next, A has n columns, so at most n of them can be pivot columns. Thus $\text{rank}(A) \le n$. And A has m rows, so $\mathsf{rref}(A)$ has m rows, and therefore $\mathsf{rref}(A)$ can have at most m leading 1's, thus at most m pivot columns. And so $\text{rank}(A) \le m$. ■

Examples 23.1 and 23.2 involved matrices $A \in \mathbb{R}^{m \times n}$ with $1 \le \text{rank}(A) < \min\{m, n\}$. Some interesting things happen in the "extreme" case of $\text{rank}(A) = \min\{m, n\}$.

25.12 Example. **(i)** Here are two matrices $A \in \mathbb{R}^{2 \times n}$ with $n \ge 2$ and $\text{rank}(A) = 2$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

I hope you see that each matrix has enough columns to span $\mathbb{R}^2$, so in each case $\mathbf{C}(A) = \mathbb{R}^2$. That is, we can always solve $A\mathbf{x} = \mathbf{b}$. However, in the second case, the third column is not a pivot column, so the columns are not independent, $\mathbf{N}(A) \ne \{\mathbf{0}_3\}$, and solutions to $A\mathbf{x} = \mathbf{b}$ aren't unique. We knew that anyway from Theorem 22.8 since the second matrix has more columns than rows.

(ii) Here are two matrices $A \in \mathbb{R}^{m \times 2}$ with $m \geq 2$ and $\text{rank}(A) = 2$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Same deal as before with I_2, of course, but in the second case, while we can't always solve $A\mathbf{x} = \mathbf{b}$, we can always do so uniquely, because every column is a pivot column, and so $\mathbf{N}(A) = \{\mathbf{0}_2\}$.

Content from Strang's *ILA* 6E. Read Example 1 on pp. 105–106 and Example 2 on p. 107.

25.13 Problem (!). Which of the six canonical forms of the RREF in Theorem 22.4 have rank m? Rank n?

Here is what these examples teach us.

25.14 Theorem. *Let $A \in \mathbb{R}^{m \times n}$.*

(i) *Suppose that $m \leq n$ and $\text{rank}(A) = m$. Then $\mathbf{C}(A) = \mathbb{R}^m$. That is, we can always solve $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^m$. In this case, we say that A has* **FULL ROW RANK**.

(ii) *Suppose that $n \leq m$ and $\text{rank}(A) = n$. Then $\mathbf{N}(A) = \{\mathbf{0}_n\}$. That is, if we can solve $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^m$, then the solution $\mathbf{x}$ is unique. In this case, we say that A has* **FULL COLUMN RANK**.

Proof. **(i)** Let $R_0 = \text{rref}(A)$ and let $E \in \mathbb{R}^{m \times m}$ be invertible with $EA = R_0$. Let $\mathbf{b} \in \mathbb{R}^m$. Then $A\mathbf{x} = \mathbf{b}$ if and only if $EA\mathbf{x} = E\mathbf{b}$, so if and only if $R_0\mathbf{x} = E\mathbf{b}$. Abbreviate $\mathbf{c} = E\mathbf{b}$, so the goal is to solve $R_0\mathbf{x} = \mathbf{c}$.

1. *An illustrative special case.* Suppose that $A \in \mathbb{R}^{3 \times 6}$ with $\text{rank}(A) = 3$ and

$$R_0 = \begin{bmatrix} \mathbf{0} & \mathbf{e}_1 & \mathbf{f}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{f}_2 \end{bmatrix}$$

with the first column and $\mathbf{f}_1$ and $\mathbf{f}_2$ all free. Abbreviate $\mathbf{c} = E\mathbf{b}$. Then we want to solve

$$\begin{bmatrix} \mathbf{0} & \mathbf{e}_1 & \mathbf{f}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{f}_2 \end{bmatrix} \mathbf{x} = \mathbf{c}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

If we multiply it out, we want

$$x_2\mathbf{e}_1 + x_3\mathbf{f}_1 + x_4\mathbf{e}_2 + x_5\mathbf{e}_3 + x_6\mathbf{f}_2 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

One way to do this is to take $x_2 = c_1$, $x_4 = c_2$, and $x_5 = c_3$ and put $x_1 = x_3 = x_6 = 0$. That is, $\mathbf{x} = (0, c_1, 0, c_2, c_3, 0)$.

2. *The general case.* If $\mathbb{R}_0 \in \mathbb{R}^{m \times n}$ has m pivot columns, then R_0 contains all m columns of I_m (at least once). Define $\mathbf{x} \in \mathbb{R}^n$ by taking x_j to be the jth entry of $\mathbf{c}$ when the jth column of R_0 is the pivot column $\mathbf{e}_j$, and otherwise let x_j be 0.

(ii) Every column of A is a pivot column, so all of the columns of A are independent, and therefore $\mathbf{N}(A) = \{\mathbf{0}_n\}$. ■

Content from Strang's *ILA* 6E. Read the rest of p. 106 starting from "This example is typical. . . ." Then read all of p. 108.

25.15 Problem ($\star$). Make and fill in a table with the following five columns. The first column contains the six forms for the RREF from Theorem 22.4. The second column contains two matrices: one a matrix whose RREF has that form (follow the guidelines of—and feel free to reuse your matrices from—Problem 22.7) and the other the exact RREF of that matrix. The third column is "Existence"; put "Always" or "Sometimes" depending on whether solutions to $A\mathbf{x} = \mathbf{b}$ always exist or only sometimes exist when A has that RREF form. The fourth column is "Uniqueness"; put "Always" or "Never" depending on whether solutions to $A\mathbf{x} = \mathbf{b}$ are unique or not when A has that RREF form. (Why is there no "Sometimes" for uniqueness?) The fifth column is "Rank"; specify how the rank relates to m and/or n (be more precise than $\operatorname{rank}(A) \le \min\{m, n\}$), and in particular indicate any RREF form corresponding to full column or row rank.

25.16 Problem ($\star$). Here is a happy example of how rank is "stable." Let $A \in \mathbb{R}^{m \times n}$ be any matrix and let $B \in \mathbb{R}^{m \times m}$ be invertible. Explain why $\operatorname{rank}(BA) = \operatorname{rank}(A)$, but give an example to show that $\mathbf{C}(BA) \ne \mathbf{C}(A)$ in general. [Hint: *how do the pivot columns of A become the pivot columns of BA when B is invertible? Think about spans and independence.*]

Content from Strang's *ILA* 6E. Read all of the "Worked Examples" on pp. 109–110.

Here is a summary of everything that we know about the rank of a matrix. Let $A \in \mathbb{R}^{m \times n}$ and $r := \operatorname{rank}(A)$.

- r is the number of pivot columns of A.
- If A is the zero matrix (all entries are 0), then $r = 0$. Otherwise, $1 \le r \le \min\{m, n\}$.

- Any list in $\mathbf{C}(A)$ of length greater than r is dependent.
- Any list in $\mathbf{C}(A)$ of length less than r cannot span $\mathbf{C}(A)$.
- Any independent list in $\mathbf{C}(A)$ of length exactly r does span $\mathbf{C}(A)$.
- If $\text{rank}(A) = m$, then $\mathbf{C}(A) = \mathbb{R}^m$.
- If $\text{rank}(A) = n$, then $\mathbf{N}(A) = \{\mathbf{0}_n\}$.

Day 26: Friday, October 17.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Subspace of $\mathbb{R}^p$ (N)

It looks like we have accomplished the major goal of the course: solve $A\mathbf{x} = \mathbf{b}$ and understand when we can't. We make the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ and use Gauss–Jordan elimination to reduce A to $\mathsf{rref}(A) = R_0$ with $\mathbf{b}$ transforming into $\mathbf{c}$ along the way. Then we study $\begin{bmatrix} R_0 & \mathbf{c} \end{bmatrix}$. That is, $A\mathbf{x} = \mathbf{b}$ and $R_0\mathbf{x} = \mathbf{c}$ have the same solutions (if any). For the problem to have a solution, if the ith row of R_0 is all zero, then the ith entry of $\mathbf{c}$ must be 0. Assuming those "solvability conditions" to be true, we then rewrite the system $R_0\mathbf{x} = \mathbf{c}$ as a system of equations and solve for the "free variables" in terms of the "pivot variables" (recall Remark 24.4). In the special case that A is square, we could just do Gaussian elimination to convert A to its upper-triangular form U; if all of the diagonal entry of U are nonzero, then we can back-substitute. If a diagonal entry is 0, then it's probably best to go all the way to RREF to have some control over the null space.

Considering all of this good work, I claim that we are now pretty adept at solving $A\mathbf{x} = \mathbf{b}$ (especially when A is square and invertible), but we could still be better at *understanding* $A\mathbf{x} = \mathbf{b}$, particularly at understanding *failure*. This good work motivates two more questions.

Question 1. If uniqueness fails and $\mathbf{N}(A) \neq \{\mathbf{0}_n\}$, can we quantify and qualify how many "degrees of freedom" the null space gives to the problem, and in particular how many "different" solutions there are beyond "infinitely many?" Can we do this in terms of m, n, and $\text{rank}(A)$?

Question 2. If existence fails and $\mathbf{C}(A) \neq \mathbb{R}^m$, can we quantify and qualify how much of $\mathbb{R}^m$ the column space "hits" versus "misses"? Can we do this in terms of m, n, and $\text{rank}(A)$?

To answer these questions—and to ask them precisely—we need some new tools. The overarching tool is *abstraction*: what are the *really* essential properties of the things that we're studying?

Our successes going forward will hinge in no small part on a new perspective: how vectors within a given set interact with each other. This may sound weird at first, but trust me that it will feel completely natural soon. Let's pause from concrete numbers and focus on the *dynamic* aspects of the null space.

Let $A \in \mathbb{R}^{m\times n}$. The null space

$$\mathbf{N}(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mathbf{0}_m\}$$

behaves well with respect to the fundamental objects of vector arithmetic.

1. Suppose $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{N}(A)$. Then $A\mathbf{v}_1 = \mathbf{0}_m$ and $A\mathbf{v}_2 = \mathbf{0}_m$, so

$$A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = \mathbf{0}_m + \mathbf{0}_m = \mathbf{0}_m.$$

Thus $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbf{N}(A)$. Like the column space, the null space is "closed under addition": adding two vectors in $\mathbf{N}(A)$ yields another vector in $\mathbf{N}(A)$.

2. Similarly, if $\mathbf{v} \in \mathbf{N}(A)$ and $c \in \mathbb{R}$, then since $A\mathbf{v} = \mathbf{0}_m$, we have

$$A(c\mathbf{v}) = c(A\mathbf{v}) = c\mathbf{0}_m = \mathbf{0}_m,$$

so $c\mathbf{v} \in \mathbf{N}(A)$. That is, the null space is "closed under scalar multiplication": multiplying a vector in $\mathbf{N}(A)$ by a real number yields another vector in $\mathbf{N}(A)$.

3. Finally, since $A\mathbf{0}_n = \mathbf{0}_m$, we have $\mathbf{0}_n \in \mathbf{N}(A)$. Thus the null space is never empty, and it contains one of the most important vectors for vector and matrix arithmetic.

Content from Strang's *ILA* 6E. These properties of the null space appear in the very last paragraph of p. 88.

The column space

$$\mathbf{C}(A) = \{A\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n\}.$$

also behaves well with respect to the fundamental objects of vector arithmetic.

1. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{C}(A)$. Then there are $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ such that $\mathbf{w}_1 = A\mathbf{v}_1$ and $\mathbf{w}_2 = A\mathbf{v}_2$. So,

$$\mathbf{w}_1 + \mathbf{w}_2 = A\mathbf{v}_1 + A\mathbf{v}_2 = A(\mathbf{v}_1 + \mathbf{v}_2) \in \mathbf{C}(A)$$

since $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{R}^n$. That is, $\mathbf{C}(A)$ is "closed under addition": adding two vectors in $\mathbf{C}(A)$ yields another vector in $\mathbf{C}(A)$.

2. Similarly, if $\mathbf{w} \in \mathbf{C}(A)$ with $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^n$, and if $c \in \mathbb{R}$, then

$$c\mathbf{w} = c(A\mathbf{v}) = A(c\mathbf{v}) \in \mathbf{C}(A),$$

since $c\mathbf{v} \in \mathbb{R}^n$. That is, $\mathbf{C}(A)$ is "closed under scalar multiplication": multiplying a vector in $\mathbf{C}(A)$ by a real number yields another vector in $\mathbf{C}(A)$.

3. Finally, since $A\mathbf{0}_n = \mathbf{0}_m$, we have $\mathbf{0}_m \in \mathbf{C}(A)$. Thus the column space is never empty, and in particular it contains one of the most important vectors for vector and matrix arithmetic alike.

Sets of vectors that have these properties—closure under vector addition and scalar multiplication and containing the zero vector—are among the most special and useful kinds of sets. They don't just exist and contain things; they are *dynamic* with respect to vector operations. We'll see just how special these sets—these *spaces*—are in the context of understanding, and maybe even solving, $A\mathbf{x} = \mathbf{b}$ for A nonsquare.

Subsets of $\mathbb{R}^n$ (or $\mathbb{R}^m$, or whatever) that have these three properties—closure under vector addition, closure under scalar multiplication, presence of the zero vector—are just particularly "nice" for linear algebra. They respect the fundamental arithmetic and algebra that we do, and they arise often in connection with our fundamental problem of solving and understanding and approximating $A\mathbf{x} = \mathbf{b}$. So, they deserve a special name that reflects their dynamism—they are not merely sets but *spaces* of vectors that interact well together.

26.1 Definition. *A subset $\mathcal{V}$ of $\mathbb{R}^p$ is a* SUBSPACE *of $\mathbb{R}^p$ if the following are true.*

(i) [Closure under vector addition] *If $\mathbf{v}$, $\mathbf{w} \in \mathcal{V}$, then $\mathbf{v} + \mathbf{w} \in \mathcal{V}$.*

(ii) [Closure under scalar multiplication] *If $\mathbf{v} \in \mathcal{V}$ and $c \in \mathbb{R}$, then $c\mathbf{v} \in \mathcal{V}$.*

(iii) [Presence of the zero vector] *$\mathbf{0}_p \in \mathcal{V}$.*

Content from Strang's *ILA* 6E. Page 86 discusses the axioms for a subspace. Examples 1 and 2 on p. 87 present concrete (non)examples of subspaces of $\mathbb{R}^p$.

26.2 Example. Let $A \in \mathbb{R}^{m \times n}$.

(i) $\mathbf{C}(A) = \{A\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$. The important thing here is that every vector in $\mathbf{C}(A)$ has the form $A\mathbf{v} \in \mathbb{R}^m$.

(ii) $\mathbf{N}(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mathbf{0}_m\}$ is a subspace of $\mathbb{R}^n$. It should be obvious from the definition of $\mathbf{N}(A)$ that every vector in the null space is a vector in $\mathbb{R}^n$. We also have methods now of writing $\mathbf{N}(A)$ as a column space of some (possibly elaborate) matrix.

(iii) Let

$$\mathcal{V} = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; v_1 + v_2 = 0 \right\}.$$

We can show that $\mathcal{V}$ is a subspace of $\mathbb{R}^3$ in two ways.

The first way is to practice the definition. Let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathcal{V}$, so $v_1+v_2 = w_1+w_2 = 0$. Then $\mathbf{v}+\mathbf{w} = (v_1+w_1, v_2+w_2, v_3+w_3)$. We want $(v_1+w_1)+(v_2+w_2) = 0$, and we rearrange $(v_1 + w_1) + (v_2 + w_2) = (v_1 + v_2) + (w_1 + w_2) = 0$. So $\mathbf{v} + \mathbf{w} \in \mathcal{V}$.

Next, let $\mathbf{v} = (v_1, v_2, v_3) \in \mathcal{V}$, so $v_1 + v_2 = 0$, and $c \in \mathbb{R}$. Then $c\mathbf{v} = (cv_1, cv_2, cv_3)$, and we want $cv_1 + cv_2 = 0$. We factor $cv_1 + cv_2 = c(v_1 + v_2) = 0$, so $c\mathbf{v} \in \mathcal{V}$.

Last, we want $\mathbf{0}_3 = (0, 0, 0) \in \mathcal{V}$, and we check $0 + 0 = 0$, so, yes, $\mathbf{0}_3 \in \mathcal{V}$.

Here is the other, sneakier way. If $\mathbf{v} = (v_1, v_2, v_3) \in \mathcal{V}$, then $v_1 + v_2 = 0$, so $v_1 = -v_2$.

Then

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 v_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} \in \mathbf{C}\left(\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Conversely, if $\mathbf{v}$ is in the column space of this matrix, then $v_1 + v_2 = 0$. (Check that, please.) So $\mathcal{V}$ is a column space and therefore a subspace by the first part of this example.

We will eventually show that every subspace is both a column space and a null space (probably for different matrices). This is a miracle of definitions and algebra: the abstract conditions of the definition of subspace realize themselves concretely in matrices. For the purposes of this course, the only important subspaces that we will study will eventually be column and null spaces. However, there will be times when working with the three axioms for a subspace will be more convenient than representing the subspace as a particular column or null space.

26.3 Problem (!). Let

$$\mathcal{V} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1, \ x_2 \in \mathbb{R} \right\}.$$

Explain how each of the three conditions for a subspace fails for $\mathcal{V}$.

26.4 Problem (!). Check that $\{\mathbf{0}_p\}$ is indeed a subspace of $\mathbb{R}^p$.

Content from Strang's *ILA* 6E. Section 3.1 discusses the much more general, and hugely important, concept of a VECTOR SPACE. This is a set of elements called VECTORS that we can add together and multiply by scalars (real or complex numbers), and for which these operations of VECTOR ADDITION and SCALAR MULTIPLICATION basically behave the way that we expect arithmetic to behave. See the eight axioms on p. 89.

Maybe the two most important vector spaces are the column vectors with n entries, which, of course, is $\mathbb{R}^n$, and, from calculus, the space of continuous functions on an interval $I \subseteq \mathbb{R}$, which we denote by $\mathcal{C}(I)$. You know from calculus that if f and g are continuous on I, then so are $f + g$ and cf for any real c. (The space $\mathcal{C}(I)$ has the additional algebraic operation of function multiplication, fg, whereas we cannot multiply vectors in $\mathbb{R}^n$ in any "natural" way to get another vector in $\mathbb{R}^n$.) The r-times continuously differentiable functions (functions whose first r derivatives exist and are continuous) form the subspace $\mathcal{C}^r(I)$ of $\mathcal{C}(I)$, which is a natural player in differential equations.

The structure of vector spaces transcends matrix problems and provide the "right" framework for understanding the linear structure that pervades calculus. See pp. 84–85 for just a little on this. We will focus mostly on subspaces of $\mathbb{R}^n$, not general vector spaces, in this course.

Day 27: Monday, October 20.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Basis for a subspace of $\mathbb{R}^p$

Content from Strang's *ILA* 6E. You should be very comfortable with the notion of independence by now. Read pp. 115–117 thoroughly. Think carefully about the "guilty" remark at the end of p. 117. Which way of saying that the columns of $A \in \mathbb{R}^{m \times n}$ are independent feels easier to you—that $A\mathbf{x} = \mathbf{0}_m$ forces $\mathbf{x} = \mathbf{0}_n$ (the "democratic" way) or that one column of A is a combination of other columns (the "guilty" way)?

You should also be very comfortable with the notion of span by now. Read "Vectors that Span a Subspace" at the start of p. 118.

27.1 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}.$$

We know from long experience (Examples 21.1, 24.2, 23.1) that columns 1 and 3 are pivot columns and therefore are independent and span $\mathbf{C}(A)$ and that

$$\mathbf{N}(A) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix}\right).$$

The columns of this matrix giving $\mathbf{N}(A)$ are also independent, for if

$$x_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = \mathbf{0}_4,$$

then

$$\begin{bmatrix} * \\ x_1 \\ * \\ x_2 \end{bmatrix} = \mathbf{0}_4,$$

thus $x_1 = x_2 = 0$. Even though the arithmetic was simple, I intentionally wrote $*$ to show how much I don't care about the other entries of that linear combination beside the "special" ones that just reveal x_1 and x_2. While $\mathbf{C}(A)$ and $\mathbf{N}(A)$ are very different spaces, we have described them in the same way: as column spaces of matrices with independent columns, as spans of lists of independent vectors.

27.2 Problem (!). Let A be as in the previous example. Example 23.1 showed that $\mathbf{b} = (b_1, b_2, b_3) \in \mathbf{C}(A)$ if and only if $b_2 = 2b_1$. Use this fact to show that

$$\mathbf{C}(A) = \operatorname{span}\left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right).$$

Check that the vectors in this span are also independent but only one is a column of A. The new result here is that we have written $\mathbf{C}(A)$ as a span of independent vectors, not all of which were columns of A. But the old result is that we still only needed two vectors to do it. And that's not surprising, since $\operatorname{rank}(A) = 2$.

Here is the pattern that we have seen throughout this course: writing column spaces and null spaces as spans of independent vectors is an efficient way of describing them. It turns out that we can always do this, which should not surprise us. The RREF teaches us that the column space is always the span of the pivot columns, which are always independent. The RREF also teaches us that we can write the null space as the column space of a "special" kind of matrix, whose columns always turn out to be independent. (Depending on the form of the RREF, there are six forms that the null space representation can take. It's okay if this feels annoying.)

An independent list of vectors that spans a subspace has a special name.

27.3 Definition. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. A list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathcal{V}$ is a* BASIS *for $\mathcal{V}$ if the following hold.*

(i) *The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are independent.*

(ii) $\mathcal{V} = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$.

27.4 Example. **(i)** The standard basis vectors for $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$. (If they weren't, it would be a pretty awful use of the word "basis.") Here's the proof for $n = 3$, which I think you know by now already. If $\mathbf{v} \in \mathbb{R}^3$, then

$$\mathbf{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$$

Thus $\mathbb{R}^3 = \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. If $x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{0}_3$, then with $\mathbf{x} = (x_1, x_2, x_3)$, we have $\mathbf{x} = \mathbf{0}_3$, thus $x_1 = x_2 = x_3 = 0$. This is the linear independence of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

(ii) The pivot columns of a matrix are a basis for that matrix's column space. No surprises here: the pivot columns are independent and they span that matrix's column space.

(iii) The columns of an invertible matrix $A \in \mathbb{R}^{m \times m}$ are a basis for $\mathbb{R}^m$. Again, unsurprising: these columns are independent and $\mathbf{C}(A) = \mathbb{R}^m$.

(iv) Let's do one more null space calculation. Let $A \in \mathbb{R}^{3\times5}$ with

$$\mathsf{rref}(A) = \widetilde{R}_0 P, \qquad \widetilde{R}_0 = \begin{bmatrix} 1 & 0 & f_1 & f_2 & f_3 \\ 0 & 1 & f_4 & f_5 & f_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for some f_1, f_2, f_3, f_4, f_5, $f_6 \in \mathbb{R}$ and a permutation matrix $P \in \mathbb{R}^{5\times5}$. We know $A\mathbf{x} = \mathbf{0}_3$ if and only if $\widetilde{R}_0 P\mathbf{x} = \mathbf{0}_3$. The permutation matrix is annoying, so here's a trick: put $\mathbf{y} = P\mathbf{x}$. Then $A\mathbf{x} = \mathbf{0}_3$ if and only if $\widetilde{R}_0\mathbf{y} = \mathbf{0}_3$. This turns into the system

$$\begin{cases} y_1 & & + & f_1y_3 & + & f_2y_4 & + & f_3y_5 & = & 0 \\ & y_2 & + & f_4y_3 & + & f_5y_4 & + & f_6y_5 & = & 0 \\ & & & & & & & 0 & = & 0, \end{cases}$$

which we easily solve as

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -f_1y_3 - f_2y_4 - f_3y_5 \\ -f_4y_3 - f_5y_4 - f_6y_5 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = y_3 \begin{bmatrix} -f_1 \\ -f_4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y_4 \begin{bmatrix} -f_2 \\ -f_5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y_5 \begin{bmatrix} -f_3 \\ -f_6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$\mathbf{x} = P^{-1}\mathbf{y} = y_3 P^{-1} \begin{bmatrix} -f_1 \\ -f_4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y_4 P^{-1} \begin{bmatrix} -f_2 \\ -f_5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y_5 P^{-1} \begin{bmatrix} -f_3 \\ -f_6 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and so

$$\mathbf{N}(A) = \mathbf{C}\left(P^{-1} \begin{bmatrix} -f_1 & -f_2 & -f_3 \\ -f_4 & -f_5 & -f_6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

How does this give a basis for the null space? We've already expressed the null space as a column space, so that takes care of the span; as for independence, if we abbreviate $\mathbf{N}(A) = \mathbf{C}(P^{-1}B)$, then we want to show that the columns of $P^{-1}B$ are independent. So, suppose $P^{-1}B\mathbf{v} = \mathbf{0}_5$; then $B\mathbf{v} = \mathbf{0}_5$. But this says

$$\begin{bmatrix} * \\ * \\ * \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0}_5,$$

from which we get $\mathbf{v} = \mathbf{0}_3$.

More generally, if for $A \in \mathbb{R}^{m \times n}$ we have

$$\mathsf{rref}(A) = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P,$$

where $1 \le r < n$, this sort of argument shows that the columns of

$$P^{-1} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}$$

are a basis for $\mathbf{N}(A)$. The great thing is that we didn't need to know anything about P! All that mattered was the *invertibility* of P, not even that P was a permutation matrix.

27.5 Problem (!). Why can't the subspace $\{\mathbf{0}_p\}$ of $\mathbb{R}^p$ have a basis according to our definition of basis? This is not as serious an issue as it might sound; some people define the "empty list" to be a basis for $\{\mathbf{0}_p\}$, with the idea that a linear combination of no vectors is defined to be $\mathbf{0}_p$. I say that if your problem depends on having a notion of basis for $\{\mathbf{0}_p\}$, give up and do a different problem.

Content from Strang's *ILA* 6E. Read all of "A Basis for a Vector Space" on pp. 118–119. Every single thing here is important. Then read Worked Example 3.4 A on p. 122. This is a very important example that you should know how to prove.

27.6 Problem (!). Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathcal{V}$ are a basis for $\mathcal{V}$ if and only if the following are both true.

(i) The matrix $\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r \end{bmatrix} \in \mathbb{R}^{p \times r}$ has full column rank.

(ii) $\mathcal{V} = \mathbf{C}\left(\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r \end{bmatrix} \right)$.

Day 28: Wednesday, October 22.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Dimension of a subspace of $\mathbb{R}^p$ (for each integer k with $1 \le k \le p$, be able to give an example of a k-dimensional subspace of $\mathbb{R}^p$)

We made the definition of a basis as a consequence of our observations that the most efficient way to describe subspaces was as spans of lists of linearly independent vectors. We can be a little more dynamic. A basis is fundamentally a "coordinate system" for a subspace:

we can "reach" every vector in the subspace in a unique way "via" the basis. We know this in our hearts with the standard basis $\mathbf{e}_1$, $\mathbf{e}_2$ for $\mathbb{R}^2$: any $\mathbf{x} = (x, y) \in \mathbb{R}^2$ has the unique form $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$.

28.1 Problem (!). Which of the two conditions in Definition 27.3 do you think encodes this "reaching" property and which encodes the uniqueness?

Here's the rigorous proof.

28.2 Theorem. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. A list $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathcal{V}$ is a basis for $\mathcal{V}$ if and only if for any $\mathbf{v} \in \mathcal{V}$, there are unique $c_1, \ldots, c_r \in \mathbb{R}$ such that*

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_d\mathbf{v}_r.$$

Proof. ($\Longrightarrow$) Let $\mathbf{v} \in \mathcal{V}$. Then $\mathbf{v} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_r)$, so there are $c_1, \ldots, c_r \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r$. What if there's another way to write $\mathbf{v}$ as the span of $\mathbf{v}_1, \ldots, \mathbf{v}_r$? Then $\mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_r\mathbf{v}_r$, too. Do a little arithmetic and algebra:

$$\mathbf{0}_p = \mathbf{v} - \mathbf{v} = (c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r) - (d_1\mathbf{v}_1 + \cdots + d_r\mathbf{v}_r) = (c_1 - d_1)\mathbf{v}_1 + \cdots + (c_r - d_r)\mathbf{v}_r.$$

The independence of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ means $c_1 - d_1 = 0, \ldots, c_r - d_r = 0$, so $c_j = d_j$ for all j. So, there's only *one* way to write $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

($\Longleftarrow$) We already know $\mathcal{V} = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$. To check independence, suppose $c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r = \mathbf{0}_p$. We already know that we can get $\mathbf{0}_p$ in the span by writing $\mathbf{0}_p = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_r$. So we have

$$c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_r,$$

and since we're assuming coefficients are unique, this gives $c_j = 0$ for all j. ∎

A basis for a subspace won't be much use as a coordinate system or an ideally efficient representation mechanism if the basis doesn't exist in the first place! We know this to be true for the most important subspaces, column spaces and null spaces, but it turns out that the three subspace axioms alone guarantee the existence of a basis. To prove that, we need an unsurprising auxiliary result.

