Reflection Activity

Submit responses to the following questions to the `Exam 2` slot on D2L by 11:59 pm on Monday, April 6. Any cogent response will earn you 5 points on the exam; you can earn the other 95 points on the exam itself.

1. (Required.) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and $\mathcal{T} \in \mathbf{L}(\mathcal{V}, \mathcal{W})$. What more do you know about the fundamental operator equation $\mathcal{T}v = w$ if you know that *at least* one of $\mathcal{V}$ or $\mathcal{W}$ is finite-dimensional (but maybe not that *both* $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional) than you do without any assumptions of finite-dimensionality?

2. (Required.) What have you found most difficult or confusing in the course on Days 13 to 29? Write it down explicitly. Then think hard about this concept for at least half an hour—go back over your notes, the daily log, and the textbook and reread and rework material related to this sticky topic. How do you feel now?

3. (Optional.) What would you like to discuss during our review in class on Wednesday, April 8? Please be as specific as possible and, if you can, point to numbered items in the daily log, problems from problem sets, or content in the textbook.

Exam Content

You will take Exam 2 on April 10. The exam will test material covered in the daily log from Days 13 to 29. You should be able to do the following.

1. Provide definitions, examples, and, as appropriate, nonexamples for all vocabulary indicated at the start of daily material in the log on Days 13through 29. (While I will not explicitly ask about vocabulary from Days 1 through 12, you still need to be competent with that material.) Not every day has required vocabulary, and not all definitions and terms within the daily log are candidates for the exam. You may wish to fill out the vocabulary template provided on the course website and continue to update it as you experience the evolving role of prior concepts in the course.

I have specified in the daily log which terms need nonexamples by marking them (N); not all do. You can probably find easy examples and (as needed) nonexamples within the daily log and the textbook, and I encourage you to memorize the ones that you find simplest and most meaningful and accessible. Your definitions should be so precise that I should be able to use them to decide whether any mathematical object I ever encounter does or does not meet the properties under consideration.

2. Determine if a given list of vectors is dependent or independent.

3. Prove fundamental results about bases and independence: from the daily log, these are Theorem 14.6, Lemma 14.10, Theorem 14.14, Lemma 14.17, and Theorem 20.1.

4. Prove results about injectivity/surjectivity/bijectivity of linear operators using dimension-counting arguments. This includes Theorem 18.8 and Problems 25.5 and 25.6 in the daily log. You should also be familiar with the techniques from Problems 15.3, 15.4, 15.5, and 15.6 in the daily log.

5. Given a basis for a finite-dimensional vector space, prove that the dual basis exists and explain the utility of the dual basis, specifically the identities in (20.3) in the daily log. Prove Lemma 20.7.

6. Prove properties of eigenvalues of operators on finite-dimensional spaces. Specifically, you should be able to prove the results of Example 23.5 and Theorem 24.1 for "small" n, say, $n = 3$ or $n = 4$.

7. Determine if an operator is finite-rank and prove properties of finite-rank operators. This corresponds to most of the results and problems from Day 24 in the daily log.

8. State and prove the rank–nullity theorem. Use rank–nullity to prove that the canonical isomorphism between a finite-dimensional vector space and its double dual is surjective.

9. Describe the vectors in the quotient space and how vector addition and scalar multiplication are defined in the quotient space. Given a specific vector space $\mathcal{V}$, subspace $\mathcal{U}$ of $\mathcal{V}$, and vector $v \in \mathcal{V}$, describe all elements of the coset $v + \mathcal{U}$. Prove Theorems 26.11 and 26.13 in the daily log.

10. Given a vector space $\mathcal{V}$, explain why a particular map from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{F}$ fails to be an inner product.

11. Prove the following results about inner products from the daily log: Theorem 27.10, Lemma 27.17, Theorem 27.19, Theorem 28.4.

12. Write out pseudocode for the Gram–Schmidt process and explain why Gram–Schmidt is an important result. (You will not have to do any actual Gram–Schmidt calculations). Interpret that pseudocode using projections (Definition 29.1).

A natural question is how many problems will be on the exam. A numerical answer to this question that does not also discuss the length and difficulty of each problem (which would, more or less, require disclosing the content of each problem) will tell you very little. I expect that most students will need the full allotted time to complete an exam. There is definitely nothing wrong with you if the exam takes you all of the available time.

I will include at least the following notational reminders on the first page of the exam.

- $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ denotes all column vectors of length n with entries in $\mathbb{F}$, and $\mathbb{F}^1 = \mathbb{F}$. $\mathbb{F}^{m \times n}$ denotes all matrices with m rows and n columns and entries in $\mathbb{F}$, and $\mathbb{F}^{m \times 1} = \mathbb{F}^m$.

- $\mathcal{C}(I)$ is the set of all continuous functions on the interval I. $\mathcal{C}^r(I)$ is the set of all r-times differentiable functions on I whose rth derivative is continuous on I. $\mathcal{C}^\infty(I)$ is the set of all infinitely differentiable functions on I.

- $\mathbb{F}^\infty$ is the set of all sequences in $\mathbb{F}$.

• $\mathbf{L}(\mathcal{V}, \mathcal{W})$ is the set of all linear operators from the vector space $\mathcal{V}$ to the vector space $\mathcal{W}$. $\mathbf{L}(\mathcal{V}) = \mathbf{L}(\mathcal{V}, \mathcal{V})$ and $\mathcal{V}' = \mathbf{L}(\mathcal{V}, \mathbb{F})$.

• If $\mathcal{V}$ is a vector space, then $\mathcal{I}_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{V} \colon v \mapsto v$ is the identity operator. We denote the zero vector in $\mathcal{V}$ by $0_{\mathcal{V}}$.

How to Prepare

Here are some questions for your consideration.

1. Have you completed all of the (!)- and ($\star$)-problems in the daily log corresponding to the material above?

2. Have you completed every problem set and checked your solutions carefully?

3. Have you completed every recommended problem from the problem sets?

4. Can you do all these problems with minimal reference to your notes, my notes, the textbook, or any other source?

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