Set Theory and Quantifiers

This document outlines essential concepts, vocabulary, and notation that we will use frequently and without much comment.

Content from *Linear Algebra* by Meckes & Meckes. This material corresponds to pp. 378–379 of Appendix A.1 on sets (skip functions for now) and pp. 384–386 of Appendix A.1 on logical connectives and quantifiers; see also the bullet points on p. 388 on negating statements. We will review the material on functions (pp. 379–382) of Appendix A.1 as we need it. You may find the material on pp. 386–389 of Appendix A.3 on proof techniques helpful; we will highlight and emphasize the techniques of contrapositive, contradiction, and induction as they arise.

Set theory.

1 Undefinition. *A* **SET** *is a collection of objects, called* **ELEMENTS**. *If x is an element of the set U, then we write $x \in U$, and if y is not an element of the set A, then we write $y \notin U$. We assume that there is a notion of equality among elements of U such that the expression $x = y$ makes sense for $x, y \in U$.*

This is an undefinition, not a definition, because we have not defined what "collections" or "objects" or "equals" really means. And we will not. If a set U consists of only finitely many elements, then we may denote U by listing those elements between curly braces. For example, the set consisting precisely of the numbers 1, 2, and 3 is $\{1, 2, 3\}$; the set consisting precisely of the number 1 is $\{1\}$, and $1 \in \{1\}$.

2 Example. Let $U = \{1, 2, 3\}$. Then $1 \in U$ but $4 \notin U$.

If U is a set, and if $P(x)$ is a statement that is either true or false for each $x \in U$, then we denote the set of all elements x of U for which $P(x)$ is true by

$$\{x \in U \mid P(x)\}.$$

We read the expression $P(x)$ as "it is the case that $P(x)$" or "it is the case that $P(x)$ is true."

3 Example. If $U = \{1, 2, 3, 4\}$, then

$$\{x \in U \mid x \text{ is even}\} = \{2, 4\}.$$

Here $P(x)$ is the statement "x is even."

One of the most important actions that we can perform on a set is to compare it to another set. Frequently we want to show that every element of a given set is an element of another set; this encapsulates the logic of showing that if a given property is true, then another property is true (and this logic is almost all of math).

4 Definition. *A set A is a* **SUBSET** *of a set B if for each $x \in A$, it is the case that $x \in B$. That is, every element of A is an element of B. If A is a subset of B, we write $A \subseteq B$; if A is not a subset of B, then we write $A \nsubseteq B$.*

In symbols,

$$A \subseteq B \iff (x \in A \Longrightarrow x \in B).$$

5 Example. $\{1,2\} \subseteq \{1,2,3\}$ and $\{1,2,3\} \subseteq \{1,2,3\}$, but $\{1,2\} \nsubseteq \{1,3\}$.

A special instance of comparing sets is determining when they are the same, or equal.

6 Definition. *Two sets A and B are* **EQUAL**, *written $A = B$, if every element of A is an element of B, and if every element of B is an element of A. (And so $A = B$ precisely when both $A \subseteq B$ and $B \subseteq A$.) In symbols,*

$$A = B \iff A \subseteq B \text{ and } B \subseteq A \iff (x \in A \iff x \in B).$$

7 Hypothesis. **(i)** *There exists a set $\varnothing$ that contains no element. That is, if x is an element of any set U, then $x \notin \varnothing$. We call this set the* **EMPTY SET**.

(ii) *Let U be a set. An element $x \in U$ cannot be equal to the set $\{x\} \subseteq U$ whose only element is x. That is, $x \neq \{x\}$.*

(iii) *If we define a set by listing its elements within curly braces, repetition or reordering of the elements does not change the set. For example, $\{1,2,3\} = \{1,2,3,1\} = \{1,3,2\}$.*

There are several fundamental, almost "algebraic" ways in which two or more sets interact.