28.3 Lemma. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. If $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{V}$ and $c_1, \ldots, c_n \in \mathbb{R}$, then $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \in \mathcal{V}$. That is, $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is contained in $\mathcal{V}$.*

Proof. This is really an induction argument on n. Here's why it's true for $n = 3$. Since $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3 \in \mathcal{V}$, the subspace axioms guarantee $c_1\mathbf{v}_1$, $c_2\mathbf{v}_2$, $c_3\mathbf{v}_3 \in \mathcal{V}$. Then the axioms guarantee $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \in \mathcal{V}$, and so $(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + c_3\mathbf{v}_3 \in \mathcal{V}$. ∎

28.4 Theorem. *Let $\mathcal{V} \neq \{\mathbf{0}_p\}$ be a subspace of $\mathbb{R}^p$. Then $\mathcal{V}$ has a basis.*

Proof. This is a proof by exhaustion, which means that we'll exhaust all possible cases and also ourselves. Since $\mathcal{V} \neq \{\mathbf{0}_p\}$, there is $\mathbf{v}_1 \in \mathcal{V}$ such that $\mathbf{v}_1 \neq \mathbf{0}_p$. One of two things is true:

either $\mathcal{V} = \text{span}(\mathbf{v}_1)$ or $\mathcal{V} \neq \text{span}(\mathbf{v}_1)$. In the first case, $\mathbf{v}_1$ by itself is a basis for $\mathcal{V}$ since it's nonzero and therefore independent and also spans $\mathcal{V}$.

In the second case, there is $\mathbf{v}_2 \in \mathcal{V}$ such that $\mathbf{v}_2 \notin \text{span}(\mathbf{v}_1)$. Since $\mathbf{v}_1 \neq \mathbf{0}_p$, the list $\mathbf{v}_1$, $\mathbf{v}_2$ is independent. Once again, one of two things is true: either $\mathcal{V} = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ or $\mathcal{V} \neq \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. In the first case, $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for $\mathcal{V}$.

In the second case, there is $\mathbf{v}_3 \in \mathcal{V}$ such that $\mathbf{v}_3 \notin \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. I think you know what to do...

Assuming $\mathcal{V} \neq \text{span}(\mathbf{v}_1)$, one of two things has to happen in the end. First, we could have a list $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathcal{V}$ such that $\mathcal{V} = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$, $\mathbf{v}_1 \neq \mathbf{0}_p$ and $\mathbf{v}_j \notin \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$ for $j = 2, \ldots, r$. This list is therefore independent and so is a basis for $\mathcal{V}$.

Or, we've turned the crank far enough to have an independent list $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathcal{V}$ of p (necessarily distinct!) vectors. Put $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix}$, so $A \in \mathbb{R}^{p \times p}$ has independent columns and therefore is invertible. Thus $\mathbf{C}(A) = \mathbb{R}^p$. I claim this means $\mathcal{V} = \mathbb{R}^p$. Here's why. If $\mathbf{b} \in \mathbb{R}^p$, then $\mathbf{b} \in \mathbf{C}(A)$ since $\mathbf{C}(A) = \mathbb{R}^p$. And Lemma 28.3 says that if $\mathbf{b} \in \mathbf{C}(A)$, then $\mathbf{b} \in \mathcal{V}$. ■

28.5 Problem (!). Reread the preceding proof and identify exactly where we used the assumption that $\mathcal{V}$ was a subspace.

The point of a basis is efficient representation. Theorem 28.2 gives us part of that efficiency: there is only one way to represent vectors with respect to a basis. And now we know there is always a basis. One more big thing remains: the amount of data in a basis is effectively always the same in that any basis contains the same number of vectors. (Deeper question: is there a "best" basis for a subspace? What more could we want? Think about it...)

28.6 Theorem. *Let $\mathcal{V} \neq \{\mathbf{0}_p\}$ be a subspace of $\mathbb{R}^p$. All bases for $\mathcal{V}$ have the same length.*

Proof. This is basically (sorry I'm not sorry) the proof of Theorem 25.1. We know that at least one basis for $\mathcal{V}$ exists by Theorem 28.4. Call that basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ for some $r \geq 1$. Put $A := \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r \end{bmatrix} \in \mathbb{R}^{p \times r}$. So $\mathcal{V} = \mathbf{C}(A)$.

The columns of A are independent, so $\text{rank}(A) = r$ by Problem 25.5. Now let the list $\mathbf{w}_1, \ldots, \mathbf{w}_s$ also be a basis for $\mathcal{V}$. We want to show $s = r$. This list $\mathbf{w}_1, \ldots, \mathbf{w}_s$ is therefore independent in $\mathbf{C}(A)$, so it can't be longer than $\text{rank}(A)$ by Theorem 25.1. Thus $s \leq r$. This list $\mathbf{w}_1, \ldots, \mathbf{w}_s$ also spans $\mathbf{C}(A)$, so it has to be at least as long as $\text{rank}(A)$ by Theorem 25.6. Thus $s \geq r$. Since both $s \leq r$ and $s \geq r$, we have $s = r$. ■

Content from Strang's *ILA* 6E. Page 120 proves Theorem 28.6. Read the paragraphs after the boxed "Definition." Then read Worked Example 3.4 B on p. 122.

Since any (nontrivial) subspace has a basis, and any basis for a (nontrivial) subspace has the same length, it's fair to give a name to that length.

28.7 Definition. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. If $\mathcal{V} \neq \{\mathbf{0}_p\}$, then the* DIMENSION *of $\mathcal{V}$, denoted $\dim(\mathcal{V})$, is the length of any basis for that subspace. We define $\dim(\{\mathbf{0}_p\}) := 0$.*

28.8 Example. **(i)** Since the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for $\mathbb{R}^n$ contains n vectors, $\dim(\mathbb{R}^n) = n$.

(ii) Let $A \in \mathbb{R}^{m \times n}$ have rank $r \geq 1$. Then $\dim(\mathbf{C}(A)) = r$, since A has r pivot columns and those pivot columns form a basis for A by Theorem 24.5. *That is, rank is the dimension of the column space.*

(iii) Let $A \in \mathbb{R}^{m \times n}$ have rank $r \geq 1$. If $r = n$, then $\mathbf{N}(A) = \{\mathbf{0}_n\}$, so in that case $\dim(\mathbf{N}(A)) = 0$. If $1 \leq r < n$, then A has $n - r$ free variables, and we should expect that this means $\dim(\mathbf{N}(A)) = n - r$. (If $r = n$, we recover the $r = 0$ result.) More precisely, we worked out in part (iv) of Example 27.4 that if

$$\mathsf{rref}(A) = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P,$$

then the columns of

$$P^{-1} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}$$

are a basis for $\mathbf{N}(A)$. There are $n - r$ such columns.

Part (iii) of the preceding example, and the specific version of it in Example 27.4, are hugely important. These results relate the dimension r of the column space of a matrix $A \in \mathbb{R}^{m \times n}$ to the dimension $n - r$ of its null space. And $r + n - r = n$, which is the number of columns of A. We will celebrate this shortly, but you have some work to do first.

There are a lot of useful properties of bases and dimension that are basically analogues of things that we (or you) figured out for column spaces relative to rank.

28.9 Problem ($\star$). Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$ with dimension $\dim(\mathcal{V}) = r \geq 1$.

(i) Show that $r \leq p$. [Hint: *if $r > p$, what contradicts Problem 22.9?*]

(ii) Explain why any list of more than r vectors in $\mathcal{V}$ is dependent. [Hint: *write $\mathcal{V}$ as the column space of a matrix of rank r. Then use Theorem 25.1.*]

(iii) Explain why any list of r independent vectors in $\mathcal{V}$ is a basis for $\mathcal{V}$. [Hint: *write $\mathcal{V}$ as the column space of a matrix of rank r. Then use Problem 25.8.*]

(iv) Explain why any list of r vectors in $\mathcal{V}$ that spans $\mathcal{V}$ is a basis for $\mathcal{V}$. [Hint: *write $\mathcal{V}$ as the column space of a matrix of rank r. If a spanning list of r vectors is not a basis, apply Lemma 9.6. What contradiction to Theorem 25.6 results?*]

(v) Explain why any list of s independent vectors can be extended to a basis for $\mathcal{V}$ if $s < r$. That is, if the list $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \mathcal{V}$ is independent, then there are vectors $\mathbf{w}_{s+1}, \ldots, \mathbf{w}_r \in \mathcal{V}$

such that $\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{w}_{s+1}, \ldots, \mathbf{w}_r$ is a basis for $\mathcal{V}$. [Hint: *mimic the "exhaustive" construction in the proof of Theorem 28.4. I'll get you started: if* $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_s) \neq \mathcal{V}$*, then there has to be* $\mathbf{v}_{s+1} \in \mathcal{V}$ *such that* $\mathbf{v}_{s+1} \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_s)$*. What does that say about the (in)dependence of the list* $\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}$*?*]

28.10 Problem ($\star$). It's useful to be able to compare dimensions of related subspaces.

(i) Suppose that $\mathcal{V}$ and $\mathcal{W}$ are subspaces of $\mathbb{R}^p$ such that every vector in $\mathcal{V}$ is also in $\mathcal{W}$. (We might think that $\mathcal{V}$ is a subspace of the subspace $\mathcal{W}$!) Prove that $\dim(\mathcal{V}) \le \dim(\mathcal{W})$. [Hint: *if* $\dim(\mathcal{W}) = r \le p$*, what goes wrong, per part (ii) of Problem 28.9 if there is a list of more than* r *independent vectors in* $\mathcal{V}$*?*]

(ii) In the previous part, if $\dim(\mathcal{V}) = \dim(\mathcal{W})$, explain why $\mathcal{V} = \mathcal{W}$. [Hint: *use part (iii) of Problem 28.9 to explain why any basis for* $\mathcal{V}$ *is then a basis for* $\mathcal{W}$.]

28.11 Problem ($\star$). It's also useful to see how a subspace does, or does not, "change" under the "action" of a matrix. Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$ with $\dim(\mathcal{V}) \ge 1$. Let $A \in \mathbb{R}^{p \times p}$ be invertible and let

$$\mathcal{W} = \{A\mathbf{v} \mid \mathbf{v} \in \mathcal{V}\}.$$

We can think of $\mathcal{W}$ as the part of $\mathbf{C}(A)$ where the inputs (the $\mathbf{v}$) are "coming from" $\mathcal{V}$.

(i) Prove that $\mathcal{W}$ is a subspace of $\mathbb{R}^p$. [Hint: *modify the proof that* $\mathbf{C}(A)$ *is a subspace of* $\mathbb{R}^p$.]

(ii) Prove that $\dim(\mathcal{W}) = \dim(\mathcal{V})$. [Hint: *let* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *be a basis for* $\mathcal{V}$*. Why is* $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ *a basis for* $\mathcal{W}$*?*]

28.12 Problem ($\star$). And it's also useful to be able to compare ranks of related matrices.

(i) Prove that $\operatorname{rank}(AB) \le \operatorname{rank}(A)$. [Hint: *Problems 10.6 and 28.10.*]

(ii) Prove that if B is invertible (so here we're assuming $n = p$), then $\mathbf{C}(A) = \mathbf{C}(AB)$ and so $\operatorname{rank}(AB) = \operatorname{rank}(A)$. [Hint: *this is one of the rare times when we need to work at both parts of Definition 5.1. One trick is to write* $A\mathbf{v} = (AB)(B^{-1}\mathbf{v})$*. And do you agree that* $AB\mathbf{v} \in \mathbf{C}(A)$*?*]

(iii) Give an example to show that if B is not invertible, then we may have $\operatorname{rank}(AB) < \operatorname{rank}(A)$.

(iv) Prove that $\operatorname{rank}(AB) \le \operatorname{rank}(B)$. [Hint: *let* $r := \operatorname{rank}(B)$*. If* $r = 0$*, then* B *is the zero matrix. If* $r \ge 1$*, suppose that* $\operatorname{rank}(AB) > \operatorname{rank}(B)$ *and find vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_{r+1}$ *that are columns of* B *with* $A\mathbf{v}_1, \ldots, A\mathbf{v}_{r+1}$ *independent. Find something that goes wrong if* $\mathbf{v}_1, \ldots, \mathbf{v}_{r+1}$ *is dependent, which is what you expect since this is a list of columns of* B *longer than* $\operatorname{rank}(B)$.]

(v) Prove that if A is invertible (so here $m = n$), then $\text{rank}(AB) = \text{rank}(B)$. [Hint: *apply Problem 28.11 with* $\mathcal{W} = \mathbf{C}(AB)$ *and* $\mathcal{V} = \mathbf{C}(B)$.]

(vi) Give an example to show that if A is not invertible, then we may have $\text{rank}(AB) < \text{rank}(B)$.

Content from Strang's *ILA* 6E. Look at the matrix R_0 on p. 130. Then read #2 and #3 on pp. 130–131 on the dimensions of its column and null spaces. Next, look at the matrix A at the bottom of p. 131 and read about its column and null spaces in #2 and #3 on pp. 132–133. We'll come back to the row space and left null space shortly. Read Worked Example 3.5 B on p. 137.

28.13 Problem (!). You might wonder why, in considering the "size" of a subspace, we never just try to count its elements. Explain why if $\mathcal{V} \neq \{\mathbf{0}_p\}$ is a subspace of $\mathbb{R}^p$, then $\mathcal{V}$ contains infinitely many vectors. [Hint: *explain why there is* $\mathbf{v} \in \mathcal{V}$ *with* $\mathbf{v} \neq \mathbf{0}_p$ *and then* $c\mathbf{v} \in \mathcal{V}$ *for all* $c \in \mathbb{R}$.]

Day 29: Friday, October 24.

Now we can answer our lingering questions on how to quantify and qualify failure for the problem $A\mathbf{x} = \mathbf{b}$. We proved the following in part (iii) of Example 28.8.

29.1 Theorem (Rank–nullity). *Let* $A \in \mathbb{R}^{m \times n}$. *Then*

$$\dim(\mathbf{N}(A)) + \dim(\mathbf{C}(A)) = n.$$

The point is that if you know one of these dimensions, then you know the other. It's interesting, and maybe a bit weird, that even though $\mathbf{C}(A)$ is not a subspace of $\mathbb{R}^n$ (it's a subspace of $\mathbb{R}^m$), its dimension still talks to the dimension of $\mathbf{N}(A)$ (which is a subspace of $\mathbb{R}^n$) and the dimension of $\mathbb{R}^n$ itself.

29.2 Example. We finally need a larger matrix than our most frequently used, beloved example. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 & 0 \\ 2 & 4 & 2 & 14 & 0 \\ 0 & 0 & 2 & 8 & 0 \end{bmatrix}.$$

I'll leave it to you to check that

$$\mathsf{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that A has 2 pivot columns, so $\text{rank}(A) = \dim(\mathbf{C}(A)) = 2$, and therefore

$\dim(\mathbf{N}(A)) = 5 - 2 = 3$. We can verify this explicitly by computing (I'll let you do that)

$$\mathbf{N}(A) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

and seeing the three independent columns spanning $\mathbf{N}(A)$ right there.

Here are some ways that I interpret rank–nullity and the quantitative (and morally qualitative) extents to which solutions to $A\mathbf{x} = \mathbf{b}$ exist and are unique.

If $A \in \mathbb{R}^{m \times n}$ with $m \neq n$ (so A is not square), then either existence or uniqueness of solutions fails. Of course, either or both could fail even when A is square! There's just a little more hope to avoid failure in the square case. Rank–nullity allows us to quantify this failure.

First, if existence fails and $\mathbf{C}(A) \neq \mathbb{R}^m$, then $\dim(\mathbf{C}(A))$ is not as "large" as it could be. If $\dim(\mathbf{C}(A))$ is "too small," then $\dim(\mathbf{N}(A))$ will have to be "large enough' to make $\dim(\mathbf{N}(A)) + \dim(\mathbf{C}(A)) = n$ true. This means that if existence fails "sufficiently much," then uniqueness also fails with a "certain large degree of freedom."

Second, if uniqueness fails and $\mathbf{N}(A) \neq \{\mathbf{0}_n\}$, then $\dim(\mathbf{N}(A))$ is not as "small" as it could be. If $\dim(\mathbf{N}(A))$ is "too big," then $\dim(\mathbf{C}(A))$ will have to be "small enough" to make, again, $\dim(\mathbf{N}(A)) + \dim(\mathbf{C}(A)) = n$ true. This means that if uniqueness fails with a "certain large degree of freedom," then uniqueness also fails "sufficiently much."

29.3 Problem ($\star$). Let $A \in \mathbb{R}^{m \times m}$. Use rank–nullity to prove that $\mathbf{N}(A) = \{\mathbf{0}_m\}$ if and only if $\mathbf{C}(A) = \mathbb{R}^m$. This gives another proof of some parts of the invertible matrix theorem (which parts?).

The notion of dimension allows us to quantify how much of $\mathbb{R}^m$ the column space of $A \in \mathbb{R}^{m \times n}$ misses: it misses $m - \dim[\mathbf{C}(A)]$ "dimensions" of $\mathbb{R}^m$. But what is going on elsewhere in $\mathbb{R}^m$ beyond $\mathbf{C}(A)$? The deeper question is not just *how much* of $\mathbb{R}^m$ does $\mathbf{C}(A)$ miss but rather *what exactly* in $\mathbb{R}^m$ does $\mathbf{C}(A)$ miss. Is there a simpler way to characterize and describe $\mathbf{C}(A)$ than just its definition?

We could also ask about $\mathbb{R}^n$ and $\mathbf{N}(A)$. When $\mathbf{v} \in \mathbf{N}(A)$, we have $A\mathbf{v} = \mathbf{0}_m$. For what $\mathbf{w} \in \mathbb{R}^n$ does A "act nontrivially" with $A\mathbf{w} \neq \mathbf{0}_m$? On what parts of $\mathbb{R}^n$ is A "more interesting"?

It turns out that these questions are "dual" to each other in that if we know how to handle one of them, we can understand the other pretty quickly. And it also turns out (I think) that asking how $\mathbf{N}(A)$ interacts with the rest of $\mathbb{R}^n$ is an easier thing to control.

29.4 Example. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\mathbf{N}(A) = \mathbf{C}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

I hope it's glaringly obvious what's missing from $\mathbf{N}(A)$: $\mathbf{e}_1$.

More precisely, let

$$\mathcal{W} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

Then any $\mathbf{x} \in \mathbb{R}^3$ can be written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}.$$

That is, any $\mathbf{x} \in \mathbb{R}^3$ can be written (or, more evocatively, "decomposed") in the form $\mathbf{x} = \mathbf{v} + \mathbf{w}$ for $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathcal{W}$. Think for a moment about why those $\mathbf{v}$ and $\mathbf{w}$ are unique; that is, why there is only one way to achieve this "decomposition" of $\mathbf{x}$.

Can we see $\mathcal{W}$ directly from A itself, without passing to the null space? Sure: $\mathbf{e}_1$ is the first row of A. We're more used to thinking about columns than rows, so let's flip every row of A to a column and every column of A to a row by taking the **TRANSPOSE**:

$$A^{\mathsf{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then we see

$$\mathcal{W} = \mathbf{C}(A^{\mathsf{T}}).$$

And so we have written any $\mathbf{x} \in \mathbb{R}^3$ uniquely as a sum of the form $\mathbf{x} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^{\mathsf{T}})$.

Did we just get really lucky, since the bases for $\mathbf{N}(A)$ and $\mathbf{C}(A^{\mathsf{T}})$ in the previous example just involved the coordinate axes for three-dimensional space? Here's a more complicated situation. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This was the RREF of the original matrix from Example 29.2. Then

$$\mathbf{N}(A) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

and (flipping columns to rows and rows to columns)

$$A^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so (ignoring that third column, but keeping those first two independent columns)

$$\mathbf{C}(A^{\mathsf{T}}) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}\right).$$

Can we write each $\mathbf{x} \in \mathbb{R}^5$ as $\mathbf{x} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^{\mathsf{T}})$, and is such a decomposition of $\mathbf{x}$ unique?

Some notation will compress things helpfully. Let

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \tag{29.1}$$

so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbf{N}(A)$, and $\mathbf{w}_1$ and $\mathbf{w}_2$ form a basis for $\mathbf{C}(A^{\mathsf{T}})$. Let

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix}, \tag{29.2}$$

so $M \in \mathbb{R}^{5 \times 5}$. If M is invertible, then we can write each $\mathbf{x} \in \mathbb{R}^5$ as $\mathbf{x} = M\mathbf{y}$ for some (unique!) $\mathbf{y} \in \mathbb{R}^5$. Then

$$\mathbf{x} = (y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3) + (y_4\mathbf{w}_1 + y_5\mathbf{w}_2) \tag{29.3}$$

with $y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3 \in \mathbf{N}(A)$ and $y_4\mathbf{w}_1 + y_5\mathbf{w}_2 \in \mathbf{C}(A^{\mathsf{T}})$. That gives the decomposition of $\mathbf{x}$ that we want, and we could probably push it further with an independence argument to get uniqueness.

So, is M invertible? Do you really want to do the row operations to find out? This matrix M is a beast! There's a less obvious approach that will teach us some valuable new things, and it all hinges on a deeper notion of the dot product. First, I encourage you to review some hopefully unsurprising dot product arithmetic.

29.5 Problem ($\star$). Show that the dot product has the following properties. All vectors below are in the same space, e.g., $\mathbb{R}^p$. (If it makes things more concrete for you, do it for $p = 3$.)

(i) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

(ii) $\mathbf{v} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = (\mathbf{v} \cdot \mathbf{w}_1) + (\mathbf{v} \cdot \mathbf{w}_2)$.

(iii) $\mathbf{v} \cdot (c\mathbf{w}) = c(\mathbf{v} \cdot \mathbf{w})$.

(iv) $\mathbf{v} \cdot \mathbf{v} \geq 0$.

(v) If $\mathbf{v} \cdot \mathbf{v} = 0$, then $\mathbf{v} = \mathbf{0}$. [Hint: *if* $\mathbf{v} \neq \mathbf{0}$, *explain why* $\mathbf{v} \cdot \mathbf{v} > 0$.]

Day 30: Monday, October 27.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Orthogonal vectors (N), transpose of a matrix

Here's a more complicated situation. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This was the RREF of the original matrix from Example 29.2. Then

$$\mathbf{N}(A) = \mathbf{C}\left(\begin{bmatrix} -2 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

and (flipping columns to rows and rows to columns)

$$A^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so (ignoring that third column, but keeping those first two independent columns)

$$\mathbf{C}(A^{\mathsf{T}}) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}\right).$$

Can we write each $\mathbf{x} \in \mathbb{R}^5$ as $\mathbf{x} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$, and is such a decomposition of $\mathbf{x}$ unique?

Some notation will compress things helpfully. Let

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \tag{30.1}$$

so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbf{N}(A)$, and $\mathbf{w}_1$ and $\mathbf{w}_2$ form a basis for $\mathbf{C}(A^\mathsf{T})$. Since $\dim(\mathbb{R}^5) = 5$ and $3 + 2 = 5$, we might wonder if the list $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{w}_1$, $\mathbf{w}_2$ is a basis for $\mathbb{R}^5$. And since this list has length 5, all that we have to do is check its independence. So, suppose that

$$\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3}_{\mathbf{v}} + \underbrace{c_4\mathbf{w}_1 + c_5\mathbf{w}_2}_{\mathbf{w}} = \mathbf{0}_5. \tag{30.2}$$

Our goal is $c_j = 0$ for $j = 1, \ldots, 5$.

We could turn this into a 5×5 linear system and slug it out. Or we could be clever and do something new. The right idea—and, I grant you, maybe the surprising idea—is to exploit how the vectors $\mathbf{v}_i$ and $\mathbf{w}_j$ talk to each other under the dot product. I suggest that you review the following unsurprising ideas first.

30.1 Problem ($\star$). Show that the dot product has the following properties. All vectors below are in the same space, e.g., $\mathbb{R}^p$. (If it makes things more concrete for you, do it for $p = 3$.)

(i) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

(ii) $\mathbf{v} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = (\mathbf{v} \cdot \mathbf{w}_1) + (\mathbf{v} \cdot \mathbf{w}_2)$.

(iii) $\mathbf{v} \cdot (c\mathbf{w}) = c(\mathbf{v} \cdot \mathbf{w})$.

(iv) $\mathbf{v} \cdot \mathbf{v} \geq 0$.

(v) If $\mathbf{v} \cdot \mathbf{v} = 0$, then $\mathbf{v} = \mathbf{0}$. [Hint: *if* $\mathbf{v} \neq \mathbf{0}$, *explain why* $\mathbf{v} \cdot \mathbf{v} > 0$.]

Next, take a look at the dot products of the vectors in (30.1). I'll get you started:

$$\mathbf{v}_1 \cdot \mathbf{w}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} = (-2 \cdot 1) + (1 \cdot 2) + (0 \cdot 0) + (0 \cdot 3) + (0 \cdot 0) = -2 + 2 + 0 + 0 + 0 = 0.$$

30.2 Problem (!). Check that

$$\mathbf{v}_i \cdot \mathbf{w}_j = 0$$

for $i = 1, 2, 3$ and $j = 1, 2$. Then, with $\mathbf{v}$ and $\mathbf{w}$ defined in (30.2), check that

$$\mathbf{v} \cdot \mathbf{w} = 0.$$

[Hint: *use* $\mathbf{v}_i \cdot \mathbf{w}_j = 0$ *and results from Problem 30.1.*]

Why would we think to look for these relations between the bases of $\mathbf{N}(A)$ and $\mathbf{C}(A^{\mathsf{T}})$? One reason might be the transparent relations among the bases from Example 29.4. Those bases talk to each other so well via dot products—maybe other bases do, too.

Now we can check that the list $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{w}_1$, $\mathbf{w}_2$ is independent. From (30.2), we are assuming $\mathbf{v}+\mathbf{w} = \mathbf{0}_5$, and we now know $\mathbf{v}\cdot\mathbf{w} = 0$. The great trick is to take the dot product of both sides with $\mathbf{v}$ (although $\mathbf{w}$ would also work):

$$(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{0}_5 \cdot \mathbf{v}.$$

Then we get

$$(\mathbf{v} \cdot \mathbf{v}) + (\mathbf{w} \cdot \mathbf{v}) = 0$$

and since $\mathbf{w} \cdot \mathbf{v} = 0$ this simplifies to

$$\mathbf{v} \cdot \mathbf{v} = 0.$$

Dot product properties tell us $\mathbf{v} = \mathbf{0}_5$, and so

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}_5.$$

The independence of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ implies $c_1 = c_2 = c_3 = 0$, thus $\mathbf{v} = \mathbf{0}_5$, and so the equation $\mathbf{v} + \mathbf{w} = \mathbf{0}_5$ reduces to $\mathbf{w} = \mathbf{0}_5$. Then $c_4\mathbf{w}_1 + c_5\mathbf{w}_2 = \mathbf{0}_5$, and so $c_4 = c_5 = 0$ by the independence of $\mathbf{w}_1$, $\mathbf{w}_2$.

The list $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{w}_1$, $\mathbf{w}_2$ in $\mathbb{R}^5$ is therefore independent and thus a basis for $\mathbb{R}^5$. We can therefore write any $\mathbf{b} \in \mathbb{R}^5$ in the form

$$\mathbf{b} = \underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3}_{\mathbf{v}} + \underbrace{c_4\mathbf{w}_1 + c_5\mathbf{w}_2}_{\mathbf{w}},$$

where $\mathbf{v} \cdot \mathbf{w} = 0$. The coefficients c_j are, of course, unique.

But $\mathbf{v}$ and $\mathbf{w}$ are unique, too. Say that we go coordinate-free and can write $\mathbf{b} \in \mathbb{R}^5$ as both

$$\mathbf{b} = \mathbf{v} + \mathbf{w} \quad \text{and} \quad \mathbf{b} = \widetilde{\mathbf{v}} + \widetilde{\mathbf{w}}$$

for some $\mathbf{v}$, $\widetilde{\mathbf{v}} \in \mathbf{N}(A)$ and $\mathbf{w}$, $\widetilde{\mathbf{w}} \in \mathbf{C}(A^{\mathsf{T}})$. Then

$$\mathbf{0}_5 = \mathbf{b} - \mathbf{b} = (\mathbf{v} + \mathbf{w}) - (\widetilde{\mathbf{v}} + \widetilde{\mathbf{w}}) = (\mathbf{v} - \widetilde{\mathbf{v}}) + (\mathbf{w} - \widetilde{\mathbf{w}})$$

and so

$$\mathbf{v} - \widetilde{\mathbf{v}} = \widetilde{\mathbf{w}} - \mathbf{w}.$$

Here is where subspace dynamics helps, and $\mathbf{N}(A)$ and $\mathbf{C}(A^{\mathsf{T}})$ are subspaces. Since $\mathbf{v}$, $\widetilde{\mathbf{v}} \in \mathbf{N}(A)$, we have $\mathbf{v} - \widetilde{\mathbf{v}} \in \mathbf{N}(A)$, and likewise $\mathbf{w} - \widetilde{\mathbf{w}} \in \mathbf{C}(A^{\mathsf{T}})$. Hence $\mathbf{v} - \widetilde{\mathbf{v}} \in \mathbf{N}(A)$ and $\mathbf{v} - \widetilde{\mathbf{v}} = \widetilde{\mathbf{w}} - \mathbf{w} \in \mathbf{C}(A^{\mathsf{T}})$.

