Reflection Activity

Submit responses to the following questions to the `1` slot on D2L by 11:59 pm on Monday, February 16. Any cogent response will earn you 5 points on the exam; you can earn the other 95 points on the exam itself.

1. (Required.) I like to say that the fundamental problem of the course is solving, or at least understanding, the operator equation $\mathcal{T}v = w$, where $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$ for some vector spaces $\mathcal{V}$ and $\mathcal{W}$ and $w \in \mathcal{W}$. Provide a thorough but concise summary of what we have learned in the course so far about this problem. You might consider discussing the definitions of linear operators and vector spaces, properties of the operator space $\mathbf{L}(\mathcal{V}, \mathcal{W})$, how the set-theoretic properties of injectivity and surjectivity address the solvability of the operator equation and how these properties manifest themselves linearly via kernels and ranges, and the role of eigenvalues in the special case $\mathcal{V} = \mathcal{W}$. Your answer should be accessible to a classmate who has been keeping up with the material and to you if you reread it an hour after you wrote it.

2. (Required.) What have you found most difficult or confusing in the course so far? Write it down explicitly. Then think hard about this concept for at least half an hour—go back over your notes, the daily log, and the textbook and reread and rework material related to this sticky topic. How do you feel now?

3. (Optional.) What would you like to discuss during our review in class on Wednesday, February 18? Please be as specific as possible and, if you can, point to numbered items in the daily log, problems from problem sets, or content in the textbook.

Exam Content

The exam will cover material from Days 1 through 13 in the daily log (but not material about finite-dimensional vector spaces starting on Day 13; the last topic from Day 13 for you to review is Problem 13.6). Specifically, the exam will test your ability to do the following. All numbered references below are to the daily log.

1. Provide definitions, examples, and, as appropriate, nonexamples for all vocabulary indicated at the start of daily material in the log on Days 1 through 12. Not every day has required vocabulary, and not all definitions and terms within the daily log are candidates for the exam.

I have specified in the daily log which terms need nonexamples by marking them (N); not all do. You can probably find easy examples and (as needed) nonexamples within the daily log and the textbook, and I encourage you to memorize the ones that you find simplest and most meaningful and accessible. Your definitions should be so precise that I should be able to use them to decide whether any mathematical object I ever encounter does or does not meet the properties under consideration.

You may wish to fill out the vocabulary template provided on the course website and continue to update it as you experience the evolving role of prior concepts in the course.

2. Determine if a given subset of a vector space is a subspace of that vector space, and otherwise explain why it is not. Give an example of a subset that is not a subspace.

3. Prove that a given function between vector spaces is a linear operator, and otherwise explain why it is not. Give examples of linear operators and of functions that are not linear operators.

4. Find the eigenvalues and corresponding eigenvectors of a linear operator, or explain why it does not have any.

5. Compute the composition of operators, when defined, and, if applicable, discuss if they commute.

6. Use properties of operator composition and matrix-vector multiplication to discuss the motivation for the definition of matrix multiplication.

7. Determine the matrix representation of a linear operator from $\mathbb{F}^n$ to $\mathbb{F}^m$ with respect to the standard basis vectors.

8. Show that a given linear operator is invertible, or otherwise explain why it is not. There are many ways to do this, including Theorem 11.5 in the daily log, or discussing injectivity and surjectivity separately. You can use any approach that you want.

9. Determine if a given linear operator is injective or surjective, and otherwise explain why it is not. Give examples of operators that are injective but not surjective, surjective but not injective, and neither.

10. Describe as precisely as possible the kernel and range of a given linear operator.

11. Show that two vector spaces are isomorphic by finding an explicit isomorphism between them (and checking that it is indeed an isomorphism).

12. Prove the following results from the daily log: Theorem 9.1, Corollary 9.3, Theorem 9.6, Theorem 11.5, Theorem 12.1, Theorem 12.5, Theorem 13.1. Prove that $\mathbb{F}^{m\times n}$ and $\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ are isomorphic as in part (ii) of Example 12.15.

13. I will not ask any questions about algebras (Definition 9.14 in the daily log). I will not ask you to prove fundamental properties of vector spaces from the axioms as in Example 3.9 in the daily log.

I will provide at least the following four reminders about notation on the front cover of the exam.

- $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ denotes all column vectors of length n with entries in $\mathbb{F}$, and $\mathbb{F}^1 = \mathbb{F}$. $\mathbb{F}^{m\times n}$ denotes all matrices with m rows and n columns and entries in $\mathbb{F}$, and $\mathbb{F}^{m\times 1} = \mathbb{F}^m$.

- $\mathcal{C}(I)$ is the set of all continuous functions on the interval I. $\mathcal{C}^r(I)$ is the set of all r-times differentiable functions on I whose rth derivative is continuous on I. $\mathcal{C}^\infty(I)$ is the set of all infinitely differentiable functions on I.

- $\mathbb{F}^\infty$ is the set of all sequences in $\mathbb{F}$.

- $\mathbf{L}(\mathcal{V}, \mathcal{W})$ is the set of all linear operators from the vector space $\mathcal{V}$ to the vector space $\mathcal{W}$. $\mathbf{L}(\mathcal{V}) = \mathbf{L}(\mathcal{V}, \mathcal{V})$ and $\mathcal{V}' = \mathbf{L}(\mathcal{V}, \mathbb{F})$.

A natural question is how many problems will be on the exam. A numerical answer to this question that does not also discuss the length and difficulty of each problem (which would, more or less, require disclosing the content of each problem) will tell you very little. I expect that most students will need the full allotted time to complete an exam. There is definitely nothing wrong with you if the exam takes you all of the available time.

How to Prepare

Here are some questions for your consideration.

1. Have you completed all of the (!)- and ($\star$)-problems in the lecture notes corresponding to the material above?

2. Have you completed every problem set and checked your solutions carefully?

3. Have you completed every recommended problem from the problem sets?

4. Can you do all these problems with minimal reference to your notes, my notes, the textbook, or any other source?