8 Definition. *Let U be a set and A, $B \subseteq U$. The* **UNION** *of A and B is the set*

$$A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\},$$

the **INTERSECTION** *of A and B is the set*

$$A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\},$$

and the **COMPLEMENT** *of A in B is the set*

$$B \setminus A := \{x \in B \mid x \notin A\}.$$

That is, $A \cup B$ is the set of all elements in either A or B (or both), $A \cap B$ is the set of all elements in both A and B, and $B \setminus A$ is the set of all elements in B but not in A.

9 Example. Let

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{2, 4, 6\}.$$

Then

$$A \cup B = \{1, 2, 3, 4, 6\},$$

$$A \cap B = \{2\},$$

and

$$B \setminus A = \{4, 6\}.$$

10 Problem (!). Let A and B be as in Example 9. Determine the elements of the following sets.

(i) $A \setminus B$

(ii) $(A \setminus B) \cup B$

(iii) $(A \cap B) \setminus A$

(iv) $A \setminus \varnothing$

(v) $\varnothing \setminus B$

We have agreed that when listing the elements of a (finite) set in curly braces, order does not matter: $\{1, 2\} = \{2, 1\}$. However, there are situations in which a notion of order is essential. One way to accomplish this is via the concept of the ordered pair, which is fundamental to a rigorous definition of function and to constructing many interesting and useful sets out of existing sets (often these interesting and useful constructs are sets of functions!).

Quantifiers.

Quantifiers tell us how elements of different sets may interact with each other and with other overarching properties of these sets. We use the symbols $\forall$, $\exists$, and $\exists!$ to abbreviate three very common phrases. Let A be a set and, for $x \in A$, let $P(x)$ be a property that is either true or false.

- The string of symbols $\forall x \in A : P(x)$ is read as "for all $x \in A$ it is the case that $P(x)$ is true." For the statement $\forall x \in A : P(x)$ to be true, we need to show that picking any $x \in A$ results in $P(x)$ being true.

- The string of symbols $\exists x \in A : P(x)$ is read as "there exists $x \in A$ such that $P(x)$ is true." For the statement $\exists x \in A : P(x)$ to be true, we just need to show that $P(x)$ is true for at least one $x \in A$. Perhaps $P(x)$ is true for all $x \in A$; then, certainly, $P(x)$ is true for one $x \in A$. (And so $\forall x \in A : P(x) \Longrightarrow \exists x \in A : P(x)$.)

- The string of symbols $\exists! x \in A : P(x)$ is read as "there exists a unique $x \in A$ such that

$P(x)$ is true." For $\exists! x \in A : P(x)$ to be true, we need to show that $P(x)$ is true for one $x \in A$ and no other. Often we do this in two steps. First prove existence: show that $P(x)$ is true for at least one $x \in A$. Then prove uniqueness: assume that $P(x_1)$ and $P(x_2)$ are true for some $x_1,\ x_2 \in A$, and then show that $x_1 = x_2$, whatever "=" means in the context at hand. Thus

$$(\exists! x \in A : P(x)) \iff \\ ((\exists x \in A : P(x) \text{ and } P(x_1)) \text{ and } (P(x_2) \text{ are true for } x_1,\ x_2 \in A \Longrightarrow x_1 = x_2).$$

11 Example. **(i)** The statement "For all real numbers x it is the case that x^2 is nonnegative" compresses to

$$\forall x \in \mathbb{R} : x^2 \geq 0.$$

Here $\mathbb{R}$ denotes the set of all real numbers. This is a true statement.

(ii) The statement "There exists a real number x such that $x^2 = 4$" compresses to

$$\exists x \in \mathbb{R} : x^2 = 4.$$

This too is true.

(iii) The statement "There exists a unique positive real number x such that $x^2 = 4$" compresses to

$$\exists! x \in (0, \infty) : x^2 = 4.$$

And this is also true, although the statement $\exists! x \in \mathbb{R} : x^2 = 4$ is false. In the first part of the compressed symbolic form, we could also have written

$$\exists! x > 0 : x^2 = 4$$

and used the equivalence of $x \in (0, \infty)$ and $x > 0$ to phrase things differently.