What happens if there is $\mathbf{y} \in \mathbb{R}^5$ such that both $\mathbf{y} \in \mathbf{N}(A)$ and $\mathbf{y} \in \mathbf{C}(A^\mathsf{T})$? Recall that if $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$, then $\mathbf{v} \cdot \mathbf{w} = 0$. Taking $\mathbf{v} = \mathbf{y}$ and $\mathbf{w} = \mathbf{y}$, we get $\mathbf{y} \cdot \mathbf{y} = 0$, thus $\mathbf{y} = \mathbf{0}_5$.

In the situation above, this means $\mathbf{v} - \widetilde{\mathbf{v}} = \mathbf{0}_5$, so $\mathbf{v} = \widetilde{\mathbf{v}}$, but then also $\widetilde{\mathbf{w}} - \mathbf{w} = \mathbf{0}_5$, so $\mathbf{w} = \widetilde{\mathbf{w}}$. This is our desired uniqueness.

Now here is the great thing: all of this generalizes far beyond the specific matrices A just considered. First, let's name the feature of dot products that we've been using.

30.3 Definition. *The vectors $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$ are* **ORTHOGONAL** *if $\mathbf{v} \cdot \mathbf{w} = 0$.*

Next, let's formalize the notion of transpose.

30.4 Definition (What the transpose is). *The* **TRANSPOSE** *of $A \in \mathbb{R}^{m \times n}$ is the matrix $A^\mathsf{T} \in \mathbb{R}^{n \times m}$ such that the (i, j)-entry of A^T is the (j, i)-entry of A. We write*

$$A^\mathsf{T}_{ij} = A_{ji}.$$

30.5 Remark. *Recall that we write $\mathbf{v} \in \mathbb{R}^n$ as*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = (v_1, \ldots, v_n).$$

If we think of $\mathbf{v}$ as an $n \times 1$ matrix, which is sometimes convenient, then

$$\mathbf{v}^\mathsf{T} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix},$$

but we never write $\mathbf{v}^\mathsf{T} = (v_1, \ldots, v_n)$. Parentheses and square brackets are different kinds of notation!

30.6 Example. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

then

$$A^\mathsf{T} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Now we can generalize how $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$ interact in general.

30.7 Theorem. *Let $A \in \mathbb{R}^{m \times n}$, $\mathbf{v} \in \mathbf{N}(A)$, and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$. Then $\mathbf{v} \cdot \mathbf{w} = 0$.*

Proof. If $\mathbf{v} \in \mathbf{N}(A)$, then $A\mathbf{v} = \mathbf{0}_m$. One way to compute $A\mathbf{v}$ is by taking dot products of $\mathbf{v}$

with the rows of A viewed as columns in $\mathbb{R}^n$. That $A\mathbf{v} = \mathbf{0}_m$ says that each such dot product is 0. Say that the ith row of A, viewed as a column in $\mathbb{R}^n$, is $\mathbf{b}_i \in \mathbb{R}^n$. Then $\mathbf{v} \cdot \mathbf{b}_i = 0$ for all i. Any vector in $\mathbf{C}(A^\mathsf{T})$ is a linear combination of the rows of A viewed as columns in $\mathbb{R}^n$. Say that $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$ has the form

$$\mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_m\mathbf{b}_m.$$

Then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot (c_1\mathbf{b}_1 + \cdots + c_m\mathbf{b}_m) = c_1(\mathbf{v} \cdot \mathbf{b}_1) + \cdots + c_m(\mathbf{v} \cdot \mathbf{b}_m) = 0.$$ ■

This is a perfectly adequate proof based on what A^T is: the matrix formed by swapping the rows and columns of A. This is a "static" way to think about A^T: it's an array of data. Nothing wrong with that. But we can be dynamic: *what things do defines what things are.*

Day 31: Wednesday, October 29.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Row space of a matrix

Here is what A^T does. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors for $\mathbb{R}^n$ and $\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_m$ be the standard basis vectors for $\mathbb{R}^m$. (I'm putting tildes on the vectors for $\mathbb{R}^m$ because the notation $\mathbf{e}_j$ doesn't otherwise indicate what space it's in. I guess we could stack extra subscripts or superscripts to indicate n and m but, ew.) So, if $n = 4$ and $m = 3$, then

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \qquad \text{and} \qquad \mathbf{e}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix},$$

while

$$\widetilde{\mathbf{e}}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \widetilde{\mathbf{e}}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \text{and} \qquad \widetilde{\mathbf{e}}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Now recall that multiplying a matrix by standard basis vectors extracts its columns, while taking the dot product of a vector with standard basis vectors extracts its entries. Since $A \in \mathbb{R}^{m\times n}$, the jth column of A is $A\mathbf{e}_j \in \mathbb{R}^m$, and then $A\mathbf{e}_j \cdot \widetilde{\mathbf{e}}_i$ is the ith entry in that column. That is,

$$A_{ij} = A\mathbf{e}_j \cdot \widetilde{\mathbf{e}}_i.$$

31.1 Problem (!). Suppose that $A, B \in \mathbb{R}^{m\times n}$. Certainly if $A = B$, then

$$A\mathbf{v} \cdot \mathbf{w} = B\mathbf{v} \cdot \mathbf{w} \tag{31.1}$$

for all $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. Just substitute A for B and plug and chug. Conversely, prove that if (31.1) holds for all $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$, then $A = B$. [Hint: *get the standard basis vectors to show up.*]

Likewise, since $A^{\mathsf{T}} \in \mathbb{R}^{n \times m}$, the jth column of A^{T} is $A^{\mathsf{T}}\widetilde{\mathbf{e}}_j \in \mathbb{R}^n$, and then $A^{\mathsf{T}}\widetilde{\mathbf{e}}_j \cdot \mathbf{e}_i$ is the ith entry in that column. That is,

$$A^{\mathsf{T}}_{ij} = A^{\mathsf{T}}\widetilde{\mathbf{e}}_j \cdot \mathbf{e}_i.$$

Thus

$$A^{\mathsf{T}}\widetilde{\mathbf{e}}_j \cdot \mathbf{e}_i = A^{\mathsf{T}}_{ij} = A_{ji} = A\mathbf{e}_i \cdot \widetilde{\mathbf{e}}_j,$$

and so

$$A\mathbf{e}_i \cdot \widetilde{\mathbf{e}}_j = A^{\mathsf{T}}\widetilde{\mathbf{e}}_j \cdot \mathbf{e}_i.$$

The commutativity of the dot product ($\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$) gives

$$A\mathbf{e}_i \cdot \widetilde{\mathbf{e}}_j = \mathbf{e}_i \cdot A^{\mathsf{T}}\widetilde{\mathbf{e}}_j. \tag{31.2}$$

This is what A^{T} *does*: it pops across the dot product.

31.2 Remark. *There are actually two dot products in* (31.2)*. The one on the left is in* $\mathbb{R}^m$*, since* $A\mathbf{e}_i$*,* $\widetilde{\mathbf{e}}_j \in \mathbb{R}^m$*. The one on the right is in* $\mathbb{R}^n$*, since* $\mathbf{e}_i$*,* $A^{\mathsf{T}}\widetilde{\mathbf{e}}_j \in \mathbb{R}^n$*.*

This "popping" behavior of the transpose is not limited to standard basis vectors.

31.3 Theorem (What the transpose does). *Let* $A \in \mathbb{R}^{m \times n}$*,* $\mathbf{v} \in \mathbb{R}^n$*, and* $\mathbf{w} \in \mathbb{R}^m$*. Then*

$$A\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot A^{\mathsf{T}}\mathbf{w}.$$

Moreover, the transpose is the only matrix in $\mathbb{R}^{n \times m}$ *to do this: if there is* $B \in \mathbb{R}^{n \times m}$ *such that*

$$A\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot B\mathbf{w} \tag{31.3}$$

for all $\mathbf{v} \in \mathbb{R}^n$ *and* $\mathbf{w} \in \mathbb{R}^m$*, then* $B = A^{\mathsf{T}}$*.*

31.4 Problem (+). Prove it.

(i) First, with $\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_m$ as the standard basis vectors for $\mathbb{R}^m$, show

$$A\mathbf{v} \cdot \widetilde{\mathbf{e}}_i = \mathbf{v} \cdot A^{\mathsf{T}}\widetilde{\mathbf{e}}_i$$

by expanding $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$ and using linearity of matrix-vector multiplication and dot product arithmetic from Problem 30.1. Then show the general result by expanding $\mathbf{w} = w_1\widetilde{\mathbf{e}}_1 + \cdots + w_m\widetilde{\mathbf{e}}_m$.

(ii) Use Problem 31.1 to show that (31.3).

Here are some nice properties of the transpose that can easily be deduced from what it does, rather than what it is. These are important, and I expect that you're going to know them, but I think they'll be more meaningful if you prove them yourself.

31.5 Problem ($\star$). **(i)** Let $A \in \mathbb{R}^{m \times n}$. First explain why $(A^{\mathsf{T}})^{\mathsf{T}} \in \mathbb{R}^{m \times n}$, too. Then use Problem 31.1 to prove that $(A^{\mathsf{T}})^{\mathsf{T}} = A$ by showing that

$$(A^{\mathsf{T}})^{\mathsf{T}}\mathbf{v} \cdot \mathbf{w} = A\mathbf{v} \cdot \mathbf{w}$$

for all $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$.

(ii) Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$. First explain why $(AB)^{\mathsf{T}}$, $B^{\mathsf{T}}A^{\mathsf{T}} \in \mathbb{R}^{p \times m}$. Then show that

$$(B^{\mathsf{T}}A^{\mathsf{T}})\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot AB\mathbf{w}$$

for all $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^p$. Use the uniqueness result of Theorem 31.3 to conclude $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$.

(iii) Let $A \in \mathbb{R}^{m \times m}$ be invertible. Prove that A^{T} is also invertible with inverse $(A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}$ by computing $(AA^{-1})^{\mathsf{T}} = I_m^{\mathsf{T}}$ and $(AA^{-1})^{\mathsf{T}} = A^{\mathsf{T}}(A^{-1})^{\mathsf{T}}$. What does this tell you?

(iv) Let $P \in \mathbb{R}^{m \times m}$ be a permutation matrix, so P contains all of the columns of the identity matrix I_m (each column appearing once, and only once) in some order. Argue that $P\mathbf{v} \cdot P\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^m$. [Hint: *maybe do this for the only non-identity* 2×2 *permutation to get a feel for what's going on, then generalize.*] Conclude $P^{\mathsf{T}}P\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^m$. Why does this imply that P is invertible with $P^{-1} = P^{\mathsf{T}}$? (Good news: no more writing P^{-1} when doing calculations with the abstract form of the RREF!)

Content from Strang's *ILA* 6E. Pages 67–68 discuss fundamental properties of the transpose. Pages 68–69 show how the transpose interacts with dot products (I wholly disagree that $\cdot$ is "unprofessional"—I like that it emphasizes how the dot product takes in two inputs and how it's "linear in each input." I like dot products.). If you have seen integration by parts in calculus, read Example 2 on p. 69.

Now we can give another proof of Theorem 30.7 that relies on what the transpose does, rather than is. Let $A \in \mathbb{R}^{m \times n}$, $\mathbf{v} \in \mathbf{N}(A)$, and $\mathbf{w} \in \mathbf{C}(A^{\mathsf{T}})$. Then $A^{\mathsf{T}}\mathbf{v} = \mathbf{0}_m$ and there is $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{w} = A\mathbf{y}$. We compute

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot A^{\mathsf{T}}\mathbf{y} = A\mathbf{v} \cdot \mathbf{y} = \mathbf{0}_m \cdot \mathbf{y} = 0. \tag{31.4}$$

So slick! The first two dot products were dot products in $\mathbb{R}^n$, but the second two were in $\mathbb{R}^m$.

Content from Strang's *ILA* 6E. Read p. 144 starting with the box "The **nullspace of** A…." Then read the "Important" paragraph on p. 145 about the orthogonality of $\mathbf{C}(A)$ and $\mathbf{N}(A^{\mathsf{T}})$, which we will discuss in greater detail later, and Example 1. Note that Strang typically likes to write the dot product as $\mathbf{x}^{\mathsf{T}}\mathbf{y}$, not $\mathbf{x} \cdot \mathbf{y}$.

31.6 Problem (!). Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}.$$

Use results from Examples 21.1 and 23.2 to give bases for $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$, and check directly that the vectors in the basis for $\mathbf{N}(A)$ are orthogonal to the vectors in the basis for $\mathbf{C}(A^\mathsf{T})$.

31.7 Problem (+). Here is a much less slick way to show the orthogonality of vectors in $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$ that reinforces properties of the RREF. Let $A \in \mathbb{R}^{m\times n}$ with $r = \operatorname{rank}(A)$.

(i) If $r = n$, why do you have very little work to do? Do it.

(ii) From now on suppose $r < n$. Suppose $EA = R_0$ with $E \in \mathbb{R}^{m\times m}$ invertible and $R_0 = \mathsf{rref}(A)$. Use Problem 14.12 to explain why $\mathbf{C}(A^\mathsf{T}) = \mathbf{C}(R_0)$. (Don't get too excited: remember that in general $\mathbf{C}(A) \neq \mathbf{C}(R_0)$.)

(iii) Write R_0 in the very general form

$$R_0 = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P,$$

where maybe the F- and/or zero blocks are not present, and P is a permutation matrix that could be the identity. Then compute

$$R_0^\mathsf{T} = P^\mathsf{T} \begin{bmatrix} I_r & 0 \\ F^\mathsf{T} & 0 \end{bmatrix}. \tag{31.5}$$

(iv) Combine some old ideas with part (iv) of Problem 31.5 to show

$$\mathbf{N}(A) = \mathbf{C}\left(P^\mathsf{T} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}\right).$$

Since $r < n$, that block I_{n-r} will always genuinely be present.

(v) Conclude that if $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$, then there are $\mathbf{x} \in \mathbb{R}^{n-r}$ and $\mathbf{y} \in \mathbb{R}^r$ such that

$$\mathbf{v} = P^\mathsf{T} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{w} = P^\mathsf{T} \begin{bmatrix} I_r & 0 \\ F^\mathsf{T} & 0 \end{bmatrix} \mathbf{y}.$$

Use this to show $\mathbf{v} \cdot \mathbf{w} = 0$. [Hint: *an identity from part (iv) of Problem 31.5 will be help with those common factors of* P^T.]

It will save us some time (like 3 seconds) to give a different name to $\mathbf{C}(A^\mathsf{T})$ other than "column space of A transpose."

31.8 Definition. *The* ROW SPACE *of* $A \in \mathbb{R}^{m \times n}$ *is* $\mathbf{C}(A^\mathsf{T})$.

The orthogonality of vectors in $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$ is the key to generalizing the decomposition of $\mathbb{R}^n$ that we did for some special matrices and some special n. Let $A \in \mathbb{R}^{m \times n}$. For each $\mathbf{x} \in \mathbb{R}^n$, we want to find unique $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$ such that $\mathbf{x} = \mathbf{v} + \mathbf{w}$. We need one more fact.

Informally, "row rank = column rank." The rank of A is the number of pivot columns and the number of pivot rows of A. If you look at the RREF, pivot rows are independent. I mean, just look at

$$R_0 = \begin{bmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Two pivot columns, two pivot rows. Then

$$R_0^\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 \\ * & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix}.$$

Those first two columns in R_0^T are definitely independent (even though R_0^T isn't in RREF anymore). So $\operatorname{rank}(R_0) = 2$ and $\operatorname{rank}(R_0^\mathsf{T}) = 2$. Basically, you can see from the structure of the RREF that

$$\operatorname{rank}[\mathsf{rref}(A)] = \operatorname{rank}[\mathsf{rref}(A)^\mathsf{T}].$$

Then you use the fact that $A = E^{-1}\mathsf{rref}(A)$ for some invertible matrix E, compute

$$A^\mathsf{T} = \mathsf{rref}(A)^\mathsf{T}(E^{-1})^\mathsf{T},$$

and use the fact (from Problem 28.12) that multiplying on the right by an invertible matrix doesn't change rank. Donezo.

31.9 Theorem. *Let* $A \in \mathbb{R}^{m \times n}$. *Then* $\operatorname{rank}(A) = \operatorname{rank}(A^\mathsf{T})$.

Proof. Here is an argument different from the one above. Let $\operatorname{rank}(A) = r$. If $r = 0$, then A is the zero matrix in $\mathbb{R}^{m \times n}$, so A^T is the zero matrix in $\mathbb{R}^{n \times m}$. Then $\mathbf{C}(A^\mathsf{T}) = \{\mathbf{0}_n\}$, so $\operatorname{rank}(A^\mathsf{T}) = 0$.

Otherwise, suppose $r \ge 1$, and let $A = CR$ be the CR-factorization of A. We have $C \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$, so $R^\mathsf{T} \in \mathbb{R}^{n \times r}$. Then $A^\mathsf{T} = (CR)^\mathsf{T} = R^\mathsf{T}C^\mathsf{T}$, so $\operatorname{rank}(A^\mathsf{T}) \le \operatorname{rank}(R^\mathsf{T})$ by Problem 28.12. And since $R^\mathsf{T} \in \mathbb{R}^{n \times r}$, we can estimate $\operatorname{rank}(R^\mathsf{T}) \le \min\{n, r\} \le r$.

All of this is to say that $\operatorname{rank}(A^\mathsf{T}) \le \operatorname{rank}(A)$. This is true for any matrix A, so replace A with A^T:

$$\operatorname{rank}(A) = \operatorname{rank}((A^\mathsf{T})^\mathsf{T}) \le \operatorname{rank}(A^\mathsf{T}).$$

Put the inequalities $\operatorname{rank}(A^\mathsf{T}) \le \operatorname{rank}(A)$ and $\operatorname{rank}(A) \le \operatorname{rank}(A^\mathsf{T})$ together to get $\operatorname{rank}(A) = \operatorname{rank}(A^\mathsf{T})$. ■

31.10 Problem (+). Here is a more painfully precise version of the RREF-style proof. Let $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$ and $R_0 = \mathsf{rref}(A)$.

(i) Use part (ii) of Problem 31.7 to show that $\operatorname{rank}(A^{\mathsf{T}}) = \operatorname{rank}(R_0^{\mathsf{T}})$.

(ii) Suppose that R_0 has the very general form

$$R_0 = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} P,$$

where maybe the F- and/or zero blocks are not present, and P is a permutation matrix that could be the identity. Use Problem 25.16 and the invertibility of P from part (iv) of Problem 31.5 to show that

$$\operatorname{rank}(R_0^{\mathsf{T}}) = \operatorname{rank}\left(\begin{bmatrix} I_r & 0 \\ F^{\mathsf{T}} & 0 \end{bmatrix}\right).$$

(iii) Put all of this together to conclude that $\operatorname{rank}(A) = \operatorname{rank}(A^{\mathsf{T}})$.

Content from Strang's *ILA* 6E. Read #1 on p. 130, #4 on p. 131, #1 on p. 132, and #4 on p. 133. Actually, probably best to reread all of pp. 130–133 and see all four subspaces talk to each other.

Day 32: Friday, October 31.

You took Exam 2.

Day 33: Monday, November 3.

Now we can build a basis for $\mathbb{R}^n$. Suppose that $A \in \mathbb{R}^{m \times n}$ has rank r. Then $\dim[\mathbf{N}(A)] = n - r$ by rank-nullity and $\dim[\mathbf{C}(A^{\mathsf{T}})] = r$ by the result above. Since $(n - r) + r = n$, this should make us feel optimistic.

33.1 Problem (!). **(i)** Prove that if A has full column rank ($r = n$), then every $\mathbf{x} \in \mathbb{R}^n$ can be written uniquely in the form $\mathbf{x} = \mathbf{v} + \mathbf{w}$ for $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^{\mathsf{T}})$. [Hint: *you don't have many choices for* $\mathbf{v}$.]

(ii) What happens if $r = 0$?

Going forward, suppose $1 \le r < n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}$ be a basis for $\mathbf{N}(A)$ and let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis for $\mathbf{C}(A^{\mathsf{T}})$.

33.2 Problem (!). Suppose you know that $\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}, \mathbf{w}_1, \ldots, \mathbf{w}_r$ are independent. Why do they form a basis for $\mathbb{R}^n$? [Hint: *what is* $\mathbf{C}\big(\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_{n-r} & \mathbf{w}_1 & \cdots & \mathbf{w}_r\end{bmatrix}\big)$*? Or look at part (iii) of Problem 28.9.*]

I claim that we can check independence using the orthogonality argument that we used on (30.2), and I think you'll learn more by doing that yourself.

33.3 Problem ($\star$). Suppose that

$$y_1\mathbf{v}_1 + \cdots + y_{n-r}\mathbf{v}_{n-r} + z_1\mathbf{w}_1 + \cdots + z_r\mathbf{w}_r = \mathbf{0}_n$$

for some y_j, $z_j \in \mathbb{R}$. Put

$$\mathbf{v} = y_1\mathbf{v}_1 + \cdots + y_{n-r}\mathbf{v}_{n-r} \quad \text{and} \quad \mathbf{w} = z_1\mathbf{w}_1 + \cdots + z_r\mathbf{w}_r.$$

Explain why $\mathbf{v}\cdot\mathbf{w} = 0$ and $\mathbf{v}+\mathbf{w} = \mathbf{0}_n$. Obtain $\mathbf{v}\cdot\mathbf{v} = 0$, thus $\mathbf{v} = 0$, and therefore $y_j = 0$ for all j. From this, obtain $\mathbf{w} = \mathbf{0}_n$, thus $z_j = 0$ for all j.

This implies that $\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}, \mathbf{w}_1, \ldots, \mathbf{w}_r$ form a basis for $\mathbb{R}^n$, and so we can write each $\mathbf{x} \in \mathbb{R}^n$ as

$$\mathbf{x} = y_1\mathbf{v}_1 + \cdots + y_{n-r}\mathbf{v}_{n-r} + z_1\mathbf{w}_1 + \cdots + z_r\mathbf{w}_r$$

for some y_j, $z_j \in \mathbb{R}$. With

$$\mathbf{v} = y_1\mathbf{v}_1 + \cdots + y_{n-r}\mathbf{v}_{n-r} \quad \text{and} \quad \mathbf{w} = z_1\mathbf{w}_1 + \cdots + z_r\mathbf{w}_r,$$

this shows that $\mathbf{x} = \mathbf{v}+\mathbf{w}$ for some $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$.

33.4 Problem ($\star$). Prove uniqueness of this decomposition by generalizing the argument that preceded Definition 30.3.

33.5 Remark. *This is one of those times when having a basis for a subspace in the abstract was very useful. Without knowing precisely the forms of the bases for* $\mathbf{N}(A)$ *(which we could extract from the RREF of* A*) or* $\mathbf{C}(A^\mathsf{T})$ *(which we could extract from the pivot rows of the RREF of* A*), we built a basis for* $\mathbb{R}^n$ *and used that to get our desired decomposition.*

Content from Strang's *ILA* 6E. "Combining Bases from Subspaces" on p. 147 contains these "counting" arguments that lead to a basis for all of $\mathbb{R}^n$ out of bases for $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$. Read Examples 3 and 4. Then go back to the box on p. 145 with the inequality $\dim(\mathcal{V})+\dim(\mathcal{W}) \le n$. Can you prove this? [Hint: *start with bases for* $\mathcal{V}$ *and* $\mathcal{W}$*, and show that together, the vectors in both bases are still independent.*] Can you give an example of orthogonal subspaces for which the inequality is strict? [Hint: *look at some, but not all, of the standard basis vectors.*]

And so we (mostly you, but also me) have proved a pretty big result.

33.6 Theorem. *Let $A \in \mathbb{R}^{m \times n}$. For each $\mathbf{x} \in \mathbb{R}^n$ there exist unique $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$ such that $\mathbf{x} = \mathbf{v} + \mathbf{w}$. Also, $\mathbf{v} \cdot \mathbf{w} = 0$. We summarize this symbolically by writing*

$$\mathbb{R}^n = \mathbf{N}(A) \oplus \mathbf{C}(A^\mathsf{T}),$$

and we call $\mathbb{R}^n$ the ORTHOGONAL DIRECT SUM *of $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$.*

33.7 Problem (!). Let $A \in \mathbb{R}^{m \times n}$. Prove that we can write any $\mathbf{b} \in \mathbb{R}^m$ uniquely in the form $\mathbf{b} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$, and that $\mathbf{v} \cdot \mathbf{w} = 0$. We summarize this symbolically by writing

$$\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T}).$$

[Hint: *replace A in Theorem 33.6 with A^T.*]

We should not interpret the dual results $\mathbb{R}^n = \mathbf{N}(A) \oplus \mathbf{C}(A^\mathsf{T})$ and $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$ as saying that any vector in $\mathbb{R}^n$ or $\mathbb{R}^m$ is in one or another of these FOUR FUNDAMENTAL SUBSPACES $\mathbf{N}(A)$, $\mathbf{C}(A^\mathsf{T})$, $\mathbf{C}(A)$, and $\mathbf{N}(A^\mathsf{T})$ associated with A. Rather, we can build $\mathbb{R}^n$ and $\mathbb{R}^m$ out of the four fundamental subspaces.

33.8 Problem ($\star$). Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Give an example of a vector $\mathbf{v} \in \mathbb{R}^4$ such that $\mathbf{v} \notin \mathbf{N}(A)$ and $\mathbf{v} \notin \mathbf{C}(A^\mathsf{T})$. Then give an example of $\mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{w} \notin \mathbf{C}(A)$ and $\mathbf{w} \notin \mathbf{N}(A^\mathsf{T})$. Feel free to refer to Example 23.1 (which tells you about $\mathbf{C}(A)$) and Example 23.2 (which tells you about $\mathbf{N}(A^\mathsf{T})$).

The decomposition $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$ will tell us something valuable about the solvability of $A\mathbf{x} = \mathbf{b}$, if we work at it. I claim that

$$\mathbf{N}(A^\mathsf{T}) = \{\mathbf{v} \in \mathbb{R}^m \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in \mathbf{C}(A)\}. \tag{33.1}$$

We basically saw this when we proved that all vectors in $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$ are orthogonal, but let's do it again for practice.

First suppose $A^\mathsf{T}\mathbf{v} = \mathbf{0}_n$ and let $\mathbf{w} \in \mathbf{C}(A)$. Then $\mathbf{w} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, and we compute

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot A\mathbf{x} = A^\mathsf{T}\mathbf{v} \cdot \mathbf{x} = \mathbf{0}_n \cdot \mathbf{x} = 0.$$

Now suppose $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in \mathbf{C}(A)$. We want to show $A^\mathsf{T}\mathbf{v} = \mathbf{0}_n$. When all else fails, rewrite what you know. Since $\mathbf{C}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$, we have

$$0 = \mathbf{v} \cdot A\mathbf{x} = A^\mathsf{T}\mathbf{v} \cdot \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Gloriously, this is enough to get us $A^\mathsf{T}\mathbf{v} = \mathbf{0}_n$.

Here's why.

33.9 Problem (!). Suppose that $\mathbf{v} \in \mathbb{R}^n$ satisfies $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in \mathbb{R}^n$. Prove that $\mathbf{v} = \mathbf{0}_n$. [Hint: *take advantage of that generous quantifier "for all" and let* $\mathbf{w}$ *be one of the standard basis vectors.*]

I view this problem as another way that the dot product extracts information about vectors. If you test or measure a given vector against all vectors under the lens of the dot product and you always get 0, then that given vector is the zero vector. (This makes me feel like a real scientist using lab instruments.)

Day 34: Wednesday, November 5.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Orthogonal complement of a subspace of $\mathbb{R}^p$

From Problem 33.9, we conclude the equality (33.1), which motivates a new kind of structure.

34.1 Definition. *Let* $\mathcal{V}$ *be a subset of* $\mathbb{R}^p$ *(not necessarily a subspace). The* **ORTHOGONAL COMPLEMENT** *of* $\mathcal{V}$ *in* $\mathbb{R}^p$ *is*

$$\mathcal{V}^{\perp} := \{\mathbf{w} \in \mathbb{R}^p \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{v} \in \mathcal{V}\}.$$

We pronounce the symbol $\mathcal{V}^{\perp}$ as "vee perp."

Content from Strang's *ILA* 6E. The last paragraph on p. 145 defines orthogonal complements.

34.2 Problem ($\star$). Let $\mathcal{V}$ be a subset of $\mathbb{R}^p$. Prove that $\mathcal{V}^{\perp}$ is a subspace of $\mathbb{R}^p$. Convince yourself that you didn't need $\mathcal{V}$ to be a subspace.

34.3 Example. **(i)** The equality (33.1) says that

$$\mathbf{C}(A)^{\perp} = \mathbf{N}(A^{\mathsf{T}}) \tag{34.1}$$

for any $A \in \mathbb{R}^{m \times n}$.

(ii) Let $\mathcal{V} = \mathbb{R}^p$ and suppose that $\mathbf{w} \in \mathbb{R}^p$ with $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{v} \in \mathbb{R}^p$. Problem 33.9 says that $\mathbf{w} = \mathbf{0}_p$, so $(\mathbb{R}^p)^{\perp} = \{\mathbf{0}_p\}$.