We can chain quantifiers together as much as necessary, and we will not be too picky about saying "such that" every single time we write in English words.

12 Example. **(i)** The statement "For all real numbers x, there is a real number y such that there is a unique positive real number z with $z^2 = xy$" compresses to

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \exists! z > 0 : z^2 = xy.$$

This turns out to be a true statement (think about the three separate cases $x > 0$, $x = 0$, and $x < 0$).

(ii) The statement "For all real numbers x, if x is nonnegative, then there exists a unique nonnegative y such that $y^2 = x$" compresses to

$$\forall x \in \mathbb{R} : x \geq 0 \Longrightarrow \exists! y \geq 0 : y^2 = x.$$

However, a shorter, and (importantly!) still logically equivalent version of the above is

$$\forall x \geq 0 \ \exists! y \geq 0 : y^2 = x.$$

When writing a statement with symbolic quantifiers, we do not always need to replicate verbatim every part of that statement, as long as we have a logically equivalent form.

13 Problem (⋆). Translate each of the following statements into (a logically equivalent) symbolic form using ∀, ∃, and/or ∃! whenever possible.

(i) For all $x \in \mathbb{R}$ and all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in I$ with $|x - y| < \delta$, then

$$|x^2 - y^2| < \epsilon.$$

(ii) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \mathbb{R}$ with $|x - y| < \delta$, then

$$|x^2 - y^2| < \epsilon.$$

(iii) There exist $x \in \mathbb{R}$ and $\delta > 0$ such that if $y \in I$ with $|x - y| < \delta$, then

$$|y|^2 > \frac{x^2}{2}.$$

(iv) For all $a, b \in \mathbb{R}$ and $c \in \mathbb{R}$ such that $a^2 < c < b^2$, there exists $x \in \mathbb{R}$ such that $x^2 = c$.

(v) For all $y \in \mathbb{R}$ there exists a unique $x \in I$ such that $x^2 = y$.

When *negating* quantified statements, it can be helpful to write them out symbolically and then formally flip each ∀ to ∃ and each ∃ to ∀. More precisely, the negation of the statement $\forall x : P(x)$ is $\exists x :\sim P(x)$, where $\sim P(x)$ is an abbreviation for the statement "it is not the case that $P(x)$" or "it is not the case that $P(x)$ is true" (or "it is the case that $P(x)$ is false"). Likewise, the negation of $\exists x : P(x)$ is $\forall x :\sim P(x)$. There is no standard way to flip ∃! in a negation, as the negation of unique existence is either nonunique existence or nonexistence.

14 Example. We abbreviate the statement "For all real numbers x, there is a real number y such that the product xy is nonnegative" by "$\forall x \in \mathbb{R} \ \exists y \in \mathbb{R} : xy \geq 0$." Symbolically, its negation is $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : xy < 0$." In words, this negation reads "There is $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ the product xy is negative." (Which is true, the original statement or its negation?)

Frequently we need to negate quantified statements that are couched in "if-then" language. Recall that the statement "If P, then Q" is true if the statement "It is not the case that P is true and Q is false." (This reflects the intuition behind the compression $P \implies Q$: we want the truth of P to force the truth of Q.) So, the negation of "If P, then Q" is "P and not Q." Negating that Q often involves manipulating quantifiers.

15 Example. Consider the statement "If x is a real number, then there exists a real number y such that $y^3 = x$." (This number y is actually unique, but we will not include the uniqueness quantifier here to make negation easier.) One way to compress this symbolically is

$$x \in \mathbb{R} \Longrightarrow \exists y \in \mathbb{R} : y^3 = x.$$

The negation of this statement is then

$$x \in \mathbb{R} \text{ and } \forall y \in \mathbb{R} : y^3 \neq x.$$

16 Problem ($\star$). Negate all but the last quantified symbolic statements from Problem 13 and then write out those negations as complete sentences using English words and none of the symbols $\forall$, $\exists$, and/or $\exists!$.