(iii) Let $\mathcal{V} = \{\mathbf{0}_p\}$. Then $\mathbf{0}_p \cdot \mathbf{w} = 0$ for any $\mathbf{w} \in \mathbb{R}^p$, so $\{\mathbf{0}_n\}^{\perp} = \mathbb{R}^p$.

34.4 Problem (!). For $A \in \mathbb{R}^{m \times n}$, prove that $\mathbf{N}(A) = \mathbf{C}(A^{\mathsf{T}})^{\perp}$.

We just saw the extreme cases of $(\mathbb{R}^p)^{\perp} = \{\mathbf{0}_p\}$ and $\{\mathbf{0}_p\}^{\perp} = \mathbb{R}^p$. Thus

$$\big((\mathbb{R}^p)^{\perp}\big)^{\perp} = \{\mathbf{0}_p\}^{\perp} = \mathbb{R}^p \quad \text{and} \quad \big(\{\mathbf{0}_p\}^{\perp}\big)^{\perp} = (\mathbb{R}^p)^{\perp} = \{\mathbf{0}_p\}.$$

34.5 Problem (!). Here is a less extreme case in $\mathbb{R}^2$. Let $\mathcal{V} = \operatorname{span}(\mathbf{e}_1)$, $\mathbf{e}_1 = (1, 0)$. Draw pictures to convince yourself that $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$ and then prove it.

These examples might make us wonder the following.

34.6 Conjecture. *$(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$ for any subset $\mathcal{V}$ of $\mathbb{R}^p$.*

In particular, if true Conjecture 34.6 would imply

$$\mathbf{C}(A) = \big(\mathbf{C}(A)^{\perp}\big)^{\perp} = \mathbf{N}(A^{\mathsf{T}})^{\perp}. \tag{34.2}$$

This would give us a new way of deciding solvability of $A\mathbf{x} = \mathbf{b}$: check that $\mathbf{b}$ is orthogonal to everything in $\mathbf{N}(A^{\mathsf{T}})$. Or that $\mathbf{b}$ is orthogonal to a basis for $\mathbf{N}(A^{\mathsf{T}})$.

34.7 Problem (!). Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for $\mathcal{V}$. Suppose that $\mathbf{w} \in \mathbb{R}^p$ satisfies $\mathbf{w} \cdot \mathbf{v}_j = 0$ for $j = 1, \ldots, r$. Prove that $\mathbf{w} \in \mathcal{V}^{\perp}$.

The point is that if (34.2) is true, then it would give us a different way of describing the column space. In particular, we might get an easier way of checking that a vector is *not* in the column space than doing elementary row operations and going to the RREF.

34.8 Problem (!). Check that (34.2) is true for the familiar matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 2 & 14 \\ 0 & 0 & 2 & 8 \end{bmatrix}.$$

Feel free to refer to Example 23.1 (which tells you about $\mathbf{C}(A)$) and Example 23.2 (which tells you about $\mathbf{N}(A^{\mathsf{T}})$).

34.9 Problem ($\star$). Assuming Conjecture 34.6 to be true, prove the **Fredholm alternative**: if $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, then one, and only one, of the following is true.

(i) $\mathbf{b} \in \mathbf{C}(A)$ and so the problem $A\mathbf{x} = \mathbf{b}$ has a solution.

(ii) There is $\mathbf{v} \in \mathbb{R}^m$ such that $A^{\mathsf{T}}\mathbf{v} = \mathbf{0}_n$ and $\mathbf{b} \cdot \mathbf{v} \neq 0$.

[Hint: *Either $\mathbf{b} \in \mathbf{C}(A)$ or $\mathbf{b} \notin \mathbf{C}(A)$. If $\mathbf{b} \in \mathbf{C}(A)$, then the first condition is true; show* *that the second condition must be false. If* $\mathbf{b} \notin \mathbf{C}(A)$*, then the first condition is false; show that the second condition must be true.*] So who cares? If you want to consider the problem $A\mathbf{x} = \mathbf{b}$ for a bunch of $\mathbf{b}$, I think it might be easier to figure out $\mathbf{N}(A^{\mathsf{T}})$ and then see if the $\mathbf{b}$ are orthogonal to every vector in a basis for $\mathbf{N}(A^{\mathsf{T}})$. Those orthogonality relations give "solvability conditions" for $A\mathbf{x} = \mathbf{b}$.

Let's prove it.

34.10 Theorem. *Let* $A \in \mathbb{R}^{m \times n}$*. Then* $\mathbf{C}(A) = \mathbf{N}(A^{\mathsf{T}})^{\perp}$*.*

Proof. This is one of those times when we really need the long-ago definition of set equality from Definition 5.1.

1. *The proof of* $\mathbf{w} \in \mathbf{C}(A) \Longrightarrow \mathbf{w} \in \mathbf{N}(A^{\mathsf{T}})^{\perp}$. Let $\mathbf{w} \in \mathbf{C}(A)$, so $\mathbf{w} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. We want to show $\mathbf{w} \in \mathbf{N}(A^{\mathsf{T}})^{\perp}$, so we need to prove that $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbf{N}(A^{\mathsf{T}})$.

That is, we want to show that if $A^{\mathsf{T}}\mathbf{v} = \mathbf{0}_n$, then $A\mathbf{x} \cdot \mathbf{v} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. This isn't so bad:

$$A\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot A^{\mathsf{T}}\mathbf{v} = \mathbf{x} \cdot \mathbf{0}_n = 0.$$

We've done something like this calculation at least once, right? It all comes down to how A and A^{T} "pop" across the dot product.

2. *The proof of* $\mathbf{w} \in \mathbf{N}(A^{\mathsf{T}})^{\perp} \Longrightarrow \mathbf{w} \in \mathbf{C}(A)$. Now let $\mathbf{w} \in \mathbf{N}(A^{\mathsf{T}})^{\perp}$. We want to show $\mathbf{w} \in \mathbf{C}(A)$. This is harder. Think about why: we know that if $A^{\mathsf{T}}\mathbf{v} = \mathbf{0}_n$, then $\mathbf{w} \cdot \mathbf{v} = 0$, and somehow we want to use this to summon up $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{w}$. Isn't that the whole point of the course, finding $\mathbf{x}$?

The trick here is contradiction. What goes wrong if $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{R}^m$ with $A^{\mathsf{T}}\mathbf{v} = \mathbf{0}_n$ but $\mathbf{w} \notin \mathbf{C}(A)$? Consider the matrix $B = \begin{bmatrix} A & \mathbf{w} \end{bmatrix} \in \mathbb{R}^{m \times (n+1)}$. Since $\mathbf{w} \notin \mathbf{C}(A)$, the independent columns of B are the independent columns of A along with $\mathbf{w}$. So, if $\operatorname{rank}(A) = r$, then $\operatorname{rank}(B) = r + 1$.

Now think about

$$B^{\mathsf{T}} = \begin{bmatrix} A^{\mathsf{T}} \\ \mathbf{w}^{\mathsf{T}} \end{bmatrix} \in \mathbb{R}^{(n+1) \times m}.$$

Here I am thinking of $\mathbf{w} \in \mathbb{R}^m$ as an $m \times 1$ block in B, so $\mathbf{w}^{\mathsf{T}}$ is a $1 \times m$ block in B^{T}. We know $\operatorname{rank}(B^{\mathsf{T}}) = \operatorname{rank}(B) = r + 1$, too. And we know

$$\operatorname{rank}(B^{\mathsf{T}}) + \dim[\mathbf{N}(B^{\mathsf{T}})] = m,$$

so

$$\dim[\mathbf{N}(B^{\mathsf{T}})] = m - (r + 1).$$

We're going to show $\mathbf{N}(B^{\mathsf{T}}) = \mathbf{N}(A^{\mathsf{T}})$. (Recall that $A^{\mathsf{T}} \in \mathbb{R}^{n \times m}$, so both $\mathbf{N}(B^{\mathsf{T}})$ and $\mathbf{N}(A^{\mathsf{T}})$ are subspaces of $\mathbb{R}^m$, so it's at least possible that they're equal.) Then we get

$$m = \operatorname{rank}(A^{\mathsf{T}}) + \dim[\mathbf{N}(A^{\mathsf{T}})] = r + \dim[\mathbf{N}(B^{\mathsf{T}})],$$

and so

$$\dim[\mathbf{N}(B^{\mathsf{T}})] = m - r.$$

This is the contradiction! This is why dimension-counting is helpful!

Here we go. If $\mathbf{v} \in \mathbf{N}(B^{\mathsf{T}})$, then $B^{\mathsf{T}}\mathbf{v} = \mathbf{0}_{n+1}$ and so

$$\begin{bmatrix} \mathbf{0}_n \\ 0 \end{bmatrix} = \mathbf{0}_{n+1} = B^{\mathsf{T}}\mathbf{v} = \begin{bmatrix} A^{\mathsf{T}}\mathbf{v} \\ \mathbf{w}^{\mathsf{T}}\mathbf{v} \end{bmatrix} = \begin{bmatrix} A^{\mathsf{T}}\mathbf{v} \\ \mathbf{v}\cdot\mathbf{w} \end{bmatrix}.$$

Comparing components, we get $A^{\mathsf{T}}\mathbf{v} = \mathbf{0}_n$, so $\mathbf{v} \in \mathbf{N}(A^{\mathsf{T}})$.

Conversely, suppose $\mathbf{v} \in \mathbf{N}(A^{\mathsf{T}})$, so both $A^{\mathsf{T}}\mathbf{v} = \mathbf{0}_n$ and $\mathbf{w}\cdot\mathbf{v} = 0$. Then

$$B^{\mathsf{T}}\mathbf{v} = \begin{bmatrix} A^{\mathsf{T}}\mathbf{v} \\ \mathbf{v}\cdot\mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n \\ 0 \end{bmatrix} = \mathbf{0}_{n+1},$$

so $\mathbf{v} \in \mathbf{N}(B^{\mathsf{T}})$. ■

34.11 Problem ($\star$). Now we can prove Conjecture 34.6. We already know it's true if $\mathcal{V} = \mathbb{R}^p$ or $\mathcal{V} = \{\mathbf{0}_p\}$, so assume that $\mathcal{V}$ is any subspace of $\mathbb{R}^p$ with $d := \dim(\mathcal{V}) \geq 1$. Then $\mathcal{V} = \mathbf{C}(A)$ for some $A \in \mathbb{R}^{p\times d}$. Combine (33.1) and Theorem 34.10 to get $(\mathbf{C}(A)^{\perp})^{\perp} = \mathbf{C}(A)$.

Here is another proof of Conjecture 34.6 that does not rely so explicitly on dimension counting. First, it is not all that hard to show that any vector in $\mathcal{V}$ is also in $(\mathcal{V}^{\perp})^{\perp}$. Let $\mathbf{v} \in \mathcal{V}$. We want to show $\mathbf{v}\cdot\mathbf{w} = 0$ for all $\mathbf{w} \in \mathcal{V}^{\perp}$. But that is exactly what it means for $\mathbf{w}$ to be a vector in $\mathcal{V}^{\perp}$! So we have $\mathcal{V} \subseteq (\mathcal{V}^{\perp})^{\perp}$.

However, once again, showing $(\mathcal{V}^{\perp})^{\perp} \subseteq \mathcal{V}$ is a bit harder, and we need a trick. Any subspace $\mathcal{V}$ of $\mathbb{R}^p$ has the form $\mathcal{V} = \mathbf{C}(B)$ for some matrix $B \in \mathbb{R}^{p\times d}$, where $d = \dim(\mathcal{V})$. Let $A = B^{\mathsf{T}}$, so $A \in \mathbb{R}^{d\times p}$ and $\mathcal{V} = \mathbf{C}(A^{\mathsf{T}})$. Then

$$\mathcal{V}^{\perp} = \mathbf{C}(A^{\mathsf{T}})^{\perp} = \mathbf{N}(A) \tag{34.3}$$

by Problem 34.4. And by Theorem 33.6, we can write any $\mathbf{x} \in \mathbb{R}^p$ as $\mathbf{x} = \mathbf{v} + \mathbf{w}$ for unique $\mathbf{v} \in \mathbf{C}(A^{\mathsf{T}}) = \mathcal{V}$ and $\mathbf{w} \in \mathbf{N}(A) = \mathcal{V}^{\perp}$. (I realize I'm flipping the roles of $\mathbf{v}$ and $\mathbf{w}$ from that theorem, but I want to keep $\mathbf{v}$ as the label for things in $\mathcal{V}$, which is $\mathbf{C}(A^{\mathsf{T}})$. Sue me.) Here is what we have proved.

34.12 Lemma. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. For each $\mathbf{x} \in \mathbb{R}^p$, there exist unique $\mathbf{v} \in \mathcal{V}$ and $\mathbf{w} \in \mathcal{V}^{\perp}$ such that $\mathbf{x} = \mathbf{v} + \mathbf{w}$.*

34.13 Problem (!). If $\mathcal{V}$ is a subspace of $\mathbb{R}^p$, prove that $\dim(\mathcal{V}^{\perp}) = p - \dim(\mathcal{V})$.

Here is why all of this jumping around between subspaces and matrices matters. Start with a subspace $\mathcal{V}$ of $\mathbb{R}^p$ and $\mathbf{x} \in (\mathcal{V}^{\perp})^{\perp}$. We want to show $\mathbf{x} \in \mathcal{V}$. The key thing is that $\mathcal{V}^{\perp}$ is also a subspace of $\mathbb{R}^p$. This was Problem 34.2.

Now apply Lemma 34.12 with $\mathcal{V}^{\perp}$ in place of $\mathcal{V}$. So, we can write any $\mathbf{x} \in (\mathcal{V}^{\perp})^{\perp}$ as $\mathbf{x} = \mathbf{v} + \mathbf{w}$ for (unique) $\mathbf{v} \in \mathcal{V}$ and $\mathbf{w} \in \mathcal{V}^{\perp}$. If we can show $\mathbf{w} = \mathbf{0}_p$, then we'll have

$\mathbf{x} = \mathbf{v} \in \mathcal{V}$. The trick is subtraction:

$$\mathbf{w} = \mathbf{x} - \mathbf{v} \in (\mathcal{V}^\perp)^\perp.$$

This is because we're assuming $\mathbf{x} \in (\mathcal{V}^\perp)^\perp$, and just above we showed that if $\mathbf{v} \in \mathcal{V}$, then $\mathbf{v} \in (\mathcal{V}^\perp)^\perp$. But then $\mathbf{w} \in \mathcal{V}^\perp$ and $\mathbf{w} \in (\mathcal{V}^\perp)^\perp$. I claim this means $\mathbf{w} = \mathbf{0}_p$.

34.14 Problem ($\star$). Prove that if $\mathcal{V}$ is a subspace of $\mathbb{R}^p$ and $\mathbf{v} \in \mathbb{R}^p$ with both $\mathbf{v} \in \mathcal{V}$ and $\mathbf{v} \in \mathcal{V}^\perp$, then $\mathbf{v} = \mathbf{0}_p$. Draw a picture illustrating this phenomenon in $\mathbb{R}^2$.

This completes the second proof of Conjecture 34.6, which we now upgrade the conjecture to a theorem.

34.15 Theorem. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. Then $(\mathcal{V}^\perp)^\perp = \mathcal{V}$.*

Here is a summary of all of our work. This answers the question "What is missing beyond the null space or the column space?" and provides a complete overview of how a matrix in $\mathbb{R}^{m\times n}$ determines the structure of $\mathbb{R}^n$ and $\mathbb{R}^m$.

34.16 Theorem (Fundamental theorem of linear algebra). *Let $A \in \mathbb{R}^{m\times n}$.*

(i) $\mathbb{R}^n = \mathbf{N}(A) \oplus \mathbf{C}(A^\mathsf{T})$

(ii) $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$

(iii) $\mathbf{N}(A) = \mathbf{C}(A^\mathsf{T})^\perp$

(iv) $\mathbf{C}(A) = \mathbf{N}(A^\mathsf{T})^\perp$

(v) $\dim[\mathbf{N}(A)] = n - \operatorname{rank}(A)$

(vi) $\operatorname{rank}(A) = \operatorname{rank}(A^\mathsf{T})$

Content from Strang's *ILA* 6E. Figure 4.1 on p. 146 says all of this. Study the figure carefully and read the paragraph following its caption. I like to start reading the figure by beginning with $\mathbf{b}$, then tracking it back to $\mathbf{x}_r$ and $\mathbf{x}_n$. (The subscript n there is for "null space," not the n in $\mathbb{R}^n$.)

There is just one major problem with our fundamental theorem: all of these results are highly existential. We developed those existential results by starting with the null space and asking "What else is missing from $\mathbb{R}^n$?" Now we'll start with the column space. Specifically, it's great at a theoretical level to say that $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$ in the sense that each $\mathbf{b} \in \mathbb{R}^m$ can be written uniquely as $\mathbf{b} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$ and that $\mathbf{v} \cdot \mathbf{w} = 0$, but how do we find those $\mathbf{v}$ and $\mathbf{w}$ explicitly and easily?

First, we only need one of them. For if we know $\mathbf{b} = \mathbf{v} + \mathbf{w}$, then $\mathbf{w} = \mathbf{b} - \mathbf{v}$. So how do we get $\mathbf{v}$?

Well, how do we do anything in this course? *We multiply by a matrix.* Can we find $P \in \mathbb{R}^{m \times m}$ such that if $\mathbf{b} \in \mathbb{R}^m$, then $P\mathbf{b} \in \mathbf{C}(A)$ and $\mathbf{b} - P\mathbf{b} \in \mathbf{N}(A^\mathsf{T})$. Then we have $\mathbf{b} = P\mathbf{b} + (\mathbf{b} - P\mathbf{b})$ as our decomposition.

34.17 Problem (!). Explain why you expect $P^2 = P$. (Have we talked about matrix powers? Just in case: $P^2 = PP$.) [Hint: *what does P do? We have $P\mathbf{b} \in \mathbf{C}(A)$ and any $\mathbf{b}$ can be written uniquely as $\mathbf{b} = P\mathbf{b} + \mathbf{w}$ with $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$. If $\mathbf{b} \in \mathbf{C}(A)$ already, what is that $\mathbf{w}$, and so what should $P\mathbf{b}$ be? Then what is $P^2\mathbf{b}$?*]

Content from Strang's *ILA* 6E. Read all of pp. 151–152 up to, but not including, "Projection Onto a Line." This is the mission statement of Section 4.2, and it's a very helpful overview of where we're going.

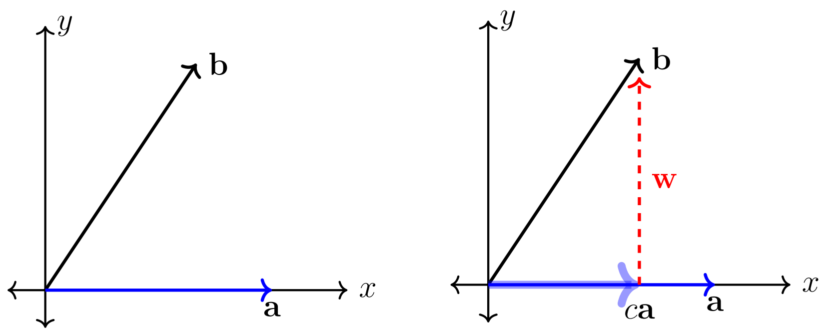
It turns out to be very helpful to assume that A has full column rank (= all of its columns are independent = all of its columns are pivot columns). This is not as huge a restriction as you might initially think. After all, $\mathbf{C}(A) = \mathbf{C}(\widetilde{A})$, where $\widetilde{A}$ is the matrix containing just the pivot columns of A, and $\widetilde{A}$ has full column rank. So, if we are going to understand the decomposition $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$, we may as well do it when A has full column rank.

We'll do this first in the case that A has only one column, in which case $\mathbf{C}(A) = \operatorname{span}(\mathbf{a})$ for some $\mathbf{a} \neq \mathbf{0}_m$. This has some transparent geometry and will give a useful auxiliary result for later.

So here is what we want: given $\mathbf{b} \in \mathbb{R}^m$, there are (necessarily unique) $\mathbf{v} \in \mathbf{C}(A) = \operatorname{span}(\mathbf{a})$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$ such that $\mathbf{b} = \mathbf{v} + \mathbf{w}$. Since $\mathbf{v} \in \operatorname{span}(\mathbf{a})$, we can write $\mathbf{v} = c\mathbf{a}$ for some $c \in \mathbb{R}$. Then

$$\mathbf{b} = c\mathbf{a} + \mathbf{w}.$$

Here is a picture of what's going on when $m = 2$ and $\mathbf{a}$ is a multiple of $\mathbf{e}_1 = (1, 0)$.



The two unknowns $c \in \mathbb{R}$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$ have to satisfy this one equation—not a recipe for success—but remember that we have an orthogonality condition:

$$0 = \mathbf{v} \cdot \mathbf{w} = (c\mathbf{a}) \cdot \mathbf{w} = c(\mathbf{a} \cdot \mathbf{w}).$$

Actually, since $\mathbf{a} \in \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$, we always have

$$\mathbf{a} \cdot \mathbf{w} = 0,$$

so forget about c there. Now we have two equations and two unknowns:

$$\begin{cases} \mathbf{b} = c\mathbf{a} + \mathbf{w} \\ \mathbf{a} \cdot \mathbf{w} = 0. \end{cases} \tag{34.4}$$

A little algebraic trickery will reduce this to one equation: rewrite

$$\mathbf{w} = \mathbf{b} - c\mathbf{a}$$

and plug in to get

$$0 = \mathbf{a} \cdot (\mathbf{b} - c\mathbf{a}).$$

Rearrange a little:

$$0 = (\mathbf{a} \cdot \mathbf{b}) - c(\mathbf{a} \cdot \mathbf{a}),$$

and a little more:

$$c(\mathbf{a} \cdot \mathbf{a}) = \mathbf{a} \cdot \mathbf{b},$$

and divide:

$$c = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}.$$

This division is perfectly legal since $\mathbf{a} \neq \mathbf{0}_m$, and therefore $\mathbf{a} \cdot \mathbf{a} \neq 0$.

We worked backwards, so we should check our work. Certainly

$$\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} \in \operatorname{span}(\mathbf{a}).$$

34.18 Problem (!). Let

$$\mathbf{w} = \mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}.$$

Check that $\mathbf{a} \cdot \mathbf{w} = 0$, so $\mathbf{w} \in \mathbf{C}(A)^{\perp} = \mathbf{N}(A^{\mathsf{T}})$.

34.19 Problem ($\star$). Where is P? This requires a bit of sleight-of-hand. We want

$$P\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}.$$

Here it is helpful to think of column vectors as $m \times 1$ matrices and the dot product as the matrix product

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathsf{T}}\mathbf{b}.$$

It's also helpful to break our usual convention of how we write scalar multiplication and allow $c\mathbf{a} = \mathbf{a}c$. I don't like it, either. Show then that

$$\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} = \frac{1}{\mathbf{a} \cdot \mathbf{a}}(\mathbf{a}\mathbf{a}^{\mathsf{T}})\mathbf{b},$$

so

$$P = \frac{1}{\mathbf{a} \cdot \mathbf{a}}(\mathbf{a}\mathbf{a}^{\mathsf{T}}).$$

34.20 Example. Computations with this sort of "projection" onto span($\mathbf{a}$) can become bulky, so let's check how this respects our intuition. Say $\mathbf{a} = \mathbf{e}_1$ in $\mathbb{R}^2$. Of course, we expect

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1\mathbf{e}_1 + \begin{bmatrix} 0 \\ b_2 \end{bmatrix}.$$

We compute

$$\left(\frac{\mathbf{e}_1 \cdot \mathbf{b}}{\mathbf{e}_1 \cdot \mathbf{e}_1}\right)\mathbf{e}_1 = \frac{b_1}{1}\mathbf{e}_1 = b_1\mathbf{e}_1.$$

How nice it was that $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$.

Content from Strang's *ILA* 6E. Read "Projection Onto a Line" from pp. 152–154. Check Examples 1 and 2.

Day 35: Friday, November 7.

Before, when A had just one column, we rewrote $\mathbf{v}$ as a scalar multiple of that column. Now we can say $\mathbf{v} = A\widehat{\mathbf{x}}$ for some $\widehat{\mathbf{x}} \in \mathbb{R}^n$. (The hat is sort of traditional.) The analogue of (34.4) is now

$$\begin{cases} \mathbf{b} = A\widehat{\mathbf{x}} + \mathbf{w} \\ \mathbf{a}_j \cdot \mathbf{w} = 0, \ j = 1, \ldots, n. \end{cases}$$

That second (set of) equation(s) is the orthogonality of $\mathbf{w}$ to everything in $\mathbf{C}(A)$, equivalently, to the columns of A.

This reduces to n equations:

$$0 = \mathbf{a}_j \cdot \mathbf{w} = \mathbf{a}_j \cdot (\mathbf{b} - A\widehat{\mathbf{x}}), \ j = 1, \ldots, n.$$

And now for the trick: rewrite $\mathbf{a}_j = A\mathbf{e}_j$, so

$$0 = (A\mathbf{e}_j) \cdot (\mathbf{b} - A\widehat{\mathbf{x}}) = \mathbf{e}_j \cdot A^{\mathsf{T}}(\mathbf{b} - A\widehat{\mathbf{x}}), \ j = 1, \ldots, n.$$

The powerful Problem 33.9 implies that

$$A^{\mathsf{T}}(\mathbf{b} - A\widehat{\mathbf{x}}) = \mathbf{0}_n,$$

which rearranges to

$$A^{\mathsf{T}}A\widehat{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}.$$

If only $A^{\mathsf{T}}A$ were invertible, we could peel it off to solve for $\widehat{\mathbf{x}}$:

$$\widehat{\mathbf{x}} = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\mathbf{b}.$$

If only. Then we would have

$$\mathbf{v} = A\widehat{\mathbf{x}} = A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\mathbf{b},$$

and so putting

$$P := A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}$$

would give $P\mathbf{b} \in \mathbf{C}(A)$, $\mathbf{b} - P\mathbf{b} \in \mathbf{N}(A^{\mathsf{T}})$, and $\mathbf{b} = P\mathbf{b} + (\mathbf{b} - P\mathbf{b})$.

Good news: $A^{\mathsf{T}}A$ is invertible here.

35.1 Lemma. *Let $A \in \mathbb{R}^{m \times n}$ have full column rank. Then $A^{\mathsf{T}}A$ is invertible.*

Proof. We might initially think $A^{\mathsf{T}}A$ = independent rows in A^{T} dotted with independent columns in A has to give us something good. It does, but the trick is to show $\mathbf{N}(A^{\mathsf{T}}A) = \{\mathbf{0}_n\}$. For if $A^{\mathsf{T}}A\mathbf{x} = \mathbf{0}_n$, then

$$0 = \mathbf{x} \cdot \mathbf{0}_n = \mathbf{x} \cdot (A^{\mathsf{T}}A\mathbf{x}) = (A\mathbf{x}) \cdot (A\mathbf{x}),$$

and so $A\mathbf{x} = \mathbf{0}_n$, thus $\mathbf{x} \in \mathbf{N}(A)$. Since A has full column rank, $\mathbf{N}(A) = \{\mathbf{0}_n\}$, so $\mathbf{x} = \mathbf{0}_n$. This is just one of those classical tricks that I never would have thought of myself if someone else hadn't shown it to me, but now it feels like instinct. ■

Content from Strang's *ILA* 6E. This lemma is proved on p. 157. Read the warning at the top of the page and then the calculations at the bottom of the page of how this breaks when A has dependent columns.

Bad news: this was a lot of working backward.

35.2 Problem ($\star$). Let $A \in \mathbb{R}^{m \times n}$ have full column rank and set

$$P_A := A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}.$$

(i) Explain why, just from looking at P_A, every vector in $\mathbf{C}(P_A)$ is in $\mathbf{C}(A)$.

(ii) Show that if $\mathbf{b} \in \mathbf{C}(A)$, then $P_A\mathbf{b} = \mathbf{b}$, and so $\mathbf{b} \in \mathbf{C}(P_A)$ as well. Thus $\mathbf{C}(A) = \mathbf{C}(P_A)$.

(iii) Show that $P_A^2 = P_A$.

(iv) Show that $P_A^{\mathsf{T}} = P_A$.

(v) Justify each of the following equalities:

$$\mathbf{N}(P_A) = \mathbf{N}(P_A^{\mathsf{T}}) = \mathbf{C}(P_A)^{\perp} = \mathbf{C}(A)^{\perp} = \mathbf{N}(A^{\mathsf{T}}). \tag{35.1}$$

(vi) Explain why $\mathbf{b} - P_A\mathbf{b} \in \mathbf{N}(P_A)$ for each $\mathbf{b} \in \mathbb{R}^m$. [Hint: *just compute it.*] So, it's also true that $\mathbf{b} - P_A\mathbf{b} \in \mathbf{N}(A^{\mathsf{T}})$.

The fruit of this problem is that we can write any $\mathbf{b} \in \mathbb{R}^m$ as

$$\mathbf{b} = \mathbf{v} + \mathbf{w}, \qquad \mathbf{v} = P_A\mathbf{b}, \qquad \mathbf{w} = \mathbf{b} - P_A\mathbf{b},$$

and we'll have $\mathbf{v} \in \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(A^{\mathsf{T}})$. Problem 33.4 assures us that this decomposition is unique.

35.3 Problem ($\star$). If you wanted to find the decomposition of some $\mathbf{x} \in \mathbb{R}^n$ as a sum of vectors in $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$, discuss how you would use P_{A^T}. Is there any relation between P_{A^T} and P_A^T, or is that just wishful, and inappropriate, juggling of the symbol T?

Content from Strang's *ILA* 6E. Pages 155–156 develop all of this. I don't think memorizing equations (5), (6), and (7) is a good idea, or even memorizing the structure of our P_A above. I think it's more important to be able to *replicate* the derivation of P_A on your own. Check Worked Example 4.2 A on p. 158.

Day 36: Monday, November 10.

We can accomplish the orthogonal decomposition $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$ for $A \in \mathbb{R}^{m \times n}$ whether or not A has full column rank. The decomposition for the zero matrix is easy, since then $\mathbf{C}(A) = \{\mathbf{0}_m\}$ and $\mathbf{N}(A^\mathsf{T}) = \mathbb{R}^m$. So, assume $\text{rank}(A) = r \ge 1$. Let $B \in \mathbb{R}^{m \times r}$ be a matrix whose columns form a basis for $\mathbf{C}(A)$, so $\mathbf{C}(A) = \mathbf{C}(B)$ and B has full column rank. (For example, we could choose $B = C$ from the CR-factorization.) The matrix $P_B = B(B^\mathsf{T}B)^{-1}B^\mathsf{T}$ is defined since B has full column rank. Every $\mathbf{b} \in \mathbb{R}^m$ can therefore be written in the form $\mathbf{b} = P_B\mathbf{b} + \mathbf{w}$, where $P_B\mathbf{b} \in \mathbf{C}(B) = \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(B^\mathsf{T})$. And

$$\mathbf{N}(B^\mathsf{T}) = \mathbf{C}(B)^\perp = \mathbf{C}(A)^\perp = \mathbf{N}(A^\mathsf{T}).$$

Thus every $\mathbf{b} \in \mathbb{R}^m$ has the form $\mathbf{b} = P_B\mathbf{b} + \mathbf{w}$ with $P_B\mathbf{b} \in \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$.

36.1 Example. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 7 \\ 0 & 2 & 4 & 2 & 14 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

We have studied variations on this matrix often, and now is a good time to review everything all in one go.

(i) Since $\mathbf{a}_1 = \mathbf{0}_3$, column $\mathbf{a}_2$ is the first nonzero column of A. If we are constructing a list of independent columns of A, starting with $\mathbf{a}_2$ is probably a good idea. Since $\mathbf{a}_3 = 2\mathbf{a}_2$, we should ignore that column if we're looking for an independent list. We can check that $\mathbf{a}_4 \notin \text{span}(\mathbf{a}_2)$ because of how the third entries of $\mathbf{a}_2$ and $\mathbf{a}_4$ interact. And $\mathbf{a}_5 = 3\mathbf{a}_2 + 4\mathbf{a}_4$. So an independent list of columns of A is the list $\mathbf{a}_2$, $\mathbf{a}_4$, thus $\text{rank}(A) \ge 2$, and any longer list of columns of A is dependent, so $\text{rank}(A) = 2$.

(ii) Here is how the columns of A interact:

$$\mathbf{a}_1 = 0\mathbf{a}_2 + 0\mathbf{a}_4, \quad \mathbf{a}_2 = 1\mathbf{a}_2 + 0\mathbf{a}_4, \quad \mathbf{a}_3 = 2\mathbf{a}_2 + 0\mathbf{a}_4, \quad \mathbf{a}_4 = 0\mathbf{a}_2 + 1\mathbf{a}_4, \quad \text{and} \quad \mathbf{a}_5 = 3\mathbf{a}_2 + 4\mathbf{a}_4.$$

This gives the CR-factorization of A:

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 7 \\ 0 & 2 & 4 & 2 & 14 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 12 & 20 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

The columns of $C = \begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_4 \end{bmatrix} \in \mathbb{R}^{3\times 2}$ are independent columns of A, and the columns of $R \in \mathbb{R}^{2\times 5}$ are the "recipes" that tell us how to build columns of A from columns of C.

(iii) We could also compute the RREF of A, and we would find

$$R_0 := \mathsf{rref}(A) = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns of R_0 are columns 2 and 4, so the pivot columns of A are $\mathbf{a}_2$ and $\mathbf{a}_4$; thus the list $\mathbf{a}_2$, $\mathbf{a}_4$ is a basis for $\mathbf{C}(A)$. As is typical, $\mathbf{C}(A) \neq \mathbf{C}(R_0)$.

By using elimination to transform the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ into $\begin{bmatrix} R_0 & \mathbf{c} \end{bmatrix}$, we can see that $\mathbf{b} \in \mathbf{C}(A)$ if and only if $b_2 = 2b_1$. Thus $\mathbf{b} \in \mathbf{C}(A)$ if and only if $\mathbf{b} = (b_1, 2b_1, b_3)$, and this shows that another basis for $\mathbf{C}(A)$ is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Also, if we remove the row of zeros from R_0, then we get the factor R from the CR-factorization:

$$R_0 = \begin{bmatrix} R \\ 0 \end{bmatrix}.$$

(iv) Since $\mathbf{N}(A) = \mathbf{N}(R_0)$, from the RREF we could solve $A\mathbf{x} = \mathbf{0}_3$ by solving the easier problem $R_0\mathbf{x} = \mathbf{0}_3$. From this work, we can conclude that a basis for $\mathbf{N}(A)$ is the list

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

In particular, $\dim(\mathbf{N}(A)) = 3$, and so $\operatorname{rank}(A) + \dim(\mathbf{N}(A)) = 5$, as rank–nullity predicts.

(v) We have

$$A^{\mathsf{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 2 \\ 7 & 14 & 8 \end{bmatrix}.$$

We know that $\mathbf{C}(A^{\mathsf{T}}) = \mathbf{C}(R_0^{\mathsf{T}})$ and that a basis for $\mathbf{C}(R_0^{\mathsf{T}})$ consists of the pivot rows of R_0 transposed. Thus a basis for $\mathbf{C}(A^{\mathsf{T}})$ is

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}.$$

We can also compute directly

$$\text{rref}(A^\mathsf{T}) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\text{rank}(A^\mathsf{T}) = 2 = \text{rank}(A)$, as expected, and a basis for the row space $\mathbf{C}(A^\mathsf{T})$ is the list of pivot columns of A^T:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 8 \end{bmatrix}.$$

We should be careful in that $\text{rref}(A^\mathsf{T}) \neq \text{rref}(A)^\mathsf{T}$ here. What we do have is $\text{rank}(A^\mathsf{T}) = \text{rank}(A)$ and $\mathbf{C}(A^\mathsf{T}) = \mathbf{C}(\text{rref}(A))$.

We can also find $\mathbf{N}(A^\mathsf{T})$ by solving $A^\mathsf{T}\mathbf{x} = \mathbf{0}_5$, which we do by solving $\text{rref}(A^\mathsf{T})\mathbf{x} = \mathbf{0}_5$. Doing so tells us that a basis for $\mathbf{N}(A^\mathsf{T})$ is the very short list

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Thus $\text{rank}(A^\mathsf{T}) + \dim(\mathbf{N}(A^\mathsf{T})) = 2 + 1 = 3$, as rank–nullity predicts.

(vi) Part of proving the general orthogonal direct sum decomposition $\mathbb{R}^n = \mathbf{N}(A) \oplus \mathbf{C}(A^\mathsf{T})$ for a general $A \in \mathbb{R}^{m \times n}$ involved combining bases for $\mathbf{N}(A)$ and $\mathbf{C}(A^\mathsf{T})$, counting their lengths, and using orthogonality to get a basis for $\mathbb{R}^n$. The same idea shows that combining a basis for $\mathbf{C}(A)$ and a basis for $\mathbf{N}(A^\mathsf{T})$ gives a basis for $\mathbb{R}^m$. You can check that the lists

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

are independent in $\mathbb{R}^3$ and thus each list is a basis for $\mathbb{R}^3$. The first two vectors in each list form a basis for $\mathbf{C}(A)$ in this particular example, and the third vector by itself is a basis for $\mathbf{N}(A^\mathsf{T})$.

(vii) We can write any $\mathbf{b} \in \mathbb{R}^3$ uniquely in the form $\mathbf{b} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$. Since $\text{rank}(A) = 2 < 5$, A does not have full column rank, and so we cannot form the matrix P_A and say $\mathbf{v} = P_A\mathbf{b}$. However, we can just take a matrix $B \in \mathbb{R}^{3 \times 2}$ such that $\mathbf{C}(A) = \mathbf{C}(B)$ and $\operatorname{rank}(B) = 2$ and then use $\mathbf{v} = P_B\mathbf{b}$. It turns out to be helpful to take

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The columns of B form a basis for $\mathbf{C}(A)$, although they are not the pivot columns.

We compute

$$B^{\mathsf{T}}B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(B^{\mathsf{T}}B)^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$P_B = B(B^{\mathsf{T}}A)^{-1}B^{\mathsf{T}}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/5 & 2/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It was nice that $B^{\mathsf{T}}B$ turned out to be diagonal, as that made computing its inverse very easy. This in turn is a consequence of the orthogonality of the columns of B.

We note that the second row of P_B is twice the first row; this reflects our observation above that $\mathbf{b} \in \mathbf{C}(A)$ if and only if $b_2 = 2b_1$. Also, $P_B^{\mathsf{T}} = P_B$; this can't be an accident.

Day 37: Wednesday, November 12.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Projection, orthogonal projection, orthogonal projection onto the column space of a matrix of rank n, norm of a vector, least squares solution

When $A \in \mathbb{R}^{m \times n}$ has full column rank, the matrix $P_A := A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is defined and satisfies $P_A\mathbf{b} \in \mathbf{C}(A)$ and $\mathbf{b} - P_A\mathbf{b} \in \mathbf{N}(A^\mathsf{T})$ for all $\mathbf{b} \in \mathbb{R}^m$. Taking $\mathbf{v} = P_A\mathbf{b}$ and $\mathbf{w} = \mathbf{b} - P_A\mathbf{b}$ allows us to write $\mathbf{b} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in \mathbf{C}(A)$ and $\mathbf{w} \in \mathbf{N}(A^\mathsf{T})$. This makes explicit the previously existential orthogonal decomposition $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$.

The matrix P_A has many nice properties, chief among them that $P_A^2 = P_A$ and $P_A^\mathsf{T} = P_A$.

37.1 Definition. *Let $P \in \mathbb{R}^{m \times m}$.*

(i) *P is a* **PROJECTION** *if $P^2 = P$.*

(ii) *P is an* **ORTHOGONAL PROJECTION** *if $P^2 = P$ and $P^\mathsf{T} = P$.*

(iii) *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^m$. The matrix P is an* **ORTHOGONAL PROJECTION ONTO** *$\mathcal{V}$ if P is an orthogonal projection with $\mathbf{C}(P) = \mathcal{V}$.*

37.2 Problem (!). Without doing any matrix calculations, in $\mathbb{R}^3$, what do you expect an orthogonal projection onto $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$ to be? Now do those calculations.

37.3 Problem ($\star$). Let $\mathcal{V}$ be a subspace of $\mathbb{R}^{m \times m}$ and let $P \in \mathbb{R}^{m \times m}$ be an orthogonal projection onto $\mathcal{V}$. Explain why $I_m - P$ is an orthogonal projection onto $\mathcal{V}^\perp$.

37.4 Example. **(i)** The zero matrix in $\mathbb{R}^{m \times m}$ and I_m are both orthogonal projections.

(ii) Problem 35.2 shows that $P_A = A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is an orthogonal projection onto $\mathbf{C}(A)$ when $A \in \mathbb{R}^{m \times n}$ has full column rank.

(iii) Let $\mathcal{V}$ be a subspace of $\mathbb{R}^m$. If $\mathcal{V} = \{\mathbf{0}_m\}$, then the matrix $P \in \mathbb{R}^{m \times m}$ whose entries are all 0 is an orthogonal projection onto $\mathcal{V}$. Otherwise, let $r = \dim(\mathcal{V})$ and let $A \in \mathbb{R}^{m \times r}$ be a matrix whose columns are a basis for $\mathcal{V}$. Then $P_A = A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is an orthogonal projection onto $\mathcal{V}$.

After all of our work, we probably want to call P_A *the* orthogonal projection *onto* $\mathbf{C}(A)$ when A has full column rank. Is it unique? What an insult it would be if it were not. This is asking if there is only one $P \in \mathbb{R}^{m \times m}$ such that $P^2 = P$, $P^\mathsf{T} = P$, and $\mathbf{C}(P) = \mathbf{C}(A)$. And this is true.

37.5 Problem (!). Let $P \in \mathbb{R}^{m \times m}$ be an orthogonal projection. Show that for each $\mathbf{x} \in \mathbb{R}^m$, there is $\mathbf{w} \in \mathbf{N}(P)$ such that

$$\mathbf{x} = P\mathbf{x} + \mathbf{w}.$$

[Hint: *the equality suggests taking $\mathbf{w} = \mathbf{x} - P\mathbf{x}$; show that with this definition of $\mathbf{w}$, we* *have* $\mathbf{w} \in \mathbf{N}(P)$.]

37.6 Problem ($\star$). Let P_1, $P_2 \in \mathbb{R}^{m \times m}$ be orthogonal projections with $\mathbf{C}(P_1) = \mathbf{C}(P_2)$. We prove $P_1 = P_2$ by showing $P_1\mathbf{x} = P_2\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$. (*What things do defines what things are.*)

(i) Show that $\mathbf{N}(P_1) = \mathbf{N}(P_2)$. [Hint: $\mathbf{N}(P) = \mathbf{C}(P^\mathsf{T})^\perp$ *for any* $P \in \mathbb{R}^{m \times m}$. *What does the equality* $\mathbf{C}(P_1) = \mathbf{C}(P_2)$ *and the assumption that* P_1 *and* P_2 *are orthogonal projections say about* $\mathbf{C}(P_1^\mathsf{T})$ *and* $\mathbf{C}(P_2^\mathsf{T})$ *and thus about their orthogonal complements?*]

(ii) Use Problem 37.5 to explain why we can write any $\mathbf{x} \in \mathbb{R}^m$ as both

$$\mathbf{b} = P_1\mathbf{x} + \mathbf{w}_1 \quad \text{and} \quad \mathbf{b} = P_2\mathbf{x} + \mathbf{w}_2$$

for some $\mathbf{w}_1 \in \mathbf{N}(P_1)$ and $\mathbf{w}_2 \in \mathbf{N}(P_2)$. Conclude that

$$P_1\mathbf{x} - P_2\mathbf{x} \in \mathbf{C}(P_1) \quad \text{and} \quad P_1\mathbf{x} - P_2\mathbf{x} \in \mathbf{N}(P_1).$$

Then invoke Problem 34.14.

We now possess a much deeper understanding of how a matrix induces *structure* from the decompositions $\mathbb{R}^n = \mathbf{N}(A) \oplus \mathbf{C}(A^\mathsf{T})$ and $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$ for $A \in \mathbb{R}^{m \times n}$, and how to perform those decompositions via matrix multiplication. We also have the characterization $\mathbf{C}(A) = \mathbf{N}(A^\mathsf{T})^\perp$ and the resulting "solvability conditions" from the Fredholm alternative (Problem 34.9).

What else do we gain from these results? There has been something of a dichotomy in our approach to linear systems. Either we can solve $A\mathbf{x} = \mathbf{b}$ (uniquely or not) or we cannot We have focused on the solving part. Otherwise, if $A\mathbf{x} = \mathbf{b}$ has no solution, what is the point in talking about it?

Very often in life, mathematically or otherwise, we cannot solve the problems that we face. The next best thing is to solve an easier problem. (If your question is too hard, give up and ask a different question.) If we cannot solve $A\mathbf{x} = \mathbf{b}$, could we solve a related problem $A\widehat{\mathbf{x}} = \mathbf{p}$ and view that related problem as an approximation to our desired problem? Yes! If we pick the right problem.

If we are going to solve $A\widehat{\mathbf{x}} = \mathbf{p}$, we need $\mathbf{p} \in \mathbf{C}(A)$. Is there some "ideal" $\mathbf{p}$ to pick relative to the $\mathbf{b}$ that will not work? Again, yes! The answer is something of a miracle, and we have already done most of the hard work with projections.

Approximating requires a new concept: the notion of *size*, which is really a notion of *length*. The following definition generalizes the notion that the length of the line segment in two dimensions from the origin $(0, 0)$ to a point (x, y) is $\sqrt{x^2 + y^2}$.

37.7 Definition. *The* NORM *of* $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbb{R}^m$ *is*

$$\|\mathbf{v}\| := \left(v_1^2 + \cdots + v_m^2\right)^{1/2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

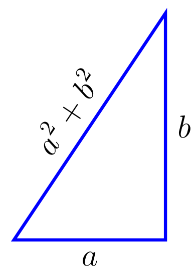
37.8 Example. If $\mathbf{v} = (1, 2, 3)$, then

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

37.9 Problem (!). Let $c \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^m$. Prove that $\|c\mathbf{v}\| = |c| \|\mathbf{v}\|$ and interpret this as a statement about "stretching" vectors.

Content from Strang's *ILA* 6E. Reread all of p. 9 right now. There are plenty of other meaningful ways of measuring the length of a vector in $\mathbb{R}^m$ that we won't need. You might enjoy reading pp. 355–356 up to and including Figure 9.8.

Length and orthogonality interact in a helpful way. You know this already because you believe the Pythagorean theorem, which the definition of $\|\cdot\|$ is basically designed to respect.



37.10 Theorem (Pythagorean theorem). *Let* $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^m$ *with* $\mathbf{v} \cdot \mathbf{w} = 0$. *Then*

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

37.11 Problem (!). Prove it. [Hint: *use the definition* $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$ *to compute* $\|\mathbf{v} + \mathbf{w}\|^2$ *and get* $\mathbf{v} \cdot \mathbf{w} = 0$ *to show up somewhere.*]

37.12 Problem ($\star$). Let $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^m$ with $\mathbf{v} \cdot \mathbf{w} = 0$. Prove that

$$\|\mathbf{v}\| \le \|\mathbf{v} + \mathbf{w}\|.$$

Draw a picture illustrating this in $\mathbb{R}^2$. [Hint: *use the Pythagorean theorem to get an expression for* $\|\mathbf{v} + \mathbf{w}\|$, *and then use the facts that the square root function is increasing and* $\|\mathbf{w}\| \ge 0$.]

Here is how we use this new tool of the norm. We think that two vectors $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^m$ are "close" if the difference $\|\mathbf{v} - \mathbf{w}\|$ is "small."

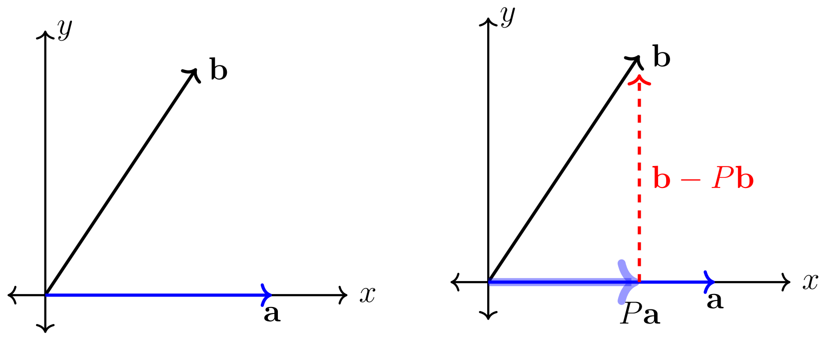
37.13 Remark. *And what exactly does "small" mean? Say* $\|\mathbf{v}\| < \epsilon$ *for some* $\epsilon > 0$. *Then* *since the square root is increasing,*

$$|v_j| = \sqrt{v_j^2} \leq \sqrt{v_1^2 + \cdots + v_m^2} = \|\mathbf{v}\| < \epsilon.$$

So if $\|\mathbf{v}\|$ *is "small" in the sense that it's less than some threshold* $\epsilon > 0$*, then each component* v_j *is "small" in the same way:* $|v_j| < \epsilon$ *for all* j*. A vector with small components is probably a small vector.*

Given $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ with $\mathbf{b} \notin \mathbf{C}(A)$, can we find $\mathbf{p} \in \mathbf{C}(A)$ such that $\mathbf{b}$ and $\mathbf{b}$ are "close"? Then maybe solving $A\widehat{\mathbf{x}} = \mathbf{p}$ will be an adequate substitute for failing to solve $A\mathbf{x} = \mathbf{b}$.

Gloriously, finding this $\mathbf{p}$ is quite easy when we have the orthogonal projection $P_A\mathbf{b}$. Here is a picture that we drew before, more or less.



Here

$$A = \begin{bmatrix} \mathbf{a} \end{bmatrix} \in \mathbb{R}^{m \times 1}, \qquad \mathbf{a} \neq \mathbf{0}_m, \qquad \text{and} \qquad P = P_A = P_{[\mathbf{a}]}.$$

Hopefully the picture makes it clear that the closest vector in $\mathbf{C}([\mathbf{a}]) = \text{span}(\mathbf{a})$ to $\mathbf{b}$ is $P\mathbf{b}$.

Here is the general result. Going forward, as before, we assume that $A \in \mathbb{R}^{m \times n}$ has full column rank. This is what makes the projection P_A exist. And, as before, we will handle the case $\text{rank}(A) < n$ later.

Take any $\mathbf{v} \in \mathbf{C}(A)$. The following inequality encodes the idea that $P_A\mathbf{b}$ is the closest vector in $\mathbf{C}(A)$ to $\mathbf{b}$:

$$\|\mathbf{b} - P_A\mathbf{b}\| \leq \|\mathbf{b} - \mathbf{v}\| \ \text{ for all } \mathbf{v} \in \mathbf{C}(A). \tag{37.1}$$

Our goal is to prove (37.1).

This inequality is equivalent to

$$\|\mathbf{b} - \mathbf{v}\|^2 \geq \|\mathbf{b} - P_A\mathbf{b}\|^2, \tag{37.2}$$

and so this is what we will really prove. The upshot to (37.2) is that the square roots from the norm are gone.

We can make $P_A\mathbf{b}$ show up on the left side of (37.2) by adding and subtracting:

$$\|\mathbf{b} - \mathbf{v}\|^2 = \|\mathbf{b} - P_A\mathbf{b} + P_A\mathbf{b} - \mathbf{v}\|^2. \tag{37.3}$$

Now we group terms cleverly:

$$\|\mathbf{b} - P_A\mathbf{b} + P_A\mathbf{b} - \mathbf{v}\|^2 = \|(\mathbf{b} - P_A\mathbf{b}) + (P_A\mathbf{b} - \mathbf{v})\|^2. \tag{37.4}$$

We know $\mathbf{b} - P_A\mathbf{b} \in \mathbf{N}(A^\mathsf{T})$ by (35.1), $P_A\mathbf{b} \in \mathbf{C}(A)$, and $\mathbf{v} \in \mathbf{C}(A)$. So, $P_A\mathbf{b} - \mathbf{v} \in \mathbf{C}(A) = \mathbf{N}(A^\mathsf{T})^\perp$. That is, $\mathbf{b} - P_A\mathbf{b}$ and $P_A\mathbf{b} - \mathbf{v}$ are orthogonal. The Pythagorean theorem implies

$$\|(\mathbf{b} - P_A\mathbf{b}) + (P_A\mathbf{b} - \mathbf{v})\|^2 = \|\mathbf{b} - P_A\mathbf{b}\|^2 + \|P_A\mathbf{b} - \mathbf{v}\|^2 \ge \|\mathbf{b} - P_A\mathbf{b}\|^2. \tag{37.5}$$

Combining (37.3), (37.4), and (37.5) gives (37.2).

37.14 Theorem (Least squares). *Let $A \in \mathbb{R}^{m\times n}$ have full column rank. Then for any $\mathbf{b} \in \mathbb{R}^m$, the closest vector in $\mathbf{C}(A)$ to $\mathbf{b}$ is $P_A\mathbf{b}$:*

$$\|\mathbf{b} - P_A\mathbf{b}\| \le \|\mathbf{b} - \mathbf{v}\| \text{ for all } \mathbf{v} \in \mathbf{C}(A). \tag{37.6}$$

No other vector in $\mathbf{C}(A)$ is "as close" to $\mathbf{b}$ as $P_A\mathbf{b}$ in the sense that the inequality in (37.6) is strict for $\mathbf{v} \ne P_A\mathbf{b}$. Moreover, with $\widehat{\mathbf{x}} := (A^\mathsf{T}A)^{-1}A^\mathsf{T}\mathbf{b}$, we have

$$\|A\widehat{\mathbf{x}} - \mathbf{b}\| \le \|A\mathbf{x} - \mathbf{b}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{37.7}$$

The LEAST SQUARES SOLUTION *$\widehat{\mathbf{x}}$ is the best "approximate solution" to the (possibly unsolvable) problem $A\mathbf{x} = \mathbf{b}$, and $\widehat{\mathbf{x}}$ solves*

$$A\widehat{\mathbf{x}} = P_A\mathbf{b}, \tag{37.8}$$

which is the "best approximation to" the (possibly unsolvable) problem $A\mathbf{x} = \mathbf{b}$.

Proof. We proved this in the discussion above, but here is a terser recap. Let $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbf{C}(A)$. Then

$$\begin{aligned}\|\mathbf{b} - \mathbf{v}\|^2 &= \|(\mathbf{b} - P_A\mathbf{b}) + (P_A\mathbf{b} - \mathbf{v})\|^2 \\ &= \|\mathbf{b} - P_A\mathbf{b}\|^2 + \|P_A\mathbf{b} - \mathbf{v}\|^2 \\ &\ge \|\mathbf{b} - P_A\mathbf{b}\|^2.\end{aligned}$$

If $\mathbf{v} \ne P_A\mathbf{b}$, then $\|P_A\mathbf{b} - \mathbf{v}\|^2 > 0$, and so the inequality above is strict, which gives the strict inequality in (37.6).

The second inequality (37.7) is just (37.6) with

$$P_A\mathbf{b} = A\widehat{\mathbf{x}}, \qquad \widehat{\mathbf{x}} := (A^\mathsf{T}A)^{-1}A^\mathsf{T}\mathbf{b},$$

and $\mathbf{v} \in \mathbf{C}(A)$ replaced by $\mathbf{v} = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. ■

37.15 Remark. *Let $A \in \mathbb{R}^{m\times n}$ have full column rank. The closest vector in $\mathbf{C}(A)$ to $\mathbf{b}$ is $P_A\mathbf{b}$, but can another vector $\mathbf{v}_\star \in \mathbf{C}(A)$ be "equally closest" to $\mathbf{b}$ with $\mathbf{v}_\star \ne P_A\mathbf{b}$? That is,* *we know that*

$$\|\mathbf{b} - P_A\mathbf{b}\| < \|\mathbf{b} - \mathbf{v}\| \text{ for all } \mathbf{v} \in \mathbf{C}(A) \text{ with } \mathbf{v} \neq P_A\mathbf{b}, \tag{37.9}$$

but can $\mathbf{v}_\star \in \mathbf{C}(A)$ *satisfy*

$$\|\mathbf{b} - \mathbf{v}_\star\| < \|\mathbf{b} - \mathbf{w}\| \text{ for all } \mathbf{w} \in \mathbf{C}(A) \text{ with } \mathbf{w} \neq \mathbf{v}_\star? \tag{37.10}$$

No. For then we could take $\mathbf{v} = \mathbf{v}_\star$ *in* (37.9) *and* $\mathbf{w} = P_A\mathbf{b}$ *in* (37.10) *to find*

$$\|\mathbf{b} - P_A\mathbf{b}\| < \|\mathbf{b} - \mathbf{v}_\star\| < \|\mathbf{b} - P_A\mathbf{b}\|,$$

and that is impossible.

We use the phrase "least squares solution" because the sum of the squares in $\|A\widehat{\mathbf{x}} - \mathbf{b}\|$ is the smallest of all sums of squares of the form $\|A\mathbf{x} - \mathbf{b}\|$.

37.16 Problem (!). If $A \in \mathbb{R}^{m \times m}$ is invertible, what is $\widehat{\mathbf{x}}$? Are you surprised?

Content from Strang's *ILA* 6E. Read p. 163 up to and including the box before Example 1. Then read "Minimizing the Error" on pp. 164–165. Skip the "By calculus" section on pp. 165–166 if you haven't taken multivariable calculus. Then read "The Big Picture for Least Squares" on pp. 166–167. Spend some time contrasting Figure 4.7 on p. 166 with Figure 4.1 back on p. 146. How is $\mathbf{b}$ behaving differently between the two figures?

Here is the lesson of least squares: when we cannot solve $A\mathbf{x} = \mathbf{b}$ because $\mathbf{b} \notin \mathbf{C}(A)$, we first find the best approximation to $\mathbf{b} \in \mathbf{C}(A)$, which we call $\mathbf{p}$, and then we solve $A\widehat{\mathbf{x}} = \mathbf{p}$. So far, this approach requires A to have full column rank.

37.17 Example. Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

If $\mathbf{y} \in \mathbf{C}(A)$, then $y_2 = 2y_1$; here $\mathbf{b} \notin \mathbf{C}(A)$. We could just use the formula from Theorem 37.14 to find the least squares solution $\widehat{\mathbf{x}} \in \mathbb{R}^2$ that makes $\|A\widehat{\mathbf{x}} - \mathbf{b}\|$ as small as possible, but here it might be enlightening to see how the structure of $P_A\mathbf{b}$ allows us to solve $A\widehat{\mathbf{x}} = P_A\mathbf{b}$ directly.

From Example 36.1, the orthogonal projection onto $\mathbf{C}(A)$ is

$$P_A = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We compute

$$P_A\mathbf{b} = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 0 \end{bmatrix}.$$

Then the problem $A\widehat{\mathbf{x}} = P_A\mathbf{b}$ becomes

$$\begin{cases} \widehat{x}_1 & & = & 1/5 \\ 2\widehat{x}_1 & & = & 2/5 \\ & \widehat{x}_2 & = & 0, \end{cases}$$

which gives $\widehat{x}_1 = 1/5$ and $\widehat{x}_2 = 0$, so the least squares solution is

$$\widehat{\mathbf{x}} = \begin{bmatrix} 1/5 \\ 0 \end{bmatrix}.$$

We can overthink this. Saying that this is the least squares solution means $\|A\widehat{\mathbf{x}} - \mathbf{b}\| \le \|A\mathbf{x} - \mathbf{b}\|$ for all $\mathbf{x} \in \mathbb{R}^2$. We are never going to make $\|A\mathbf{x} - \mathbf{b}\|$ smaller than when we choose $\mathbf{x} = \widehat{\mathbf{x}}$. We compute (squaring for simplicity)

$$\|A\mathbf{x} - \mathbf{b}\|^2 = \left\| \begin{bmatrix} x_1 \\ 2x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} x_1 - 1 \\ 2x_1 \\ x_2 \end{bmatrix} \right\|^2 = (x_1-1)^2 + (2x_1)^2 + x_2^2 \ge (x_1-1)^2 + 4x_1^2.$$

That last inequality holds because $x_2^2 \ge 0$. What this says is that $\|A\mathbf{x} - \mathbf{b}\|^2$ is always at least as large as $(x_1-1)^2 + 4x_1^2$. And what is that? A function of x_1 alone! A little calculus, or graphing the parabola, will show that the minimum of $f(x_1) = (x_1-1)^2 + 4x_1^2$ occurs at $x_1 = 1/5$. Thus

$$\|A\mathbf{x} - \mathbf{b}\|^2 \ge (1/5-1)^2 + 4(1/5)^2 + 0^2 = \|A\widehat{\mathbf{x}} - \mathbf{b}\|^2,$$

exactly as least squares predicts.

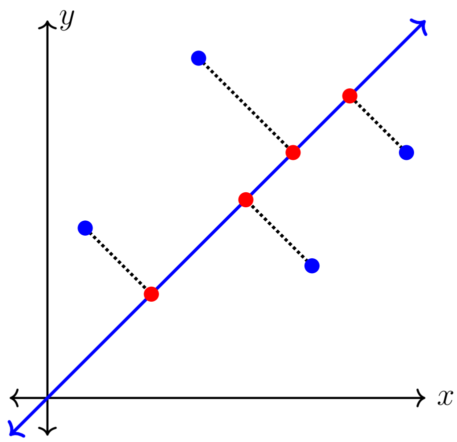
37.18 Problem (!). Use the formula for $\widehat{\mathbf{x}}$ from Theorem 37.14 to compute $\widehat{\mathbf{x}}$ in the previous example directly, and check that the result is, indeed, $\widehat{\mathbf{x}} = (1/5, 0)$.

Day 38: Friday, November 14.

The following is probably the most legitimate "application" of linear algebra that we will ever do, beyond the overarching application of solving and understanding the fundamental problem $A\mathbf{x} = \mathbf{b}$.

38.1 Example. We can find a line between any two points in the plane, but no line is guaranteed to exist between any three or more points. Suppose that we have m sample points of data: $(x_1, y_1), \ldots, (x_m, y_m)$. We probably cannot find a line that passes through all of them, but can we find the line that is "closest" to all of them? What does "closest" even mean here?

Here is a picture with $m = 4$.



Perhaps "closest" should mean the line whose "perpendicular distance" from each point is the smallest. This line has the form $y = mx+b$. Ideally, we would have have $y_k = mx_k+b$ for $k = 1, \ldots, 4$. This really becomes the system of equations

$$\begin{cases} mx_1 & + & b & = & y_1 \\ mx_2 & + & b & = & y_2 \\ mx_3 & + & b & = & y_3 \\ mx_4 & + & b & = & y_4, \end{cases}$$

and that is the matrix-vector equation

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}.$$

Remember that in the notation of this problem, x_k and y_k are given, while m and b are unknown.

Let

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \end{bmatrix}.$$

We want to do least squares, so A better have independent columns. For that, the first column should not be a multiple of the second. The first column *is* a multiple of the second precisely when all of the x_k's are the same number. But in that case, all of the data points have the same x-coordinate, in which case they all lie on the same vertical line, and that is boring.

So, assume that at least one of the x_k's is not equal to the other. Then we can do least squares and say that the best choice of slope and y-intercept is

$$\begin{bmatrix} \widehat{m} \\ \widehat{b} \end{bmatrix} = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}.$$

We could compute what $\widehat{m}$ and $\widehat{b}$ are explicitly, or we could go to a computer and replace thinking with typing.

This example suggests that an extremely natural, and important, application results in a matrix A that transparently has full column rank but not full row rank. This justifies our emphasis on A having full column rank in the least squares developments so far.

Content from Strang's *ILA* 6E. Least squares for data fitting to lines appears in Example 1 on pp. 163–164, Figure 4.6, and pp. 167–168. Pay careful attention to the utility of orthogonal columns in A in Example 2 on p. 168. There's no reason to stop with lines. What if you wanted to find the "best" parabola approximating a set of data? Add one more column to A to account for the extra coefficient in the parabola and read p. 170.

Nonetheless, a careful review of the work leading to Theorem 37.14 will convince you that we did not need A to have full column rank to find a best approximation to $\mathbf{b}$.

38.2 Problem ($\star$). Let $\mathcal{V}$ be a subspace of $\mathbb{R}^m$ and let $\mathbf{b} \in \mathbb{R}^m$. Find $\mathbf{v}_\star \in \mathcal{V}$ such that

$$\|\mathbf{b} - \mathbf{v}_\star\| \le \|\mathbf{b} - \mathbf{v}\|$$

for any $\mathbf{v} \in \mathcal{V}$. [Hint: *if* $\mathcal{V} = \{\mathbf{0}_m\}$, *there are not many options for* $\mathbf{v}_\star$. *Otherwise, start by writing* $\mathcal{V} = \mathbf{C}(A)$ *for some* $A \in \mathbb{R}^{m\times r}$ *with* $\operatorname{rank}(A) = r \ge 1$.]

So, if A does not have full column rank, we could still find the closest point $\mathbf{p} \in \mathbf{C}(A)$ to $\mathbf{b}$ and then try to solve $A\widehat{\mathbf{x}} = \mathbf{p}$. We will definitely succeed in solving this because $\mathbf{p} \in \mathbf{C}(A)$! The challenge is that because A does not have full column rank, we will succeed with too many degrees of freedom: $\mathbf{N}(A) \ne \{\mathbf{0}_n\}$, and so we have many choices for $\widehat{\mathbf{x}}$. Which is best?

38.3 Remark. *Here is one way of proceeding, motivated by the notion that less complicated data is probably better than complicated data.*

Let $A \in \mathbb{R}^{m\times n}$ *and* $\mathbf{b} \in \mathbb{R}^m$, *and let* $\mathbf{p} \in \mathbf{C}(A)$ *be the closest point in* $\mathbf{C}(A)$ *to* $\mathbf{b}$. *Let* $\widehat{\mathbf{x}} \in \mathbb{R}^n$ *satisfy* $A\widehat{\mathbf{x}} = \mathbf{p}$. *By Theorem 33.6, write* $\widehat{\mathbf{x}} = \widehat{\mathbf{v}} + \widehat{\mathbf{w}}$, *where* $\widehat{\mathbf{v}} \in \mathbf{C}(A^{\mathsf{T}})$ *and* $\widehat{\mathbf{w}} \in \mathbf{N}(A)$. *Then* $A\widehat{\mathbf{v}} = \mathbf{p}$, *and so by Theorem 23.4 any other solution* $\widehat{\mathbf{y}}$ *to* $A\widehat{\mathbf{y}} = \mathbf{p}$ *also has the form* $\widehat{\mathbf{y}} = \widehat{\mathbf{v}} + \widehat{\mathbf{z}}$ *for some* $\widehat{\mathbf{z}} \in \mathbf{N}(A)$.

By Problem 37.12, $\|\widehat{\mathbf{v}}\| \le \|\widehat{\mathbf{y}}\|$. *That is,* $\widehat{\mathbf{v}}$ *has the smallest norm of any solution* $\widehat{\mathbf{y}}$ *to* $A\widehat{\mathbf{y}} = \mathbf{p}$. *We might call* $\widehat{\mathbf{v}}$ *the* MINIMUM-NORM LEAST SQUARES SOLUTION.

But how do we find $\widehat{\mathbf{v}}$? This requires first finding that $\widehat{\mathbf{x}}$ that solves $A\widehat{\mathbf{x}} = \mathbf{p}$, which requires knowing $\mathbf{p}$; this requires the orthogonal projection onto $\mathbf{C}(A)$. Then to get $\widehat{\mathbf{v}}$ from $\widehat{\mathbf{x}}$, we need the orthogonal projection onto $\mathbf{C}(A^\mathsf{T})$ This seems like a lot of work.

It would be nice if there were a simpler formula for $\widehat{\mathbf{v}}$ in terms of A and $\mathbf{b}$, and experience teaches us that such a formula probably involves multiplying $\mathbf{b}$ by a special matrix. This turns out to be true: there is a matrix $A^+ \in \mathbb{R}^{n \times m}$ such that $\widehat{\mathbf{v}} = A^+\mathbf{b}$, and this A^+ is the PSEUDOINVERSE *of A.*

Content from Strang's *ILA* 6E. Page 169 gives a concrete example of what to do when A doesn't have full column rank. The construction of the pseudoinverse is best resolved via the glorious tool of the singular value decomposition. Read the comment at the bottom of p. 169 for a nice review of the three possibilities for solutions to linear systems.

Optionally (this is wholly, totally optional), read Section 4.5, which details what the pseudoinverse does. You can skip the example from the "Incidence Matrix of a Graph" on p. 194. Ideally we will develop the SVD, so the formula for A^+ on p. 195 will eventually make sense.

38.4 Problem ($\star$). Here is the opposite question: what is the best solution when we have too many solutions? Suppose that $A \in \mathbb{R}^{m \times n}$ has full row rank, so we can always solve $A\mathbf{x} = \mathbf{b}$. However, perhaps A is not square, in which case A won't have full column rank as well, and so solutions are not unique. This resembles the situation in Remark 38.3, which floated the idea of choosing the "minimum norm solution."

First reread that remark carefully. Since $\mathbf{b} \in \mathbf{C}(A)$ here, we can assume $\mathbf{p} = \mathbf{b}$ throughout, and we may as well dispense with the hats since there is actually a solution to $A\mathbf{x} = \mathbf{b}$ now; call this solution $\mathbf{x} = \mathbf{x}_0$, so $A\mathbf{x}_0 = \mathbf{b}$. Use Problem 35.2 to get the orthogonal projection P_{A^T} onto $\mathbf{C}(A^\mathsf{T})$. Write $\mathbf{x}_\star = P_{A^\mathsf{T}}\mathbf{x}_0$. Check the following.

(i) $\mathbf{x}_\star = A^\mathsf{T}(AA^\mathsf{T})^{-1}\mathbf{b} \in \mathbf{C}(A^\mathsf{T})$. [Hint: *what is $A\mathbf{x}_0$?*]

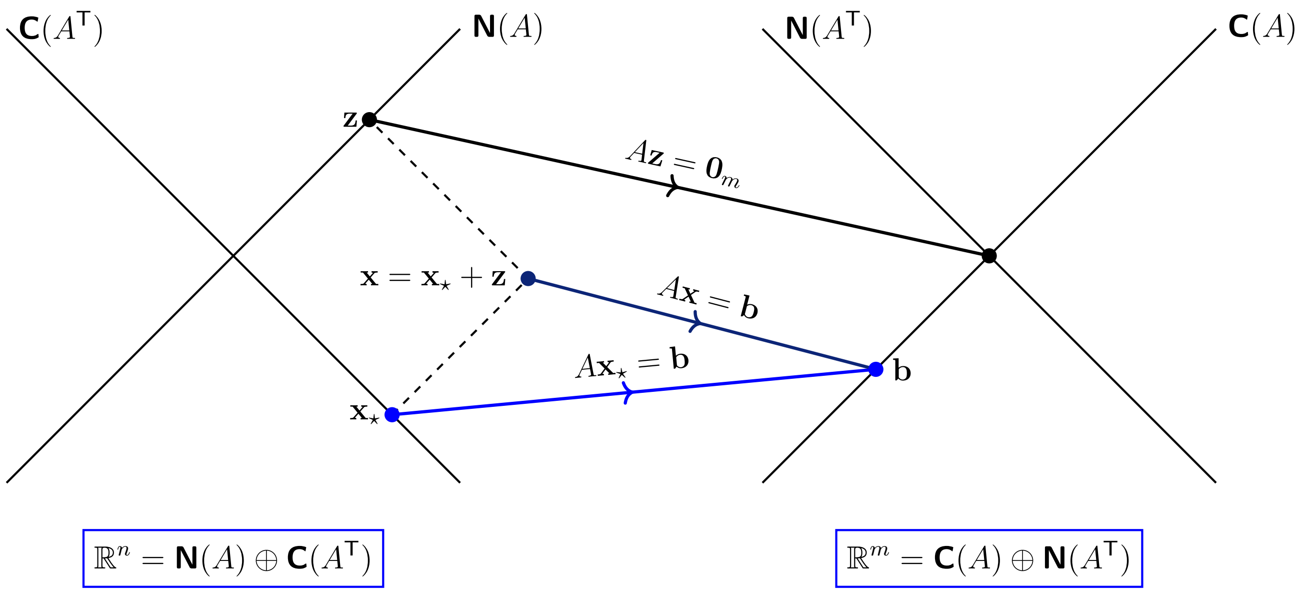
(ii) $A\mathbf{x}_\star = \mathbf{b}$.

(iii) If $A\mathbf{x} = \mathbf{b}$, then $\|\mathbf{x}_\star\| \le \|\mathbf{x}\|$. [Hint: *use Theorem 23.4 to write $\mathbf{x} = P_{A^\mathsf{T}}\mathbf{x} + \mathbf{z}$ for some $\mathbf{z} \in \mathbf{N}(A)$; show $P_{A^\mathsf{T}}\mathbf{x} = \mathbf{x}_\star$ and then use Problem 37.12 to estimate $\|\mathbf{x}\| \ge \|\mathbf{x}_\star\|$.*]

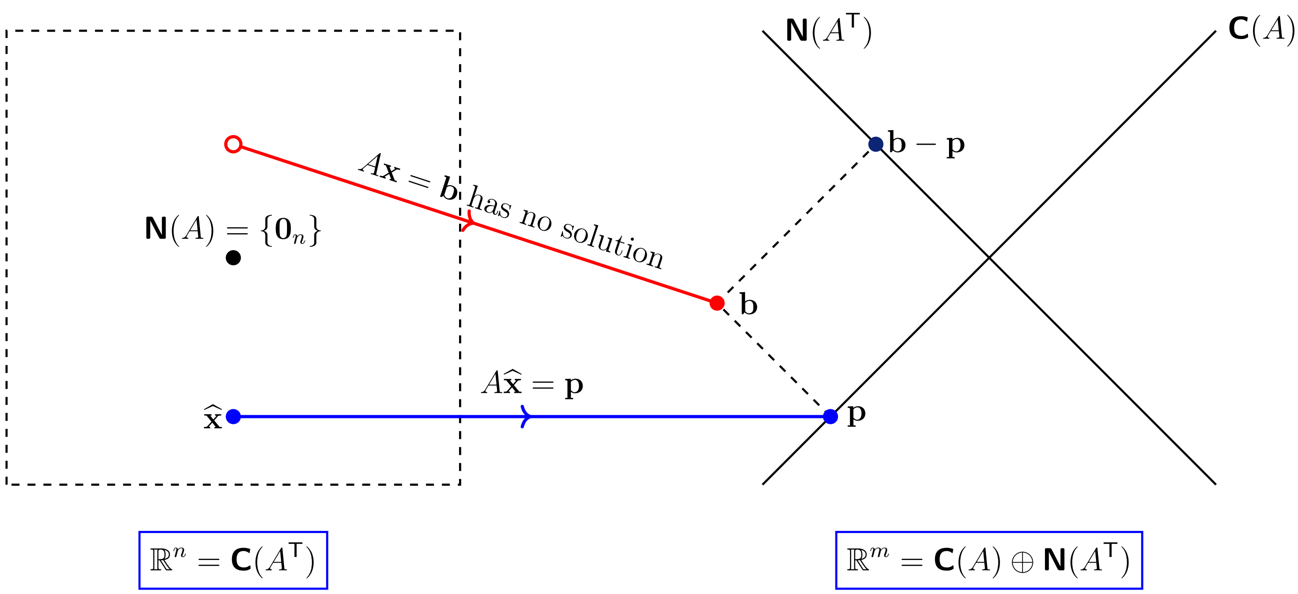
Here is a summary, in pictures, of everything that we have done. Literally: these three pictures encapsulate most of the ideas of the course. In these pictures, which are really fake cartoons, we should imagine that all of the four fundamental subspaces are one-dimensional (except in the second, where the null space of A is trivial), and so we can picture them as coordinate axes in a two-dimensional plane.

- The best case is that we can solve $A\mathbf{x} = \mathbf{b}$, although maybe $\mathbf{N}(A) \ne \{\mathbf{0}_n\}$ and we have infinitely many solutions. In that case perhaps the "best" solution is the "minimum norm solution," i.e., the $\mathbf{x}_\star \in \mathbb{R}^n$ such that $A\mathbf{x}_\star = \mathbf{b}$ and if $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$, then $\|\mathbf{x}\star\| \le \|\mathbf{x}\|$. And in this case, we would need $\mathbf{x}_\star \in \mathbf{C}(A^\mathsf{T})$. Here is why: since $\mathbb{R}^n = \mathbf{N}(A) \oplus \mathbf{C}(A^\mathsf{T})$, we could write $\mathbf{x}_\star = \mathbf{v} + \mathbf{w}$ for $\mathbf{v} \in \mathbf{N}(A)$ and $\mathbf{w} \in \mathbf{C}(A^\mathsf{T})$. If $\mathbf{x}_\star \notin \mathbf{C}(A^\mathsf{T})$, then $\mathbf{v} \ne \mathbf{0}_n$ (otherwise

$\mathbf{x}_\star = \mathbf{0}_n + \mathbf{w} = \mathbf{w} \in \mathbf{C}(A^\mathsf{T})$, and then $\|\mathbf{x}_\star\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 > \|\mathbf{w}\|^2$, thus $\|\mathbf{x}_\star\| > \|\mathbf{w}\|$. But $A\mathbf{w} = A(\mathbf{v}+\mathbf{w}) = A\mathbf{x}_\star = \mathbf{b}$ since $A\mathbf{v} = \mathbf{0}_n$. Such a minimum norm solution can be constructed with the pseudoinverse; Problem 38.4 outlines an approach when A has full row rank.



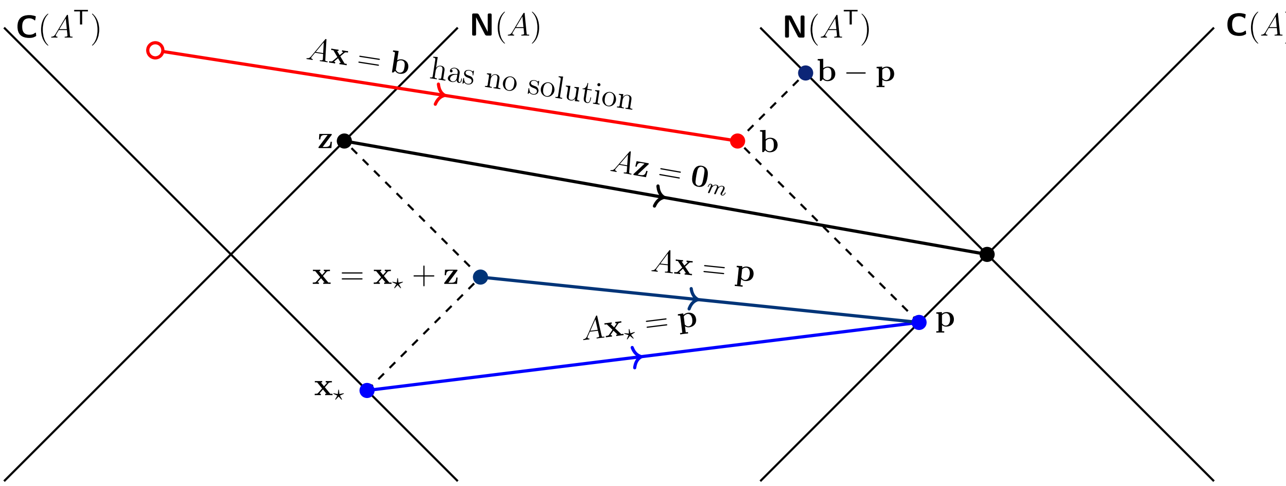
• The next best case is that while we cannot solve $A\mathbf{x} = \mathbf{b}$, since $\mathbf{b} \notin \mathbf{C}(A)$, A does have full column rank, so we can do a least squares approximation.



• The worst case is that we cannot solve $A\mathbf{x} = \mathbf{b}$ and A does not have full column rank. Then while we can approximate $A\mathbf{x} = \mathbf{b}$ with the problem $A\widehat{\mathbf{x}} = \mathbf{p}$, where $\mathbf{p}$ is the projection of $\mathbf{b}$ onto $\mathbf{C}(A)$, this new approximate problem will not have a unique solution, since $\mathbf{N}(A) \neq \{\mathbf{0}_n\}$.

So, can we find $\mathbf{x}_\star \in \mathbb{R}^n$ such that $A\mathbf{x}_\star$ is the closest vector in $\mathbf{C}(A)$ to $\mathbf{b}$ and that $\mathbf{x}_\star$ is the smallest vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x}$ is the closest vector in $\mathbf{C}(A)$ to $\mathbf{b}$? Saying that aloud sounds awful, but here is what $\mathbf{x}_\star$ should do:

$$\begin{cases} \mathbf{x}_\star \in \mathbf{C}(A^\mathsf{T}) \\ \\ \mathbf{v} \in \mathbf{C}(A) \Longrightarrow \|\mathbf{b} - A\mathbf{x}_\star\| \le \|\mathbf{b} - \mathbf{v}\| \\ \\ \begin{pmatrix} \mathbf{x} \in \mathbb{R}^n \\ \mathbf{v} \in \mathbf{C}(A^\mathsf{T}) \Longrightarrow \|\mathbf{b} - A\mathbf{x}\| \le \|\mathbf{b} - \mathbf{v}\| \end{pmatrix} \Longrightarrow \|\mathbf{x}_\star\| \le \|\mathbf{x}\|. \end{cases}$$



$\mathbb{R}^n = \mathbf{N}(A) \oplus \mathbf{C}(A^\mathsf{T})$ $\qquad$ $\mathbb{R}^m = \mathbf{C}(A) \oplus \mathbf{N}(A^\mathsf{T})$

As in the first, possibly best, case, we would want $\mathbf{x}_\star \in \mathbf{C}(A^\mathsf{T})$ here. Again, this can be accomplished with the pseudoinverse.

We continue to accumulate victories. We know how the fundamental subspaces associated with a matrix fit into the surrounding Euclidean spaces, and we know how to compute explicitly orthogonal decompositions of vectors. Unfortunately, "explicitly" does not mean "easily." Actually calculating the orthogonal projection onto the column space of $A \in \mathbb{R}^{m\times n}$ can be annoying, because we have to invert a matrix of the form $B^\mathsf{T}B$ (where $B = A$ if A has full column rank). This could make solving a least squares problem hard, and we know anyway that numerically computing inverses is rarely a good idea.

There is actually a way to avoid inverses if $A \in \mathbb{R}^{m\times n}$ has full column rank. Per Theorem 37.14, if we cannot solve $A\mathbf{x} = \mathbf{b}$, we consider the approximate problem $A\widehat{\mathbf{x}} = P_A\mathbf{b}$ with $P_A = A(A^\mathsf{T}A)^{-1}A^\mathsf{T}\mathbf{b}$, and the least squares solution is $\widehat{\mathbf{x}} = (A^\mathsf{T}A)^{-1}A^\mathsf{T}\mathbf{b}$. This is equivalent to the so-called **NORMAL EQUATION**

$$A^\mathsf{T}A\widehat{\mathbf{x}} = A^\mathsf{T}\mathbf{b}. \tag{38.1}$$

And the normal equation is just a matrix-vector equation, which we could solve with Gaussian elimination and back-substitution, or perhaps an LU-factorization; the point is that we do not really have to compute $(A^{\mathsf{T}}A)^{-1}$ to get $\widehat{\mathbf{x}}$ if we are open to exploring other avenues of solution.

Content from Strang's *ILA* 6E. Reread the first three paragraphs on p. 163.

Day 39: Monday, November 17.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Orthogonal list of vectors (N), orthonormal list of vectors (N), orthogonal matrix(N)

We continue to accumulate victories. We know how the fundamental subspaces associated with a matrix fit into the surrounding Euclidean spaces, and we know how to compute explicitly orthogonal decompositions of vectors. Unfortunately, "explicitly" does not mean "easily." Actually calculating the orthogonal projection onto the column space of $A \in \mathbb{R}^{m \times n}$ can be annoying, because we have to invert a matrix of the form $B^{\mathsf{T}}B$ (where $B = A$ if A has full column rank). This could make solving a least squares problem hard, and we know anyway that numerically computing inverses is rarely a good idea.

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And the normal equation is just a matrix-vector equation, which we could solve with Gaussian elimination and back-substitution, or perhaps an LU-factorization; the point is that we do not really have to compute $(A^{\mathsf{T}}A)^{-1}$ to get $\widehat{\mathbf{x}}$ if we are open to exploring other avenues of solution.

Content from Strang's *ILA* 6E. Reread the first three paragraphs on p. 163.

It turns out that if we ask a little more of our matrix, the projection onto its column space becomes much nicer. The right thing to do is exploit geometry further. Long ago (in Problem 4.2) we saw why the standard basis vectors in $\mathbb{R}^m$ were so nice. Since

$$\mathbf{e}_j \cdot \mathbf{e}_k = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

and $\mathbb{R}^m = \operatorname{span}(\mathbf{e}_1, \ldots, \mathbf{e}_m) = \mathbf{C}(I_m)$, we have the representation

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{v} \cdot \mathbf{e}_m)\mathbf{e}_m$$

for any $\mathbf{v} \in \mathbb{R}^m$. What is special here is not the formulas for the standard basis vectors but how they interact under the dot product. And what is most important is their mutual orthogonality.

39.1 Definition. *A list* $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^m$ *is* **ORTHOGONAL** *if*

$$\mathbf{u}_j \cdot \mathbf{u}_k = 0$$

for $j \neq k$.

To keep things simple, look at an orthogonal list $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3 \in \mathbb{R}^m$ and say $\mathbf{v} \in \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. Then $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ for some c_1, c_2, $c_3 \in \mathbb{R}$. The trick is to compute

$$\begin{aligned}\mathbf{v} \cdot \mathbf{u}_1 &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) \cdot \mathbf{u}_1 \\ &= \big((c_1\mathbf{u}_1) \cdot \mathbf{u}_1\big) + \big((c_2\mathbf{u}_2) \cdot \mathbf{u}_1\big) + \big((c_3\mathbf{u}_3) \cdot \mathbf{u}_1\big) \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_1) + c_3(\mathbf{u}_3 \cdot \mathbf{u}_1) \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1).\end{aligned}$$

If $\mathbf{u}_1 \neq \mathbf{0}_m$, then $\mathbf{u}_1 \cdot \mathbf{u}_1 = \|\mathbf{u}_1\|^2 \neq 0$, and so we have

$$c_1 = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}.$$

Assume that none of the $\mathbf{u}_j$ are $\mathbf{0}_m$; otherwise, they contribute nothing worthwhile to the span. Taking dot products of $\mathbf{v}$ against the other $\mathbf{u}_j$ then yields

$$c_j = \frac{\mathbf{v} \cdot \mathbf{u}_j}{\|\mathbf{u}_j\|^2}.$$

This generalizes to an arbitrary orthogonal list.

39.2 Theorem. *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^m$ *be orthogonal and let* $\mathbf{v} \in \operatorname{span}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$*. Then*

$$\mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{v} \cdot \mathbf{u}_n}{\|\mathbf{u}_n\|^2}\right)\mathbf{u}_n.$$

A nice consequence is that any orthogonal list of *nonzero* vectors is independent. For if $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = \mathbf{0}_m$, then each c_j must be 0. (In the theorem above, take $\mathbf{v} = \mathbf{0}_m$, so the dot products collapse to 0.)

39.3 Problem (!). What is the maximum length of any list of orthogonal vectors in $\mathbb{R}^m$?

All that division, however, gets annoying, and it s much more efficient to assume $\|\mathbf{u}_j\| = 1$ for all j.

39.4 Definition. *A list* $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbb{R}^m$ *is* ORTHONORMAL *if*

$$\mathbf{q}_j \cdot \mathbf{q}_k = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

The $j \neq k$ condition means that an orthonormal list is orthogonal, while the $j = k$ condition gives $\|\mathbf{q}_j\| = \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j} = 1$.

39.5 Problem ($\star$). Let $\mathbf{q}_1$, $\mathbf{q}_2 \in \mathbb{R}^m$ be orthonormal and let $\mathbf{v} \in \operatorname{span}(\mathbf{q}_1, \mathbf{q}_2)$. Use the Pythagorean theorem (Theorem 37.10) to show that

$$\|\mathbf{v}\|^2 = |\mathbf{v} \cdot \mathbf{q}_1|^2 + |\mathbf{v} \cdot \mathbf{q}_2|^2.$$

This generalizes: if $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbb{R}^m$ are orthonormal and $\mathbf{v} \in \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_n)$, then

$$\|\mathbf{v}\|^2 = |\mathbf{v} \cdot \mathbf{q}_1|^2 + \cdots + |\mathbf{v} \cdot \mathbf{q}_n|^2.$$

We work with matrices and column spaces as much as we do with lists of vectors and spans, so we put those orthonormal vectors into a matrix and get an unfortunate definition.

39.6 Definition. *A matrix* $Q \in \mathbb{R}^{m \times n}$ *is* ORTHOGONAL *if the columns of* Q *are orthonormal.*

39.7 Example. **(i)** The identity matrix is always orthogonal.

(ii) Let $\theta \in \mathbb{R}$ and

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Then trig and the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ say that Q is orthogonal.

Content from Strang's *ILA* 6E. Everything on pp. 176–178 is important. I am being a little more general and calling any matrix, square or not, with orthogonal columns an "orthogonal matrix."

Day 40: Wednesday, November 19.

Vocabulary from today

You should memorize the definition of each term, phrase, or concept below and be able to provide a concrete example of each and a nonexample for those marked "N."

Orthonormal basis for a subspace

Here is a nice consequence of definitions. Let $Q = \begin{bmatrix}\mathbf{q}_1 & \cdots & \mathbf{q}_n\end{bmatrix} \in \mathbb{R}^{m\times n}$ be orthogonal. Then

$$\mathbf{q}_i \cdot \mathbf{q}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Remember that row i of Q^T is just column i of Q, and that the (i,j)-entry of $Q^\mathsf{T}Q$ is the dot product of row i of Q^T and column j of Q. That is, the (i,j)-entry of $Q^\mathsf{T}Q$ is $\mathbf{q}_i \cdot \mathbf{q}_j$, and so this (i,j)-entry is 1 when $i = j$ (= on the diagonal) and 0 otherwise (= off the diagonal). This sounds a lot like an identity matrix, and it is! Since $Q \in \mathbb{R}^{m\times n}$, we have $Q^\mathsf{T} \in \mathbb{R}^{n\times m}$, and so $Q^\mathsf{T}Q \in \mathbb{R}^{n\times n}$.

40.1 Theorem. *Let $Q \in \mathbb{R}^{m\times n}$ be orthogonal. Then $Q^\mathsf{T}Q = I_n$.*

40.2 Problem (!). Is every orthogonal matrix invertible?

40.3 Problem ($\star$). State and prove an analogue of Theorem 40.1 for the case when the columns of Q are only orthogonal, not orthonormal.

40.4 Problem (!). Is every orthogonal projection (Definition 37.1) an orthogonal matrix?

We can use Theorem 40.1 to get a slick representation of vectors in $\mathbf{C}(Q)$ for orthogonal Q. Let $Q \in \mathbb{R}^{m\times n}$ be orthogonal and $\mathbf{b} \in \mathbf{C}(Q)$. Then $\mathbf{b} = Q\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, and so $Q^\mathsf{T}\mathbf{b} = Q^\mathsf{T}Q\mathbf{x} = \mathbf{x}$. Thus

$$\mathbf{b} = Q\mathbf{x} = QQ^\mathsf{T}\mathbf{b}.$$

Now we think about matrix multiplication. One way to compute the entries of $Q^\mathsf{T}\mathbf{b}$ is to take the dot product of the rows of Q^T with $\mathbf{b}$. (This is probably how we usually compute matrix-vector products by hand.) And the rows of Q^T are the columns of Q, so

$$Q^\mathsf{T}\mathbf{b} = \begin{bmatrix}\mathbf{q}_1 \cdot \mathbf{b} \\ \vdots \\ \mathbf{q}_n \cdot \mathbf{b}\end{bmatrix}.$$

Next, one way to compute $QQ^\mathsf{T}\mathbf{b}$ is to take the linear combination of the columns of Q weighted by the entries of $Q^\mathsf{T}\mathbf{b}$:

$$\mathbf{b} = QQ^\mathsf{T}\mathbf{b} = \begin{bmatrix}\mathbf{q}_1 & \cdots & \mathbf{q}_n\end{bmatrix}\begin{bmatrix}\mathbf{q}_1 \cdot \mathbf{b} \\ \vdots \\ \mathbf{q}_n \cdot \mathbf{b}\end{bmatrix} = (\mathbf{q}_1 \cdot \mathbf{b})\mathbf{q}_1 + \mathbf{q}_1 \cdots + (\mathbf{q}_n \cdot \mathbf{b})\mathbf{q}_n.$$

This brings together two ways of looking at matrix-vector multiplication: the dot product way for quick and dirty calculations by hand, the linear combination of columns way to actually understand what happens.

These are old ideas reinterpreted in a new framework. Here is what is really new, and useful: orthonormality and orthogonal matrices make least squares so much easier. Suppose that $Q \in \mathbb{R}^{m \times n}$ is orthogonal and we want to solve $Q\mathbf{x} = \mathbf{b}$, but $\mathbf{b} \notin \mathbf{C}(Q)$. Then we would solve the least squares problem

$$Q\widehat{\mathbf{x}} = P_Q\mathbf{b},$$

where

$$P_Q = Q(Q^\mathsf{T}Q)^{-1}Q^\mathsf{T} = QI_n^{-1}Q^\mathsf{T} = QQ^\mathsf{T}. \tag{40.1}$$

Look at that: the orthogonal projection onto $\mathbf{C}(Q)$ collapses to QQ^T. No inverses needed.

40.5 Theorem. *Let $Q = \begin{bmatrix}\mathbf{q}_1 & \cdots & \mathbf{q}_n\end{bmatrix} \in \mathbb{R}^{m \times n}$ be orthogonal. Then the orthogonal projection onto $\mathbf{C}(Q)$ is QQ^T, and every $\mathbf{v} \in \mathbf{C}(Q)$ has the form*

$$\mathbf{v} = QQ^\mathsf{T}\mathbf{v} = (\mathbf{v} \cdot \mathbf{q}_1)\mathbf{q}_1 + \cdots + (\mathbf{v} \cdot \mathbf{q}_n)\mathbf{q}_n.$$

Content from Strang's *ILA* 6E. You should hold the answer to Worked Example 4.4 B on p. 185 deep within your heart.

Then the least squares problem is just

$$Q\widehat{\mathbf{x}} = QQ^\mathsf{T}\mathbf{b}.$$

Since Q has full column rank, $\mathbf{N}(Q) = \{\mathbf{0}_n\}$, and so the only solution $\widehat{\mathbf{x}}$ to $Q\widehat{\mathbf{x}} = Q(Q^\mathsf{T}\mathbf{b})$ is, basically by design, and that says

$$\widehat{\mathbf{x}} = Q^\mathsf{T}\mathbf{b}.$$

Computing the least squares solution requires no inverses, only transposing.

40.6 Problem (!). Reread Example 36.1 and explain how orthonormality made calculating the projection operator easier. How would things have been more complicated there if we used the pivot columns of A as the basis for the column space, not the columns of B?

Content from Strang's *ILA* 6E. Page 179 through the top of p. 180 discuss least squares with orthogonal matrices.

We conclude with a special, and hugely important, case. Every orthonormal list is an orthogonal list with no zero vectors in it. And every orthogonal list with no zero vectors is independent. So an orthonormal list of length m in $\mathbb{R}^m$ is an independent list of length m: thus a basis for $\mathbb{R}^m$. (No need to check spanning because we can count.) This is the best basis.

40.7 Definition. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^p$. A list $\mathbf{q}_1, \ldots, \mathbf{q}_r$ in $\mathcal{V}$ is an* ORTHONORMAL BASIS *for $\mathcal{V}$ if it is both a basis for $\mathcal{V}$ and an orthonormal list.*

Since orthonormality implies orthogonality, and orthogonality implies independence, to check that a list is an orthonormal basis for a subspace, we just have to check that the list is orthonormal and spans the subspace; we get independence for free from orthonormality.

Theorem 39.2 gives us a really slick way to represent a vector $\mathbf{v} \in \mathbb{R}^m$ in terms of an orthonormal basis $\mathbf{q}_1, \ldots, \mathbf{q}_m$ for $\mathbb{R}^m$: since $\mathbf{v} \in \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_m)$, we get

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{q}_1)\mathbf{q}_1 + \cdots + (\mathbf{v} \cdot \mathbf{q}_m)\mathbf{q}_m.$$

This is so great because we *explicitly* have the coefficients for $\mathbf{v}$ with respect to this basis. Just take the dot product.

40.8 Problem ($\star$). Let $\mathbf{q}_1, \ldots, \mathbf{q}_m$ be an orthonormal basis for $\mathbb{R}^m$ and let $1 \le r < m$. Put $\mathcal{V} = \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_r)$ and explain why $\mathcal{V}^\perp = \operatorname{span}(\mathbf{q}_{r+1}, \ldots, \mathbf{q}_m)$. [Hint: *we need Definition 5.1 here. If* $\mathbf{v} \in \operatorname{span}(\mathbf{q}_{r+1}, \ldots, \mathbf{q}_m)$, *then* $\mathbf{v} \cdot \mathbf{q}_j = 0$ *for* $j = 1, \ldots, r$, *right? How does that show* $\mathbf{v} \in \mathcal{V}^\perp$*? Next, if* $\mathbf{v} \in \mathcal{V}^\perp$, *worst case is we know* $\mathbf{v} = (\mathbf{v} \cdot \mathbf{q}_1)\mathbf{q}_1 + \cdots + (\mathbf{v} \cdot \mathbf{q}_m)\mathbf{q}_m$. *But we also know* $\mathbf{v} \cdot \mathbf{q}_j = 0$ *for* $j = 1, \ldots, r$. *What goes away in that expansion of* $\mathbf{v}$*?*]

Life is best when the vectors under consideration are orthonormal. Often they are not. How do we turn a problem governed by "ordinary" vectors into a problem controlled by orthonormal vectors? Of course this depends on the exact problem, and the problem that we have most recently studied is least squares.

Let $A \in \mathbb{R}^{m \times n}$. How can we find an orthonormal basis for $\mathbf{C}(A)$? This will make computing the orthogonal projection P_A onto $\mathbf{C}(A)$ easy. The hard way to find P_A is to start with a matrix $B \in \mathbb{R}^{m \times r}$ such that the columns of B are independent and $\mathbf{C}(A) = \mathbf{C}(B)$ and then compute $P_A = B(B^\mathsf{T}B)^{-1}B^\mathsf{T}$. But if we have an orthogonal matrix $Q \in \mathbb{R}^{m \times r}$ such that $\mathbf{C}(A) = \mathbf{C}(Q)$, then $P_A = QQ^\mathsf{T}$, per (40.1).

We therefore may as well just start with an independent list $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$. Here is our goal: given an independent list $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$, find an orthonormal list $\mathbf{q}_1, \ldots, \mathbf{q}_r$ in $\mathbb{R}^m$ such that

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_n).$$

We will figure out a very transparent, iterative algorithm for doing this. It is less important to be able to perform this algorithm by hand than it is to understand how to figure it out in the first place. We look at three increasingly more complicated cases on r to get some ideas.

1. $n = 1$. We want the very small list $\mathbf{q}_1$ to be orthonormal with $\operatorname{span}(\mathbf{v}_1) = \operatorname{span}(\mathbf{q}_1)$. We therefore want $\|\mathbf{q}_1\| = 1$, and we will get the spanning property if $\mathbf{q}_1 = c_1\mathbf{v}_1$ for some $c_1 \in \mathbb{R}$. How should we choose c_1? All that we know is

$$1 = \|\mathbf{q}_1\| = \|c_1\mathbf{v}_1\| = |c_1| \, \|\mathbf{v}_1\| \, .$$

Since the list $\mathbf{v}_1$ is independent, $\mathbf{v}_1 \ne \mathbf{0}_m$, so $\|\mathbf{v}_1\| \ne 0$. Then we may solve for

$$|c_1| = \frac{1}{\|\mathbf{v}_1\|}.$$

This suggests taking

$$c_1 = \frac{1}{\|\mathbf{v}_1\|}.$$

2. $n = 2$. We want the list $\mathbf{q}_1, \mathbf{q}_2$ to be orthonormal and $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{q}_1, \mathbf{q}_2)$. Since declaring $\mathbf{q}_1 := \mathbf{v}_1 / \|\mathbf{v}_1\|$ was enough in the $r = 1$ case, we might try doing that here, too. This gives $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{q}_1, \mathbf{v}_2)$. (Think about it...) If we then want $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{q}_1, \mathbf{q}_2)$, we need $\mathbf{q}_2 \in \text{span}(\mathbf{q}_1, \mathbf{v}_2)$. So, we must be able to write $\mathbf{q}_2 = c_1\mathbf{q}_1 + c_2\mathbf{v}_2$, and then to have $\mathbf{q}_2 \cdot \mathbf{q}_1 = 0$, we need

$$0 = \mathbf{q}_2 \cdot \mathbf{q}_1 = (c_1\mathbf{q}_1 + c_2\mathbf{v}_2) \cdot \mathbf{q}_1 = c_1(\mathbf{q}_1 \cdot \mathbf{q}_1) + c_2(\mathbf{v}_2 \cdot \mathbf{q}_1) = c_1 + c_2(\mathbf{v}_2 \cdot \mathbf{q}_1) = c_1 + c_2(\mathbf{v}_2 \cdot \mathbf{q}_1)$$

since $\mathbf{q}_1 \cdot \mathbf{q}_1 = \|\mathbf{q}_1\|^2 = 1$. Then

$$c_1 = -c_2(\mathbf{v}_2 \cdot \mathbf{q}_1)$$

and so

$$\mathbf{q}_2 = -c_2(\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1 + c_2\mathbf{v}_2 = c_2(\mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1) \tag{40.2}$$

Put

$$\mathbf{w}_2 := \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1,$$

so we want $\mathbf{q}_2 = c_2\mathbf{w}_2$. Since the list $\mathbf{v}_1, \mathbf{v}_2$ is independent and $\mathbf{q}_1$ is a nonzero scalar multiple of $\mathbf{v}_1$, the list $\mathbf{q}_1, \mathbf{v}_2$ is also independent (check that), and therefore $\mathbf{w}_2 \neq \mathbf{0}_m$. To have $\|\mathbf{q}_2\| = 1$, we need $|c_2| \|\mathbf{w}_2\| = 1$; this suggests taking $c_2 = 1/\|\mathbf{w}_2\|$, which is permissible.

To summarize, we put

$$\begin{cases} \mathbf{q}_1 := \mathbf{v}_1 / \|\mathbf{v}_1\| \\ \mathbf{w}_2 := \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1 \\ \mathbf{q}_2 := \mathbf{w}_2 / \|\mathbf{w}_2\| \end{cases} \tag{40.3}$$

to get an orthonormal list $\mathbf{q}_1$, $\mathbf{q}_2$ with the desired property that $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{q}_1, \mathbf{q}_2)$, and the added bonus that $\text{span}(\mathbf{v}_1) = \text{span}(\mathbf{q}_1)$.

3. $n = 3$. We want the list $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ to be orthonormal with

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3). \tag{40.4}$$

Based on our prior success, we might use (40.3) to define $\mathbf{q}_1$ and $\mathbf{q}_2$. Then we would have (40.4). We therefore want $\mathbf{q}_3 \in \text{span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{v}_3)$, and so we need $\mathbf{q}_3 = c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{v}_3$. To have $\mathbf{q}_3 \cdot \mathbf{q}_1 = 0$, we need

$$0 = \mathbf{q}_3 \cdot \mathbf{q}_1 = (c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{v}_3) \cdot \mathbf{q}_1 = c_1(\mathbf{q}_1 \cdot \mathbf{q}_1) + c_2(\mathbf{q}_2 \cdot \mathbf{q}_1) + c_3(\mathbf{v}_3 \cdot \mathbf{q}_1) = c_1 + c_3(\mathbf{v}_3 \cdot \mathbf{q}_1)$$

since $\mathbf{q}_1 \cdot \mathbf{q}_1 = \|\mathbf{q}_1\|^2 = 1$ and $\mathbf{q}_2 \cdot \mathbf{q}_1 = 0$. Similarly, we want

$$0 = \mathbf{q}_3 \cdot \mathbf{q}_2 = c_1(\mathbf{q}_1 \cdot \mathbf{q}_2) + c_2(\mathbf{q}_2 \cdot \mathbf{q}_2) + c_3(\mathbf{v}_3 \cdot \mathbf{q}_2) = c_2 + c_3(\mathbf{v}_3 \cdot \mathbf{q}_2)$$

since $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$ and $\mathbf{q}_2 \cdot \mathbf{q}_2 = \|\mathbf{q}_2\|^2 = 1$. Then we need

$$c_1 = -c_3(\mathbf{v}_3 \cdot \mathbf{q}_1) \quad \text{and} \quad c_2 = -c_3(\mathbf{v}_3 \cdot \mathbf{q}_2).$$

So, $\mathbf{q}_3$ must have the form

$$\mathbf{q}_3 = -c_3(\mathbf{v}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 - c_3(\mathbf{v}_3 \cdot \mathbf{q}_2)\mathbf{q}_2 + c_3\mathbf{v}_3 = c_3(\mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{v}_3 \cdot \mathbf{q}_2)\mathbf{q}_2).$$

Put $\mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{v}_3 \cdot \mathbf{q}_2)\mathbf{q}_2$. If $\mathbf{w}_3 = \mathbf{0}_m$, then $\mathbf{v}_3 \in \operatorname{span}(\mathbf{q}_1, \mathbf{q}_2) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$, which contradicts the independence of the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. So, $\mathbf{w}_3 \neq \mathbf{0}_m$, and therefore to have $\|\mathbf{q}_3\| = 1$, we could take $c_3 = 1/\|\mathbf{w}_3\|$.

To summarize, we put

$$\begin{cases} \mathbf{q}_1 := \mathbf{v}_1/\|\mathbf{v}_1\| \\ \mathbf{w}_2 := \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1 \\ \mathbf{q}_2 := \mathbf{w}_2/\|\mathbf{w}_2\| \\ \mathbf{w}_3 := \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{v}_3 \cdot \mathbf{q}_2)\mathbf{q}_2 \\ \mathbf{q}_3 := \mathbf{w}_3/\|\mathbf{w}_3\| \end{cases}$$

to get an orthonormal list $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ with the desired property that

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \operatorname{span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),$$

and the added bonus that

$$\operatorname{span}(\mathbf{v}_1) = \operatorname{span}(\mathbf{q}_1) \quad \text{and} \quad \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) = \operatorname{span}(\mathbf{q}_1, \mathbf{q}_2).$$

Content from Strang's *ILA* 6E. Pages 180–181 do Gram–Schmidt for three vectors. See in particular the 3D drawings in Figure 4.10 on p. 181.

Day 41: Friday, November 21.

We are almost ready for the general result and need one auxiliary tool first.

41.1 Problem ($\star$). Let A, $B \in \mathbb{R}^{m \times n}$ with $\mathbf{C}(A) = \mathbf{C}(B)$ and $\operatorname{rank}(B) = n$. Let $\mathbf{v} \in \mathbb{R}^m$. Show that

$$\mathbf{C}\big(\begin{bmatrix} A & \mathbf{v} \end{bmatrix}\big) = \mathbf{C}\big(\begin{bmatrix} B & (\mathbf{v} - P_B\mathbf{v}) \end{bmatrix}\big).$$

[Hint: *let* $c \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$ *and write* $A\mathbf{x} + c\mathbf{v} = A\mathbf{x} + c(\mathbf{v} - P_B\mathbf{v}) + cP_B\mathbf{v}$. *Explain why* $A\mathbf{x} + cP_B\mathbf{v} \in \mathbf{C}(B)$. *Next, let* $c \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$ *and write* $B\mathbf{x} + c(\mathbf{v} - P_B\mathbf{v}) = (B\mathbf{x} - zP_B\mathbf{v}) + c\mathbf{v}$. *Explain why* $B\mathbf{x} - cP_B\mathbf{v} \in \mathbf{C}(A)$.]

41.2 Theorem (Gram–Schmidt procedure). *Suppose that the list* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *in* $\mathbb{R}^m$ *is independent. There exists an orthonormal list* $\mathbf{q}_1, \ldots, \mathbf{q}_n$ *in* $\mathbb{R}^m$ *such that* $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j) = \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_j)$ *for* $j = 1, \ldots, n$. *Specifically,*

$$\begin{cases} \mathbf{q}_1 := \mathbf{v}_1/\|\mathbf{v}_1\| \\ \mathbf{w}_j := \mathbf{v}_j - \big((\mathbf{v}_j \cdot \mathbf{q}_1)\mathbf{q}_1 + \cdots + (\mathbf{v}_j \cdot \mathbf{q}_{j-1})\mathbf{q}_{j-1}\big), \ j \geq 2 \\ \mathbf{q}_j := \mathbf{w}_j/\|\mathbf{w}_j\|, \ j \geq 2. \end{cases} \tag{41.1}$$

With

$$Q_{j-1} := \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_{j-1} \end{bmatrix}$$

for $j \geq 2$, we can also write

$$\begin{cases} \mathbf{q}_1 := \mathbf{v}_1 / \|\mathbf{v}_1\| \\ \mathbf{w}_j := (I_m - Q_{j-1}Q_{j-1}^\mathsf{T})\mathbf{v}_j, \ j \geq 2. \mathbf{q}_j := \mathbf{w}_j / \|\mathbf{w}_j\|, \ j \geq 2. \end{cases}$$

Proof. This is really a proof by induction, but the key ideas are outlined in the $n = 3$ case above. The point is that we know how to construct $\mathbf{q}_1$, and then we assume that we have constructed through $\mathbf{q}_j$ with $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j) = \text{span}(\mathbf{q}_1, \ldots, \mathbf{q}_j)$ and $\mathbf{q}_1, \ldots, \mathbf{q}_j$ orthonormal. Then we check that the algorithm above defines $\mathbf{q}_{j+1}$ correctly.

1. We put $\mathbf{w}_{j+1} := (I_m - Q_{j+1}Q_{j+1}^\mathsf{T})\mathbf{v}_{j+1}$.

2. By Theorem 40.5, the matrix $Q_{j+1}Q_{j+1}^\mathsf{T}$ is the orthogonal projection onto

$$\mathbf{C}(Q_{j+1}) = \text{span}(\mathbf{q}_1, \ldots, \mathbf{q}_j) = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j).$$

And by Problem 37.3, $I_m - Q_{j+1}Q_{j+1}^\mathsf{T}$ is the orthogonal projection onto $\mathbf{C}(Q_{j+1})^\perp$. Thus $\mathbf{w}_{j+1}$ is orthogonal to every vector in $\mathbf{C}(Q_{j+1})$, in particular the vectors $\mathbf{q}_1, \ldots, \mathbf{q}_j$.

3. To show that $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{v}_{j+1}) = \text{span}(\mathbf{q}_1, \ldots, \mathbf{q}_j, \mathbf{w}_{j+1})$, invoke Problem 41.1 with $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix}$, $B = Q_j$, and $\mathbf{v} = \mathbf{w}_{j+1}$.

4. We check that $\mathbf{w}_{j+1} \neq \mathbf{0}_m$: if $\mathbf{w}_{j+1} = \mathbf{0}_m$, then $\mathbf{v}_{j+1} = Q_jQ_j^\mathsf{T}\mathbf{v}_{j+1} \in \mathbf{C}(Q_j) = \mathbf{C}(A)$. This contradicts the independence of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

5. And so we can renormalize $\mathbf{q}_{j+1} := \mathbf{w}_{j+1} / \|\mathbf{w}_{j+1}\|$ and obtain

$$\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{v}_{j+1}) = \text{span}(\mathbf{q}_1, \ldots, \mathbf{q}_j, \mathbf{w}_{j+1}) = \text{span}(\mathbf{q}_1, \ldots, \mathbf{q}_j, \mathbf{q}_{j+1}).$$

Since this was true for an arbitrary $j \geq 1$, we can "turn the crank" and keep going to get the full list $\mathbf{q}_1, \ldots, \mathbf{q}_n$, regardless of the value of n. ■

Content from Strang's *ILA* 6E. Page 183 presents some pseudocode for computing Gram–Schmidt. See also the confession on p. 184. Read it and take a numerical linear algebra class.

To write things out concisely, the projection $Q_jQ_j^\mathsf{T}$ condenses notation, and the formula $\mathbf{w}_j = (I_m - Q_{j-1}Q_{j-1}^\mathsf{T})\mathbf{v}_j$ is an instance of our course's theme that we do things by multiplying by matrices—in this case, the matrix $I_m - Q_{j-1}Q_{j-1}^\mathsf{T}$. To do calculations by hand, the formula for $\mathbf{w}_j$ and $\mathbf{q}_j$ in (41.1) is probably more transparent.

41.3 Problem ($\star$). Using the hypotheses and notation of the Gram–Schmidt procedure, prove that $\mathbf{v}_j \cdot \mathbf{q}_j > 0$ as follows.

(i) First explain why we just need $\mathbf{v} \cdot \mathbf{u}_j > 0$, and check that this is true in the case $j = 1$.

(ii) For $j \geq 2$, rewrite

$$\mathbf{v}_j = \mathbf{u}_j + Q_{j-1}Q_{j-1}^\mathsf{T}\mathbf{v}_j.$$

Explain why $Q_{j-1}Q_{j-1}^\mathsf{T}\mathbf{v}_j \cdot \mathbf{u}_j = 0$. [Hint: $Q_{j-1}Q_{j-1}^\mathsf{T}\mathbf{v}_j = Q_{j-1}(Q_{j-1}^\mathsf{T}\mathbf{v}_j)$ *and* $\mathbf{u}_j \in \mathbf{C}(Q_{j-1})^\perp$.]

(iii) Conclude that $\mathbf{v}_j \cdot \mathbf{u}_j = \|\mathbf{u}_j\|^2$.

41.4 Example. Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}1\\2\\2\end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix}-2\\1\\0\end{bmatrix}.$$

The list $\mathbf{v}_1$, $\mathbf{v}_2$ is the list of pivot columns of the long-suffering matrix

$$A = \begin{bmatrix}0 & 1 & 2 & 1 & 7\\ 0 & 2 & 4 & 2 & 14\\ 0 & 0 & 0 & 2 & 8\end{bmatrix},$$

so $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for $\mathbf{C}(A)$. From Example **??**, the (short) list $\mathbf{v}_3$ is a basis for $\mathbf{N}(A^\mathsf{T})$. When we combine a basis for $\mathbf{C}(A)$ with a basis for $\mathbf{N}(A^\mathsf{T})$, orthogonality gives us a basis for the whole space. That is, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is a basis for $\mathbb{R}^3$. In particular, this list is independent, so we can apply Gram–Schmidt to it.

1. Start by computing

$$\|\mathbf{v}_1\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

and put

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}.$$

2. Then we want to set

$$\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1.$$

We compute

$$\mathbf{v}_2 \cdot \mathbf{q}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix} = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5},$$

so

$$\mathbf{w}_2 = \begin{bmatrix}1\\2\\2\end{bmatrix} - \sqrt{5}\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}.$$

Then

$$\|\mathbf{w}_2\| = \sqrt{0^2 + 0^2 + 2^2} = 2,$$

so we put

$$\mathbf{q}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{2}\begin{bmatrix}0\\0\\2\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

3. Last, we want to set

$$\mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{v}_3 \cdot \mathbf{q}_2)\mathbf{q}_2.$$

We have

$$\mathbf{v}_3 \cdot \mathbf{q}_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix} = -\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} = 0$$

and

$$\mathbf{v}_3 \cdot \mathbf{q}_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = 0,$$

so there is not much work to do here: $\mathbf{w}_3 = \mathbf{v}_3$. Compute

$$\|\mathbf{w}_3\| = \|\mathbf{v}_3\| = \sqrt{(-2)^2 + 1^2 + 0^2} = \sqrt{5},$$

and set

$$\mathbf{q}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix} = \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\\0\end{bmatrix}.$$

The result is that the list $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ is orthonormal and preserves spans in the sense that

$$\text{span}(\mathbf{v}_1) = \text{span}(\mathbf{q}_1), \qquad \text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{q}_1, \mathbf{q}_2),$$
$$\text{and} \qquad \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3).$$

In particular, since there are three vectors in the list $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, it is an orthonormal basis for $\mathbb{R}^3$. The best basis. Also, since $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{q}_1, \mathbf{q}_2)$, the list $\mathbf{q}_1$, $\mathbf{q}_2$ is an orthonormal basis for $\mathbf{C}(A)$ with A from above.. Again, the best basis.

41.5 Problem ($\star$). Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$.

(i) Suppose that for some integer j with $1 \le j \le n-1$, the list $\mathbf{v}_1, \ldots, \mathbf{v}_j$ is independent but $\mathbf{v}_{j+1} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j)$, so the list $\mathbf{v}_1, \ldots, \mathbf{v}_{j+1}$ is dependent. What happens at the $(j+1)$st step in the Gram–Schmidt process? [Hint: *reread the proof of Theorem 41.2.*]

(ii) Let j be an integer with $1 \le j \le n-1$, and now suppose that the list $\mathbf{v}_1, \ldots, \mathbf{v}_{j+1}$ is independent, so we can do Gram–Schmidt through the $(j+1)$st step. Suppose that $\mathbf{v}_{j+1} \cdot \mathbf{v}_k = 0$ for $k = 1, \ldots, j$. What now happens at this $(j+1)$st step in Gram–Schmidt?

[Hint: *think about the third step in Example 41.4.*]

That the Gram–Schmidt procedure "preserves spans" is probably not a consequence that we expected when we originally started out with an independent list and wanted to get an orthonormal list with the same span as the whole list. Sometimes accidental consequences are nice.

Look at the $n = 3$ situation. We have an independent list $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^m$ and orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3 \in \mathbb{R}^m$ such that the spans are preserved:

$$\begin{cases} \mathbf{v}_1 \in \operatorname{span}(\mathbf{q}_1) \\ \mathbf{v}_2 \in \operatorname{span}(\mathbf{q}_1, \mathbf{q}_2) \\ \mathbf{v}_3 \in \operatorname{span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3). \end{cases}$$

Since the $\mathbf{q}_k$ are orthonormal, we have the expansions

$$\begin{cases} \mathbf{v}_1 = (\mathbf{v}_1 \cdot \mathbf{q}_1)\mathbf{q}_1 \\ \mathbf{v}_2 = (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{v}_2 \cdot \mathbf{q}_2)\mathbf{q}_2 \\ \mathbf{v}_3 = (\mathbf{v}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{v}_3 \cdot \mathbf{q}_2)\mathbf{q}_2 + (\mathbf{v}_3 \cdot \mathbf{q}_3)\mathbf{q}_3. \end{cases}$$

The very intentional typesetting should reveal a "triangular" structure.

Work backwards:

$$\mathbf{v}_1 = (\mathbf{v}_1 \cdot \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{q}_1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{v}_2 = (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{v}_2 \cdot \mathbf{q}_2)\mathbf{q}_2 = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_2 \cdot \mathbf{q}_1 \\ \mathbf{v}_2 \cdot \mathbf{q}_2 \end{bmatrix},$$

and

$$\mathbf{v}_3 = (\mathbf{v}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{v}_3 \cdot \mathbf{q}_2)\mathbf{q}_2 + (\mathbf{v}_3 \cdot \mathbf{q}_3)\mathbf{q}_3 = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_3 \cdot \mathbf{q}_1 \\ \mathbf{v}_3 \cdot \mathbf{q}_2 \\ \mathbf{v}_3 \cdot \mathbf{q}_3 \end{bmatrix}.$$

Then

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} (\mathbf{v}_1 \cdot \mathbf{q}_1) & (\mathbf{v}_2 \cdot \mathbf{q}_1) & (\mathbf{v}_3 \cdot \mathbf{q}_1) \\ 0 & (\mathbf{v}_2 \cdot \mathbf{q}_2) & (\mathbf{v}_3 \cdot \mathbf{q}_2) \\ 0 & 0 & (\mathbf{v}_3 \cdot \mathbf{q}_3) \end{bmatrix}.$$

Put

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}, \qquad Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix}, \qquad \text{and} \qquad R = \begin{bmatrix} (\mathbf{v}_1 \cdot \mathbf{q}_1) & (\mathbf{v}_2 \cdot \mathbf{q}_1) & (\mathbf{v}_3 \cdot \mathbf{q}_1) \\ 0 & (\mathbf{v}_2 \cdot \mathbf{q}_2) & (\mathbf{v}_3 \cdot \mathbf{q}_2) \\ 0 & 0 & (\mathbf{v}_3 \cdot \mathbf{q}_3) \end{bmatrix}$$

to see that we have factored the matrix A (which has independent columns) into the product $A = QR$, with Q orthogonal and R upper-triangular. In fact, the diagonal entries of R are positive (not just nonzero) by Problem 41.3.

We have a lot of recent knowledge about why orthogonal matrices are nice, and we have a lot of past knowledge about why upper-triangular matrices with nonzero diagonal entries are nice. We put all of that together with this "QR-factorization" to obtain the ultimate form of least squares.

Content from Strang's *ILA* 6E. Page 182 develops the QR-factorization for a matrix with three independent columns.

41.6 Theorem (QR-factorization). *Let $A \in \mathbb{R}^{m\times n}$ have independent columns (so A has full column rank: $\operatorname{rank}(A) = n$). There exist an orthogonal matrix $Q \in \mathbb{R}^{m\times n}$ and an upper-triangular matrix $R \in \mathbb{R}^{n\times n}$ such that $A = QR$. Specifically, the columns of Q are the vectors constructed from the columns of A by the Gram–Schmidt procedure, and the (i,j)-entry of R is $\mathbf{a}_j \cdot \mathbf{q}_i$, where $\mathbf{a}_j$ is the jth column of A, and $\mathbf{q}_i$ is the ith column of Q.*

We proved the $n = 3$ case of this. The general proof just hinges on (1) the orthonormality of the vectors produced by Gram–Schmidt, (2) the "span preservation property" that $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j) = \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_j)$ for each j, not just $j = n$, and (3) Problem 41.3 to get the positive diagonal entries in R.

41.7 Example. Let

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

We performed Gram–Schmidt on the columns of A in Example 41.4. Collect the Gram–Schmidt output in

$$Q = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}.$$

If we have forgotten the coefficients from the Gram–Schmidt work, we can compute them quickly (and we only do this for R_{ij} with $j \ge i$):

$$R_{11} = \mathbf{a}_1 \cdot \mathbf{q}_1 = \sqrt{5},$$
$$R_{12} = \mathbf{a}_2 \cdot \mathbf{q}_1 = \sqrt{5},$$
$$R_{13} = \mathbf{a}_3 \cdot \mathbf{q}_1 = 0,$$
$$R_{22} = \mathbf{a}_2 \cdot \mathbf{q}_2 = 2,$$
$$R_{23} = \mathbf{a}_3 \cdot \mathbf{q}_2 = 0,$$
$$R_{33} = \mathbf{a}_3 \cdot \mathbf{q}_3 = \sqrt{5}.$$

Then

$$R = \begin{bmatrix} \sqrt{5} & \sqrt{5} & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix},$$

and we have

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

41.8 Problem ($\star$). The QR-factorization is our third major matrix factorization, after CR and LU.

(i) Let $A \in \mathbb{R}^{m\times n}$ have independent columns. What is the CR-factorization of A? When is the R in that factorization the same as the R in the QR-factorization?

(ii) Trefethen & Bau's beautiful *Numerical Linear Algebra* describes Gaussian elimination and the LU-factorization as "triangular triangularization" and the QR-factorization as "triangular orthogonalization" (p. 148). Why do you think Trefethen & Bau chose those phrases to describe LU and QR?

(iii) Given $A \in \mathbb{R}^{m\times n}$, briefly summarize what guarantees the existence of the CR-, LU-, and QR-factorizations. Does a factorization always exist, or do A, m, and n need to satisfy some extra hypotheses?

Day 42: Monday, December 1.

We reviewed the story of the entire course, from $A\mathbf{x} = \mathbf{b}$ to least squares. This was a selective coverage of the topics in the final exam information document available on the course website.

Day 43: Wednesday, December 3.

Here is how the QR-factorization is useful for least squares. Start with $A \in \mathbb{R}^{m\times n}$ with independent columns and factor $A = QR$ with $Q \in \mathbb{R}^{m\times n}$ orthogonal and $R \in \mathbb{R}^{n\times n}$ upper-triangular with positive diagonal entries.

If $\mathbf{b} \notin \mathbf{C}(A)$, then solving $A\widehat{\mathbf{x}} = P_A\mathbf{b}$ is the next best thing to solving the unsolvable problem $A\mathbf{x} = \mathbf{b}$. While we do have a formula for $\widehat{\mathbf{x}}$, the annoying thing is that it requires computing the inverse $(A^\mathsf{T}A)^{-1}$. Better to solve a linear system than compute an inverse, and, from (39.1), solving $A\widehat{\mathbf{x}} = P_A\mathbf{b}$ is equivalent to solving the normal equation

$$(A^\mathsf{T}A)\widehat{\mathbf{x}} = A^\mathsf{T}\mathbf{b}. \tag{43.1}$$

We could always use Gaussian elimination to solve (43.1). Again, better to solve a linear system than compute an inverse.

But this system (43.1) is pretty nice after the QR-factorization. Since $A = QR$, we have $A^\mathsf{T} = (QR)^\mathsf{T} = R^\mathsf{T}Q^\mathsf{T}$, and since Q is orthogonal, we have $Q^\mathsf{T}Q = I_n$. Then

$$A^\mathsf{T}A = (R^\mathsf{T}Q^\mathsf{T})(QR) = R^\mathsf{T}(Q^\mathsf{T}Q)R = R^\mathsf{T}I_nR = R^\mathsf{T}R.$$

The problem (43.1) now reads

$$R^\mathsf{T} R\widehat{\mathbf{x}} = R^\mathsf{T} Q^\mathsf{T}\mathbf{b}. \tag{43.2}$$

This is great! Since R is upper-triangular with positive diagonal entries, R^T is lower-triangular with positive diagonal entries.

43.1 Problem ($\star$). Prove that. [Hint: *recall that $R^\mathsf{T}_{ij} = R_{ji}$. Since R has positive diagonal entries, $R_{ii} > 0$. Since R is upper-triangular, $R_{ij} = 0$ for $i > j$. To show that R^T is lower-triangular, you want $R^\mathsf{T}_{ij} = 0$ for $j > i$. Is this true?*]

Any triangular matrix with nonzero diagonal entries is invertible, so R^T is invertible. Then (43.2) is just

$$R\widehat{\mathbf{x}} = Q^\mathsf{T}\mathbf{b}.$$

Again, R is upper-triangular with positive diagonal entries, so we can solve this system by back-substitution (no need even for Gaussian elimination!) No inverses anywhere in the actual calculations, just in the theory.

Content from Strang's *ILA* 6E. Pages 182–183 discuss how to use the QR-factorization in least squares. Read the second half of p. 185, which summarizes everything. You don't need to read about the pseudoinverses.

Then read about the "victory of orthogonality" on p. 197. Stop with #5 for now, and there just be able to explain why if $Q \in \mathbb{R}^{n\times n}$ is orthogonal, so is any power Q^k for $k \geq 1$.

In the paragraph after that, note the "sum of squares definition of length." There are many valid, meaningful ways of defining the length of a vector (pp. 355–356), but the way that interacts best with the dot product is saying length is $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$. As you think about least squares, keep in mind how length and dot product interact so nicely.

43.2 Problem (!). Let $A \in \mathbb{R}^{m\times n}$ have full column rank, and let $A = QR$ be its QR-factorization.

(i) Suppose that $A\mathbf{x} = \mathbf{b}$. Show then that $R\mathbf{x} = Q^\mathsf{T}\mathbf{b}$, as expected.

(ii) Conversely, we can always solve $R\mathbf{x} = Q^\mathsf{T}\mathbf{b}$. (Why?) Why does multiplying by Q not necessarily get us back to $A\mathbf{x} = \mathbf{b}$?

Courses tell stories, and sometimes stories have abrupt plot twists. Most of the story of this course has been solving, then understanding, and finally approximating $A\mathbf{x} = \mathbf{b}$. We have introduced a significant amount of vocabulary, notation, and technology to do this. Now the time has come to ask a different question, mostly about A, and less about $A\mathbf{x} = \mathbf{b}$ (although we will still think about that).

Matrices are *static* and *dynamic*: they *encode* data and they *act* on data. Namely, matrices act by multiplying other matrices and vectors (matrix-matrix multiplication is, of course, matrix-vector multiplication done repeatedly). When we have to choose between what is right and what is easy in math, we always want what is easy. What is the easiest action of a matrix? This is a little subjective, perhaps the easiest action involves as little multiplication involved as possible. What if $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$? (The Greek letter λ is culturally traditional.) Then multiplying by A is just scalar multiplication by λ.

The matrix λI_n is diagonal with constant diagonal. Maybe the next simplest matrix is still diagonal but has a nonconstant diagonal, say,

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

If $\mathbf{v} = c\mathbf{e}_j$ for some $c \in \mathbb{R}$ and $j = 1, 2, 3$, then $A\mathbf{v} = c\lambda_j \mathbf{e}_j$. For these special vectors, A still acts like scalar multiplication.

The right question to ask in our search for simple matrix operations is where/when/how does the matrix act just as scalar multiplication? If $A \in \mathbb{R}^{n \times n}$, are there $\mathbf{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}?$$

43.3 Problem (!). Explain why the equality above only makes sense for square A.

The boring answer is yes when $\mathbf{v} = \mathbf{0}_n$. For this reason, we make the following restriction.

43.4 Definition. *Let $A \in \mathbb{R}^{n \times n}$. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is an* **EIGENVECTOR** *of A corresponding to the* **EIGENVALUE** *$\lambda \in \mathbb{R}$ if*

$$A\mathbf{v} = \lambda\mathbf{v}.$$

43.5 Remark. **(i)** *Here is some linguistic commentary from a footnote on p. 69 of the excellent Linear Algebra by Meckes and Meckes: "[The words 'eigenvector' and 'eigenvalue'] are halfway translated from the German words "Eigenvektor" and "Eigenwert." The German adjective 'eigen' can be translated as 'own' or 'proper,' so an eigenvector of [a matrix] is something like 'the matrix's very own vector."' Because of these German origins, Tefethen and Bau's equally excellent Numerical Linear Algebra suggests abbreviating "eigenvector" by "ev" and eigenvalue" by "ew."*

(ii) *We are back to square matrices now (Problem 43.3). Earlier in the course, when we focused on elimination and inversion, we said that our square matrices were $m \times m$. That notation was helpful because we were working row-by-row with elimination, and there are m rows. Now we are thinking about matrices very dynamically (not the first time we've done that, to be fair), and so we emphasize that the matrices are $n \times n$ because matrices act on other matrices by acting on the columns.*

The eigenvalues and eigenvectors of a square matrix encode a huge amount of information about it, and this will become apparent over time. Here is one quick application. Say that we want to compute "matrix powers": A^k for $k \geq 2$. Here $A^2 = AA$, $A^3 = A^2A = AAA$, and so on. If $\mathbf{v}$ is an eigenvector corresponding to λ, then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v},$$

and, more generally,

$$A^k\mathbf{v} = \lambda^k\mathbf{v}.$$

This is vastly easier than computing A^k and then multiplying $A^k\mathbf{v}$.

Content from Strang's *ILA* 6E. Read the first two paragraphs on p. 216 and then all of p. 217.

This is all of the groundwork that we need to appreciate eigenvalues. The best situation is when a matrix has n independent eigenvectors. This definitely happens if the matrix has n independent eigenvalues, but that is not strictly necessary (I_n has only one eigenvalue but $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are all eigenvectors).

We start small. Say that $A \in \mathbb{R}^{3\times 3}$ has 3 independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3 \in \mathbb{C}^3$. This means $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for some $\lambda_k \in \mathbb{C}$, and also $\mathbf{v}_k \neq \mathbf{0}_3$. We are not requiring λ_1, λ_2, and λ_3 to be distinct, although that does not hurt. Slap it all together in a matrix:

$$\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \lambda_3\mathbf{v}_3 \end{bmatrix}.$$

Having made it this far in life, w know how to factor:

$$A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

(Multiplying by a diagonal matrix on the right = scaling the columns.)

Now abbreviate

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

so that we have

$$AV = V\Lambda.$$

Since $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are independent, V is invertible. We conclude

$$A = V\Lambda V^{-1}.$$

Nothing here was special about $n = 3$, and the following holds more generally.

43.6 Theorem (Diagonalization). *Let $A \in \mathbb{C}^{n\times n}$ have n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{C}^n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. (That is, $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$.*

We do not require the λ_k to be distinct.) Put

$$V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \quad \textit{and} \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Then

$$A = V\Lambda V^{-1}.$$

43.7 Definition. *A matrix $A \in \mathbb{C}^{n\times n}$ is* **DIAGONALIZABLE** *if there exist an invertible matrix $V \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ such that $A = V\Lambda V^{-1}$.*

43.8 Problem ($\star$). Prove that if A is diagonalizable, then the columns of V are eigenvectors of A corresponding to the eigenvalues given by the diagonal elements of Λ. [Hint: $A = V\Lambda V^{-1}$ *means* $AV = \Lambda V$.] In particular, this means that if $A \in \mathbb{R}^{n\times n}$ is diagonalizable, then we can find an independent list of length n whose entries are eigenvectors of A.

Day 44: Friday, December 5.

All of the elementary row operations and the calculations in the Gram–Schmidt procedure have been, fundamentally, *arithmetic*: adding and multiplying numbers. Finding eigenvalues, however, will fundamentally be a *transcendental* operation that we will not be able to resolve, except in very special cases, with a neat and finite algorithm.

Our first impression might be that the "eigenproblem" $A\mathbf{v} = \lambda\mathbf{v}$ is too hard because it contains two kinds of unknowns: the vector $\mathbf{v} \in \mathbb{R}^n$ and the scalar $\lambda \in \mathbb{R}$. What usually happens is that we find the eigenvalues first and then $A\mathbf{v} = \lambda\mathbf{v}$ becomes the matrix-vector equation

$$(A - \lambda I_n)\mathbf{v} = \mathbf{0}_n. \tag{44.1}$$

We know how to check if this has a nonzero solution. The greater utility of (44.1) is that is tells us what happens when λ is an eigenvalue: $\mathbf{N}(A - \lambda I_n) \neq \{\mathbf{0}_n\}$, so $A - \lambda I_n$ is not invertible.

44.1 Example. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

We want to find $\lambda \in \mathbb{R}$ such that $A - \lambda I_2$ is not invertible. We have

$$A - \lambda I_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 1 \\ 0 & (2-\lambda) \end{bmatrix}.$$

This last matrix is upper-triangular, so we know it not invertible (precisely) when a diagonal entry is zero. This happens when $1 - \lambda = 0$ or $2 - \lambda = 0$. Thus the eigenvalues are $\lambda = 1$ and $\lambda = 2$. Surely it no coincidence that the diagonal entries are the eigenvalues.

Just for practice, we find eigenvectors corresponding to the eigenvalue 1. We want to find $\mathbf{v} \in \mathbb{R}^2$ such that $(A - I_2)\mathbf{v} = \mathbf{0}_2$, and we want $\mathbf{v} \neq \mathbf{0}_2$ to keep things interesting. We look at

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v} = 1\mathbf{v}.$$

This becomes the linear system

$$\begin{cases} v_2 = v_1 \\ v_2 = v_2. \end{cases}$$

The second equation tells us nothing useful (of course $v_2 = v_2$, what else would it equal?), while the first tells us

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So, the eigenvectors corresponding to the eigenvalue 1 are all scalar multiples of $\mathbf{v} = (1, 1)$.

Here is the generalization of the eigenvalue result from this example.

44.2 Theorem. *Let $A \in \mathbb{R}^{n \times n}$ be triangular. Then the eigenvalues of A are the diagonal entries of A.*

44.3 Problem ($\star$). Prove this theorem when A is upper-triangular.

44.4 Problem ($\star$). Prove that $A \in \mathbb{R}^{n \times n}$ is not invertible if and only if 0 is an eigenvalue of A.

The eigenvector calculation in Example 44.1 revealed the eigenvectors corresponding to a particular eigenvalue as the vectors in a certain span. This generalizes nicely.

44.5 Problem ($\star$). Suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$. The EIGENSPACE of A corresponding to λ is

$$\mathbf{E}(A, \lambda) := \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}.$$

The GEOMETRIC MULTIPLICITY of λ as an eigenvalue of A is $g(A, \lambda) := \dim[\mathbf{E}(A, \lambda)]$.

(i) Prove that $\mathbf{E}(A, \lambda)$ is a subspace of $\mathbb{R}^n$. [Hint: *for extra practice, try proving this in two ways: from Definition 43.4 alone and by thinking about null spaces.*]

(ii) Is every vector in $\mathbf{E}(A, \lambda)$ an eigenvector of A?

Now for some bad news.

44.6 Problem ($\star$). **(i)** By considering

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

explain why A and its RREF need not have the same eigenvalues.

(ii) Suppose that A, $E \in \mathbb{R}^{n \times n}$ and $U = EA$ is upper-triangular. Do you expect A and U to have the same eigenvalues?

44.7 Example. Let

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

We could think about $A - \lambda I_2$, or we could think about what an eigenvector $\mathbf{v}$ and an eigenvalue λ *do*. They satisfy $A\mathbf{v} = \lambda\mathbf{v}$. We compute

$$A\mathbf{v} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2(v_1 + v_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This almost looks like $A\mathbf{v} = \lambda\mathbf{v}$, if we pick λ and $\mathbf{v}$ correctly. The vector on the right is $(1, 1)$, so if we try $\mathbf{v} = (1, 1)$, then we get $A\mathbf{v} = 4\mathbf{v}$. This says that 4 is an eigenvalue with eigenvector $(1, 1)$.

Is this the only one? We think about (44.1) now. We want $A - \lambda I_2$ to fail to be invertible, and it looks like A is already not invertible, since its columns are dependent. We can get just A to show up in $A - \lambda I_2$ by taking $\lambda = 0$. Will this be an eigenvalue? Does $A\mathbf{v} = \mathbf{0}_2$ have a nontrivial solution? Sure: $\mathbf{v} = (1, -1)$. This says that 0 is an eigenvalue with eigenvector $(1, -1)$.

Here are more generalizations of the previous example.

44.8 Problem ($\star$). Let $A \in \mathbb{R}^{n \times n}$ such that the sum of the entries in any row of A is always the same value $s \in \mathbb{R}$.

(i) Prove that s is an eigenvalue of A. [Hint: *for what* $\mathbf{v} \in \mathbb{R}^n$ *does* $A\mathbf{v}$ *involve adding the entries in each row?*]

(ii) Is 0 always an eigenvalue of A?

Content from Strang's *ILA* 6E. Read Examples 2 and 3 on pp. 221–222.

We have seen some special cases of eigenexistence. (Every word is better with "eigen" in front.) It turns out that every matrix has at least one eigenvalue. . . it just may not be real.

This is easiest to see at the 2×2 level. Let

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $A - \lambda I_2$ is not invertible. While we have not stressed determinants in this course, a 2×2 matrix is not invertible if and only if its determinant is 0:

$$\det\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = ps - rq.$$

So, $A - \lambda I_2$ is not invertible if and only if

$$0 = \det(A - \lambda I_2) = \det\left(\begin{bmatrix} (a-\lambda) & c \\ b & (d-\lambda) \end{bmatrix}\right) = (a-\lambda)(d-\lambda) - cb.$$

The product $(a-\lambda)(d-\lambda) - cb$ is just a quadratic in λ, and we have years of experience studying that:

$$(a-\lambda)(d-\lambda) - cb = ad - a\lambda - d\lambda + \lambda^2 - cb = \lambda^2 - (a+d)\lambda + (ad - bc).$$

This has a nice structure: the coefficient $a + d$ is the sum of the diagonal entries of A, which is its **TRACE**, denoted $\mathrm{tr}(A)$. And $ad - bc$ is its determinant. So, λ is an eigenvalue of A if and only if

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0. \tag{44.2}$$

This is the **CHARACTERISTIC EQUATION** of A, and the quadratic on the left is the **CHARACTERISTIC POLYNOMIAL** of A.

44.9 Problem (!). Revisit the matrices in Examples 44.1 and 44.7 and compute their eigenvalues by finding the roots of their characteristic polynomials.

Content from Strang's *ILA* 6E. Read pp. 220–222 on determinants. For now, just assume that A is 2×2 throughout. Then read Worked Example 6.1 A on pp. 224–225. Can you prove in general the statements about the eigenvalues and eigenvectors of A^2, A^{-1}, and $A + cI$ (with $c \in \mathbb{R}$), relative to the eigenvalues and eigenvectors of A?

We have years of painful experience with solving quadratic equations. In particular, they may not always have two distinct real solutions, and so we should not expect a 2×2 matrix always to have two distinct real eigenvalues.

44.10 Example. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This is upper-triangular with only 1's on the diagonal, so the only eigenvalue is 1. If

$A\mathbf{v} = \mathbf{v}$, then

$$\begin{cases} v_1 + v_2 = v_1 \\ v_2 = v_2 \end{cases}$$

The second equation tells us nothing useful, but the first collapses to $v_2 = 0$, so

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We can see this at the level of the characteristic equation by computing

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} (1-\lambda) & 1 \\ 0 & (1-\lambda) \end{bmatrix}\right) = (1-\lambda)^2,$$

which has only $\lambda = 1$ as its (repeated real) root.

44.11 Problem (!). For an arbitrary integer $n \geq 1$, give an example of a matrix $A \in \mathbb{R}^{n \times n}$ with only one eigenvalue $\lambda \in \mathbb{R}$ but such that $\dim(\mathbf{E}(A, \lambda)) = n$.

And quadratic equations may not always have real roots, even when the coefficients are real.

44.12 Example. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1.$$

The quadratic equation $\lambda^2 + 1 = 0$ has $\lambda = \pm i$ as solutions. Here i is the **COMPLEX NUMBER** such that $i^2 = -1$.

We can find an eigenvector for i just as before: solve $A\mathbf{v} = i\mathbf{v}$. This becomes the system

$$\begin{cases} -v_2 = iv_1 \\ v_1 = iv_2. \end{cases}$$

The second equation is slightly easier and gives

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} iv_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

So, all eigenvectors of A corresponding to i are scalar multiples of $(i, 1)$.

44.13 Problem (!). With A from the previous example, find all eigenvectors corresponding to $-i$.

These examples demonstrate the need for a more generous notion of eigenvalue and eigenvector than afforded by Definition 43.4. Let

$$\mathbb{C} = \left\{ x + iy \mid x,\ y \in \mathbb{R},\ i^2 = -1 \right\}.$$

We add and multiply numbers in $\mathbb{C}$ by following our noses, combining like terms, and using $i^2 = -1$:

$$(1+2i) + (3+4i) = (1+3) + (2i+4i) = 4 + 6i$$

and

$$(1+2i)(3+4i) = 1 + 4i + 6i + 8i^2 = (1-8) + 10i = -7 + 10i.$$

If $x \in \mathbb{R}$ with $x < 0$, we follow the convention that

$$\sqrt{x} := i\sqrt{|x|}.$$

For example,

$$\sqrt{-4} = i\sqrt{|-4|} = i\sqrt{4} = 2i.$$

This is how we will use the square root in the quadratic formula with negative input.

Let $\mathbb{C}^n$ be the set of all column vectors with entries in $\mathbb{C}$. We add vectors in $\mathbb{C}^n$ componentwise and multiply by scalars in $\mathbb{C}$ componentwise, too. Let $\mathbb{C}^{m \times n}$ be the set of all $m \times n$ matrices with entries in $\mathbb{C}$. For $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$, we still multiply AB as we did when the entries were real (except now $i^2 = -1$). We define a subspace of $\mathbb{C}^n$ exactly as we did a subspace of $\mathbb{R}^n$, except we replace any use of the symbols $\mathbb{R}$ or $\mathbb{R}^n$ with $\mathbb{C}$ or $\mathbb{C}^n$. Column and null spaces of a matrix in $\mathbb{C}^{m \times n}$ are still subspaces.

Nothing changes in the arithmetic.

Except $i^2 = -1$.

44.14 Definition (Improvement of Definition 43.4). *A scalar $\lambda \in \mathbb{C}$ is an* **EIGENVALUE** *of the matrix $A \in \mathbb{C}^{n \times n}$ corresponding to the* **EIGENVECTOR** $\mathbf{v} \in \mathbb{C}^n$ *if*

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Content from Strang's *ILA* 6E. For a refresher on complex numbers, look at pp. 262–263. Then Read "Imaginary Eigenvalues on p. 223 and Worked Example 6.1 B on p. 225.